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# NEARLY KÄHLER MANIFOLDS OF CONSTANT ANTIHOLOMORPHIC SECTIONAL CURVATURE

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In this paper we consider nearly Kähler manifolds of pointwise constant antiholomorphic sectional curvature. An analogue of Schur's theorem is proved and a classification theorem for such manifolds is given.

Let $M$ be an almost Hermitian manifold with $\dim M = 2n$, a metric tensor $g$, an almost complex structure $J$ and a curvature tensor $R$. A 2-plane $\alpha$ in the tangential space $T_pM$, $p \in M$ is said to be holomorphic (antiholomorphic) if $J\alpha = \alpha$ ($J\alpha \perp \alpha$). The sectional curvature $K(\alpha, p)$ of the 2-plane $\alpha$ in $T_pM$ with an orthonormal basis $\{x, y\}$ is given by the equality $K(\alpha, p) = R(x, y, y, x)$. The manifold $M$ is said to be of pointwise constant holomorphic (antiholomorphic) sectional curvature if the sectional curvature $K(\alpha, p)$ of an arbitrary holomorphic (antiholomorphic) 2-plane $\alpha$ in $T_pM$ for every $p \in M$ does not depend on $\alpha$. An almost Hermitian manifold $M$ is called an $RK$-manifold if $R(x, y, z, u) = R(Jx, Jy, Jz, Ju)$ for all $x, y, z, u \in T_pM$, $p \in M$. The nearly Kähler manifolds are characterized by the equality $(\nabla_X J)X = 0$, where $X$ is an arbitrary vector field. Every nearly Kähler manifold is an $RK$-manifold [4].

For Kähler manifolds the following analogue of the classical Schur theorem is well known:

**Theorem.** Let $M$ be a connected Kähler manifold with $\dim M \geq 4$. If $M$ is of pointwise constant holomorphic sectional curvature $\mu(p)$, then $\mu$ is a constant. Moreover, $M$ is of constant antiholomorphic sectional curvature $\mu/4$.

Conversely, if the Kähler manifold $M$ is of pointwise constant antiholomorphic sectional curvature $\mu/4$, then $M$ is of constant holomorphic sectional curvature $\mu$ [1]. Such a manifold is locally isometric to $\mathbb{C}^n$, $\mathbb{CP}^n$ or $\mathbb{CD}^n$.

For nearly Kähler manifolds the following statements are known:

**Theorem,** [5]. If $M$ is a connected nearly Kähler manifold of pointwise constant holomorphic sectional curvature $\mu(p)$ and $\dim M \geq 4$, then $\mu$ is a constant.

**Theorem,** [3]. If $M$ is a nearly Kähler manifold of constant holomorphic sectional curvature and $\dim M \geq 4$, then $M$ is locally isometric to $\mathbb{C}^n$, $\mathbb{CP}^n$, $\mathbb{CD}^n$ or $\mathbb{S}^6$.

We shall consider nearly Kähler manifolds of pointwise constant antiholomorphic sectional curvature. We need the following lemma:

**Lemma,** [2]. Let $M$ be an almost Hermitian manifold and $T$ be a tensor of type (0,4) in $T_pM$ satisfying the conditions:

1) $T(x, y, z, u) = -T(y, x, z, u)$;

2) $T(x, y, z, u) + T(y, z, x, u) + T(z, x, y, u) = 0$;

3) $T(x, y, z, u) = -T(x, y, u, z)$;

4) $T(x, y, z, u) = T(Jx, Jy, Jz, Ju)$;

5) $T(x, y, y, x) = 0$, where $\{x, y\}$ is a basis of an arbitrary holomorphic or antiholomorphic 2-plane in $T_pM$.

Then $T = 0$.

Let $R'(x,y,z,u) = R(x,y,Jz,Ju)$. We denote by $S$, $S'$ and $\tau$, $\tau'$ the Ricci tensors and scalar curvatures with respect to the tensors $R$, $R'$ respectively. If $\{E_1,\dots,E_{2n}\}$ is an arbitrary orthonormal frame field, we have

$$\sum_{i=1}^{2n}(\nabla_{E_i}R)(X,Y,Z,E_i) = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \; ; \tag{1}$$

$$\sum_{i=1}^{2n}(\nabla_{E_i}S)(X,E_i) = \frac{1}{2}X\tau \tag{2}$$

for arbitrary vector fields $X$, $Y$, $Z$. The following identities are valid for a nearly Kähler manifold [4]:

$$\sum_{i=1}^{2n}(\nabla_{E_i}S')(X,E_i) = \frac{1}{2}X\tau' \; ; \tag{3}$$

$$X(\tau - \tau') = 0 \; ; \tag{4}$$

$$2(\nabla_X(S-S'))(Y,Z) = (S-S')((\nabla_X J)Y, JZ) + (S-S')(JY,(\nabla_X J)Z) \; . \tag{5}$$

Let the tensors $R_1$, $R_2$, $\psi$ be defined by

$$R_1(x,y,z,u) = g(y,z)g(x,u) - g(x,z)g(y,u) \; ;$$

$$R_2(x,y,z,u) = g(Jy,z)g(Jx,u) - g(Jx,z)g(Jy,u) - 2g(Jx,y)g(Jz,u) \; ;$$

$$\begin{aligned}\psi(x,y,z,u) &= g(Jy,z)S(Jx,u) - g(Jx,z)S(Jy,u) - 2g(Jx,y)S(Jz,u) \\ &\quad + g(Jx,u)S(Jy,z) - g(Jy,u)S(Jx,z) - 2g(Jz,u)S(Jx,y) \; .\end{aligned}$$

**Proposition 1.** Let $M$ be an $RK$-manifold of pointwise constant antiholomorphic sectional curvature $\nu$ and $\dim M = 2n \geq 4$. Then the curvature tensor $R$ has the form

$$R = \frac{1}{6}\psi + \nu R_1 - \frac{2n-1}{3}\nu R_2 \tag{6}$$

and

$$3S' - (n+1)S = \frac{1}{2n}(3\tau' - (n+1)\tau)g \; , \tag{7}$$

$$\nu = \frac{(2n+1)\tau - 3\tau'}{8n(n^2-1)} \; . \tag{8}$$

**Proof.** Let $\{x,y\}$ be an orthonormal basis for an arbitrary antiholomorphic 2-plane in $T_pM$. From the condition $R(x,y,y,x) = \nu$ immediately follows

$$S(x,x) - R(x,Jx,Jx,x) = 2(n-1)\nu \; , \tag{9}$$

where $x$ is an arbitrary unit vector. Now let $T = R - (1/6)\psi - \nu R_1 + ((2n-1)/3)\nu R_2$. From the given condition, (9) and the Lemma it follows (6). By direct computation we find (7) and (8).

**Theorem 2.** Let $M$ be a connected nearly Kähler manifold of pointwise constant antiholomorphic sectional curvature $\nu$ and $\dim M = 2n > 4$. Then $\nu$ is a constant.

**Proof.** From (7), (3) and (2) it follows that $X(3\tau' - (n+1)\tau) = 0$ for an arbitrary vector field $X$. Let $n > 2$. Taking into account (4), we obtain that $\tau$ and $\tau'$ are constants and hence $\nu$ is a constant.

**Theorem 3.** Let $M$ be a connected nearly Kähler manifold of pointwise constant antiholomorphic sectional curvature $\nu$ and $\dim M = 2n > 4$. Then $M$ is locally isometric to one of the following manifolds:

1) The complex Euclidean space $\mathbb{C}^n$;
2) The complex projective space $\mathbb{CP}^n$;
3) The complex hyperbolic space $\mathbb{CD}^n$;
4) The six sphere $\mathbb{S}^6$.

**Proof.** From (5) and (7) it follows that

$$2(\nabla_X S)(Y,Z) = S((\nabla_X J)Y, JZ) + S(JY, (\nabla_X J)Z)\ ; \tag{10}$$

$$(\nabla_X S)(Y,Z) + (\nabla_{JX} S)(JY,Z) = 0\ ; \tag{11}$$

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0\ . \tag{12}$$

Let $\{E_1,\dots,E_{2n}\}$ be an arbitrary orthonormal frame field. Taking into account (10) and (11), from (6) and Theorem 2 we obtain

$$\sum_{i=1}^{2n}(\nabla_{E_i}R)(X,Y,Z,E_i) = -\frac{1}{6}\{(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\}.$$

This equality and (1) imply $(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)$. Then (12) gives that the Ricci tensor is parallel: $\nabla_X S = 0$.

If $M$ is irreducible, then it is an Einstein manifold and from (9) it follows that $M$ is of pointwise constant holomorphic sectional curvature. Now the assertion follows from the classification theorem in [3].

Let $M$ be reducible and be locally a product $M_1(\lambda_1)\times\ldots\times M_k(\lambda_k)$ $(k\geq 2)$ where $S=\lambda_i g$ on $M_i(\lambda_i)$ and $\lambda_i$ are different constants $(i=1,\dots,k)$. All $M_i$ are nearly Kähler manifolds [4]. Let $X$ on $M_i$, $Y$ on $M_j$ $(i\neq j)$ be unit vector fields. From $R(X,Y,Y,X)=0$ we find $\nu=0$. From $R(X,JX,JY,Y)=0$ and (6) it follows that

$$S(X,X) + S(Y,Y) = 0\ . \tag{13}$$

If $k>2$, (13) implies $\lambda_i=0$ for $i=1,\dots,k$. Hence $M$ is of zero Ricci curvature. The equality (9) gives that $M$ is of zero holomorphic sectional curvature. According to [3], $M$ is locally isometric to $\mathbb{C}^n$. Let $k=2$ and $\dim M_1 = 2n_1 \geq 4$. From (9) and $\nu=0$ it follows that $M_1$ is of constant holomorphic sectional curvature. Hence $M_1$ is locally isometric to $\mathbb{C}^{n_1}$. Now (13) gives that $M$ is of zero Ricci curvature and consequently $M$ is locally isometric to $\mathbb{C}^n$.

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