On the metric dimension of corona product graphs

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Abstract

Given a set of vertices $S = \{v_1, v_2, ..., v_k\}$ of a connected graph G, the metric representation of a vertex v of G with respect to S is the vector $r(v|S) = (d(v, v_1), d(v, v_2), ..., d(v, v_k))$, where $d(v, v_i)$, $i \in \{1, ..., k\}$ denotes the distance between v and v_i. S is a resolving set for G if for every pair of vertices u, v of G, $r(u|S) \neq r(v|S)$. The metric dimension of G, $dim(G)$, is the minimum cardinality of any resolving set for G. Let G and H be two graphs of order n_1 and n_2, respectively. The corona product $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and n_1 copies of H and joining by an edge each vertex from the i^{th}-copy of H with the i^{th}-vertex of G. For any integer $k \geq 2$, we define the graph $G \odot^k H$ recursively from $G \odot H$ as $G \odot^k H = (G \odot^{k-1} H) \odot H$. We give several results on the metric dimension of $G \odot^k H$. For instance, we show that given two connected graphs G and H of order $n_1 \geq 2$ and $n_2 \geq 2$, respectively, if the diameter of H is at most two, then $dim(G \odot^k H) = n_1(n_2 + 1)^{k-1} dim(H)$. Moreover, if $n_2 \geq 7$ and the diameter of H is greater than five or H is a cycle graph, then $dim(G \odot^k H) = n_1(n_2+1)^{k-1}dim(K_1 \odot H)$.

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1 Introduction

The concepts of resolvability and location in graphs were described independently by Harary and Melter [10] and Slater [19], to define the same structure in a graph. After these papers were published several authors developed diverse theoretical works about this topic [3, 4, 5, 6, 7, 16, 18, 20]. Slater described the usefulness of these ideas into long range aids to navigation [19]. Also, these concepts have some applications in chemistry for representing chemical compounds [14, 15] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [17]. Other applications of this concept to navigation of robots in networks and other areas appear in [6, 12, 16]. Some variations on resolvability or location have been appearing in the literature, like those about conditional resolvability [18], locating domination [11], resolving domination [1] and resolving partitions [5, 8, 9, 21]. In this article we study the metric dimension of corona product graphs.

We begin by giving some basic concepts and notations. Let $G = (V, E)$ be a simple graph of order $n = |V|$. Let $u, v \in V$ be two different vertices in G, the distance $d_G(u, v)$ between two vertices u and v of G is the length of a shortest path between u and v. If there is no ambiguity, we will use the notation $d(u, v)$ instead of $d_G(u, v)$. The diameter of G is defined as $D(G) = \max_{u,v \in V}\{d(u, v)\}$. Given $u, v \in V$, $u \sim v$ means that u and v are adjacent vertices. Given a set of vertices $S = \{v_1, v_2, ..., v_k\}$ of a connected graph G, the *metric representation* of a vertex $v \in V$ with respect to S is the vector $r(v|S) = (d(v, v_1), d(v, v_2), ..., d(v, v_k))$. We say that S is a *resolving set* for G if for every pair of distinct vertices $u, v \in V$, $r(u|S) \neq r(v|S)$. The *metric dimension* of G is the minimum cardinality of any resolving set for G, and it is denoted by $dim(G)$.

Let G and H be two graphs of order n_1 and n_2, respectively. The corona product $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and n_1 copies of H and joining by an edge each vertex from the

i^{th}-copy of H with the i^{th}-vertex of G. We will denote by $V = \{v_1, v_2, ..., v_n\}$ the set of vertices of G and by $H_i = (V_i, E_i)$ the copy of H such that $v_i \sim v$ for every $v \in V_i$. Notice that the corona graph $K_1 \odot H$ is isomorphic to the join graph $K_1 + H$. For any integer $k \geq 2$, we define the graph $G \odot^k H$ recursively from $G \odot H$ as $G \odot^k H = (G \odot^{k-1} H) \odot H$. We also note that the order of $G \odot^k H$ is $n_1(n_2 + 1)^k$.

2 Metric dimension of corona product graphs

We begin by presenting the following useful facts.

Lemma 1. *Let $G = (V, E)$ be a connected graph of order $n \geq 2$ and let H be a graph of order at least two. Let $H_i = (V_i, E_i)$ be the subgraph of $G \odot H$ corresponding to the i^{th}-copy of H.*

(i) *If $u, v \in V_i$, then $d_{G\odot H}(u, x) = d_{G\odot H}(v, x)$ for every vertex x of $G \odot H$ not belonging to V_i.*

(ii) *If S is a resolving set for $G \odot H$, then $V_i \cap S \neq \emptyset$ for every $i \in \{1, ..., n\}$.*

(iii) *If S is a resolving set for $G \odot H$ of minimum cardinality, then $V \cap S = \emptyset$.*

(iv) *If H is a connected graph and S is a resolving set for $G \odot H$, then for every $i \in \{1, .., n\}$, $S \cap V_i$ is a resolving set for H_i.*

Proof. (i) Let $y = v_i \in V$. The result directly follows from the fact that $d_{G\odot H}(u, x) = d_{G\odot H}(u, y) + d_{G\odot H}(y, x) = d_{G\odot H}(v, y) + d_{G\odot H}(y, x) = d_{G\odot H}(v, x)$.

(ii) We suppose $V_i \cap S = \emptyset$ for some $i \in \{1, ..., n\}$. Let $x, y \in V_i$. By (i) we have $d_{G\odot H}(x, u) = d_{G\odot H}(y, u)$ for every vertex $u \in S$, which is a contradiction.

(iii) We will show that $S' = S - V$ is a resolving set for $G \odot H$. Now let x, y be two different vertices of $G \odot H$. We have the following cases.

Case 1: $x, y \in V_i$. By (i) we conclude that there exist $v \in V_i \cap S'$ such that $d_{G\odot H}(x, v) \neq d_{G\odot H}(y, v)$.

Case 2: $x \in V_i$ and $y \in V_j$, $i \neq j$. Let $v \in V_i \cap S'$. Then we have $d_{G\odot H}(x, v) \leq 2 < 3 \leq d_{G\odot H}(y, v)$.

Case 3: $x, y \in V$. Let $x = v_i$ and let $v \in V_i \cap S'$. Then we have $d_{G\odot H}(x, v) = 1 < 1 + d_{G\odot H}(y, x) = d_{G\odot H}(y, v)$.

Case 4: $x \in V_i$ and $y \in V$. If $x \sim y$, then $y = v_i$. Let $v_j \in V$, $j \neq i$, and let $v \in V_j \cap S'$. Then we have $d_{G \odot H}(x, v) = 1 + d_{G \odot H}(y, v) > d_{G \odot H}(y, v)$. For $x \not\sim y = v_l$ we take $v \in V_l \cap S'$ and we obtain $d_{G \odot H}(x, v) = d_{G \odot H}(x, y) + d_{G \odot H}(y, v) > d_{G \odot H}(y, v)$.

Therefore, S' is a resolving set for $G \odot H$.

(iv) Let $S_i = S \cap V_i$. For $x \in S_i$ or $y \in S_i$ the result is straightforward. We suppose $x, y \in V_i - S_i$. Since S is a resolving set for $G \odot H$, we have $r(x|S) \neq r(y|S)$. By (i), $d_{G \odot H}(x, u) = d_{G \odot H}(y, u)$ for every vertex u of $G \odot H$ not belonging to V_i. So, there exists $v \in S_i$ such that $d_{G \odot H}(x, v) \neq d_{G \odot H}(y, v)$. Thus, either ($v \sim x$ and $v \not\sim y$) or ($v \not\sim x$ and $v \sim y$). In the first case we have $d_{G \odot H}(x, v) = d_{H_i}(x, v) = 1$ and $d_{G \odot H}(y, v) = 2 \leq d_{H_i}(y, v)$. The case $v \not\sim x$ and $v \sim y$ is analogous. Therefore, S_i is a resolving set for H_i. □

Theorem 2. *Let G and H be two connected graphs of order $n_1 \geq 2$ and $n_2 \geq 2$, respectively. Then,*

$$dim(G \odot^k H) \geq n_1(n_2 + 1)^{k-1} dim(H).$$

Proof. Let S be a resolving set of minimum cardinality in $G \odot H$. From Lemma 1 (iii) we have that $S \cap V = \emptyset$. Moreover, by Lemma 1 (ii) we have that for every $i \in \{1, ..., n_1\}$ there exist a nonempty set $S_i \subset V_i$ such that $S = \bigcup_{i=1}^{n_1} S_i$. Now, by using Lemma 1 (iv) we have that S_i is a resolving set for H_i. Hence, $dim(G \odot H) = |S| = \sum_{i=1}^{n_1} |S_i| \geq \sum_{i=1}^{n_1} dim(H) = n_1 dim(H)$. As a result, the lower bound follows. □

Theorem 3. *Let G be a connected graph of order $n_1 \geq 2$ and let H be a graph of order $n_2 \geq 2$. If $D(H) \leq 2$, then*

$$dim(G \odot^k H) = n_1(n_2 + 1)^{k-1} dim(H).$$

Proof. Let $S_i \subset V_i$ be a resolving set for H_i and let $S = \bigcup_{i=1}^{n_1} S_i$. We will show that S is a resolving set for $G \odot H$. Let us consider two different vertices x, y of $G \odot H$. We have the following cases.

Case 1: $x, y \in V_i$. Since $D(H_i) \leq 2$, we have that $r(x|S_i) \neq r(y|S_i)$ leads to $r(x|S) \neq r(y|S)$.

Case 2: $x \in V_i$ and $y \in V_j$, $i \neq j$. Let $v \in S_i$. Hence we have $d(x, v) \leq 2 < 3 \leq d(y, v)$.

Case 3: $x, y \in V$. Let $x = v_i$. Then for every vertex $v \in S_i$ we have $d(x, v) = 1 < d(y, x) + 1 = d(y, v)$.

Case 4: $x \in V_i$ and $y \in V$. If $x \sim y$, then let $v \in S_j$, for some $j \neq i$. So we have $d(x,v) = 1 + d(y,v) > d(y,v)$. Moreover, if $x \not\sim y = v_j$, for $v \in S_j$ we have $d(x,v) = d(x,y) + d(y,v) > d(y,v)$.

Thus, for every different vertices x, y of $G \odot H$, we have $r(x|S) \neq r(y|S)$, as a consequence, $dim(G \odot H) \leq n_1 dim(H)$. Therefore, we have $dim(G \odot^k H) \leq n_1(n_2+1)^{k-1} dim(H)$. By Theorem 2 we conclude the proof. □

In order to show a consequence of the above theorem we present the following well known result, where K_t denotes a complete graph of order t, $K_{s,t}$ denotes a complete bipartite graph of order $s+t$ and N_t denotes an empty graph of order t.

Lemma 4. [6] *Let G be a connected graph of order $n \geq 4$. Then $dim(G) = n-2$ if and only if $G = K_{s,t}$, $(s,t \geq 1)$, $G = K_s + N_t$, $(s \geq 1,\ t \geq 2)$, or $G = K_s + (K_1 \cup K_t)$, $(s,t \geq 1)$.*

Corollary 5. *Let G be a connected graph of order $n_1 \geq 2$ and let H be a graph of order $n_2 \geq 4$ and diameter $D(H) \leq 2$. Then*

$$dim(G \odot^k H) = n_1(n_2+1)^{k-1}(n_2-2)$$

if and only if $H = K_{s,t}$, $(s,t \geq 1)$; $H = K_s + N_t$, $(s \geq 1,\ t \geq 2)$, or $H = K_s + (K_1 \cup K_t)$, $(s,t \geq 1)$.

We recall that the wheel graph of order $n+1$ is defined as $W_{1,n} = K_1 \odot C_n$, where K_1 is the singleton graph and C_n is the cycle graph of order n. The metric dimension of the wheel $W_{1,n}$ was obtained by Buczkowski et. al. in [2].

Remark 6. [2] *Let $W_{1,n}$ be a wheel graph. Then*

$$dim(W_{1,n}) = \begin{cases} 3 & \text{for } n = 3,6, \\ 2 & \text{for } n = 4,5, \\ \left\lfloor \frac{2n+2}{5} \right\rfloor & \text{otherwise.} \end{cases}$$

The fan graph F_{n_1,n_2} is defined as the graph join $N_{n_1} + P_{n_2}$, where N_{n_1} is the empty graph of order n_1 and P_{n_2} is the path graph of order n_2. The case $n_1 = 1$ corresponds to the usual fan graphs. Notice that, for the metric dimension of fan graphs, it is possible to find an equivalent result to Remark 6 which was obtained by Caceres et. al. in [4].

Remark 7. [4] *Let $F_{1,n}$ be a fan graph. Then*

$$dim(F_{1,n}) = \begin{cases} 1 & for\ n = 1, \\ 2 & for\ n = 2, 3, \\ 3 & for\ n = 6, \\ \left\lfloor \frac{2n+2}{5} \right\rfloor & otherwise. \end{cases}$$

As a particular case of the Theorem 3 we obtain the following results.

Corollary 8. *Let G be a connected graph of order $n_1 \geq 2$. If H is a wheel graph or a fan graph of order $n_2 \geq 8$, then*

$$dim(G \odot^k H) = n_1(n_2+1)^{k-1} \left\lfloor \frac{2n_2}{5} \right\rfloor.$$

Theorem 9. *Let G be a connected graph of order $n_1 \geq 2$ and let H be a graph of order $n_2 \geq 2$. Let α be the number of connected components of H of order greater than one and let β be the number of isolated vertices of H. Then*

$$dim(G \odot^k H) \leq \begin{cases} n_1(n_2+1)^{k-1}(n_2 - \alpha - 1) & for\ \alpha \geq 1\ and\ \beta \geq 1, \\ \\ n_1(n_2+1)^{k-1}(n_2 - \alpha) & for\ \alpha \geq 1\ and\ \beta = 0, \\ \\ n_1(n_2+1)^{k-1}(n_2 - 1) & for\ \alpha = 0. \end{cases}$$

Proof. We suppose $\alpha \geq 1$ and $\beta \geq 1$. Let A_i be the set of vertices of $G \odot H$ formed by all but one of the vertices per each of the α connected components of H_i. If $\beta \geq 2$ we define B_i to be the set of vertices of $G \odot H$ formed by all but one of the isolated vertices of H_i. If $\beta = 1$ we assume $B_i = \emptyset$. Let us show that $S = \cup_{j=1}^{n_1}(A_j \cup B_j)$ is a resolving set for $G \odot H$. Let x, y be two different vertices of $G \odot H$. We suppose $x, y \notin S$. We have the following cases.

Case 1. $x = v_i \in V$ and $y \in V_i$. For every vertex $u \in V_j \cap S$, $j \neq i$, we obtain $d(y,u) = d(y,x) + d(x,u) > d(x,u)$.

case 2. $x = v_i \in V$ and $y \notin V_i$. For every $v \in S \cap V_i$ we have $d(x,v) = 1 < d(y,v)$.

Case 3. $x \in V_i$ and $y \in V_j$, $j \neq i$. For every $u \in V_i \cap S$ we have $d(x,u) \leq 2 < 3 \leq d(y,u)$.

Case 4. $x, y \in V_i$. We consider, without loss of generality, that x is not an isolated vertex in H_i. Then there exists $v \in V_i \cap S$ such that $v \sim x$, so $d(x,v) = 1 < 2 = d(y,v)$.

Thus, for every two different vertices x, y of $G \odot H$, we obtain $r(x|S) \neq r(y|S)$ and, as a consequence, $dim(G \odot H) \leq n_1(n_2 - \alpha - 1)$.

As above, if $\beta = 0$ then we take $S = \cup_{j=1}^{n_1} A_j$ and we obtain $dim(G \odot H) \leq n_1(n_2 - \alpha)$ and if $\alpha = 0$, then we take $S = \cup_{j=1}^{n_1} B_j$ and we obtain $dim(G \odot H) \leq n_1(n_2 - 1)$. Note that if $\alpha = 0$, then it is not necessary to consider Case 4. Thus, the result follows. □

Corollary 10. *Let G be a connected graphs of order $n_1 \geq 2$ and let H be an unconnected graph of order $n_2 \geq 2$. Then*

$$dim(G \odot^k H) = n_1(n_2 + 1)^{k-1}(n_2 - 1)$$

if and only if $H \cong N_{n_2}$.

Proof. In [13] the authors showed that $dim(G \odot N_{n_2}) = n_1(n_2 - 1)$. Hence, $dim(G \odot^k N_{n_2}) = n_1(n_2+1)^{k-1}(n_2-1)$. Moreover, by the above theorem, if H is unconnected and $H \not\cong N_{n_2}$, then $dim(G \odot^k H) \leq n_1(n_2+1)^{k-1}(n_2-2)$. □

Theorem 11. *Let G and H be two connected graphs of order $n_1 \geq 2$ and $n_2 \geq 3$, respectively. Then*

$$dim(G \odot^k H) = n_1(n_2 + 1)^{k-1}(n_2 - 1)$$

if and only if $H \cong K_{n_2}$. Moreover, if $H \not\cong K_{n_2}$, then

$$dim(G \odot^k H) \leq n_1(n_2 + 1)^{k-1}(n_2 - 2).$$

Proof. Since $dim(K_{n_2}) = n_2 - 1$, by Theorem 3 we conclude $dim(G \odot^k K_{n_2}) = n_1(n_2 + 1)^{k-1}(n_2 - 1)$. On the contrary, we suppose $H \not\cong K_{n_2}$. Given a set X of vertices of H and a vertex v of H, $N_X(v)$ denotes the set of neighbors that v has in X: $N_X(v) = \{u \in X : u \sim v\}$. Given two vertices a, b of H, let $X_{a,b}$ be the set formed by all vertices of H different from a and b. Since H is a connected graph and $H \neq K_{n_2}$, there exist at least two vertices a, b of H such that $N_{X_{a,b}}(a) \neq N_{X_{a,b}}(b)$. Let a_i, b_i be the vertices corresponding to a, b, respectively, in the i^{th}-copy $H_i = (V_i, E_i)$ of H. Let $S = \cup_{i=1}^{n_2}(V_i - \{a_i, b_i\})$. We will show that S is a resolving set for $G \odot H$. Let x, y be two different vertices of $G \odot H$ such that $x, y \notin S$. We have the following cases.

Case 1. $x = a_i$ and $y = b_i$. Since $N_{X_{a,b}}(a) \neq N_{X_{a,b}}(b)$ we have $r(x|S) \neq r(y|S)$.

Case 2. $x = v_i \in V$ and $y \in V_i$. For every $v \in V_j - \{a_j, b_j\}$, $j \neq i$, we have $d(y, v) = d(y, x) + d(x, v) > d(x, v)$. If $x \in V_i$ and $y \in V_j$, $j \neq i$, then for every $v \in V_i - \{a_i, b_i\}$ we have $d(x, v) \leq 2 < 3 \leq d(y, v)$.

Case 3. $x, y \in V$. Say $x = v_i$. Then for every $v \in V_i - \{a_i, b_i\}$ we have $d(x, v) = 1 < d(y, v)$.

Hence, for every two different vertices x, y of $G \odot H$, we obtain $r(x|S) \neq r(y|S)$. Thus, $dim(G \odot H) \leq n_1(n_2 - 2)$. Therefore, the result follows. □

As we have shown in Corollary 5, the above bound is tight.

Theorem 12. *Let G be a connected graph of order $n_1 \geq 2$ and let H be a graph of order $n_2 \geq 2$. Then*

$$dim(G \odot^k H) \leq n_1(n_2 + 1)^{k-1} dim(K_1 \odot H).$$

Proof. We denote by $K_1 \odot H_i$ the subgraph of $G \odot H$, obtained by joining the vertex $v_i \in V$ with all vertices of H_i. For every $v_i \in V$, let B_i be a resolving set of minimum cardinality of $K_1 \odot H_i$ and let $B = \bigcup_{i=1}^{n_1} B_i$. By Lemma 1 (iii) we have that v_i does not belong to any resolving set of minimum cardinality for $K_1 \odot H_i$. So, B does not contain any vertex from G. We will show that B is a resolving set for $G \odot H$. Let x, y be two different vertices in $G \odot H$. We consider the following cases.

Case 1: $x, y \in V_i$. There exists $u \in B_i$ such that $d_{K_1 \odot H_i}(x, u) \neq d_{K_1 \odot H_i}(y, u)$, which leads to $d_{G \odot H}(x, u) \neq d_{G \odot H}(y, u)$.

Case 2: $x \in V_i$ and $y \in V_j$, $i \neq j$. Let $v \in B_i$. We have $d_{G \odot H}(x, v) \leq 2 < 3 \leq d_{G \odot H}(y, v)$.

Case 3: $x, y \in V$. Suppose now that x is adjacent to the vertices of H_i. Hence, for every vertex $v \in B_i$ we have $d_{G \odot H}(x, v) = 1 < d_{G \odot H}(y, x) + 1 = d_{G \odot H}(y, v)$.

Case 4: $x \in V_i$ and $y \in V$. If $x \sim y$, then for every vertex $v \in B_j$, with $j \neq i$, we have $d_{G \odot H}(x, v) = 1 + d_{G \odot H}(y, v) > d_{G \odot H}(y, v)$. Now, let us assume that $x \not\sim y$. Hence, there exists $v \in B_j$ adjacent to y, with $j \neq i$. So, we have $d_{G \odot H}(x, v) = d_{G \odot H}(x, y) + 1 = d_{G \odot H}(x, y) + d_{G \odot H}(y, v) > d_{G \odot H}(y, v)$.

Thus, for every two different vertices x, y of $G \odot H$, we have $r(x|S) \neq r(y|S)$ and, as a consequence, $dim(G \odot H) \leq n_1 dim(K_1 \odot H)$. Therefore, the result follows. □

Theorem 13. *Let G be a connected graph of order $n_1 \geq 2$ and let H be a graph of order $n_2 \geq 7$. If $D(H) \geq 6$ or H is a cycle graph, then*

$$dim(G \odot^k H) = n_1(n_2+1)^{k-1} dim(K_1 \odot H).$$

Proof. Let S be a resolving set of minimum cardinality in $G \odot H$. By Lemma 1 (iii) we have $S \cap V = \emptyset$, as a consequence, $S = \cup_{i=1}^{n_1} S_i$, where $S_i \subset V_i$. Notice that, by Lemma 1 (ii), $S_i \neq \emptyset$ for every $i \in \{1, ..., n_1\}$. Now we differentiate two cases in order to show that $r(x|S_i) \neq (1, ..., 1)$ for every $x \in V_i - S_i$.

Case 1. H is a cycle graph of order $n_2 \geq 7$. If $r(a|S_i) = (1,1)$ for some $a \in V_i - S_i$, then, since $n_2 \geq 7$, there exist two vertices $x, y \in V_i - S_i$ such that $d_{H_i}(x, v) > 1$ and $d_{H_i}(y, v) > 1$, for every $v \in S_i$. Hence, $d_{G \odot H}(x, v) = d_{G \odot H}(y, v) = 2$ for every $v \in S_i$, which is a contradiction because, by Lemma 1 (i), $d_{G \odot H}(x, v) = d_{G \odot H}(y, v)$ for every vertex u of S not belonging to S_i.

Case 2. $D(H) \geq 6$. Let $x, y \in V_i - S_i$. Since S is a resolving set for $G \odot H$, we have $r(x|S) \neq r(y|S)$. As we have noted before, by Lemma 1 (i) we have that $d_{G \odot H}(x, u) = d_{G \odot H}(y, u)$ for every vertex u of $G \odot H$ not belonging to V_i. So, there exists $v \in S_i$ such that $d_{G \odot H}(x, v) \neq d_{G \odot H}(y, v)$ and, as a consequence, either ($v \sim x$ and $v \not\sim y$) or ($v \not\sim x$ and $v \sim y$). Now we suppose that there exists a vertex $a \in V_i - S_i$ such that $r(a|S_i) = (1, 1, ...1)$. If there exists a vertex $b \in V_i - S_i$ such that $d_{H_i}(b, u) > 1$, for every $u \in S_i$, then for every $w \in V_i - (S_i \cup \{a, b\})$, there exists $v \in S_i$ such that $w \sim v$. Then $D(H_i) \leq 5$. Moreover, if for every $b \in V_i - S_i$ there exists $v_b \in S_i$ such that $v_b \sim b$, then $D(H) \leq 4$. Therefore, if $D(H) \geq 6$, then $r(a|S_i) \neq (1, 1, ...1)$ for every $a \in V_i - S_i$.

Now, we denote by $K_1 \odot H_i$ the subgraph of $G \odot H$, obtained by joining the vertex $v_i \in V$ with all vertices of the i^{th}-copy of H. In both the above cases we have $r(v_i|S_i) = (1, 1, ..., 1) \neq r(x|S_i)$ for every $x \in V_i - S_i$, so S_i is a resolving set for $K_1 \odot H_i$. Hence, $dim(K_1 \odot H_i) \leq |S_i|$, for every $i \in \{1, ..., n_1\}$. Thus, $dim(G \odot H) \geq n_1 dim(K_1 \odot H_i)$ and, as a consequence, $dim(G \odot^k H) \geq n_1(n_2+1)^{k-1} dim(K_1 \odot H)$. We conclude the proof by Theorem 12. □

Corollary 14. *Let G be a connected graph of order $n_1 \geq 2$.*

(i) *If $n_2 \geq 7$, then $dim(G \odot^k C_{n_2}) = n_1(n_2+1)^{k-1} \left\lfloor \frac{2n_2+2}{5} \right\rfloor$.*

(ii) *If $n_2 \geq 7$, then $dim(G \odot^k P_{n_2}) = n_1(n_2+1)^{k-1} \left\lfloor \frac{2n_2+2}{5} \right\rfloor$.*

All our previous results concern to $G \odot H$ for H of order at least two. Now we consider the case $H \cong K_1$. We obtain a general bound for $dim(G \odot^k K_1)$ and, when G is a tree, we give the exact value for this parameter.

Claim 15. *Let G be a simple graph. If v is a vertex of degree greater than one in G, then for every vertex u adjacent to v there exists a vertex $x \neq u, v$ of G, such that $d(v, x) \neq d(u, x) + 1$.*

The following lemma obtained in [2] is useful to obtain the next result.

Lemma 16. [2] *If G_1 is a graph obtained by adding a pendant edge to a nontrivial connected graph G, then $dim(G) \leq dim(G_1) \leq dim(G) + 1$.*

Theorem 17. *For every connected graph G of order $n \geq 2$,*

$$dim(G \odot^k K_1) \leq 2^{k-1}n - 1.$$

Proof. If $G \cong K_2$, then $dim(K_2 \odot K_1) = dim(P_4) = 1$. So, let us suppose $G \ncong K_2$. Let us suppose, without loss of generality, that v_n is a vertex of degree greater than one in G and let $S = V - \{v_n\}$. For every $i \in \{1, ..., n\}$, let u_i be the pendant vertex of v_i in $G \odot K_1$. We will show that S is a resolving set for $G \odot K_1$. Let x, y be two different vertices of $G \odot K_1$. If $x = u_i$ and $y = u_j$, $i \neq j$, then we have either $i \neq n$ or $j \neq n$. Let us suppose for instance $i \neq n$. So, we obtain that $d(x, v_i) = 1 \neq d(y, v_i)$. On the other hand, if $x = v_n$ and $y = u_i$, then let us suppose $d(x, v_i) = 1$. Since v_n is a vertex of degree greater than one in G, by Claim 15, there exists a vertex $v_j \in S$ such that $d(x, v_j) \neq d(v_i, v_j) + 1$. So, we have $d(x, v_j) \neq d(v_i, v_j) + 1 = d(v_i, v_j) + d(u_i, v_i) = d(y, v_i) + d(v_i, v_j) = d(y, v_j)$. Therefore, for every different vertices x, y of $G \odot K_1$ we have $r(x|S) \neq r(y|S)$ and, as a consequence, $dim(G \odot K_1) \leq n - 1$. Therefore, $dim(G \odot^k K_1) \leq 2^{k-1}n - 1$. □

By Lemma 16 we have $dim(K_n \odot K_1) \geq dim(K_n) = n - 1$. Thus, for $k = 1$ the above bound is achieved for the graph $G = K_n$.

To present the next result, we need additional definitions. A vertex of degree at least 3 in a graph G will be called a *major vertex* of G. Any vertex u of degree one is said to be a *terminal vertex* of a major vertex v if $d(u, v) < d(u, w)$ for every other major vertex w of G. The *terminal degree* of a major vertex v is the number of terminal vertices of v. A major vertex v is an *exterior major vertex* if it has positive terminal degree. Given a graph G, $n_1(G)$ denotes the number of vertices of degree one and $ex(G)$ denotes the number of exterior major vertices of G.

Lemma 18. [6, 10, 19] *If T is a tree that is not a path, then $dim(T) = n_1(T) - ex(T)$.*

Theorem 19. *For any tree T of order $n \geq 3$,*

$$dim(T \odot^k K_1) = \begin{cases} n_1(T) & \text{for } k = 1, \\ 2^{k-2}n & \text{for } k \geq 2. \end{cases}$$

Proof. If T is a path of order $n \geq 3$, then we have $dim(T \odot K_1) = 2 = n_1(T)$. Now, if T is not a path, then by using Lemma 18, since $T \odot K_1$ is a tree, $n_1(T \odot K_1) = n$ and $ex(T \odot K_1) = n - n_1(T)$, we obtain the result for $k = 1$. Since for every tree T of order n we have $n_1(T \odot K_1) = n$, we obtain the result for $k \geq 2$. □

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