

# Composition Functionals in Fractional Calculus of Variations

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## Abstract

We prove Euler–Lagrange and natural boundary necessary optimality conditions for fractional problems of the calculus of variations which are given by a composition of functionals. Our approach uses the recent notions of Riemann–Liouville fractional derivatives and integrals in the sense of Jumarie. As an application, we get optimality conditions for the product and the quotient of fractional variational functionals.

**Keywords:** Fractional calculus of variations; Composition of functionals; Fractional Euler–Lagrange equations; Fractional natural boundary conditions; Modified Riemann–Liouville derivative and integral.

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## 1 Introduction

The present work is dedicated to the study of general (non-classical) fractional problems of calculus of variations. As a particular case, when  $\alpha \rightarrow 1$ , one gets the generalized calculus of variations [1] with functionals of the form

$$H \left( \int_a^b f(t, x(t), x'(t)) dt \right),$$

where  $f$  has  $n$  components and  $H$  has  $n$  independent variables. Problems of calculus of variations as these appear in practical applications (see [1–3] and the references given therein) but cannot be solved using the classical theory. Therefore, an extension of this theory is needed.

The fractional calculus of variations started in 1996 with the work of Riewe [4]. Riewe formulated the problem of the calculus of variations with fractional derivatives and obtained the respective Euler–Lagrange equations, combining both conservative and nonconservative cases. Nowadays the fractional calculus of variations is a subject under strong research. Different definitions for fractional derivatives and integrals are used, depending on the purpose under study. Investigations cover problems depending on Riemann–Liouville fractional derivatives (see, e.g., [5–8]), the Caputo fractional derivative (see, e.g., [9–11]), the symmetric fractional derivative (see, e.g., [12, 13]), the Jumarie fractional derivative (see, e.g., [14–21]), and others [22–26]. For applications of the fractional calculus of variations we refer the reader to [12, 14, 18, 25, 27–31]. Here we use the fractional calculus proposed by Jumarie. This modified Riemann–Liouville calculus has shown recently to be very useful in the fractional calculus of variations for multiple integrals [14], and provides an efficient tool to solve fractional differential equations [21].

The paper is organized as follows. In Section 2 we present some preliminaries on the fractional calculus proposed by Jumarie. Our results are then given in Section 3. We begin Section 3 by formulating the general (non-classical) fractional problem of calculus of variations (3). The problem is defined via the fractional derivative and the fractional integral in the sense of Jumarie. We obtain Euler–Lagrange equations and natural boundary conditions for the general problem (Theorem 3.2), which are then applied to the product (Corollary 3.4) and the quotient (Corollary 3.6). In Section 4 we provide an example illustrating our results.

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## 2 Preliminaries

For an introduction to the classical fractional calculus we refer the reader to [32–34]. In this section we briefly review the main notions and results from the recent fractional calculus proposed by Jumarie [16–19].

**Definition 2.1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous (but not necessarily differentiable) function. The Jumarie fractional derivative of  $f$  is defined by

$$f^{(\alpha)}(t) := \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\tau)^{-\alpha-1} (f(\tau) - f(a)) d\tau, \quad \alpha < 0,$$

where  $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ . For a positive  $\alpha$ ,

$$f^{(\alpha)}(t) = (f^{(\alpha-1)}(t))' = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} (f(\tau) - f(a)) d\tau,$$

in the case  $0 < \alpha < 1$ , and

$$f^{(\alpha)}(t) := (f^{(\alpha-n)}(t))^{(n)}, \quad n \leq \alpha < n+1, \quad n \geq 1.$$

The Jumarie fractional derivative has the following properties:

- (i) The  $\alpha$ th derivative of a constant is zero.
- (ii) If  $0 < \alpha \leq 1$ , then the Laplace transform of  $f^{(\alpha)}$  is given by

$$\mathfrak{L}\{f^{(\alpha)}(t)\} = s^\alpha \mathfrak{L}\{f(t)\} - s^{\alpha-1} f(0).$$

- (iii)  $(g(t)f(t))^{(\alpha)} = g^{(\alpha)}(t)f(t) + g(t)f^{(\alpha)}(t)$ ,  $0 < \alpha < 1$ .

**Example 2.2.** Let  $f(t) = t^\gamma$ ,  $\gamma > 0$ , and  $0 < \alpha < 1$ . Then  $f^{(\alpha)}(x) = \Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-\alpha)t^{\gamma-\alpha}$ .

**Example 2.3.** Let  $c$  and  $x_0$  be given constants. The solution of the fractional differential equation

$$x^{(\alpha)}(t) = c, \quad x(0) = x_0,$$

is given by

$$x(t) = \frac{c}{\alpha!} t^\alpha + x_0,$$

where  $\alpha! := \Gamma(1+\alpha)$ .

The integral with respect to  $(dt)^\alpha$  is defined as the solution of the fractional differential equation

$$dy = f(x)(dx)^\alpha, \quad x \geq 0, \quad y(0) = y_0, \quad 0 < \alpha \leq 1. \quad (1)$$

Such solution is provided by the following result:

**Lemma 2.4.** Let  $f$  denote a continuous function. The solution of the equation (1) is

$$\int_0^t f(\tau)(d\tau)^\alpha := \alpha \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad 0 < \alpha \leq 1.$$

**Example 2.5.** Let  $f(t) \equiv 1$ , and  $0 < \alpha \leq 1$ . Then,  $\int_0^t (d\tau)^\alpha = t^\alpha$ .

**Example 2.6.** The solution of the fractional differential equation  $x^{(\alpha)}(t) = f(t)$ ,  $x(0) = x_0$ , is

$$x(t) = x_0 + \Gamma^{-1}(\alpha) \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

In the discussion to follow, we will need the following formula of integration by parts:

$$\int_a^b u^{(\alpha)}(t)v(t)(dt)^\alpha = \alpha! [u(t)v(t)]_a^b - \int_a^b u(t)v^{(\alpha)}(t)(dt)^\alpha, \quad (2)$$

where  $\alpha! := \Gamma(1+\alpha)$ .

### 3 Main Results

The general (non-classical) problem of the fractional calculus of variations under our consideration consists of extremizing (i.e., minimizing or maximizing)

$$\mathcal{L}[x] = H \left( \int_a^b f_1(t, x(t), x^{(\alpha_1)}(t))(dt)^{\alpha_1}, \dots, \int_a^b f_n(t, x(t), x^{(\alpha_n)}(t))(dt)^{\alpha_n} \right) \quad (3)$$

$$(x(a) = x_a) \quad (x(b) = x_b)$$

over all  $x \in \mathbf{D}$  with

$$\mathbf{D} := \{x \in C^0 : x^{(\alpha_i)}, i = 1, \dots, n, \text{ exists and is continuous on the interval } [a, b]\}.$$

Using parentheses around the end-point conditions means that these conditions may or may not be present. We assume that:

- (i) the function  $H : \mathbb{R}^n \rightarrow \mathbb{R}$  has continuous partial derivatives with respect to its arguments and we denote them by  $H'_i$ ,  $i = 1, \dots, n$ ;
- (ii) functions  $(t, y, v) \rightarrow f_i(t, y, v)$  from  $[a, b] \times \mathbb{R}^2$  to  $\mathbb{R}$ ,  $i = 1, \dots, n$ , have partial continuous derivatives with respect to  $y, v$  for all  $t \in [a, b]$  and we denote them by  $f_{iy}, f_{iv}$ ;
- (iii)  $f_i$ ,  $i = 1, \dots, n$ , and their partial derivatives are continuous in  $t$  for all  $x \in \mathbf{D}$ .

A function  $x \in \mathbf{D}$  is said to be an admissible function provided that it satisfies the end-points conditions (if any is given). The following norm in  $\mathbf{D}$  is considered:

$$\|x\| = \max_{t \in [a, b]} |x(t)| + \sum_{i=1}^n \max_{t \in [a, b]} |x^{(\alpha_i)}(t)|.$$

**Definition 3.1.** An admissible function  $\tilde{x}$  is said to be a *weak local minimizer* (resp. *weak local maximizer*) for (3) if there exists  $\delta > 0$  such that  $\mathcal{L}[\tilde{x}] \leq \mathcal{L}[x]$  (resp.  $\mathcal{L}[\tilde{x}] \geq \mathcal{L}[x]$ ) for all admissible  $x$  with  $\|x - \tilde{x}\| < \delta$ .

For simplicity of notation we introduce the operator  $\langle x \rangle_i$ ,  $i = 1, \dots, n$ , defined by

$$\langle x \rangle_i(t) = (t, x(t), x^{(\alpha_i)}(t)).$$

Then,

$$\mathcal{L}[x] = H \left( \int_a^b f_1 \langle x \rangle_1(t)(dt)^{\alpha_1}, \dots, \int_a^b f_n \langle x \rangle_n(t)(dt)^{\alpha_n} \right).$$

The next theorem gives necessary optimality conditions for problem (3).

**Theorem 3.2.** *If  $\tilde{x}$  is a weak local solution to problem (3), then the Euler–Lagrange equation*

$$\sum_{i=1}^n \alpha_i H'_i(\mathcal{F}_1[\tilde{x}], \dots, \mathcal{F}_n[\tilde{x}]) (b-t)^{\alpha_i-1} \left( f_{iy} \langle \tilde{x} \rangle_i(t) - f_{iv}^{(\alpha_i)} \langle \tilde{x} \rangle_i(t) \right) = 0$$

holds for all  $t \in [a, b)$ , where  $\mathcal{F}_i[\tilde{x}] = \int_a^b f_i \langle \tilde{x} \rangle_i(t)(dt)^{\alpha_i}$ ,  $i = 1, \dots, n$ . Moreover, if  $x(a)$  is not specified, then

$$\sum_{i=1}^n \alpha_i H'_i(\mathcal{F}_1[\tilde{x}], \dots, \mathcal{F}_n[\tilde{x}]) f_{iv} \langle \tilde{x} \rangle_i(a) = 0; \quad (4)$$

if  $x(b)$  is not specified, then

$$\sum_{i=1}^n \alpha_i H'_i(\mathcal{F}_1[\tilde{x}], \dots, \mathcal{F}_n[\tilde{x}]) f_{iv} \langle \tilde{x} \rangle_i(b) = 0. \quad (5)$$

*Proof.* Suppose that  $\mathcal{L}[x]$  has a weak local extremum at  $\tilde{x}$ . For an admissible variation  $h \in \mathbf{D}$  we define a function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  by  $\phi(\varepsilon) = \mathcal{L}[\tilde{x} + \varepsilon h]$ . We do not require  $h(a) = 0$  or  $h(b) = 0$  in case  $x(a)$  or  $x(b)$ , respectively, is free (it is possible that both are free). A necessary condition for  $\tilde{x}$  to be an extremizer for  $\mathcal{L}[x]$  is given by  $\phi'(\varepsilon)|_{\varepsilon=0} = 0$ . Using the chain rule to obtain the derivative of a composed function, we get

$$\phi'(\varepsilon)|_{\varepsilon=0} = \sum_{i=1}^n H'_i(\mathcal{F}_1[\tilde{x}], \dots, \mathcal{F}_n[\tilde{x}]) \int_a^b [f_{iy}\langle \tilde{x} \rangle_i(t)h(t) + f_{iv}\langle \tilde{x} \rangle_i(t)h^{(\alpha_i)}(t)] (dt)^{\alpha_i}.$$

Integration by parts (see equation (2)) of the second term of the integrands, gives

$$\int_a^b f_{iv}\langle \tilde{x} \rangle_i(t)h^{(\alpha_i)}(t)(dt)^{\alpha_i} = [\alpha_i! f_{iv}\langle \tilde{x} \rangle_i(t)h(t)]_{t=a}^{t=b} - \int_a^b f_{iv}^{(\alpha_i)}\langle \tilde{x} \rangle_i(t)h(t)(dt)^{\alpha_i}.$$

The necessary condition  $\phi'(\varepsilon)|_{\varepsilon=0} = 0$  can be written as

$$\begin{aligned} 0 = \sum_{i=1}^n H'_i(\mathcal{F}_1[\tilde{x}], \dots, \mathcal{F}_n[\tilde{x}]) \int_a^b \left( f_{iy}\langle \tilde{x} \rangle_i(t) - f_{iv}^{(\alpha_i)}\langle \tilde{x} \rangle_i(t) \right) h(t)(dt)^{\alpha_i} \\ + \sum_{i=1}^n H'_i(\mathcal{F}_1[\tilde{x}], \dots, \mathcal{F}_n[\tilde{x}]) [\alpha_i! f_{iv}\langle \tilde{x} \rangle_i(t)h(t)]_{t=a}^{t=b}. \end{aligned}$$

Taking into account Lemma 2.4, we have

$$\begin{aligned} 0 = \int_a^b \sum_{i=1}^n \alpha_i H'_i(\mathcal{F}_1[\tilde{x}], \dots, \mathcal{F}_n[\tilde{x}]) (b-t)^{\alpha_i-1} \left( f_{iy}\langle \tilde{x} \rangle_i(t) - f_{iv}^{(\alpha_i)}\langle \tilde{x} \rangle_i(t) \right) h(t) dt \\ + \sum_{i=1}^n H'_i(\mathcal{F}_1[\tilde{x}], \dots, \mathcal{F}_n[\tilde{x}]) [\alpha_i! f_{iv}\langle \tilde{x} \rangle_i(t)h(t)]_{t=a}^{t=b}. \quad (6) \end{aligned}$$

In particular, equation (6) holds for all variations which are zero at both ends. For all such  $h$ 's, the second term in (6) is zero and by the Dubois-Reymond Lemma (see, e.g., [35]), we have that

$$\sum_{i=1}^n \alpha_i H'_i(\mathcal{F}_1[\tilde{x}], \dots, \mathcal{F}_n[\tilde{x}]) (b-t)^{\alpha_i-1} \left( f_{iy}\langle \tilde{x} \rangle_i(t) - f_{iv}^{(\alpha_i)}\langle \tilde{x} \rangle_i(t) \right) = 0 \quad (7)$$

holds for all  $t \in [a, b)$ . Equation (6) must be satisfied for all admissible values of  $h(a)$  and  $h(b)$ . Consequently, equations (6) and (7) imply that

$$0 = \sum_{i=1}^n H'_i(\mathcal{F}_1[\tilde{x}], \dots, \mathcal{F}_n[\tilde{x}]) \alpha_i! f_{iv}\langle \tilde{x} \rangle_i(t)h(b) - \sum_{i=1}^n H'_i(\mathcal{F}_1[\tilde{x}], \dots, \mathcal{F}_n[\tilde{x}]) \alpha_i! f_{iv}\langle \tilde{x} \rangle_i(t)h(a). \quad (8)$$

If  $x$  is not preassigned at either end-point, then  $h(a)$  and  $h(b)$  are both completely arbitrary and we conclude that their coefficients in (8) must each vanish. It follows that condition (4) holds when  $x(a)$  is not given, and condition (5) holds when  $x(b)$  is not given.  $\square$

Note that in the limit, when  $\alpha_i \rightarrow 1$ ,  $i = 1, \dots, n$ , Theorem 3.2 implies the following result:

**Corollary 3.3** (Th. 3.1 and Eq. (4.1) in [1]). *If  $\tilde{x}$  is a solution to problem*

$$\begin{aligned} \mathcal{L}[x] = H \left( \int_a^b f_1(t, x(t), x'(t)) dt, \dots, \int_a^b f_n(t, x(t), x'(t)) dt \right) \longrightarrow \text{extr}, \\ (x(a) = x_a) \quad (x(b) = x_b) \end{aligned}$$

*then the Euler–Lagrange equation*

$$\sum_{i=1}^n H'_i(\mathcal{F}_1[\tilde{x}], \dots, \mathcal{F}_n[\tilde{x}]) \left( f_{iy}(t, \tilde{x}(t), \tilde{x}'(t)) - \frac{d}{dx} f_{iv}(t, \tilde{x}(t), \tilde{x}'(t)) \right) = 0$$

holds for all  $t \in [a, b]$ , where  $\mathcal{F}_i[\tilde{x}] = \int_a^b f_i(t, \tilde{x}(t), \tilde{x}'(t)) dt$ ,  $i = 1, \dots, n$ . Moreover, if  $x(a)$  is not specified, then

$$\sum_{i=1}^n H'_i(\mathcal{F}_1[\tilde{x}], \dots, \mathcal{F}_n[\tilde{x}]) f_{i_v}(a, \tilde{x}(a), \tilde{x}'(a)) = 0;$$

if  $x(b)$  is not specified, then

$$\sum_{i=1}^n H'_i(\mathcal{F}_1[\tilde{x}], \dots, \mathcal{F}_n[\tilde{x}]) f_{i_v}(b, \tilde{x}(b), \tilde{x}'(b)) = 0.$$

**Corollary 3.4.** *If  $\tilde{x}$  is a solution to problem*

$$\mathcal{L}[x] = \left( \int_a^b f_1 \langle \tilde{x} \rangle_1(t) (dt)^{\alpha_1} \right) \left( \int_a^b f_2 \langle \tilde{x} \rangle_2(t) (dt)^{\alpha_2} \right) \longrightarrow \text{extr},$$

$$(x(a) = x_a) \quad (x(b) = x_b)$$

then the Euler–Lagrange equation

$$\alpha_1 \mathcal{F}_2[\tilde{x}] (b-t)^{\alpha_1-1} \left( f_{1_y} \langle \tilde{x} \rangle_1(t) - f_{1_v}^{(\alpha_1)} \langle \tilde{x} \rangle_1(t) \right) + \alpha_2 \mathcal{F}_1[\tilde{x}] (b-t)^{\alpha_2-1} \left( f_{2_y} \langle \tilde{x} \rangle_2(t) - f_{2_v}^{(\alpha_2)} \langle \tilde{x} \rangle_2(t) \right) = 0$$

holds for all  $t \in [a, b]$ . Moreover, if  $x(a)$  is not specified, then

$$\alpha_1! \mathcal{F}_2[\tilde{x}] f_{1_v} \langle \tilde{x} \rangle_1(a) + \alpha_2! \mathcal{F}_1[\tilde{x}] f_{2_v} \langle \tilde{x} \rangle_2(a) = 0;$$

if  $x(b)$  is not specified, then

$$\alpha_1! \mathcal{F}_2[\tilde{x}] f_{1_v} \langle \tilde{x} \rangle_1(b) + \alpha_2! \mathcal{F}_1[\tilde{x}] f_{2_v} \langle \tilde{x} \rangle_2(b) = 0.$$

*Remark 3.5.* In the case  $\alpha_i \rightarrow 1$ ,  $i = 1, 2$ , Corollary 3.4 gives a result of [1]: the Euler–Lagrange equation associated with the product functional

$$\mathcal{L}[x] = \left( \int_a^b f_1(t, x(t), x'(t)) dt \right) \left( \int_a^b f_2(t, x(t), x'(t)) dt \right)$$

is

$$\mathcal{F}_2[x] \left( f_{1_y}(t, x(t), x'(t)) - \frac{d}{dt} f_{1_v}(t, x(t), x'(t)) \right) + \mathcal{F}_1[x] \left( f_{2_y}(t, x(t), x'(t)) - \frac{d}{dt} f_{2_v}(t, x(t), x'(t)) \right) = 0$$

and the natural condition at  $t = a$ , when  $x(a)$  is free, becomes

$$\mathcal{F}_2[x] f_{1_v}(a, x(a), x'(a)) + \mathcal{F}_1[x] f_{2_v}(a, x(a), x'(a)) = 0.$$

**Corollary 3.6.** *If  $\tilde{x}$  is a solution to problem*

$$\mathcal{L}[x] = \frac{\int_a^b f_1 \langle \tilde{x} \rangle_1(t) (dt)^{\alpha_1}}{\int_a^b f_2 \langle \tilde{x} \rangle_2(t) (dt)^{\alpha_2}} \longrightarrow \text{extr},$$

$$(x(a) = x_a) \quad (x(b) = x_b)$$

then the Euler–Lagrange equation

$$\alpha_1 (b-t)^{\alpha_1-1} \left( f_{1_y} \langle \tilde{x} \rangle_1(t) - f_{1_v}^{(\alpha_1)} \langle \tilde{x} \rangle_1(t) \right) - \alpha_2 Q (b-t)^{\alpha_2-1} \left( f_{2_y} \langle \tilde{x} \rangle_2(t) - f_{2_v}^{(\alpha_2)} \langle \tilde{x} \rangle_2(t) \right) = 0$$

holds for all  $t \in [a, b]$ , where  $Q = \frac{\mathcal{F}_1[\tilde{x}]}{\mathcal{F}_2[\tilde{x}]}$ . Moreover, if  $x(a)$  is not specified, then  $\alpha_1! f_{1_v} \langle \tilde{x} \rangle_1(a) - \alpha_2! Q f_{2_v} \langle \tilde{x} \rangle_2(a) = 0$ ; if  $x(b)$  is not specified, then  $\alpha_1! f_{1_v} \langle \tilde{x} \rangle_1(b) - \alpha_2! Q f_{2_v} \langle \tilde{x} \rangle_2(b) = 0$ .

*Remark 3.7.* In the case  $\alpha_i \rightarrow 1$ ,  $i = 1, 2$ , Corollary 3.6 gives the following result of [1]: the Euler–Lagrange equation associated with the quotient functional

$$\mathcal{L}[x] = \frac{\int_a^b f_1(t, x(t), x'(t)) dt}{\int_a^b f_2(t, x(t), x'(t)) dt}$$

is

$$f_{1_y}(t, x(t), x'(t)) - Q f_{2_y}(t, x(t), x'(t)) - \frac{d}{dt} [(f_{1_v}(t, x(t), x'(t)) - Q f_{2_v}(t, x(t), x'(t)))] = 0$$

and the natural condition at  $t = a$ , when  $x(a)$  is free, becomes

$$f_{1_v}(a, x(a), x'(a)) - Q f_{2_v}(a, x(a), x'(a)) = 0.$$

## 4 An Example

Consider the problem

$$\begin{aligned} \text{minimize } \mathcal{L}[x] &= \left( \int_0^1 (x^{(\frac{1}{2})}(t))^2 (dt)^{\frac{1}{2}} \right) \left( \int_0^1 t^{\frac{1}{2}} x^{(\frac{1}{2})}(t) (dt)^{\frac{1}{2}} \right) \\ x(0) &= 0, \quad x(1) = 1. \end{aligned} \quad (9)$$

If  $\tilde{x}$  is a local minimizer to (9), then the fractional Euler–Lagrange equation must hold, i.e.,

$$\frac{1}{2} Q_2 (1-t)^{-\frac{1}{2}} 2(\tilde{x}^{(\frac{1}{2})}(t))^{(\frac{1}{2})} + \frac{1}{2} Q_1 (1-t)^{-\frac{1}{2}} (t^{\frac{1}{2}})^{(\frac{1}{2})} = 0,$$

where

$$Q_1 = \int_0^1 (\tilde{x}^{(\frac{1}{2})}(t))^2 (dt)^{\frac{1}{2}}, \quad Q_2 = \int_0^1 t^{\frac{1}{2}} \tilde{x}^{(\frac{1}{2})}(t) (dt)^{\frac{1}{2}}.$$

Hence,

$$Q_2 2(\tilde{x}^{(\frac{1}{2})}(t))^{(\frac{1}{2})} + Q_1 \frac{\sqrt{\pi}}{2} = 0. \quad (10)$$

If  $Q_2 = 0$ , then also  $Q_1 = 0$ . This contradicts the fact that a global minimizer to the problem

$$\begin{aligned} \text{minimize } \mathcal{F}_1[x] &= \int_0^1 (x^{(\frac{1}{2})}(t))^2 (dt)^{\frac{1}{2}} \\ x(0) &= 0, \quad x(1) = 1 \end{aligned}$$

is  $\bar{x}(t) = t^{\frac{1}{2}}$  and  $\mathcal{F}_1[\bar{x}] = (\frac{\sqrt{\pi}}{2})^2$ . This can be easily shown by the results obtained in [15]. Hence,  $Q_2 \neq 0$  and (10) implies that candidate solutions to problem (9) are those satisfying the fractional differential equation

$$(\tilde{x}^{(\frac{1}{2})}(t))^{(\frac{1}{2})} = -\frac{Q_1 \sqrt{\pi}}{4Q_2} \quad (11)$$

subject to the boundary conditions  $x(0) = 0$  and  $x(1) = 1$ . Solving equation (11) we obtain

$$x(t) = \frac{1}{\sqrt{\pi}} \int_0^t \left( \frac{Q_1 \pi + 4\sqrt{\pi} Q_2}{8Q_2} - \frac{Q_1}{2Q_2} \tau^{\frac{1}{2}} \right) (t-\tau)^{-\frac{1}{2}} d\tau. \quad (12)$$

Substituting (12) into functionals  $\mathcal{F}_1$  and  $\mathcal{F}_2$  gives

$$\begin{cases} -\frac{1}{192} \frac{-32Q_1^2 - 48\pi Q_2^2 + 3Q_1^2 \pi^2}{Q_2^2} = Q_1 \\ \frac{1}{96} \frac{-32Q_1 + 3Q_1 \pi^2 + 12\pi^{3/2} Q_2}{Q_2} = Q_2. \end{cases} \quad (13)$$

We obtain the candidate minimizer to problem (9) solving the system of equations (13):

$$\tilde{x}(t) = \frac{1}{\sqrt{\pi}} \int_0^t \left( \frac{Q_1 \pi + 4\sqrt{\pi} Q_2}{8Q_2} - \frac{Q_1}{2Q_2} \tau^{\frac{1}{2}} \right) (t-\tau)^{-\frac{1}{2}} d\tau,$$

where

$$Q_1 = \frac{4\pi(\sqrt{\pi}(\frac{1}{4}\pi^{\frac{3}{2}} + \frac{1}{4}\sqrt{\pi^3 - 8\pi}) - 4)}{-32 + 3\pi^2} \quad \text{and} \quad Q_2 = \frac{1}{12}\pi^{\frac{3}{2}} + \frac{1}{12}\sqrt{\pi^3 - 8\pi}.$$

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