A generalized Schrödinger-Poisson type system *

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Abstract

In this paper we study the boundary value problem

$$\begin{cases} -\Delta u + \varepsilon q \Phi f(u) = |u|^{p-1}u & \text{in } \Omega \\ -\Delta \Phi = 2qF(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \Phi = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $1 < p < 5$, $\varepsilon = \pm 1$, $q > 0$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and F is the primitive of f such that $F(0) = 0$. We show that, if q is sufficiently small, the problem has a solution provided that f satisfies a subcritical growth assumption.
We study also the critical case for $\varepsilon = 1$ and provide an existence result.

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1 Introduction and the main result

This paper deals with the following problem

$$\begin{cases} -\Delta u + \varepsilon q \Phi f(u) = |u|^{p-1}u & \text{in } \Omega \\ -\Delta \Phi = 2qF(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \Phi = 0 & \text{on } \partial\Omega \end{cases} \qquad (\mathcal{P})$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain, $1 < p < 5$, $q > 0$, $\varepsilon = \pm 1$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $F(s) = \int_0^s f(t)\,dt$.
When the function $f(t) = t$ and $\varepsilon = 1$, this system represents the well known "Schrödinger-Poisson (or Schrödinger-Maxwell) equations", briefly SPE, that has been widely studied in the recent past. In the pioneer paper of Benci and Fortunato [3], the linear version of SPE (where the term $|u|^{p-1}u$ does not appear) has been approached as an eigenvalue problem. In [11] the authors have proved the existence of infinitely many solutions for SPE when $p > 4$, whereas in [12] an analogous result has been found for almost any $q > 0$ and

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$p \in]2,5[$. A multiplicity result has been obtained in [13] for any $q > 0$ and p sufficiently close to critical exponent 5, by using the abstract theory of Lusternik - Schnirelmann. For the sake of completeness we mention also [10] where Neumann condition on Φ is assumed on $\partial\Omega$, and [1] and the references within for results on SPE in unbounded domains.
If $f(t) = t$ and $\varepsilon = -1$ the system is equivalent to a nonlocal nonlinear problem related with the following well known Choquard equation in the whole space $\mathbb{R}^3$

$$\Delta u + u - \left(\frac{1}{|x|} * u^2\right) u = 0.$$

We refer to [7, 8] for more details on the Choquard equation, and to [9] for a recent result on a system in $\mathbb{R}^3$ strictly related with ours.
Up to our knowledge, problem $(\mathcal{P})$ has not been investigated when a more general function f appears instead of the identity. Since, as showed in section 2, problem $(\mathcal{P})$ possesses a variational structure, our aim is to find the weakest assumptions on f in order to apply the usual variational techniques. In particular, the first step in a classical approach to such a type of systems consists in the use of the reduction method. To this end, we need to assume suitable growth conditions on f which let us invert the Laplace operator to solve the second equation of the system. Then, after we have reduced the problem to finding a critical point of a one variable functional, we check if the geometrical and compactness assumptions of the Mountain Pass Theorem are satisfied. If a suitable use of some *a priori* estimates makes almost immediate to show that geometrical hypotheses are verified (at least for small q), some technical difficulties arise in proving the boundedness of the Palais-Smale sequences. In section 3 we will overcome these difficulties by means of a truncation argument based on an idea of Berti and Bolle [4] and Jeanjean and Le Coz [5] (see also [6] and [2]) which allows us to find bounded Palais-Smale sequences taking q sufficiently small.
The first result in this paper is the following

Theorem 1.1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying*

$$|f(s)| \leqslant c_1 + c_2|s|^{r-1} \tag{$\mathbf{f_1}$}$$

for some $r < 5$ and for all $s \in \mathbb{R}$. Then, there exists $\bar{q} > 0$ such that for all $0 < q \leqslant \bar{q}$ problem $(\mathcal{P})$ has at least a nontrivial solution.

As well known, when we consider the Schrödinger equation in a bounded domain $\Omega \subset \mathbb{R}^3$

$$\begin{cases} -\Delta u + mu = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

the exponent $p = 5$ is critical, due to the fact that the space $H_0^1(\Omega)$ is compactly embedded into $L^{p+1}(\Omega)$ for $p < 5$. In order to get compactness in our system, we need to control the growth of nonlinear terms in both the equations. In particular, a condition as $(\mathbf{f_1})$ seems to be the right one to have compactness with respect to the variable Φ. Actually, the exponent $r = 5$ turns out to be critical for our system, and in this sense we are justified to refer to the limit hypothesis

$$|f(s)| \leqslant c_1 + c_2|s|^4. \tag{$\mathbf{f_2}$}$$

as the *critical growth condition* for function f.
In the second part of the paper, we are interested in studying system ($\mathcal{P}$) assuming ($\mathbf{f_2}$) instead of ($\mathbf{f_1}$). The difficulties arising in this situation can be overcome assuming a further hypothesis on the sign of the function f. We are able to get this partial result concerning ($\mathcal{P}$) only for $\varepsilon = 1$

Theorem 1.2. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying* ($\mathbf{f_2}$) *and*

$$f(s)s \geqslant 0 \tag{1}$$

for all $s \in \mathbb{R}$ and assume that $\varepsilon = 1$. Then, there exists $\bar{q} > 0$ such that for all $0 < q \leqslant \bar{q}$ problem ($\mathcal{P}$) has at least a nontrivial solution.

The paper is organized as follows:

- In section 2 we introduce the functional setting where we study the problem and the variational tools we use.

- In section 3 we provide the proof of Theorem 1.1.

- Finally, in section 4, we consider the critical case, and give a proof for Theorem 1.2.

Throughout the paper we will use the symbols H^{-1} to denote the dual space of $H_0^1(\Omega)$, $\langle \cdot, \cdot \rangle$ to denote the duality between $H_0^1(\Omega)$ and H^{-1}, and $\|u\|_{H_0^1} := \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}$ for the norm on $H_0^1(\Omega)$. Moreover $\| \cdot \|_p$ will denote the usual L^p-norm.

We point out the fact that in the sequel we will use the symbols C, C_1, C_2, C_3 and so on, to denote positive constants whose value might change from line to line.

2 Variational tools

Standard arguments can be used to prove that problem (P) is variational and the related functional $J_q : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is given by

$$J_q(u, \Phi) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\varepsilon}{4}\int_\Omega |\nabla \Phi|^2 dx + \varepsilon q \int_\Omega F(u)\Phi dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx$$

Since $F(u) \in L^{\frac{6}{5}}(\Omega) \hookrightarrow H^{-1}$ for all $u \in L^{\frac{6r}{5}}(\Omega)$, then certainly we have that for all $u \in H_0^1(\Omega)$ there exists a unique $\Phi_u \in H_0^1(\Omega)$ which solves

$$-\Delta\Phi = 2qF(u) \tag{2}$$

in H^{-1}. In particular, we are allowed to consider the following map

$$u \in L^{\frac{6r}{5}}(\Omega) \mapsto \Phi_u \in H_0^1(\Omega) \tag{3}$$

which is continuously differentiable by the implicit function theorem applied to $\frac{\partial H}{\partial \Phi}$ where, for any $(u,\Phi)\in L^{\frac{6r}{5}}(\Omega)\times H_0^1(\Omega)$, H is defined as follows

$$H(u,\Phi)=\frac{1}{4}\int_\Omega |\nabla\Phi|^2 - q\int_\Omega F(u)\Phi.$$

So, it is well defined the C^1 functional

$$I_q(u):=J_q(u,\Phi_u)=\frac{1}{2}\int_\Omega |\nabla u|^2dx-\frac{\varepsilon}{4}\int_\Omega |\nabla\Phi_u|^2dx + \varepsilon q\int_\Omega F(u)\Phi_u dx-\frac{1}{p+1}\int_\Omega |u|^{p+1}dx. \quad (4)$$

Multiplying the second equation by Φ_u and integrating, we have

$$\int_\Omega |\nabla\Phi_u|^2\,dx = 2q\int_\Omega F(u)\Phi_u\,dx \quad (5)$$

and then, combining (5) with (4), we get

$$I_q(u)=\frac{1}{2}\int_\Omega |\nabla u|^2dx+\varepsilon\frac{q}{2}\int_\Omega F(u)\Phi_u dx-\frac{1}{p+1}\int_\Omega |u|^{p+1}dx. \quad (6)$$

By using standard variational arguments as those in [3], the following result can be easily proved.

Proposition 2.1. *Let $(u,\Phi)\in H_0^1(\Omega)\times H_0^1(\Omega)$, then the following propositions are equivalent:*

(a) (u,Φ) is a critical point of functional J_q;

(b) u is a critical point of functional I_q and $\Phi=\Phi_u$.

So we are led to look for critical points of I_q. To this end, we need to investigate the compactness property of its Palais-Smale sequences.
It is easy to see that the usual arguments to prove boundedness do not work. Indeed, assuming that $(u_n)_n\in (H_0^1(\Omega))^{\mathbb{N}}$ is a Palais-Smale sequence, namely $I_q(u_n)$ is bounded and $I_q'(u_n)\to 0$ in H^{-1}, we should deduce the following inequality

$$C_1+C_2\|u_n\|\geqslant I_q(u_n)-\frac{1}{p+1}\langle I_q'(u_n),u_n\rangle. \quad (7)$$

Moreover, by the definition of map Φ_u, we have that

$$\langle \partial_\Phi J_q(u_n,\Phi_{u_n}),\Phi'(u_n)u_n\rangle=0$$

and then, since

$$\langle I_q'(u_n),u_n\rangle=\langle\partial_u J_q(u_n,\Phi_{u_n}),u_n\rangle,$$

from (7) we deduce that

$$\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_\Omega |\nabla u_n|^2dx+\varepsilon\frac{q}{2}\int_\Omega F(u_n)\Phi_{u_n}dx - \varepsilon\frac{q}{p+1}\int_\Omega f(u_n)u_n\Phi_{u_n}dx\leqslant C_1+C_2\|u_n\|.$$

In classical SPE ($\varepsilon = 1$ and $f(t) = t$) we should deduce the boundedness of the sequence $(u_n)_n$ for $p \geqslant 3$. In our general situation we need a different approach.

Let $T > 0$ and $\chi : [0, +\infty[\to [0,1]$ be a smooth function such that $\|\chi'\|_{L^\infty} \leqslant 2$ and

$$\chi(s) = \begin{cases} 1 & \text{if } 0 \leqslant s \leqslant 1 \\ 0 & \text{if } s \geqslant 2. \end{cases}$$

We define a new functional $I_q^T : H_0^1(\Omega) \to \mathbb{R}$ as follows

$$I_q^T(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \varepsilon \frac{q}{2}\chi\left(\frac{\|u\|_{H_0^1}}{T}\right)\int_\Omega F(u)\Phi_u dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx \quad (8)$$

for all $u \in H_0^1(\Omega)$. We are going to find a critical point of this new functional.

3 Subcritical case

First of all, we have to prove that functional I_q^T satisfies mountain pass geometry. More precisely, we have the following result:

Lemma 3.1. *Under hypothesis* ($\mathbf{f_1}$)*, there exists* $\bar{q} \in \mathbb{R}_+ \cup \{+\infty\}$ *such that for all* $0 < q < \bar{q}$ *functional* I_q^T *satisfies:*

(i) $I_q^T(0) = 0$;

(ii) there exist constants $\rho, \alpha > 0$ *such that*

$$I_q^T(u) \geqslant \alpha \qquad \text{for all } u \in H_0^1(\Omega) \quad \text{with } \|u\|_{H_0^1} = \rho;$$

(iii) there exists a function $\bar{u} \in H_0^1(\Omega)$ *with* $\|\bar{u}\|_{H_0^1} > \rho$ *such that* $I_q^T(\bar{u}) < 0$.

Proof

(i) It is trivial;

(ii) If $\varepsilon = 1$ we deduce our assert for $\bar{q} = +\infty$ observing that, by (5), it is

$$\varepsilon \frac{q}{2}\int_\Omega F(u)\Phi_u dx \geqslant 0.$$

If $\varepsilon = -1$ we need some estimates. By Holder inequality we have

$$\int_\Omega F(u)\Phi_u dx \leqslant \left(\int_\Omega |F(u)|^{6/5} dx\right)^{5/6}\left(\int_\Omega |\Phi_u|^6 dx\right)^{1/6}. \quad (9)$$

Then, from (2), by using Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$, we get

$$\begin{aligned}\int_\Omega |\nabla \Phi_u|^2 dx &= q\int_\Omega F(u)\Phi_u dx \\ &\leqslant q\left(\int_\Omega |F(u)|^{6/5} dx\right)^{5/6}\left(\int_\Omega |\Phi_u|^6 dx\right)^{1/6} \\ &\leqslant Cq\left(\int_\Omega |F(u)|^{6/5} dx\right)^{5/6}\left(\int_\Omega |\nabla\Phi_u|^2 dx\right)^{1/2}.\end{aligned}$$

So, we have obtained that

$$\left(\int_\Omega |\nabla\Phi_u|^2 dx\right)^{1/2} \leqslant Cq\left(\int_\Omega |F(u)|^{6/5} dx\right)^{5/6} \tag{10}$$

By using this inequality in (9), taking again into account the continuous embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$, we have

$$\int_\Omega F(u)\Phi_u dx \leqslant Cq\left(\int_\Omega |F(u)|^{6/5} dx\right)^{5/3}. \tag{11}$$

Since for $(\mathbf{f_1})$ it is

$$|F(s)|^{6/5} \leqslant C_1|s|^{6/5} + C_2|s|^{\frac{6r}{5}} \tag{12}$$

for all $s \in \mathbb{R}$, by (11) and (12) we deduce that

$$\int_\Omega F(u)\Phi_u dx \leqslant q(C_1\|u\|_{6/5}^2 + C_2\|u\|_{\frac{6r}{5}}^{2r}). \tag{13}$$

By using (13) and the immersion of $H_0^1(\Omega)$ into $L^p(\Omega)$ spaces, we have

$$\begin{aligned} I_q^T(u) &\geqslant \frac{1}{2}\|u\|_{H_0^1}^2 - \frac{q}{2}\int_\Omega F(u)\Phi_u dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx \\ &\geqslant \frac{1}{2}\|u\|_{H_0^1}^2 - \frac{q^2}{2}(C_1\|u\|_{6/5}^2 + C_2\|u\|_{\frac{6r}{5}}^{2r}) - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &\geqslant \frac{1}{2}\|u\|_{H_0^1}^2 - \frac{q^2}{2}(C_1\|u\|_{H_0^1}^2 + C_2\|u\|_{H_0^1}^{2r}) - \frac{C}{p+1}\|u\|_{H_0^1}^{p+1}. \end{aligned}$$

Thus, if q is such that $q^2C_1 < 1$ and ρ is small enough, there exists $\alpha > 0$ such that $I_q^T(u) \geqslant \alpha$ for all $u \in H_0^1(\Omega)$ with $\|u\|_{H_0^1} = \rho$.

(iii) Let $u \in H_0^1(\Omega)$, $u \neq 0$ and $t > \frac{2T}{\|u\|_{H_0^1}}$. Then, we have

$$\begin{aligned} I_q^T(tu) &= \frac{t^2}{2}\int_\Omega |\nabla u|^2 dx \\ &\qquad + \varepsilon\frac{q}{2}\chi\left(\frac{t\|u\|_{H_0^1}}{T}\right)\int_\Omega F(tu)\Phi_{tu} dx - \frac{t^{p+1}}{p+1}\int_\Omega |u|^{p+1} dx \\ &= \frac{t^2}{2}\int_\Omega |\nabla u|^2 dx - \frac{t^{p+1}}{p+1}\int_\Omega |u|^{p+1} dx \end{aligned}$$

since $\chi\left(\frac{t\|u\|_{H_0^1}}{T}\right) = 0$.

Thus, for t large enough, $I_q^T(tu)$ is negative.

□

Lemma (3.1) allows us to define, for $q < \bar{q}$,

$$m_q^T = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I_q^T(\gamma(t)) > 0 \tag{14}$$

where

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], H_0^1(\Omega)) \mid \gamma(0) = 0, I_q^T(\gamma(1)) < 0\}.$$

Certainly there exists a Palais-Smale sequence at mountain pass level m_q^T, that is a sequence $(u_n)_n$ in $H_0^1(\Omega)$ such that

$$I_q^T(u_n) \to m_q^T \tag{15}$$

and

$$(I_q^T)'(u_n) \to 0. \tag{16}$$

Proof of Theorem 1.1 We show that for a suitable small $\bar{q}$ and every $q < \bar{q}$, the Palais-Smale sequence previously obtained at mountain pass level m_q^T admits a subsequence converging to a critical point of I_q.

Let $(u_n)_n$ in $H_0^1(\Omega)$ be a Palais-Smale sequence satisfying (15) and (16). Now we show that, if q is small enough, for n large enough $(u_n)_n$ lies in the ball of $H_0^1(\Omega)$ of radius T, that is where functionals I_q^T and I_q coincide. In order to prove it, we firstly estimate the mountain pass level m_q^T.
Set $u \in H_0^1(\Omega), u \neq 0$ and let $\bar{t} > 0$ be such that the path $\bar{\gamma}(t) = t\bar{t}u$ belongs to Γ. For all $t \in [0,1]$ it is

$$\begin{aligned} I_q^T(t\bar{t}u) &= \frac{t^2}{2}\int_\Omega |\nabla \bar{t}u|^2 dx \\ &\quad + \varepsilon\frac{q}{2}\chi\left(\frac{t\|\bar{t}u\|_{H_0^1}}{T}\right)\int_\Omega F(t\bar{t}u)\Phi_{t\bar{t}u}dx - \frac{t^{p+1}}{p+1}\int_\Omega |\bar{t}u|^{p+1}dx. \end{aligned}$$

From (13) we obtain

$$\begin{aligned} \max_{t\in[0,1]} I_q^T(\gamma(t)) &\leqslant \max_{t\in[0,1]}\left(C_1 t^2\int_\Omega |\nabla u|^2\,dx - C_2 t^{p+1}\int_\Omega |u|^{p+1}\,dx\right) \\ &\quad + q^2\max_{t\in[0,1]}\chi\left(\frac{t\bar{t}\|u\|_{H_0^1}}{T}\right)\left(C_3 t^2\|\bar{t}u\|^2_{H_0^1} + C_4 t^{2r}\|\bar{t}u\|^{2r}_{H_0^1}\right) \\ &\leqslant C + q^2(C_5T^2 + C_6T^{2r}) \end{aligned}$$

Then, from (14) we get

$$m_q^T \leqslant C + q^2(C_5T^2 + C_6T^{2r}).. \tag{17}$$

We claim that, if T is sufficiently large, then $\limsup_n \|u_n\|_{H_0^1} \leqslant T$.
By contradiction, we will assume that there exists a subsequence (relabeled u_n) such that for all $n \geqslant 1$ we have $\|u_n\|_{H_0^1} > T$. From (8) we deduce that

$$\langle (I_q^T)'(u_n), u_n\rangle = \int_\Omega |\nabla u_n|^2 dx + \mathcal{A}_n + \mathcal{B}_N + \mathcal{C}_N - \int_\Omega |u_n|^{p+1}dx$$

where

$$\mathcal{A}_N = \varepsilon q\chi'\left(\frac{\|u_n\|_{H_0^1}}{T}\right)\frac{\|u_n\|_{H_0^1}}{T}\int_\Omega F(u_n)\Phi_{u_n}dx$$

$$\mathcal{B}_n = \varepsilon\frac{q}{2}\chi\left(\frac{\|u_n\|_{H_0^1}}{T}\right)\int_\Omega f(u_n)u_n\Phi_{u_n}dx$$

and

$$\mathcal{C}_n = \varepsilon \frac{q}{2} \chi \left(\frac{\|u_n\|_{H_0^1}}{T} \right) \int_\Omega F(u_n) \Phi'_{u_n}[u_n] dx.$$

Since the functional

$$S(u) = \int_\Omega |\nabla \Phi_u|^2 dx - 2q \int_\Omega F(u) \Phi_u dx, \ u \in H_0^1(\Omega)$$

is everywhere equal to 0, we have that

$$0 = \langle S'(u), u \rangle = \int_\Omega (\nabla \Phi_u | \nabla \Phi'_u[u]) dx - q \int_\Omega f(u) \Phi_u u dx - q \int_\Omega F(u) \Phi'_u[u] dx.$$

On the other hand, multiplying the second equation of the system for $\Phi'_u[u]$, we have that

$$\int_\Omega (\nabla \Phi_u | \nabla \Phi'_u[u]) dx = 2q \int_\Omega F(u) \Phi'_u[u] dx$$

and then

$$\int_\Omega F(u) \Phi'_u[u] dx = \int_\Omega f(u) \Phi_u u dx.$$

We deduce that $\mathcal{B}_n = \mathcal{C}_n$ for any $n \geqslant 1$. By using (15) and (16) we obtain

$$\begin{aligned} m_q^T + o_n(1) \|u_n\|_{H_0^1} &= I_q^T(u_n) - \frac{1}{p+1} \langle (I_q^T)'(u_n), u_n \rangle \\ &= \frac{p-1}{2(p+1)} \|u_n\|_{H_0^1}^2 + \mathcal{D}_n \\ &\quad - \frac{1}{p+1} \mathcal{A}_n - \frac{2}{p+1} \mathcal{B}_n, \end{aligned} \tag{18}$$

where

$$\mathcal{D}_n = \varepsilon \frac{q}{2} \chi \left(\frac{\|u_n\|_{H_0^1}}{T} \right) \int_\Omega F(u_n) \Phi_{u_n} dx.$$

But for $(\mathbf{f_1})$ and (13) we have that

$$\max (\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n, \mathcal{D}_n) \leqslant q^2 (C_1 T^2 + C_2 T^{2r}).$$

Thus, by our contradiction hypothesis, (17) and (18), we obtain that

$$T^2 \leqslant \|u_n\|_{H_0^1}^2 \leqslant C + C_1 T + q^2 (C_2 T^2 + C_3 T^{2r}).$$

If $T^2 > C + C_1 T$, we can find $\bar{q}$ such that for any $q < \bar{q}$ the previous inequality turns out to be a contradiction.
The contradiction arises from the assumption that $\limsup_n \|u_n\|_{H_0^1} > T$ so we have that the sequence $(u_n)_n$ is bounded in the $H_0^1(\Omega)$ norm by T, and $I_q^T(u_n)$ coincides with $I_q(u_n)$. We deduce that $\{u_n\}_n$ is a bounded Palais-Smale sequence of the functional I_q.
Let $u_0 \in H_0^1(\Omega)$ be such that, up to subsequences, $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$. The last step is to prove that this convergence is also strong. But this is an easy consequence of the fact that, since $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ compactly for any $s \in [1, 6[$, we have

$$u_n \to u_0 \text{ in } L^p(\Omega),$$

$$u_n \to u_0 \text{ in } L^{\frac{6r}{5}}(\Omega)$$

and, then, for the continuity of the map (3), it is

$$\Phi_{u_n} \to \Phi_{u_0} \text{ in } H^1_0(\Omega).$$

□

4 Critical case

In this section we are going to study the system

$$\begin{cases} -\Delta u + q\Phi f(u) = |u|^{p-1}u & \text{in } \Omega \\ -\Delta \Phi = 2qF(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \Phi = 0 & \text{on } \partial\Omega \end{cases} \tag{19}$$

assuming that f satisfies the hypotheses of Theorem 1.2.

Remark 4.1. *From hypothesis* (1) *we may deduce something more about the sign of* Φ_u. *Indeed, since* $F(s) = \int_0^s f(t)\,dt$, *certainly* F *is a nonnegative function. From this and the maximum principle, also the solutions of the second equations of the system must be nonnegative.*

Proof Assuming ($\mathbf{f_2}$) instead of ($\mathbf{f_1}$) we can repeat the same arguments of Lemma 3.1 in the previous section, and we can find $\bar{q}$ sufficiently small such that for any $q < \bar{q}$ there exists a bounded Palais-Smale sequence for I_q at the level $m_q := m_q^T$ for some T. By (5) we deduce that also $(\Phi_{u_n})_n$ is bounded in $H^1_0(\Omega)$. Up to subsequences, there exist $u_0 \in H^1_0(\Omega)$ and $\Phi_0 \in H^1_0(\Omega)$ such that

$$u_n \rightharpoonup u_0 \text{ in } H^1_0(\Omega) \tag{20}$$

$$\Phi_{u_n} \rightharpoonup \Phi_0 \text{ in } H^1_0(\Omega). \tag{21}$$

By ($\mathbf{f_2}$) and (20) we also have

$$F(u_n) \rightharpoonup F(u_0) \text{ in } L^{\frac{6}{5}}(\Omega) \tag{22}$$

$$f(u_n)u_n \rightharpoonup f(u_0)u_0 \text{ in } L^{\frac{6}{5}}(\Omega). \tag{23}$$

We show that $\Phi_0 = \Phi_{u_0}$.
Let us consider a test function $\psi \in C_0^\infty(\Omega)$. From the second equation of our problem we obtain

$$\int_\Omega (\nabla \Phi_{u_n} | \nabla \psi) dx = 2q \int_\Omega F(u_n)\psi dx.$$

Passing to the limit and using (21) and (22), we have that

$$\int_\Omega (\nabla \Phi_0 | \nabla \psi) dx = 2q \int_\Omega F(u_0)\psi dx.$$

So Φ_0 is a weak solution of $-\Delta\Phi = 2qF(u_0)$, and then, by uniqueness, it is $\Phi_0 = \Phi_{u_0}$. Since $(u_n)_n$ is a Palais-Smale sequence, for any $\psi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega (\nabla u_n | \nabla \psi) dx + q \int_\Omega \Phi_{u_n} f(u_n)\psi dx = \int_\Omega |u_n|^{p-1} u_n \psi dx + o_n(1).$$

Passing to the limit, by (20) and (23) we have

$$\int_\Omega (\nabla u_0|\nabla\psi)dx + q\int_\Omega \Phi_{u_0} f(u_0)\psi dx = \int_\Omega |u_0|^{p-1}u_0\psi dx \tag{24}$$

that is u_0 is a weak solution of (19).

It remains to prove that $u_0 \neq 0$.
By using the density of test functions in $H^1_0(\Omega)$, from (24) we get

$$\int_\Omega |\nabla u_0|^2 dx + q\int_\Omega \Phi_{u_0} f(u_0)u_0 dx - \int_\Omega |u_0|^{p+1}dx = 0. \tag{25}$$

For (20) and (21), certainly

$$f(u_n)u_n\Phi_{u_n} \to f(u_0)u_0\Phi_{u_0} \text{ a.e.}$$

up to subsequences. Taking into account Remark 4.1 and (1), Fatou's Lemma implies

$$\int_\Omega f(u_0)u_0\Phi_{u_0}dx \leq \liminf_n \int_\Omega f(u_n)u_n\Phi_{u_n}dx. \tag{26}$$

Moreover, since p is subcritical, it is

$$\lim_n \int_\Omega |u_n|^{p+1}dx = \int_\Omega |u_0|^{p+1}dx. \tag{27}$$

Since $(u_n)_n$ is bounded, we have that $\langle I'_q(u_n), u_n\rangle \to 0$, that is

$$\int_\Omega |\nabla u_n|^2 dx + q\int_\Omega f(u_n)u_n\Phi_{u_n}dx = \int_\Omega |\nabla u_n|^{p+1}dx + o_n(1). \tag{28}$$

Now we prove that $\|u_n\|_{H^1_0(\Omega)} \to \|u_0\|_{H^1_0(\Omega)}$ which, together with (20), ensures us the strong convergence.
By lower weak semicontinuity we know that

$$\int_\Omega |\nabla u_0|^2 dx \leq \liminf_n \int_\Omega |\nabla u_n|^2 dx.$$

On the other hand, for (26) and (27), from (25) and (28) we deduce

$$\begin{aligned}\limsup_n \int_\Omega |\nabla u_n|^2 dx &= \limsup_n \left(-q\int_\Omega f(u_n)u_n\Phi_{u_n}dx + \int_\Omega |\nabla u_n|^{p+1}dx\right)\\ &= -\liminf_n q\int_\Omega f(u_n)u_n\Phi_{u_n}dx + \int_\Omega |\nabla u_0|^{p+1}dx\\ &\leq -q\int_\Omega f(u_0)u_0\Phi_{u_0}dx + \int_\Omega |\nabla u_0|^{p+1}dx = \int_\Omega |\nabla u_0|^2 dx.\end{aligned}$$

Then, we have

$$\lim_n \int_\Omega |\nabla u_n|^2 dx = \int_\Omega |\nabla u_0|^2 dx.$$

By continuity of the functional I_q, we conclude that $I_q(u_0) = m_q > 0$, and thus u_0 is a nontrivial solution. □

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