ALMOST KÄHLER MANIFOLDS OF CONSTANT ANTIHOLOMORPHIC SECTIONAL CURVATURE [1]

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We prove that an AK_2-manifold of dimension $2m \geq 6$ and of pointwise constant antiholomorphic sectional curvature is either a 6-dimensional manifold of constant negative sectional curvature or is locally isometric to $\mathbb{C}^n$, $\mathbb{CP}^n$ or $\mathbb{CD}^n$.

1. Introduction. Let M be an almost Hermitian manifolds with metric tensor g, almost complex structure J and covariant differentiation ∇.

If $\nabla J = 0$, or $(\nabla_X J)X = 0$ or

$$g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y) = 0\,, \tag{1.1}$$

or

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0 \tag{1.2}$$

for all X, Y, $Z \in \mathfrak{X}(M)$, M is said to be a Kähler, or nearly Kähler, or almost Kähler, or quasi Kähler manifold, respectively. The corresponding classes of manifolds are denoted by K, NK, AK and QK, respectively.

One consider the following identities:

1) $R(X,Y,Z,U) = R(X,Y,JZ,JU)$,
2) $R(X,Y,Z,U) = R(X,Y,JZ,JU) + R(X,JY,Z,JU) + R(JX,Y,Z,JU)$,
3) $R(X,Y,Z,U) = R(JX,JY,JZ,JU)$,

where R is the curvature tensor for M. If M has the identity i, it is said to belong to the class AH_i. Then $AH_1 \subset AH_2 \subset AH_3$. For a given class L of almost Hermitian manifolds one denotes $L_i = L \cap AH_i$. It is known, that $K = K_1$, $NK = NK_2$, $NK \cap AK = K$. The inclusions $K \subset NK \subset QK_2$, $K \subset AK_2 \subset QK_2$ are strict [5].

A 2-plane α in a tangent space $T_p(M)$ is said to be holomorphic (respectively, antiholomorphic) if $\alpha = J\alpha$ (respectively, $\alpha \perp J\alpha$). If for each point $p \in M$ the curvature of an arbitrary holomorphic (respectively, antiholomorphic) 2-plane α in $T_p(M)$ doesn't depend on α, M is said to be of pointwise constant holomorphic (respectively, antiholomorphic) sectional curvature.

Suppose $M \in NK$ and $\dim M = 2m \geq 6$. If M has pointwise constant holomorphic sectional curvature it is locally isometric to $\mathbb{C}^n$, $\mathbb{CP}^n$, $\mathbb{CD}^n$ or $\mathbb{S}^6$ [4]. The corresponding result for the antiholomorphic case in proved in [2].

In section 3 we prove an analogous theorem for AK_2-manifolds of pointwise constant antiholomorphic sectional curvature.

2. Preliminaries. Let M be a $2m$-dimensional almost Hermitian manifold. We denote by R, S and $\tau(R)$ the curvature tensor, the Ricci tensor and the scalar curvature of M, respectively. One defines also a tensor S' and a function $\tau'(R)$ by

$$S'(X,Y) = \sum_{i=1}^{2m} R(X, E_i, JE_i, JY)\,, \qquad \tau'(R) = \sum_{i=1}^{2m} S'(E_i, E_i)\,,$$

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where $\{E_i;\, i = 1, \ldots, 2m\}$ is a local orthonormal frame field. From the second Bianchi identity one obtains

$$\sum_{i=1}^{2m} (\nabla_{E_i} R)(X, Y, Z, E_i) = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)\,, \tag{2.1}$$

$$\sum_{i=1}^{2m} (\nabla_{E_i} S)(X, E_i) = \frac{1}{2} X(\tau(R))\,. \tag{2.2}$$

For $M \in AK_2$ the following identities hold (see e.g. [1], [5]):

$$R(X, Y, Z, U) - R(X, Y, JZ, JU) = \frac{1}{2} g((\nabla_X J)Y - (\nabla_Y J)X, (\nabla_Z J)U - (\nabla_U J)Z)\,, \tag{2.3}$$

$$2(\nabla_X (S - S'))(Y, Z) = (S - S')((\nabla_X J)Y, JZ) + (S - S')(JY, (\nabla_X J)Z)\,. \tag{2.4}$$

Assume $M \in AH_3$ and $\dim M = 2m$. If M is of pointwise constant antiholomorphic sectional curvature ν, then [2]:

$$R = \frac{1}{6}\psi(S) + \nu\pi_1 - \frac{2m-1}{3}\nu\pi_2\,, \tag{2.5}$$

$$(m+1)S - 3S' = \frac{1}{2m}\{(m+1)\tau(R) - 3\tau'(R)\}g\,, \tag{2.6}$$

where

$$\begin{aligned} \pi_1(x, y, z, u) &= g(x, u)g(y, z) - g(x, z)g(y, u)\,, \\ \psi(Q)(x, y, z, u) &= g(x, Ju)Q(y, Jz) - g(x, Jz)Q(y, Ju) - 2g(x, Jy)Q(z, Ju) \\ &\quad + g(y, Jz)Q(x, Ju) - g(y, Ju)Q(x, Jz) - 2g(z, Ju)Q(x, Jy) \end{aligned}$$

for a tensor Q of type (0,2) and $\pi_2 = \frac{1}{2}\psi(g)$. According to (2.5) $M \in AH_2$. In the case $m > 2$, ν is a global constant [6]. If moreover $M \in QK_3$, $\tau(R)$ and $\tau'(R)$ are also global constants [3].

Consequently, if $M \in AK_3$, $\dim M = 2m \geq 6$ and M is of pointwise constant antiholomorphic sectional curvature, from (2.2), (2.4) and (2.6) we obtain:

$$\sum_{i=1}^{2m} (\nabla_{E_i} S)(X, E_i) = 0\,, \tag{2.7}$$

$$2(\nabla_X S)(Y, Z) = S((\nabla_X J)Y, JZ) + S(JY, (\nabla_X J)Z)\,. \tag{2.8}$$

In section 3 we shall use the following lemma:

Lemma. *Let $M \in AK_3$ with* $\dim M = 2m \geq 6$. *If the curvature tensor of M has the form*

$$R = f\pi_1 + h\pi_2\,, \tag{2.9}$$

where f, $h \in \mathfrak{F}(M)$, then either M is a 6-dimensional manifold of constant negative sectional curvature or M is a Kähler manifold of constant holomorphic sectional curvature.

Proof. If M is not a Kähler manifold of constant holomorphic sectional curvature, it is of constant sectional curvature [8]. If $m > 3$ this is impossible [7]. On the other hand, as easily follows from (2.3), if a non Kähler almost Kähler manifold is of constant curvature f, then $f < 0$.

3. The classification theorem for the class AK_3.

Theorem. *Let $M \in AK_3$ and $\dim M = 2m \geq 6$. If M is of poitwise constant antiholomortphic sectional curvature ν, then either M is a 6-dimensional manifold of constant sectional curvature $\nu < 0$, or M is locally isometric to one of the following manifolds:*

a) the complex Euclidian space $\mathbb{C}^n$,

b) the complex projective space $\mathbb{CP}^n(4\nu)$;

c) the complex hyperbolic space $\mathbb{CD}^n(4\nu)$.

Proof. First we prove that the Ricci tensor is parallel. Let $p \in M$, $x, y \in T_p(M)$. According to the second Bianchi identity

$$(\nabla_x R)(Jx, y, y, Jx) + (\nabla_{Jx} R)(y, x, y, Jx) + (\nabla_y R)(x, Jx, y, Jx) = 0\,. \tag{3.1}$$

Let $\{e_i,\, Je_i;\, i = 1, \ldots, m\}$ be an orthonormal basis of $T_p(M)$ such that $S(e_i) = \lambda_i e_i$ (and hence $S(Je_i) = \lambda_i Je_i$), $i = 1, \ldots, m$. In (3.1) we put $x = e_i$, $y = e_j$ $(i \neq j)$ and using (1.2), (2.5) and $\nu =$const. we obtain

$$\{3\lambda_i + \lambda_j - 4(2m-1)\nu\} g((\nabla_{e_j} J)e_j, Je_i) = 0\,. \tag{3.2}$$

Analogously, from

$$(\nabla_{Je_i} R)(e_k, e_j, e_j, Je_k) + (\nabla_{e_k} R)(e_j, Je_i, e_j, Je_k) + (\nabla_{e_j} R)(Je_i, e_k, e_j, Je_k) = 0$$

for $i \neq j \neq k \neq i$ we find

$$\{2\lambda_k + \lambda_i + \lambda_j - 4(2m-1)\nu\} g((\nabla_{e_j} J)e_j, Je_i) = 0\,. \tag{3.3}$$

If for some s $(\nabla_{e_s} J)e_s \neq 0$, without loss of generality we assume that $g((\nabla_{e_s} J)e_s, Je_i) \neq 0$ for some $i \neq s$ and from (3.2) and (3.3) if follows:

$$\begin{aligned} &3\lambda_i + \lambda_s - 4(2m-1)\nu = 0\,, \\ &2\lambda_k + \lambda_i + \lambda_s - 4(2m-1)\nu = 0\,. \end{aligned} \tag{3.4}$$

Hence $\lambda_i = \lambda_k$ for $i, k = 1, \ldots, m$; $i,\, k \neq s$. If there exists $i \neq s$ such that $(\nabla_{e_i} J)e_i \neq 0$ in the same way we conclude that $\lambda_k = \lambda_s$ for $k \neq i, s$ and consequently $\lambda_i = \lambda_j$, $i,\, j = 1, \ldots, m$. Hence, using (2.8), we find $(\nabla_x S)(y, z) = 0$ for all $x,\, y,\, z \in T_p(M)$. Let

$$(\nabla_{e_i} J)e_i = (\nabla_{Je_i} J)Je_i = 0 \tag{3.5}$$

for $i = 1, \ldots, m$, $i \neq s$. Using (2.1), (2.5), (2.7), (1.1), (2.8) and $\nu =$const. we obtain

$$\begin{aligned} &4S(Jx, (\nabla_z J)y) - 4S(Jy, (\nabla_z J)x) + 5S(Jx, (\nabla_y J)z) - 5S(Jy, (\nabla_x J)z) \\ &\quad - S((\nabla_x J)y, Jz) + S((\nabla_y J)x, Jz) - 12(2m-1)\nu g(Jx, (\nabla_z J)y) = 0\,. \end{aligned} \tag{3.6}$$

In (3.6) we put $x = e_i$, $y = e_j$, $z = e_k$:

$$\begin{aligned} &\{4\lambda_i + 4\lambda_j - \lambda_k - 12(2m-1)\nu\} g(Je_i, (\nabla_{e_k} J)e_j) \\ &\quad + 5\lambda_i g(Je_i, (\nabla_{e_j} J)e_k) - 5\lambda_j g(Je_j, (\nabla_{e_i} J)e_k) = 0 \end{aligned} \tag{3.7}$$

for arbitrary i, j, k. Let $i, j \neq s$, $k = s$. Then from (3.7), (1.1) and $\lambda_i = \lambda_j$ it follows $\{13\lambda_i - \lambda_s - 12(2m-1)\nu\}g(Je_i, (\nabla_{e_s}J)e_j) = 0$. If $g(Je_i, (\nabla_{e_s}J)e_j) \neq 0$ for some $l, j \neq s$, then

$$13\lambda_i - \lambda_s - 12(2m-1)\nu = 0$$

and using (3.4) we find $\lambda_i = \lambda_s$. Hence, $\nabla S = 0$ in p. Let

$$g(e_i, (\nabla_{e_s}J)e_j) = g(Je_i, (\nabla_{e_s}J)e_j) = 0 \tag{3.8}$$

for $i, j \neq s$. Let in (3.7) $i, k \neq s$, $j = s$. Using (3.8) and $\lambda_i = \lambda_k$, we obtain $\{3\lambda_i + 9\lambda_s - 12(2m-1)\nu\}g(Je_i, (\nabla_{e_k}J)e_s) = 0$. If $g(Je_i, (\nabla_{e_k}J)e_s) \neq 0$ for some $i, k \neq s$, we have $\lambda_i = \lambda_s$ and hence $\nabla S = 0$ in p. Let

$$g(e_i, (\nabla_{e_k}J)e_s) = g(Je_i, (\nabla_{e_k}J)e_s) = 0\,. \tag{3.9}$$

If $m > 3$, let i, j, k be different and $i, j, k \neq s$. From (3.7) it follows $\{\lambda_i - (2m-1)\nu\}g(Je_i, (\nabla_{e_k}J)e_j) = 0$. If $g(Je_i, (\nabla_{e_k}J)e_j) \neq 0$ then $\lambda_i = (2m-1)\nu$ and using (3.4) we obtain $\lambda_i = \lambda_s$. Hence and because of (3.8), (3.9) we assume that

$$g(e_i, (\nabla_{e_j}J)e_k) = g(Je_i, (\nabla_{e_j}J)e_k) = 0$$

for all different i, j, k. According to these equalities and (3.5)

$$(\nabla_{e_i}J)e_j = 0 \tag{3.10}$$

for $i \neq s$ and for arbitrary $j = 1, \dots, m$. From (2.3) and (3.10) we derive $R(e_i, e_j, e_j, e_i) - R(e_i, e_j, Je_j, Je_i) = 0$ for $i \neq j$; $i, j \neq s$. Hence, because of (2.5) and $\lambda_i = \lambda_j$ it follows $\lambda_i = 2(m+1)\nu$ for $i \neq s$ and using (3.4) we find $\lambda_s = 2(m-5)\nu$. On the other hand, from (2.3) and (3.10) we obtain

$$2R(e_i, e_s, e_s, e_i) - 2R(e_i, e_s, Je_s, Je_i) = -g((\nabla_{e_s}J)e_i), (\nabla_{e_s}J)e_i)\,.$$

From the last three equalities and (2.5) we derive

$$g((\nabla_{e_s}J)e_i), (\nabla_{e_s}J)e_i) = -4\nu\,. \tag{3.11}$$

On the other hand, because of (3.8), $(\nabla_{e_s}J)e_i = \alpha_i e_s + \beta_i Je_s$, where α_i, β_i are real constants, $i = 1, \dots, m$, $i \neq s$. Using again (2.3), (2.5) and (3.10), we derive

$$g((\nabla_{e_s}J)e_i), (\nabla_{e_s}J)e_j) = 0\,, \qquad g((\nabla_{e_s}J)e_i), (\nabla_{e_s}J)Je_j) = 0$$

for $i \neq j$; $i, j \neq s$. Consequently, $\alpha_i\alpha_j + \beta_i\beta_j = 0$, $-\beta_i\alpha_j + \alpha_i\beta_j = 0$. Hence, by using (3.11) it is easy to find $\nu = 0$ and $(\nabla_{e_s}J)e_i = 0$ for $i = 1, \dots, m$; $i \neq s$. Hence, $(\nabla_{e_s}J)e_s = 0$ which is a contradiction.

Consequently, from $(\nabla_{e_s}J)e_s \neq 0$ for some s it follows $\nabla S = 0$ in p.

Let $(\nabla_{e_i}J)e_i = 0$ for every $i = 1, \dots, m$. Then

$$S((\nabla_{e_i}J)e_i, y) = S(e_i, (\nabla_{e_i}J)y) = 0 \tag{3.12}$$

for every $i = 1, \dots, m$ and for arbitrary $y \in T_p(M)$. Now from (3.1) with $x = e_i$ by using (2.5), (2.8) and (3.12) we derive

$$3S(Je_i, (\nabla_y J)y) - S(Jy, (\nabla_y J)e_i) - 4(2m-1)\nu g(Je_i, (\nabla_y J)y) = 0\,.$$

Hence,

$$3S(Jx, (\nabla_y J)y) - S(Jy, (\nabla_y J)x) - 4(2m-1)\nu g(Jx, (\nabla_y J)y) = 0$$

for all x, $y \in T_p(M)$. On the other hand, from (3.6) it follows

$$3S(Jx, (\nabla_y J)y) - S(Jy, (\nabla_y J)x) - 2S(Jy, (\nabla_x J)y) - 4(2m-1)\nu g(Jx, (\nabla_y J)y) = 0\,.$$

By using (2.8) from the last two identities we find $(\nabla_x S)(y, y) = 0$, i.e. $\nabla S = 0$ in p.

Consequently, the Ricci tensor is parallel. If M is irreducible it is an Einsteinian manifold. Hence, the curvature tensor has the form (2.9) and the assertion follows from the Lemma and the classification theorem for Kähler manifolds of constant holomorphic sectional curvature.

Let M be reducible. Then M is locally isometric to a product $M_1(\mu_1) \times \ldots \times M_k(\mu_k)$, where $S = \mu_i g$ on $M_i(\mu_i)$ and $\mu_i \neq \mu_j$ for $i \neq j$. Then it is not difficult to prove that $M_i(\mu_i)$ is an AK_2-manifold for $i = 1, \ldots, m$. If $k = 1$, M is an Einsteinian manifold and the theorem follows. Let $k > 1$. By using (2.5) it is easy to see that $\nu = 0$ and $\mu_i = -\mu_j$ for $i \neq j$. If $k > 2$ it follows $\mu_i = 0$ for $i = 1, \ldots, k$ which is a contradiction. Let $k = 2$ and $\dim M_1(\mu_1) \geq 4$. Analogously to (3.2) $\mu_1 g(x, (\nabla_y J)y) = 0$ holds good for all x, $y \in T_p(M_1(\mu_1))$, $p \in M_1(\mu_1)$. Since $\mu_1 = 0$ is a contradiction, we assume $\mu_1 \neq 0$. Then $M_1(\mu_1)$ is a nearly Kähler manifold and hence it is a Kähler manifold of constant holomorphic sectional curvature and of zero antiholomorphic sectional curvature. Hence, $\mu_1 = 0$ which is again a contradiction.

R E F E R E N C E S

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