A complete locally convex space of countable dimension admitting an operator with no invariant subspaces

Stanislav Shkarin

Abstract

We construct a complete locally convex topological vector space X of countable algebraic dimension and a continuous linear operator $T : X \to X$ such that T has no non-trivial closed invariant subspaces.

MSC: 47A16
Keywords: Cyclic operators; invariant subspaces; topological vector spaces

1 Introduction

All vector spaces in this article are over the field $\mathbb{C}$ of complex numbers. As usual, $\mathbb{R}$ is the field of real numbers, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{N}$ is the set of positive integers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Throughout the article, all topological spaces *are assumed to be Hausdorff*. For a topological vector space X, $L(X)$ is the algebra of continuous linear operators on X and X' is the space of continuous linear functionals on X. For $T \in L(X)$, the dual operator $T' : X' \to X'$ is defined as usual: $T'f = f \circ T$.

We say that a topological vector space X has the *invariant subspace property* if every $T \in L(X)$ has a non-trivial (=different from $\{0\}$ and X) closed invariant subspace. The problem whether ℓ_2 has the invariant subspace property is known as the invariant subspace problem and remains perhaps the greatest open problem in operator theory. It is worth noting that Read [10] and Enflo [8] (see also [3]) showed independently that there are separable infinite dimensional Banach spaces, which do not have the invariant subspace property. In fact, Read [11] demonstrated that ℓ_1 does not have the invariant subspace property. All existing constructions of operators on Banach spaces with no invariant subspaces are rather sophisticated and artificial. On the other hand, examples of separable non-complete normed spaces with or without the invariant subspace property are easy to construct.

Proposition 1.1. *Every normed space of countable algebraic dimension does not have the invariant subspace property. On the other hand, in every separable infinite dimensional Banach space B, there is a dense linear subspace X such that X has the invariant subspace property.*

Thus completeness is an essential difficulty in constructing operators with no non-trivial invariant subspaces. Countable algebraic dimension is often perceived as almost incompatible with completeness. Basically, there is only one complete topological vector space of countable dimension, most analysts are aware of. Namely, the locally convex direct sum φ (see [12]) of countably many copies of the one-dimensional space $\mathbb{C}$ has countable dimension and is complete. In other words, φ is a vector space of countable dimension endowed with the topology defined by the family of *all* seminorms. It is easy to see that every linear subspace of φ is closed, which easily leads to the following observation.

Proposition 1.2. *The space φ has the invariant subspace property.*

Contrary to the common perception, there is an abundance of complete topological vector spaces of countable dimension. The main result of this paper is the following theorem.

Theorem 1.3. *There is a complete locally convex topological vector space X of countable algebraic dimension such that X does not have the invariant subspace property.*

In other words, Theorem 1.3 provides a complete locally convex topological vector space X with $\dim X = \aleph_0$ and $T \in L(X)$ such that T does not have non-trivial closed invariant subspaces.

2 Proof of Propositions 1.1 and 1.2

Although Propositions 1.1 and 1.2 are certainly known facts, we do not know whether they can be found in the literature or whether they are folklore. Granted that their proofs are fairly elementary and short, we present them for the sake of convenience.

Proof of Proposition 1.1. First, assume that X is a normed space of countable dimension. Let B be the completion of X. Then B is a separable infinite dimensional Banach space. According to Ansari [1] and Bernal–Gonzáles [4], there is a hypercyclic $T \in L(B)$. That is, there is $x \in B$ such that $\{T^n x : n \in \mathbb{Z}_+\}$ is dense in B. Let Z be the linear span of $T^n x$ for $n \in \mathbb{Z}_+$. Since Z and X are both dense linear subspaces of B, according to Grivaux [9], there is an invertible $S \in L(B)$ such that $S(Z) = X$. Since Z is invariant for T, X is invariant for STS^{-1}. That is, the restriction $A = STS^{-1}\big|_X$ belongs to $L(X)$. Let $u \in X \setminus \{0\}$. Then $S^{-1}u$ is a non-zero vector in Z and therefore $S^{-1}u = p(T)x$, where p is a non-zero polynomial. Due to Bourdon [6], $S^{-1}u = p(T)x$ is also a hypercyclic and therefore cyclic vector for T. By similarity, u is a cyclic vector for STS^{-1} and therefore for A. Thus every non-zero vector in X is cyclic for A. That is, A has no non-trivial closed invariant subspaces.

Next, let B be a separable infinite dimensional invariant subspace. Then there is a dense linear subspace X of B (X can even be chosen to be a hyperplane [5]) such that every $T \in L(X)$ has the shape $\lambda I + S$ with $\lambda \in \mathbb{C}$ and $\dim S(X) < \infty$. Trivially, such a T has a one-dimensional invariant subspace. □

Proof of Proposition 1.2. Let $T \in L(\varphi)$ and $x \in \varphi \setminus \{0\}$. Then the linear span L of $\{T^n x : n \in \mathbb{Z}_+\}$ is an (automatically closed) invariant subspace of T different from $\{0\}$. If $L \neq \varphi$, we are done. If $L = \varphi$, then the vectors $T^n x$ are linearly independent (otherwise L is finite dimensional). Hence the linear span L_1 of $\{T^n x : n \in \mathbb{N}\}$ is a hyperplane in L. Clearly L_1 is T-invariant and non-trivial. □

3 Proof of Theorem 1.3

We shall construct an operator T with no non-trivial invariant subspaces, needed in order to prove Theorem 1.3, by lifting a non-linear map on a topological space to a linear map on an appropriate topological vector space.

3.1 A class of complete countably dimensional spaces

Recall that a topological space X is called *completely regular* (or *Tychonoff*) if for every $x \in X$ and a closed subset $F \subset X$ satisfying $x \notin F$, there is a continuous $f : X \to \mathbb{R}$ such that $f(x) = 1$ and $f\big|_F = 0$. Equivalently, a topological space is completely regular, if its topology can be defined by a family of pseudometrics. Note that any subspace of a completely regular space is completely regular and that every topological group is completely regular.

Our construction is based upon the concept of the *free locally convex space* [14]. Let X be a completely regular topological space. We say that a topological vector space L_X is a free locally convex space of X if L_X is locally convex, contains X as a subset with the topology induced from L_X to X being the original topology of X and for every continuous map f from X to a locally convex space Y there is a unique continuous linear operator $T : L_X \to Y$ such that $T\big|_X = f$. It turns out that for every completely regular topological space X, there is a free locally convex space L_X unique up to an isomorphism leaving points of X invariant. Thus we can speak of *the* free locally convex space L_X of X. Note that X is always a Hamel basis in L_X. Thus, as a vector space, L_X consists of formal finite linear combinations of elements of X. Identifying $x \in X$ with the point mass measure δ_x on X ($\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ if $x \notin A$), we can also think of elements of L_X as measures with finite support on the σ-algebra of all subsets of X. Under this interpretation

$$L_X^0 = \{\mu \in L_X : \mu(X) = 0\}$$

is a closed hyperplane in the locally convex space L_X. If $f : X \to X$ is a continuous map, from the definition of the free locally convex space it follows that f extends uniquely to a continuous linear operator

$T_f \in L(L_X)$. It is also clear that L^0_X is invariant for T_f. Thus the restriction S_f of T_f to L^0_X belongs to $L(L^0_X)$.

According to Uspenskii [14], L_X is complete if and only if X is Dieudonne complete and every compact subset of X is finite. Since Dieudonne completeness follows from paracompactness, every regular countable topological space is Dieudonne complete. Since every countable compact topological space is metrizable, for a countable X, finiteness of compact subsets is equivalent to the absence of non-trivial convergent sequences (a convergent sequence is trivial if it is eventually stabilizing). Note also that a regular countable topological space is automatically completely regular and therefore we can safely replace the term 'completely regular' by 'regular' in the context of countable spaces. Thus we can formulate the following corollary of the Uspenskii theorem.

Proposition 3.1. *Let X be a regular countable topological space. Then the countably dimensional locally convex topological vector spaces L_X and L^0_X are complete if and only if there are no non-trivial convergent sequences in X.*

The above proposition provides plenty of complete locally convex spaces of countable algebraic dimension. We also need the shape of the dual space of L_X. As shown in [14], L'_X can be identified with the space $C(X)$ of continuous scalar valued functions on X in the following way. Every $f \in C(X)$ produces a continuous linear functional on L_X in the usual way:

$$\langle f, \mu\rangle = \int f\,d\mu = \sum c_j f(x_j), \quad \text{where} \quad \mu = \sum c_j \delta_{x_j}$$

and there are no other continuous linear functionals on L_X.

3.2 Operators S_f with no invariant subspaces

The following lemma is the main tool in the proof of Theorem 1.3.

Lemma 3.2. *Let τ be a regular topology on $\mathbb{Z}$ such that $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n + 1$ is a homeomorphism of $\mathbb{Z}_\tau = (\mathbb{Z}, \tau)$ onto itself, $\mathbb{Z}_+$ is dense in $\mathbb{Z}_\tau$ and for every $z \in \mathbb{C} \setminus \{0, 1\}$, $n \mapsto z^n$ is non-continuous as a map from $\mathbb{Z}_\tau$ to $\mathbb{C}$. Then the operators T_f and S_f are invertible continuous linear operators on $L_{\mathbb{Z}_\tau}$ and $L^0_{\mathbb{Z}_\tau}$ respectively and S_f has no non-trivial closed invariant subspaces.*

Proof. We already know that T_f and S_f are continuous linear operators. It is easy to see that $T_f^{-1} = T_{f^{-1}}$ and $S_f^{-1} = S_{f^{-1}}$. Since f^{-1} is also continuous, T_f and S_f have continuous inverses.

Now let $\mu \in L_{\mathbb{Z}_\tau} \setminus \{0\}$. It remains to show that μ is a cyclic vector for S_f. Assume the contrary. Then there is a non-constant $g \in C(\mathbb{Z}_\tau)$ such that $\langle S^n_f \mu, g\rangle = \langle T^n_f \mu, g\rangle = 0$ for every $n \in \mathbb{Z}_+$. Decomposing μ as a linear combination of point mass measures, we have $\mu = \sum\limits_{k=-l}^{l} c_k \delta_k$ with $c_k \in \mathbb{C}$. Then $\mu = \sum\limits_{j=0}^{2l} c_{j-k} T^j_f \delta_{-l} = p(T_f)\delta_{-l}$, where p is a non-zero polynomial. Then $0 = \langle T^n_f \mu, g\rangle = \langle T^n_f \delta_{-l}, p(T_f)' g\rangle$ for $n \in \mathbb{Z}_+$. Thus the functional $p(T_f)'g$ vanishes on the linear span of $T^n_f \delta_{-l}$ with $n \in \mathbb{Z}_+$, which contains the linear span of $\mathbb{Z}_+$ in $L_{\mathbb{Z}_\tau}$. Since $\mathbb{Z}_+$ is dense in $\mathbb{Z}_\tau$, $p(T_f)'g$ vanishes on a dense linear subspace and therefore $p(T_f)'g = 0$. It immediately follows that T'_f has an eigenvector, which is given by a non-constant function $h \in C(X)$: $T'_f h = zh$ for some $z \in \mathbb{C}$. Since T_f is invertible, so is T'_f and therefore $z \neq 0$. It is easy to see that $T'_f h(n) = h(n + 1)$ for each $n \in \mathbb{Z}$. Thus the equality $T'_f h = zh$ implies that (up to a multiplication by a non-zero constant) $h(n) = z^n$ for each $n \in \mathbb{Z}$. Since h is non-constant, $z \neq 1$. Thus the map $n \mapsto z^n$ is continuous on $\mathbb{Z}_\tau$ for some $z \in \mathbb{C} \setminus \{0, 1\}$. We have arrive to a contradiction. □

3.3 A specific countable topological space

Lemma 3.3. *There exists a regular topology τ on $\mathbb{Z}$ such that the topological space $\mathbb{Z}_\tau = (\mathbb{Z}, \tau)$ has the following properties*

(a) *$f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n + 1$ is a homeomorphism of $\mathbb{Z}_\tau$ onto itself;*

(b) $\mathbb{Z}_+$ *is dense in* $\mathbb{Z}_\tau$;

(c) *for every* $z \in \mathbb{C} \setminus \{0, 1\}$, $n \mapsto z^n$ *is non-continuous as a map from* $\mathbb{Z}_\tau$ *to* $\mathbb{C}$;

(d) $\mathbb{Z}_\tau$ *has no non-trivial convergent sequences.*

Proof. Consider the Hilbert space $\ell_2(\mathbb{Z})$ and the bilateral weighted shift $T \in L(\ell_2(\mathbb{Z}))$ given by $Te_n = e_{n-1}$ if $n \leqslant 0$ and $Te_n = 2e_{n-1}$ if $n > 0$, where $\{e_n\}_{n\in\mathbb{Z}}$ is the canonical orthonormal basis in $\ell_2(\mathbb{Z})$. Symbol $\mathcal{H}_\sigma$ stands for $\ell_2(\mathbb{Z})$ equipped with its weak topology σ. Clearly T is invertible and therefore T is a homeomorphism on $\mathcal{H}_\sigma$. According to Chan and Sanders, there is $x \in \ell_2(\mathbb{Z})$ such that the set $O = \{T^n x : n \in \mathbb{Z}_+\}$ is dense in $\mathcal{H}_\sigma$. Then $Y = \{T^n x : n \in \mathbb{Z}\}$ is also dense in $\mathcal{H}_\sigma$. We equip Y with the topology inherited from $\mathcal{H}_\sigma$ and transfer it to $\mathbb{Z}$ by declaring the bijection $n \mapsto T^n x$ from $\mathbb{Z}$ to Y a homeomorphism.

Since σ is a completely regular topology, so is the just defined topology τ on $\mathbb{Z}$. Since T is a homeomorphism on $(\ell_2(\mathbb{Z}), \sigma)$ and Y is a subspace of the topological space $(\ell_2(\mathbb{Z}), \sigma)$ invariant for both T and T^{-1}, T is a homeomorphism on Y. Since $T(T^n x) = T^{n+1}x$, it follows that f is a homeomorphism on $\mathbb{Z}_\tau$. Density of O in Y implies the density of $\mathbb{Z}_+$ in $\mathbb{Z}_\tau$.

Observe that the sequence $\{\|T^n x\|\}_{n\in\mathbb{Z}}$ is strictly increasing and $\|T^n x\| \to \infty$ as $n \to +\infty$. Indeed, the inequality $\|Tu\| \geqslant \|u\|$ for $u \in \ell_2(\mathbb{Z})$ follows from the definition of T. Hence $\{\|T^n x\|\}_{n\in\mathbb{Z}}$ is increasing. Assume that $\|T^{n+1}x\| = \|T^n x\|$ for some $n \in \mathbb{Z}$. Then, by definition of T, $T^n x$ belongs to the closed linear span L of e_n with $n < 0$. The latter is invariant for T and therefore $T^m x \in L$ for $m \geqslant n$, which is incompatible with the σ-density of O. Next, if $\|T^n x\|$ does not tend to ∞ as $n \to +\infty$, the sequence $\{\|T^n x\|\}_{n\in\mathbb{Z}_+}$ is bounded. Since every bounded subset of $\ell_2(\mathbb{Z})$ is σ-nowhere dense, we have again obtained a contradiction with the σ-density of O.

In order to show that X has no non-trivial convergent sequences, it suffices to show that Y has no non-trivial convergent sequences. Assume that $\{T^{n_k}x\}_{k\in\mathbb{Z}_+}$ is a non-trivial convergent sequence in Y. Without loss of generality, we can assume that the sequence $\{n_k\}$ of integers is either strictly increasing or strictly decreasing. If $\{n_k\}$ is strictly increasing the above observation ensures that $\|T^{n_k}x\| \to \infty$ as $k \to \infty$. Since every σ-convergent sequence is bounded, we have arrived to a contradiction. If $\{n_k\}$ is strictly decreasing, then by the above observation, the sequence $\{\|T^{n_k}x\|\}$ of positive numbers is also strictly decreasing and therefore converges to $c \geqslant 0$. Then $\|T^l x\| > c$ for every $l \in \mathbb{Z}$. Since $\{T^{n_k}x\}$ σ-converges to $T^m x \in Y$, the upper semicontinuity of the norm function with respect to σ implies that $\|T^m x\| \leqslant c$ and we have arrived to a contradiction.

Finally, let $z \in \mathbb{C} \setminus \{0, 1\}$ and $f : \mathbb{Z} \to \mathbb{C}$, $f(n) = z^n$. It remains to show that f is not continuous as a function on $\mathbb{Z}_\tau$. Equivalently, it is enough to show that the function $g : Y \to \mathbb{C}$, $g(T^n x) = z^n$ is non-continuous. Assume the contrary. There are two possibilities. First, consider the case $|z| \neq 1$. In this case the the topology on the set $M = \{z^n : n \in \mathbb{Z}\}$ inherited from $\mathbb{C}$ is the discrete topology. Continuity of the bijection $g : Y \to M$ implies then that Y is also discrete, which is apparently not the case: the density of Y in $\mathcal{H}_\sigma$ ensures that Y has no isolated points. It remains to consider the case $|z| = 1$, $z \neq 1$. In this case the closure G of $\{z^n : n \in \mathbb{Z}\}$ is a closed subgroup of the compact abelian topological group $\mathbb{T}$. Since x is a hypercyclic vector for T acting on $\mathcal{H}_\sigma$ and z generates the compact abelian topological group G, [13, Corollary 4.1] implies that $\{(T^n x, z^n) : n \in \mathbb{Z}_+\}$ is dense in $\mathcal{H}_\sigma \times G$. It follows that the graph of g is dense in $Y \times G$. Since G is not a single-point space, the latter is incompatible with the continuity of g. The proof is now complete. □

Proof of Theorem 1.3. Let τ be the topology on $\mathbb{Z}$ provided by Lemma 3.3. By Proposition 3.1, $E = L^0_{\mathbb{Z}_\tau}$ is a complete locally convex space of countable algebraic dimension. Let $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n + 1$ and $S = S_f \in L(E)$. By Lemmas 3.2 and 3.3, S is invertible and has no non-trivial invariant subspaces. The proof is complete (with an added bonus of invertibility of S). □

It is easy to see that the topology τ on $\mathbb{Z}$ constructed in the proof of Lemma 3.3 does not agree with the group structure. That is $\mathbb{Z}_\tau$ is not a topological group. Indeed, it is easy to see that the group operation $+$ is only separately continuous on $\mathbb{Z}_\tau$, but not jointly continuous. As a matter of curiosity, it would be interesting to find out whether there exists a topology τ on $\mathbb{Z}$ satisfying all conditions of Lemma 3.3 and turning $\mathbb{Z}$ into a topological group.

References

[1] S. Ansari, *Existence of hypercyclic operators on topological vector spaces,* J. Funct. Anal. **148** (1997), 384–390

[2] F. Bayart and E. Matheron, *Dynamics of linear operators,* Cambridge University Press, Cambridge, 2009

[3] B. Beauzamy, *Introduction to operator theory and invariant subspaces,* North-Holland Mathematics, Amsterdam, 1988

[4] L. Bernal-González, *On hypercyclic operators on Banach spaces,* Proc. Amer. Math. Soc. **127** (1999), 1003–1010

[5] J. Bonet and P. Pérez-Carreras, *Barrelled locally convex spaces,* North-Holland Mathematics Studies **131**, North-Holland Publishing Co., Amsterdam, 1987

[6] P. Bourdon, *Invariant manifolds of hypercyclic vectors,* Proc. Amer. Math. Soc. **118** (1993), 845–847

[7] K. Chan and R. Sanders, *A weakly hypercyclic operator that is not norm hypercyclic,* J. Operator Theory **52** (2004), 39–59

[8] P. Enflo, *On the invariant subspace problem for Banach spaces,* Acta Math. **158** (1987), 212-313

[9] S. Grivaux, *Construction of operators with prescribed behaviour,* Arch. Math. (Basel) **81** (2003), 291–299

[10] C. Read, *A solution to the invariant subspace problem,* Bull. London Math. Soc. **16** (1984), 337-401

[11] C. Read, *A solution to the invariant subspace problem on the space ℓ_1,* Bull. London Math. Soc. **17** (1985), 305-317

[12] A. Robertson and W. Robertson, *Topological vector spaces,* Cambridge University Press, Cambridge, 1980

[13] S. Shkarin, *Universal elements for non-linear operators and their applications,* J. Math. Anal. Appl. **348** (2008), 193–210

[14] V. Uspenskii, *On the topology of a free locally convex space,* Soviet Math. Dokl. **27** (1983), 771–785

Stanislav Shkarin
Queens's University Belfast
Department of Pure Mathematics
University road, Belfast, BT7 1NN, UK
E-mail address: s.shkarin@qub.ac.uk