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SCHUR'S THEOEM OF ANTIHOLOMORPHIC TYPE FOR QUASI-KÄHLER MANIFOLDS

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Let M be an almost Hermitian manifold with $\dim M = 2n$, a metric tensor g and an almost complex structure J. The Ricci tensor and the scalar curvature with regard to the curvature tensor R are denoted by S and τ respectively. If α is a 2-plane in the tangential space T_pM with an orthonormal basis $\{x, y\}$, then its sectional curvature $K(\alpha, p)$ is given by $K(\alpha, p) = R(x, y, y, x)$. A 2-plane α in T_pM is said to be holomorphic (antiholomorphic) if $J\alpha = \alpha$ ($J\alpha \perp \alpha$). The manifold M is said to be of pointwise constant holomorphic (antiholomorphic) sectional curvature if the sectional curvature $K(\alpha, p)$ of an arbitrary holomorphic (antiholomorphic) 2-plane α in T_pM does not depend on α for every $p \in M$. The classes of Kählerian ($K-$), nearly-Kählerian ($NK-$) and quasi-Kählerian ($QK-$) manifolds are characterized by the following identities for the covariant derivative of J: $(\nabla_X J)Y = 0$, $(\nabla_X J)X = 0$, $(\nabla_{JX} J)Y = -J(\nabla_X J)Y$ respectively. Every K-manifold is an NK-manifold and every NK-manifold is a QK-manifold. The curvature tensor of an NK-manifold satisfies the identity [5]

$$R(x, y, z, u) = R(x, y, Jz, Ju) + R(x, Jy, z, Ju) + R(Jx, y, z, Ju) \ . \tag{1}$$

This equality implies

$$R(x, y, z, u) = R(Jx, Jy, Jz, Ju) \ . \tag{2}$$

A QK-manifold whose curvature tensor satisfies (1) is said to be a QK_2-manifold. An almost Hermitian manifold whose curvature tensor satisfies (2) is said to be an AK_3-manifold. For a K-manifold the following analogues of the classical theorem of F. Schur hold [7, 2]:

Let M be a connected $2n-$dimensional K-manifold of pointwise constant holomorphic (antiholomorphic) sectional curvature c. If $n \geq 2$ ($n \geq 3$), then c is a global constant. Moreover M is of constant antiholomorphic (holomorphic) sectional curvature $c/4\,(c)$.

Naveira and Hervella proved [8] that if a connected NK-manifold of dimension $2n \geq 4$ has a pointwise constant holomorphic sectional curvature μ, then μ is a constant and Gray proved in [4] that such a manifold is locally isometric to one of the following manifolds: $\mathbb{C}^n$, $\mathbb{CP}^n$, $\mathbb{CD}^n$, $\mathbb{S}^6$. We proved in [3] that if M is a connected NK-manifold of pointwise constant antiholomorphic sectional curvature ν and $\dim M \geq 6$, then ν is a constant and M is locally isometric to $\mathbb{C}^n$, $\mathbb{CP}^n$, $\mathbb{CD}^n$ or $\mathbb{S}^6$.

Gray and Vanhecke proved in [6] that if a connected QK_2-manifold has a pointwise constant holomorphic sectional curvature μ, then μ is a constant.

In the present paper we consider QK_2-manifolds of pointwise constant antiholomorphic sectional curvature.

Let $\{e_1, \ldots, e_{2n}\}$ be an orthonormal basis of T_pM, $p \in M$, and

$$S'(x,y) = \sum_{i=1}^{2n} R(x, e_i, Je_i, Jy) \ ,$$

$$\tau' = \sum_{i=1}^{2n} S'(e_i, e_i) \ .$$

The tensors R_1, R_2 and ψ are defined by

$$R_1(x,y,z,u) = g(y,z)g(x,u) - g(x,z)g(y,u) \ ;$$

$$R_2(x,y,z,u) = g(Jy,z)g(Jx,u) - g(Jx,z)g(Jy,u) - 2g(Jx,y)g(Jz,u) \ ;$$

$$\begin{aligned}\psi(x,y,z,u) = {} & g(Jy,z)S(Jx,u) - g(Jx,z)S(Jy,u) - 2g(Jx,y)S(Jz,u) \\ & + g(Jx,u)S(Jy,z) - g(Jy,u)S(Jx,z) - 2g(Jz,u)S(Jx,y) \ .\end{aligned}$$

In [3] we proved

Proposition. Let M be a $2n$-dimensional AH_3-manifold of pointwise constant antiholomorphic sectional curvature ν. Then

$$R = \frac{1}{6}\psi + \nu R_1 - \frac{2n-1}{3}\nu R_2 \ , \tag{3}$$

$$(n+1)S - 3S' = \frac{(n+1)\tau - 3\tau'}{2n} g \ , \tag{4}$$

$$\nu = \frac{(2n+1)\tau - 3\tau'}{8n(n^2-1)} \ . \tag{5}$$

In particular from (3) it follows that R satisfies (1).

From the second Bianchi's identity

$$(\nabla_w R)(x,y,z,u) + (\nabla_x R)(y,w,z,u) + (\nabla_y R)(w,x,z,u) = 0 \tag{6}$$

the next identities follow in a straightforward way:

$$(\nabla_x S)(y,z) - (\nabla_y S)(x,z) = \sum_{i=1}^{2n} (\nabla_{e_i} R)(x,y,z,e_i) \ , \tag{7}$$

$$\sum_{i=1}^{2n} (\nabla_{e_i} S)(x, e_i) = \frac{1}{2} x(\tau) \ . \tag{8}$$

The first autor proved in [1] that for a QK_2-manifold the following analogue of (8) holds:

$$\sum_{i=1}^{2n}(\nabla_{e_i}S')(x,e_i)=\frac{1}{2}x(\tau')\ . \tag{9}$$

Lemma. Let M be a connected $2n$-dimensional QK_2-manifold of pointwise constant antiholomorphic sectional curvature. If $n\geq 2$, the function $(n+1)\tau-3\tau'$ is a constant.

Proof. Using (8) and (9) from (4) we obtain $(n-1)x((n+1)\tau-3\tau')=0$ for an arbitrary vector x and hence the function $(n+1)\tau-3\tau'$ is a constant.

Now let M be an AH_3-manifold of pointwise constant antiholomorphic sectional curvature ν. Let $\{x,y\}$ be unit orthogonal vectors and $x\perp Jy$. Putting in (6) $w=x$, $x=y$, $y=Jy$, $z=Jy$, $u=y$ and taking into account (3), we obtain

$$\begin{aligned}4(n-1)x(\nu)&=(\nabla_xS)(y,y)+(\nabla_xS)(Jy,Jy)-(\nabla_yS)(x,y)\\&-(\nabla_{Jy}S)(x,Jy)-S(JBy,x)-g(JBy,x)S(y,y)+2(2n-1)\nu g(JBy,x)\ ,\end{aligned} \tag{10}$$

where $By=(\nabla_yJ)y+(\nabla_{Jy}J)Jy$. Putting in (7) $y=Jx$, $z=Jx$, and using (3) we obtain

$$\begin{aligned}(\nabla_xS)(Jx,Jx)-(\nabla_{Jx}S)(x,Jx)&=\frac{1}{2}\Big\{\frac{1}{2}x(\tau)-\sum_{i=1}^{2n}S((\nabla_{e_i}J)Jx,e_i)\\&+g(\delta F,Jx)S(x,x)+(\nabla_xS)(Jx,Jx)+S((\nabla_xJ)x,Jx)\Big\}\\&-2(n-1)x(\nu)-(2n-1)\nu g(\delta F,Jx)\ ,\end{aligned} \tag{11}$$

where $\delta F=\sum_{i=1}^{2n}(\nabla_{e_i}J)e_i$.

Now we shall prove the following analogue of the theorem of Schur for QK_2-manifolds.

Theorem. Let M be a connected $2n$-dimenssional QK_2-manifold of pointwise constant antiholomorphic sectional curvature ν. If $n\geq 3$, the functions ν, τ and τ' are constants.

Proof. By the conditions of the theorem, (10) and (11) give respectively

$$\begin{aligned}4(n-1)x(\nu)&=(\nabla_xS)(y,y)+(\nabla_xS)(Jy,Jy)\\&-(\nabla_yS)(x,y)-(\nabla_{Jy}S)(x,Jy)\ ,\end{aligned} \tag{12}$$

$$4(n-1)x(\nu)=\frac{1}{2}x(\tau)-(\nabla_xS)(Jx,Jx)+(\nabla_{Jx}S)(x,Jx)\ . \tag{13}$$

Using the adapted basis $\{u_1,\dots,u_n;Ju_1,\dots,Ju_n\}$, from (12) by summing up we find

$$4(n-1)^2x(\nu)=\frac{1}{2}x(\tau)-(\nabla_xS)(Jx,Jx)+(\nabla_{Jx}S)(x,Jx)\ .$$

The last equality and (13) give $(n-1)(n-2)x(\nu)=0$ and hence ν is a constant. Taking into account the lemma and (5), we obtain that τ and τ' are also constants.

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