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An $4n$-point Interpolation Formula for Certain Polynomials

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Abstract By using some techniques of the divided difference operators, we establish an $4n$-point interpolation formula. Certain polynomials, such as Jackson's ${}_8\phi_7$ terminating summation formula, are special cases of this formula. Based on Krattenthaler's identity, we also give Jackson's formula a determinantal interpretation.

1 Introduction

Recall that the i-th divided difference operator ∂_i, acting on functions $f(x_1,\ldots,x_n)$ of several variables, is defined by

$$f(x_1,\ldots,x_i,x_{i+1},\ldots)\partial_i = \frac{f(x_1,\ldots,x_i,x_{i+1},\ldots)-f(x_1,\ldots,x_{i+1},x_i,\ldots)}{(x_i-x_{i+1})}.$$

To be more general, we introduce the operator ${}_c\partial_i$ which we call the *i-th c-divided difference operator*:

$$f(x_1,\ldots,x_i,x_{i+1},\ldots)_c\partial_i = \frac{f(x_1,\ldots,x_i,x_{i+1},\ldots)-f(x_1,\ldots,x_{i+1},x_i,\ldots)}{(x_i-x_{i+1})(1-c/x_ix_{i+1})}. \tag{1.1}$$

Note that ${}_0\partial_i = \partial_i$.

Several properties of the c-divided difference operators will be given in next section, see Lemmas 2.1 to 2.5. Employing those lemmas, we obtain the main result of this paper.

Theorem 1.1 *Given two sets of variables*

$$A = \{a_1, c/a_1, \ldots, a_n, c/a_n\}, \quad B = \{b_1, c/b_1, \ldots, b_n, c/b_n\},$$

we have the following $4n$-point interpolation formula for certain polynomials $f(y)$ of degree $2n$ with symmetry $y^{-n}f(y) = (c/y)^{-n}f(c/y)$ when $c \neq 0$:

$$\begin{aligned} f(y) \;=\; & \frac{f(b_1)}{\prod_{i=1}^n(b_1-a_i)(b_1-c/a_i)}\prod_{i=1}^n(y-a_i)(y-c/a_i) \\ & +\frac{f(a_1)}{\prod_{i=1}^n(a_1-b_i)(a_1-c/b_i)}\prod_{i=1}^n(y-b_i)(y-c/b_i) \\ & +\sum_{j=1}^{n-1} C_j\cdot\prod_{i=1}^{j}(y-b_i)(y-c/b_i)\prod_{i=1}^{n-j}(y-a_i)(y-c/a_i), \end{aligned} \tag{1.2}$$

where

$$C_j = \frac{f(b_1)b_1^{1-j}}{\prod_{i=1}^{n-j+1}(b_1-a_i)(b_1-c/a_i)}\, {}_c\partial_1 \cdots {}_c\partial_j (b_{j+1}-a_{n-j+1})(1-c/a_{n-j+1}b_{j+1}).$$

Note that Theorem 1.1 leads to the following $2n$-point interpolation formula given in [1] if $c=0$:

$$f(y) = \frac{f(b_1)}{\prod_{i=1}^{n}(b_1-a_i)}\prod_{i=1}^{n}(y-a_i) + \frac{f(a_1)}{\prod_{i=1}^{n}(a_1-b_i)}\prod_{i=1}^{n}(y-b_i)$$
$$+ \sum_{j=1}^{n-1}\frac{f(b_1)}{\prod_{i=1}^{n-j+1}(b_1-a_i)}\partial_1\cdots\partial_j(b_{j+1}-a_{n-j+1})\cdot\prod_{i=1}^{j}(y-b_i)\prod_{i=1}^{n-j}(y-a_i).$$

The symmetry $y^{-n}f(y) = (c/y)^{-n}f(c/y)$ when $c \neq 0$ implies that $f(y)$ can be written as a product $\prod_{i=1}^{n}(y-x_i)(c-x_iy)$. Considering the case

$$A = \{a, c/a, \ldots, aq^{1-n}, cq^{n-1}/a\}, \quad B = \{b, c/b, \ldots, bq^{1-n}, cq^{n-1}/b\}$$

and $y = x_{n+1}$, one can check that Theorem 1.1 implies the following identity by expanding the determinant with respect to the last row:

$$\det\left(P_{n-j+1}(x_i, aq^{j-n})P_{n-j+1}(x_i, c/a)P_{j-1}(x_i, bq^{1-n})P_{j-1}(x_i, cq^{n-j+1}/b)\right)_{i,j=1}^{n+1}$$
$$= \prod_{1\le i<j\le n+1}(x_i-x_j)(c-x_ix_j)b^{\binom{n+1}{2}}q^{-(n+1)n(n-1)/3}\prod_{i=1}^{n+1}(a/b, cq^{2n+2-2i}/ab; q)_{i-1}, \quad (1.3)$$

where $(a;q)_n$ is the q-shifted factorial defined by

$$(a;q)_0 = 1, \quad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n = 1, 2, \ldots$$

and we use $P_n(a,b)$ called Cauchy polynomials [2] to denote $a^n(b/a;q)_n$ for convenience.

After some rearrangement of (1.3), we may get Krattenthaler's identity [5]:

$$\det\left(\frac{(ax_i, ac/x_i; q)_{n-j}}{(bx_i, bc/x_i; q)_{n-j}}\right)_{i,j=1}^{n}$$
$$= \prod_{1\le i<j\le n}(x_j-x_i)(1-c/x_ix_j)a^{\binom{n}{2}}q^{\binom{n}{3}}\prod_{i=1}^{n}\frac{(b/a, abcq^{2n-2i}; q)_{i-1}}{(bx_i, bc/x_i; q)_{n-1}}.$$

Through the specializations $f(y) = \prod_{i=1}^{n}(uq^{i-1}-y)(c-uq^{i-1}y)$ and

$$A = \{a, c/a, \ldots, aq^{1-n}, cq^{n-1}/a\}, \quad B = \{b, c/b, \ldots, bq^{1-n}, cq^{n-1}/b\}$$

in Theorem 1.1, we have the following variation of Jackson's ${}_8\phi_7$ terminating formula.

Corollary 1.2 *We have*

$$\prod_{i=1}^{n}(uq^{i-1}-y)(c-uq^{i-1}y)$$
$$= q^{\binom{n}{2}}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{P_{n-k}(b,uq^{n-1})P_k(a,uq^{n-k})P_{n-k}(u,c/b)P_k(c/a,u)}{P_n(b,a)(cq^{n-k-1}/ab;q)_{n-k}(cq^{2n-2k}/ab;q)_k}$$
$$\times P_{n-k}(y,aq^{k+1-n})P_{n-k}(y,c/a)P_k(y,bq^{1-n})P_k(y,cq^{n-k}/b), \tag{1.4}$$

where $\begin{bmatrix}n\\k\end{bmatrix}$ *is the q-binomial coefficient defined by*

$$\begin{bmatrix}n\\k\end{bmatrix}=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

In section 3, we shall give Corollary 1.2 a different approach by considering a special case of (1.3). The key step of our approach is to evaluate the cofactor of each entry in the last row of the determinant in (1.3).

Note that if we write cq^{-1}/ab as a, y/a as b, c/bu as c, c/ay as d and uq^{n-1}/b as e in the above corollary, we get Jackson's ${}_8\phi_7$ formula [3]:

$$\frac{(aq,aq/bc,aq/bd,aq/cd;q)_n}{(aq/b,aq/c,aq/d,aq/bcd;q)_n}$$
$$=\sum_{k=0}^{n}\frac{(1-aq^{2k})(a;q)_k(b;q)_k(c;q)_k(d;q)_k(e;q)_k(q^{-n};q)_kq^k}{(1-a)(q;q)_k(aq/b;q)_k(aq/c;q)_k(aq/d;q)_k(aq/e;q)_k(aq^{n+1};q)_k}.$$

2 The $4n$-point interpolation formula

In this section, we shall focus our attention on the proof of Theorem 1.1. To this aim, we shall first introduce several elementary properties of the c-divided difference operators.

Lemma 2.1 *c-divided difference operators satisfy the following Leibnitz rules:*

$$f(x_1)g(x_1)_c\partial_1 = f(x_1)g(x_1)_c\partial_1 + f(x_1)_c\partial_1 g(x_2)$$
$$f(x_1)g(x_1)_c\partial_1\cdots{}_c\partial_n = \sum_{k=0}^{n} f(x_1)_c\partial_1\cdots{}_c\partial_k g(x_{k+1})_c\partial_{k+1}\cdots{}_c\partial_n.$$

Lemma 2.2 *If* $f(x_1,x_2,\ldots,x_n)$ *is a symmetric function of* x_i,x_{i+1}*, then*

$$f(x_1,x_2,\ldots,x_n)_c\partial_i = 0.$$

Lemma 2.3 *If $p_n(y)$ is a polynomial of degree $2n$ such that*

$$y^{-n}p_n(y) = (c/y)^{-n}p_n(c/y),$$

then we have

$$y_1^{-n}p_n(y_1)_c\partial_1\cdots{}_c\partial_m = \begin{cases} 0, & m>n, \\ 1, & m=n. \end{cases} \tag{2.1}$$

Proof. Clearly, $y_1^{-n}p_n(y_1)$ can be rewritten as $\prod_{i=1}^n(y_1-x_1)(1-c/y_1x_1)$. When $m=1$ and $n=0$ or $n=1$, it is easy to verify that (2.1) holds. In view of Lemma 2.1, we can prove Lemma 2.3 by induction on the length of the operators. ∎

Lemma 2.4 *We have*

$$\prod_{k=1}^{j}(y_1-b_k)(1-c/y_1b_k)_c\partial_1\cdots{}_c\partial_i\Big|_{y_k=b_k,1\le k\le i+1} = \begin{cases} 0, & j\neq i; \\ 1, & j=i. \end{cases} \tag{2.2}$$

Proof. For $j\le i$, Lemma 2.4 is a direct consequence of Lemma 2.3. For $j>i$, we have

$$\begin{aligned}
&\prod_{k=1}^{j}(y_1-b_k)(1-c/y_1b_k)_c\partial_1\cdots{}_c\partial_i\Big|_{y_k=b_k,1\le k\le i+1}\\
&\quad=\prod_{k=2}^{j}(y_2-b_k)(1-c/y_2b_k)_c\partial_2\cdots{}_c\partial_i\Big|_{y_k=b_k,2\le k\le i+1}\\
&\quad=\cdots=\prod_{k=i+1}^{j}(y_{i+1}-b_k)(1-c/y_{i+1}b_k)\Big|_{y_k=b_k,i+1\le k\le i+1}=0.
\end{aligned}$$

We complete the proof. ∎

Lemma 2.5 *We have*

$$\begin{aligned}
\prod_{k=1}^{j}\frac{(y_1-b_k)(1-c/y_1b_k)}{(y_1-a_k)(1-c/y_1a_k)}\prod_{k=1}^{i-1}(y_1-a_k)(1-c/y_1a_k)_c\partial_1\cdots{}_c\partial_i\Big|_{y_k=b_k,1\le k\le i+1}&\\
=\begin{cases} 0, & j\neq i; \\ \frac{1}{(b_{j+1}-a_j)(1-c/a_jb_{j+1})}, & j=i. \end{cases}&
\end{aligned} \tag{2.3}$$

Proof. For $j<i$, we have

$$\begin{aligned}
&\prod_{k=1}^{j}(y_1-b_k)(1-c/y_1b_k)\prod_{k=j+1}^{i-1}(y_1-a_k)(1-c/y_1a_k)_c\partial_1\cdots{}_c\partial_i\Big|_{y_k=b_k,1\le k\le i+1}\\
&=\sum_{l=0}^{i}\prod_{k=1}^{j}(y_1-b_k)(1-c/y_1b_k)_c\partial_1\cdots{}_c\partial_l\Big|_{y_k=b_k,1\le k\le l+1}\\
&\times\prod_{k=l+1}^{i-1}(y_{l+1}-a_k)(1-c/y_{l+1}a_k)_c\partial_{l+1}\cdots{}_c\partial_i\Big|_{y_k=b_k,l+1\le k\le i+1}.
\end{aligned}$$

From Lemma 2.3 and Lemma 2.4, either the first product inside the sum or the second one vanishes, so does the sum.

For $j \geq i$, we have

$$\frac{\prod_{k=1}^{j}(y_1-b_k)(1-c/y_1b_k)}{\prod_{k=i}^{j}(y_1-a_k)(1-c/y_1a_k)}{}_c\partial_1\cdots{}_c\partial_i\Big|_{y_k=b_k,1\leq k\leq i+1}$$
$$=(y_1-b_1)(1-c/yb_1){}_c\partial_1\Big|_{y_1=b_1}\frac{\prod_{k=2}^{j}(y_2-b_k)(1-c/y_2b_k)}{\prod_{k=i}^{j}(y_2-a_k)(1-c/y_2a_k)}{}_c\partial_2\cdots{}_c\partial_i\Big|_{y_k=b_k,2\leq k\leq i+1}$$
$$=\frac{\prod_{k=2}^{j}(y_2-b_k)(1-c/y_2b_k)}{\prod_{k=i}^{j}(y_2-a_k)(1-c/y_2a_k)}{}_c\partial_2\cdots{}_c\partial_i\Big|_{y_k=b_k,2\leq k\leq i+1}$$
$$=\cdots=\frac{\prod_{k=i}^{j}(y_i-b_k)(1-c/y_ib_k)}{\prod_{k=i}^{j}(y_i-a_k)(1-c/y_ia_k)}{}_c\partial_i\Big|_{y_k=b_k,i\leq k\leq i+1}$$
$$=\begin{cases}0, & j>i,\\ 1/(b_{j+1}-a_j)(1-c/a_jb_{j+1}), & j=i.\end{cases}$$

We complete the proof. ∎

Proof of Theorem 1.1

Given a polynomial $f(y)$ of degree $2n$ with symmetry $y^{-n}f(y)=(c/y)^{-n}f(c/y)$, we assume that

$$f(y)=\sum_{j=0}^{n}C_j\prod_{k=1}^{j}(y-b_k)(y-c/b_k)\prod_{k=j+1}^{n}(y-a_k)(y-c/a_k). \tag{2.4}$$

Taking $y=b_1$ in (2.4), one has

$$f(b_1)=C_0\prod_{k=1}^{n}(b_1-a_k)(b_1-c/a_k).$$

Therefore,

$$C_0=\frac{f(b_1)}{\prod_{k=1}^{n}(b_1-a_k)(b_1-c/a_k)}.$$

Setting $y=a_n$ in (2.4) leads to

$$C_n=\frac{f(a_n)}{\prod_{k=1}^{n}(a_n-b_k)(a_n-c/b_k)}.$$

Let $g(y)=f(y)/\prod_{k=1}^{n}(y-a_k)(y-c/a_k)$. Rewrite (2.4) as

$$g(y)=g(b_1)+\sum_{j=1}^{n-1}C_j\frac{\prod_{k=1}^{j}(y-b_k)(1-c/yb_k)}{\prod_{k=1}^{j}(y-a_k)(1-c/ya_k)}$$
$$+\frac{f(a_n)}{\prod_{i=1}^{n}(a_n-b_i)(a_n-c/b_i)}\prod_{i=1}^{n}\frac{(y-b_k)(1-c/yb_k)}{(y-a_k)(1-c/ya_k)}.$$

Multiplying both sides by $\prod_{k=1}^{i-1}(y-a_k)(1-c/ya_k)$, then applying the operator ${}_c\partial_1\cdots{}_c\partial_i$, one has

$$\begin{aligned}
&g(y_1)\prod_{k=1}^{i-1}(y_1-a_k)(1-c/y_1a_k)_c\partial_1\cdots{}_c\partial_i\Big|_{y_k=b_k,1\le k\le i+1}\\
&=\sum_{j=1}^{i-1}C_j\prod_{k=1}^{j}(y_1-b_k)(1-c/y_1b_k)\prod_{k=j+1}^{i-1}(y_1-a_k)(1-c/y_1a_k)_c\partial_1\cdots{}_c\partial_i\Big|_{y_k=b_k,1\le k\le i+1}\\
&+\sum_{j=i}^{n-1}C_j\frac{\prod_{k=1}^{j}(y_1-b_k)(1-c/y_1b_k)}{\prod_{k=i}^{j}(y_1-a_k)(1-c/y_1a_k)}{}_c\partial_1\cdots{}_c\partial_i\Big|_{y_k=b_k,1\le k\le i+1}.
\end{aligned}$$

By Lemma 2.5, we have

$$g(y_1)\prod_{k=1}^{i-1}(y_1-a_k)(1-c/y_1a_k)_c\partial_1\cdots{}_c\partial_i\Big|_{y_k=b_k,1\le k\le i+1}=\frac{C_i}{(b_{i+1}-a_i)(1-c/b_{i+1}a_i)}.$$

Thus

$$\begin{aligned}
C_i&=g(y_1)\prod_{k=1}^{i-1}(y_1-a_k)(1-c/y_1a_k)_c\partial_1\cdots{}_c\partial_i\Big|_{y_k=b_k,1\le k\le i+1}(b_{i+1}-a_i)(1-c/b_{i+1}a_i)\\
&=\frac{f(b_1)b_1^{1-i}}{\prod_{k=i}^{n}(b_1-a_k)(b_1-c/a_k)}{}_c\partial_1\cdots{}_c\partial_i(b_{i+1}-a_i)(1-c/b_{i+1}a_i).
\end{aligned}$$

Replacing a_i by a_{n-i+1}, we complete the proof. ∎

3 Jackson's ${}_8\phi_7$ terminating summation formula

Letting $x_i=uq^{i-1}$ for $1\le i\le n$ and $x_{n+1}=y$, we shall show that (1.3) in this case is equivalent to Corollary 1.2. In other words, we shall give Jackson's ${}_8\phi_7$ terminating summation formula a determinantal interpretation.

Our proofs in this section involve the following well-known symmetric functions. Given two sets of variables X and Y, the i-th supersymmetric complete function $h_i(X-Y)$ is defined by

$$h_i(X-Y)=[t^i]\frac{\prod_{y\in Y}(1-yt)}{\prod_{x\in X}(1-xt)}=\sum_{i=0}^{n}(-1)^ie_i(Y)h_{n-i}(X),\tag{3.1}$$

where $[t^i]f(t)$ means the coefficient of t^i in $f(t)$, $e_i(X)$ and $h_i(Y)$ are i-th elementary symmetric function and i-th complete symmetric function, respectively.

Expanding the determinant of (1.3) along the last row in the case $x_i = uq^{i-1}$ for $1 \le i \le n$ and $x_{n+1} = y$, we have

$$\prod_{i=1}^{n}(uq^{i-1}-y)(c-uq^{i-1}y)b^{\binom{n+1}{2}}q^{-(n+1)n(n-1)/3}$$
$$\times \prod_{1\le i<j\le n}(uq^{i-1}-uq^{j-1})(c-u^2q^{i+j-2})\prod_{i=1}^{n+1}(a/b, cq^{2n+2-2i}/ab;q)_{i-1}$$
$$= \sum_{k=1}^{n+1} C_{n,k}P_{n-k+1}(y,aq^{k-n})P_{n-k+1}(y,c/a)P_{k-1}(y,bq^{1-n})P_{k-1}(y,cq^{n-k+1}/b), \tag{3.2}$$

where $C_{n,k}$ is the cofactor of the entry

$$P_{n-k+1}(y,aq^{k-n})P_{n-k+1}(y,c/a)P_{k-1}(y,bq^{1-n})P_{k-1}(y,cq^{n-k+1}/b).$$

It is easy to verify that $C_{n,k}$ can be rewritten in terms of the supersymmetric complete functions:

$$C_{n,k} = \prod_{1\le i<j\le n}(uq^{i-1}-uq^{j-1})\det(h_{2n-i+1}(U-Y_{j,k})), \tag{3.3}$$

where the set

$$Y_{j,k} = \begin{cases} \{a, c/a, \ldots, aq^{j-n}, cq^{n-j}/a, bq^{1-n}, cq^{n-1}/b, \ldots, bq^{j-n-1}, cq^{n-j+1}/b\}, & \text{if } 1 \le j < k, \\ \{a, c/a, \ldots, aq^{j-n+1}, cq^{n-j-1}/a, bq^{1-n}, cq^{n-1}/b, \ldots, bq^{j-n}, cq^{n-j}/b\}, & \text{if } k \le j \le n. \end{cases}$$

For convenience, we denote by $F_{n,k}(U,A,B)$ the determinant in (3.3). Now, in order to prove Corollary 1.2, we are left to evaluate these determinants $F_{n,k}(U,A,B)$ for $1 \le k \le n+1$.

Theorem 3.1 *For $1 \le k \le n+1$, we have*

$$F_{n,k}(U,A,B) = \begin{bmatrix} n \\ k-1 \end{bmatrix} q^{-\frac{n(n-1)(2n-1)}{6}} b^{\binom{n}{2}} \prod_{1\le i<j\le n}(c-u^2q^{i+j-2})$$
$$\times P_{n-k+1}(b,uq^{n-1})P_{k-1}(a,uq^{n-k+1})P_{n-k+1}(u,c/b)P_{k-1}(c/a,u)$$
$$\times \prod_{i=1}^{n-1}(1-aq^{i-1}/b)^{n-i} \prod_{\substack{j=0\\ j\ne n-k+1}}^{n-1} \prod_{\substack{i=j+1\\ i\ne n-k+1}}^{n}(1-cq^{i+j-1}/ab). \tag{3.4}$$

To make the proof clear, we shall first give two lemmas.

Lemma 3.2 *[4] Let $\{j_1, j_2, \ldots, j_n\}$ be a sequence of integers, and let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be sets of variables. The following relation holds*

$$\det\left(h_{j_k+k-l}(X_k-Y_k)\right)_{k,l=1}^{n} = \det\left(h_{j_k+k-l}(X_k-Y_k-D_{n-k})\right)_{k,l=1}^{n},$$

where D_0, D_1, …, D_{n-1} are sets of indeterminates such that the cardinality of D_i is equal to or less than i.

The second lemma is a special case of Theorem 3.1.

Lemma 3.3 *For $1 \le k \le n+1$, we have*

$$\det(e_{2n-i+1}(Y_{j,k}))_{i,j=1}^{n} = \begin{bmatrix} n \\ k-1 \end{bmatrix} c^{\binom{n+1}{2}} q^{\binom{n-k+1}{2} - \frac{n(n-1)(2n-1)}{6}} \prod_{i=1}^{n-1} (b - aq^{i-1})^{n-i} \prod_{\substack{j=0 \\ j \neq n-k+1}}^{n-1} \prod_{\substack{i=j+1 \\ i \neq n-k+1}}^{n} (1 - cq^{i+j-1}/ab), \tag{3.5}$$

where the sets of $2n$ variables $Y_{j,k}$, $1 \le j \le n$, are defined as above.

Proof. Obviously, $e_{2n}(Y_{j,k}) = c^n$. We shall use induction on n. When $n = 1$, we have

$$\det(e_2(a, c/a)) = \det(e_2(b, c/b)) = c.$$

Thus (3.5) is true for $n = 1$. Assume that (3.5) holds for $1 \le m \le n-1$, where $n \ge 2$. We now proceed to check that (3.5) is true for $m = n$.

When $k = 1$, the substraction of two successive columns of the determinant gives

$$e_i(Y_{j,1}) - e_i(Y_{j-1,1}) = q^{j-n}(b-a)(1 - cq^{2n-2j}/ab) e_{i-1}(Y'_{j-1,1}),$$

where $Y'_{j-1,1} = Y_{j,1} \setminus \{bq^{j-n}, cq^{n-j}/b\}$.

It is easy to verify that

$$\det(e_{2n-i+1}(Y_{j,1}))_{i,j=1}^{n} = c^{\binom{n+1}{2}} q^{\binom{n}{2} - \frac{n(n-1)(2n-1)}{6}} \prod_{i=1}^{n-1} (b - aq^{i-1})^{n-i} \prod_{j=0}^{n-1} \prod_{i=j+1}^{n} (1 - cq^{i+j-1}/ab),$$

which is equal to the right side of (3.5). The case $k = n+1$ is similar.

We now consider the case $2 \le k \le n$. According to

$$\begin{aligned} &e_i(Y_{k,k}) - e_i(Y_{k-1,k}) \\ &= q^{k-n}(b-a)(1-cq^{2n-2k}/ab) e_{i-1}(Y'_{k-1,k-1}) + q^{k-n-1}(b-a)(1-cq^{2n-2k+2}/ab) e_{i-1}(Y'_{k-1,k}), \end{aligned}$$

we have

$$\begin{aligned} \det(e_{2n-i+1}(Y_{j,k})) &= c^n q^{-\binom{n}{2}} (b-a)^{n-1} \prod_{i=0}^{n-1} (1 - cq^{2i}/ab) \\ &\quad \times \left(\frac{\det(e_{2n-i-1}(Y'_{j,k-1})) q^{n-k}}{1 - cq^{2n-2k}/ab} + \frac{\det(e_{2n-i-1}(Y'_{j,k})) q^{n-k+1}}{1 - cq^{2n-2k+2}/ab} \right), \end{aligned} \tag{3.6}$$

where for $i = k-1$ or $i = k$, we have

$$Y'_{j,i} = \begin{cases} Y_{j,k} \setminus \{aq^{j-n}, cq^{n-j}/a\}, & 1 \le j < i, \\ Y_{j+1,k} \setminus \{bq^{j+1-n}, cq^{n-j-1}/b\}, & i \le j \le n-1. \end{cases}$$

Our induction hypothesis implies that

$$\det(e_{2n-i-1}(Y'_{j,k}) = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} c^{\binom{n}{2}} q^{\binom{n-k}{2} - \frac{(n-1)(n-2)(2n-3)}{6}} \times \prod_{i=1}^{n-2} (bq^{-1} - aq^{i-1})^{n-i-1} \prod_{\substack{j=0 \\ j \ne n-k}}^{n-2} \prod_{\substack{i=j+1 \\ i \ne n-k}}^{n-1} (1 - cq^{i+j}/ab),$$

and

$$\det(e_{2n-i-1}(Y'_{j,k-1}) = \begin{bmatrix} n-1 \\ k-2 \end{bmatrix} c^{\binom{n}{2}} q^{\binom{n-k+1}{2} - \frac{(n-1)(n-2)(2n-3)}{6}} \times \prod_{i=1}^{n-2} (bq^{-1} - aq^{i-1})^{n-i-1} \prod_{\substack{j=0 \\ j \ne n-k+1}}^{n-2} \prod_{\substack{i=j+1 \\ i \ne n-k+1}}^{n-1} (1 - cq^{i+j}/ab).$$

Therefore,

$$\det(e_{2n-i+1}(Y_{j,k}))_{i,j=1}^{n} = c^{\binom{n+1}{2}} q^{\binom{n-k+1}{2} - \frac{n(n-1)(2n-1)}{6}} \prod_{i=1}^{n-1} (b - aq^{i-1})^{n-i} \prod_{\substack{j=0 \\ j \ne n-k+1}}^{n-1} \prod_{\substack{i=j+1 \\ i \ne n-k+1}}^{n} (1 - cq^{i+j-1}/ab) \times \left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \frac{1 - cq^{2n-k}/ab}{1 - cq^{2n-2k+1}/ab} + \begin{bmatrix} n-1 \\ k-2 \end{bmatrix} \frac{1 - cq^{n-k}/ab}{1 - cq^{2n-2k+1}/ab} q^{n-k+1} \right).$$

With the aid of the following recurrence

$$\begin{bmatrix} n \\ k-1 \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^{n-k+1} \begin{bmatrix} n-1 \\ k-2 \end{bmatrix} = q^{k-1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k-2 \end{bmatrix},$$

we complete the proof. ∎

We are now ready to complete the proof of Theorem 3.1.

Proof of Theorem 3.1.

View $F_{n,k}(U, A, B)$ as a polynomial in u of degree $n^2 + n$ with coefficients expressed in terms of the other variables. Applying Lemma 3.2, we first prove that $F_{n,k}(U, A, B)$ has $2n$ roots:

$$aq^{1-n}, \ldots, aq^{k-1-n}, cq^{2-k}/a, \ldots, c/a, bq^{k-2n+1}, \ldots, bq^{1-n}, c/b, \ldots, cq^{n-k}/b.$$

Let $u = aq^{i-n}$ in $F_{n,k}(U,A,B)$, where $1 \le i \le k-1$. We take

$$D_0 = \varnothing, \quad D_1 = \{aq\}, \ldots, D_{i-1} = \{aq, aq^2, \ldots, aq^{i-1}\},$$

$$D_i = \{aq^{i-n}, aq, aq^2, \ldots, aq^{i-1}\}, \ldots, D_{n-1} = \{aq^{i-n}, \ldots, aq^{-1}, aq, aq^2, \ldots, aq^{i-1}\},$$

then apply Lemma 3.2.

Since $e_k(X) = 0$ if the cardinality of X is less than k, $F_{n,k}(U,A,B)$ can be transformed into a determinant whose (i,j)-th entry is equal to 0 if

$$(i,j) \in \{(i,j) : 1 \le j \le k-1 \text{ and } 1 \le i \le n-k+2, \text{ or } k \le j \le n-1 \text{ and } 1 \le i \le n-j\}.$$

Thus $F_{n,k}(U,A,B) \mid_{u=aq^{i-n}} = 0$ for $1 \le i \le k-1$.

The case $u = cq^{1-i}/a$, where $1 \le i \le k-1$, is similar to the above if we take

$$D_0 = \varnothing, \quad D_1 = \{cq^{-1}/a\}, \ldots, D_{i-1} = \{cq^{1-i}/a, \ldots, cq^{-1}/a\},$$

$$D_i = \{cq^{1-i}/a, \ldots, cq^{-1}/a, cq^{n-i}/a\}, \ldots, D_{n-1} = \{cq^{1-i}/a, \ldots, cq^{-1}/a, cq/a, \ldots, cq^{n-i}/a\}.$$

For the cases $u = bq^{2-n-i}$ and $u = cq^{i-1}/b$, where $1 \le i \le n-k+1$, we take

$$D_0 = \varnothing, \quad D_1 = \{bq^{-n}\}, \ldots, D_{i-1} = \{bq^{2-i-n}, \ldots, bq^{-n}\},$$

$$D_i = \{bq^{2-i-n}, \ldots, bq^{-n}, bq^{1-i}\}, \ldots, D_{n-1} = \{bq^{2-i-n}, \ldots, bq^{-n}, bq^{2-n}, \ldots, bq^{1-i}\}$$

and

$$D_0 = \varnothing, \quad D_1 = \{cq^n/b\}, \ldots, D_{i-1} = \{cq^n/b, \ldots, cq^{n+i-2}/b\},$$

$$D_i = \{cq^{i-1}/b, cq^n/b, \ldots, cq^{n+i-2}/b\}, \ldots, D_{n-1} = \{cq^{i-1}/b, \ldots, cq^{n-2}/b, cq^n/b, \ldots, cq^{n+i-2}/b\},$$

respectively.

In view of Lemma 3.2, $F_{n,k}(U,A,B)$ in both cases becomes a determinant whose (i,j)-th entry is equal to 0 if

$$(i,j) \in \{(i,j) : 2 \le j \le k-1 \text{ and } 1 \le i \le j-1, \text{ or } k \le j \le n \text{ and } 1 \le i \le k\}.$$

Therefore $F_{n,k}(U,A,B) \mid_{u=bq^{2-n-i}} = F_{n,k}(U,A,B) \mid_{u=cq^{i-1}/b} = 0$ for $1 \le i \le n-k+1$.

Secondly, we show that $\prod_{1 \le i < j \le n}(c - u^2 q^{i+j-2})$ is a factor of $F_{n,k}(U,A,B)$. This is because the determinant vanishes if we set $x_i = c/x_j$ in (1.3) for each pair i,j where $1 \le i < j \le n$.

Based on the above, we may assume that

$$F_{n,k}(U,A,B) = C \times \prod_{1 \le i < j \le n} (c - u^2 q^{i+j-2}) \\ \times P_{n-k+1}(b, uq^{n-1}) P_{k-1}(a, uq^{n-k+1}) P_{n-k+1}(u, c/b) P_{k-1}(c/a, u).$$

To complete the proof, we need to determine C. Setting $u=0$, then applying (3.5), we have

$$C = (-1)^{n-k+1} \frac{\det(e_{2n-i+1}(-Y_{j,k}))_{i,j=1}^{n}}{c^{\binom{n+1}{2}} q^{\binom{n-k+1}{2}}} = \begin{bmatrix} n \\ k-1 \end{bmatrix} q^{-\frac{n(n-1)(2n-1)}{6}} \times \prod_{i=1}^{n-1} (b-aq^{i-1})^{n-i} \prod_{\substack{j=0 \\ j\neq n-k+1}}^{n-1} \prod_{\substack{i=j+1 \\ i\neq n-k+1}}^{n} (1-cq^{i+j-1}/ab),$$

as desired. ∎

Putting (3.3) into (3.2), then divided both sides of (3.2) by

$$\prod_{1\leq i<j\leq n} (uq^{i-1}-uq^{j-1})(c-u^2q^{i+j-2}) b^{\binom{n+1}{2}} q^{-(n+1)n(n-1)/3} \prod_{i=1}^{n+1} (a/b, cq^{2n+2-2i}/ab; q)_{i-1},$$

we complete the proof of Corollary 1.2.

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