On Lossless Approximations, the Fluctuation-Dissipation Theorem, and Limitations of Measurements

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September 16, 2010

Abstract

In this paper, we take a control-theoretic approach to answering some standard questions in statistical mechanics, and use the results to derive limitations of classical measurements. A central problem is the relation between systems which appear macroscopically dissipative but are microscopically lossless. We show that a linear system is dissipative if, and only if, it can be approximated by a linear lossless system over arbitrarily long time intervals. Hence lossless systems are in this sense dense in dissipative systems. A linear active system can be approximated by a nonlinear lossless system that is charged with initial energy. As a by-product, we obtain mechanisms explaining the Onsager relations from time-reversible lossless approximations, and the fluctuation-dissipation theorem from uncertainty in the initial state of the lossless system. The results are applied to measurement devices and are used to quantify limits on the so-called observer effect, also called *back action*, which is the impact the measurement device has on the observed system. In particular, it is shown that deterministic back

action can be compensated by using active elements, whereas stochastic back action is unavoidable and depends on the temperature of the measurement device.

1   Introduction

Analysis and derivation of limitations on what is achievable are at the core of many branches of engineering, and thus of tremendous importance. Examples can be found in estimation, information, and control theories. In estimation theory, the Cramér-Rao inequality gives a lower bound on the covariance of the estimation error, in information theory Shannon showed that the channel capacity gives an upper limit on the communication rate, and in control theory Bode's sensitivity integral bounds achievable control performance. For an overview of limitations in control and estimation, see the book [1]. Technology from all of these branches of engineering is used in parallel in modern networked control systems [2]. Much research effort is currently spent on understanding how the limitations from these fields interact. In particular, much effort has been spent on merging limitations from control and information theory, see for example [3–5]. This has yielded insight about how future control systems should be designed to maximize their performance and robustness.

Derivation of limitations is also at the core of physics. Well-known examples are the laws of thermodynamics in classical physics and the uncertainty principle in quantum mechanics [6–8]. The exact implications of these physical limitations on the performance of control systems have received little attention, even though all components of a control system, such as actuators, sensors, and computers, are built from physical components which are constrained by physical laws. Control engineers discuss limitations in terms of location of unstable plant poles and zeros, saturation limits of actuators, and more recently channel capacity in feedback loops. But how does the amount of available energy limit the possible bandwidth of a control system? How does the ambient temperature affect the estimation error of an observer? How well can you implement a desired ideal behavior using physical components? The main goal of this paper is to develop a theoretical framework where questions such as these can be answered, and initially to derive limitations on measurements using basic laws from classical physics. Quantum mechanics is not used in this paper.

The derivation of physical limitations broaden our understanding of control engineering, but these limitations are also potentially useful outside of

the traditional control-engineering community. In the physics community, the rigorous error analysis we provide could help in the analysis of far-from-equilibrium systems when time, energy, and degrees of freedom are limited. For Micro-Electro-Mechanical Systems (MEMS), the limitation we derive on measurements can be of significant importance since the physical scale of micro machines is so small. In systems biology, limits on control performance due to molecular implementation have been studied [9]. It is hoped that this paper will be a first step in a unified theoretical foundation for such problems.

1.1 Related work

The derivation of thermodynamics as a theory of large systems which are microscopically governed by lossless and time-reversible fundamental laws of physics (classical or quantum mechanics) has a large literature and tremendous progress for over a century within the field of statistical physics. See for instance [10–13] for physicists' account of how dissipation can appear from time-reversible dynamics, and the books [6–8] on traditional statistical physics. In non-equilibrium statistical mechanics, the focus has traditionally been on dynamical systems close to equilibrium. A result of major importance is the *fluctuation-dissipation theorem*, which plays an important role in this paper. The origin of this theorem goes back to Nyquist's and Johnson's work [14, 15] on thermal noise in electrical circuits. In its full generality, the theorem was first stated in [16]; see also [17]. The theorem shows that thermal fluctuations of systems close to equilibrium determines how the system dissipates energy when perturbed. The result can be used in two different ways: By observing the fluctuation of a system you can determine its dynamic response to perturbations; or by making small perturbations to the system you can determine its noise properties. The result has found wide-spread use in many areas such as fluid mechanics, but also in the circuit community, see for example [18, 19]. A recent survey article about the fluctuation-dissipation theorem is [20]. Obtaining general results for dynamical systems far away from equilibrium (far-from-equilibrium statistical mechanics) has proved much more difficult. In recent years, the so-called *fluctuation theorem* [21, 22], has received a great deal of interest. The fluctuation theorem quantifies the probability that a system far away from equilibrium violates the second law of thermodynamics. Not surprisingly, for longer time intervals, this probability is exceedingly small. A surprising fact is that the fluctuation theorem implies the fluctuation-dissipation theorem when applied to systems close to equilibrium [22]. The fluctuation

theorem is not treated in this paper, but is an interesting topic for future work.

From a control theorist's perspective, it remains to understand what these results imply in a control-theoretical setting. One contribution of this paper is to highlight the importance of the fluctuation-dissipation theorem in control engineering. Furthermore, additional theory is needed that is both mathematically more rigorous and applies to systems not merely far-from-equilibrium, but maintained there using active control. More quantitative convergence and error analysis is also needed for systems not asymptotically large, such as arise in biology, microelectronics, and micromechanical systems.

Substantial work has already been done in the control community in formulating various results of classical thermodynamics in a more mathematical framework. In [23, 24], the second law of thermodynamics is derived and a control-theoretic heat engine is obtained (in [25] these results are generalized). In [26], a rigorous dynamical systems approach is taken to derive the laws of thermodynamics using the framework of dissipative systems [27, 28]. In [29], it is shown how the entropy flows in Kalman-Bucy filters, and in [30] Linear-Quadratic-Gaussian control theory is used to construct heat engines. In [31–33], the problem of how lossless systems can appear dissipative (compare with [10–12] above) is discussed using various perspectives. In [34], how the direction of time affects the difficulty of controlling a process is discussed.

1.2   Contribution of the paper

The first contribution of the paper is that we characterize systems that can be approximated using linear or nonlinear lossless systems. We develop a simple, clear control-theoretic model framework in which the only assumptions on the nature of the physical systems are conservation of energy and causality, and all systems are of finite dimension and act on finite time horizons. We construct high-order lossless systems that approximate dissipative systems in a systematic manner, and prove that a linear model is dissipative if, and only if, it is arbitrarily well approximated by lossless causal linear systems over an arbitrary long time horizon. We show how the error between the systems depend on the number of states in the approximation and the length of the time horizon (Theorems 1 and 2). Since human experience and technology is limited in time, space, and resolution, there are limits to directly distinguishing between a low-order macroscopic dissipative system and a high-order lossless approximation. This result is important since it

shows exactly what macroscopic behaviors we can implement with linear lossless systems, and how many states are needed. In order to approximate an *active* system, even a linear one, with a lossless system, we show that the approximation *must be nonlinear*. Note that active components are at the heart of biology and all modern technology, in amplification, digital electronics, signal transduction, etc. In the paper, we construct one class of low-order lossless nonlinear approximations and show how the approximation error depends on the initial available energy (Theorems 4 and 5). Thus in this control-theoretic context, nonlinearity is not a source of complexity, but rather an essential and valuable resource for engineering design. These result are all of theoretical interest, but should also be of practical interest. In particular, the results give constructive methods for implementing desired dynamical systems using finite number of lossless components when resources such as time and energy are limited.

As a by-product of this contribution, the fluctuation-dissipation theorem (Propositions 2 and 3) and the Onsager reciprocal relations (Theorem 3) easily follows. The lossless systems studied here are consistent with classical physics since they conserve energy. If time reversibility (see [28] and also Definition 2) of the linear lossless approximation is assumed, the Onsager relations follow. Uncertainty in the initial state of linear lossless approximations give a simple explanation for noise that can be observed at a macroscopic level, as quantified by the fluctuation-dissipation theorem. The fluctuation-dissipation theorem and the Onsager relations are well know and have been shown in many different settings. Our contribution here is to give alternative explanations that use the language and tools familiar to control theorists.

The second contribution of the paper is that we highlight the importance of the fluctuation-dissipation theorem for deriving limitations in control theory. As an application of control-theoretic relevance, we apply it on models of measurement devices. With idealized measurement devices that are not lossless, we show that measurements can be done without perturbing the measured system. We say these measurement devices have no *back action*, or alternatively, no *observer effect*. However, if these ideal measurement devices are implemented using lossless approximations, simple limitations on the back action that depends on the surrounding temperature and available energy emerge. We argue that these lossless measurement devices and the resulting limitations are better models of what we can actually implement physically.

We hope this paper is a step towards building a framework for understanding fundamental limitations in control and estimation that arise due

to the physical implementation of measurement devices and, eventually, actuation. We defer many important and difficult issues here such as how to actually model such devices realistically. It is also clear that this framework would benefit from a behavioral setting [35]. However, for the points we make with this paper, a conventional input-output setting with only regular interconnections is sufficient. Aficionados will easily see the generalizations, the details of which might be an obstacle to readability for others. Perhaps the most glaring unresolved issue is how to best motivate the introduction of stochastics. In conventional statistical mechanics, a stochastic framework is taken for granted, whereas we ultimately aim to explain if, where, and why stochastics arise naturally. We hope to address this in future papers. The paper [33] is an early version of this paper.

1.3 Organization

The organization of the paper is as follows: In Section 2, we derive lossless approximations of various classes of systems. First we look at memoryless dissipative systems, then at dissipative systems with memory, and finally at active systems. In Section 3, we look at the influence of the initial state of the lossless approximations, and derive the fluctuation-dissipation theorem. In Section 4, we apply the results to measurement devices, and obtain limits on their performance.

1.4 Notation

Most notation used in the paper is standard. Let $f(t) \in \mathbb{R}^{n \times n}$ and $f_{ij}(t)$ be the (i, j)-th element. Then $f(t)^T$ denotes the transpose of $f(t)$, and $f(t)^*$ the complex conjugate transpose of $f(t)$. We define $\|f(t)\|_1 := \sum_{i,j=1}^{n} |f_{ij}(t)|$, $\|f(t)\|_2 := \sqrt{\sum_{i,j=1}^{n} |f_{ij}(t)|^2}$, and $\bar{\sigma}(f(t))$ is the largest singular value of $f(t)$. Furthermore, $\|f\|_{L_1[0,t]} := \int_0^t \|f(s)\|_1 ds$, and $\|f\|_{L_2[0,t]} := \sqrt{\int_0^t \|f(s)\|_2^2 ds}$. I_n is the n-dimensional identity matrix.

2 Lossless Approximations

2.1 Lossless systems

In this paper, linear systems in the form

$$
\begin{aligned}
\dot{x}(t) &= Jx(t) + Bu(t), & x(t) &\in \mathbb{R}^n, \\
y(t) &= B^T x(t) + Du(t), & u(t), y(t) &\in \mathbb{R}^p,
\end{aligned}
\tag{1}
$$

where J and D are anti symmetric ($J = -J^T$, $D = -D^T$) and (J, B) is controllable are of special interest. The system (1) is a *linear lossless system*. We define the *total energy $E(x)$* of (1) as

$$E(x) := \frac{1}{2} x^T x. \tag{2}$$

Lossless [27, 28] means that the total energy of (1) satisfies

$$\frac{dE(x(t))}{dt} = x(t)^T \dot{x}(t) = y(t)^T u(t) =: w(t), \tag{3}$$

where $w(t)$ is the *work rate* on the system. If there is no work done on the system, $w(t) = 0$, then the total energy $E(x(t))$ is constant. If there is work done on the system, $w(t) > 0$, the total energy increases. The work, however, can be extracted again, $w(t) < 0$, since the energy is conserved and the system is controllable. In fact, all finite-dimensional linear minimal lossless systems with supply rate $w(t) = y(t)^T u(t)$ can be written in the form (1), see [28, Theorem 5]. Nonlinear lossless systems will also be of interest later in the paper. They will also satisfy (2)–(3), but their dynamics are nonlinear. Conservation of energy is a common assumption on microscopic models in statistical mechanics and in physics in general [6]. The systems (1) are also time reversible if, and only if, they are also reciprocal, see [28, Theorem 8] and also Definitions 1–2 in Section 2.3. Hence, we argue the systems (1) have desirable "physical" properties.

Remark 1. *In this paper, we only consider systems that are lossless and dissipative with respect to the supply rate $w(t) = y(t)^T u(t)$. This supply rate is of special importance because of its relation to passivity theory. Indeed, there is a theory for systems with more general supply rates, see for example [27, 28], and it is an interesting problem to generalize the results here to more general supply rates.*

Remark 2. *The system (1) is a linear port-Hamiltonian system, see for example [36], with no dissipation. Note that the Hamiltonian of a linear port-Hamiltonian system is identical to the total energy E.*

There are well-known necessary and sufficient conditions for when a transfer function can be exactly realized using linear lossless systems: All the poles of the transfer function must be simple, located on the imaginary axis, and with positive semidefinite residues, see [28]. In this paper, we show that linear dissipative systems can be arbitrarily well approximated by linear lossless systems (1) over arbitrarily large time intervals. Indeed,
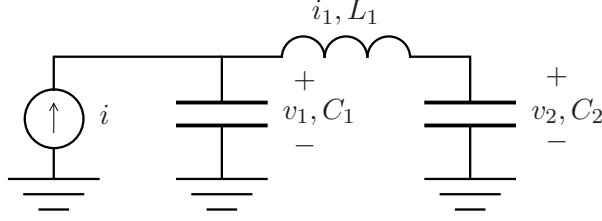
Figure 1: The inductor-capacitor circuit in Example 1.

if we believe that energy is conserved, then all macroscopic models should be realizable using lossless systems of possibly large dimension. The linear lossless systems are rather abstract but have properties that we argue are reasonable from a physical point of view, as illustrated by the following example.

Example 1. *It is a simple exercise to show that the circuit in Fig. 1 with the current $i(t)$ through the current source as input $u(t)$, and the voltage $v_1(t)$ across the current source as output $y(t)$ is a lossless linear system. We have*

$$\dot{x}(t) = \begin{pmatrix} 0 & -1/\sqrt{L_1C_1} & 0 \\ 1/\sqrt{L_1C_1} & 0 & -1/\sqrt{L_1C_2} \\ 0 & 1/\sqrt{L_1C_2} & 0 \end{pmatrix} x(t)$$
$$+ \begin{pmatrix} 1/\sqrt{C_1} \\ 0 \\ 0 \end{pmatrix} u(t),$$
$$y(t) = \begin{pmatrix} 1/\sqrt{C_1} & 0 & 0 \end{pmatrix} x(t),$$
$$x(t)^T = \begin{pmatrix} \sqrt{C_1}v_1(t) & \sqrt{L_1}i_1(t) & \sqrt{C_2}v_2(t) \end{pmatrix},$$
$$E(x(t)) = \frac{1}{2}x(t)^T x(t) = \frac{1}{2}(C_1v_1(t)^2 + L_1i_1(t)^2 + C_2v_2(t)^2),$$
$$w(t) = y(t)u(t) = v_1(t)i(t).$$

Note that $E(x(t))$ coincides with the energy stored in the circuit, and that $w(t)$ is the power into the circuit. Electrical circuits with only lossless components (capacitors and inductors) can be realized in the form (1), see [37]. Circuits with resistors can always be approximated *by systems in the form (1), as is shown in this paper.*

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2.2 Lossless approximation of dissipative memoryless systems

Many times macroscopic systems, such as resistors, are modeled by simple static (or memoryless) input-output relations

$$y(t) = ku(t), \tag{4}$$

where $k \in \mathbb{R}^{p \times p}$. If k is positive semidefinite, this system is dissipative since work can never be extracted and the work rate is always nonnegative, $w(t) = y(t)^T u(t) = u(t)^T k u(t) \geq 0$, for all t and $u(t)$. Hence, (4) is not lossless. Next, we show how we can approximate (4) arbitrarily well with a lossless linear system (1) over *finite*, but arbitrarily long, time horizons $[0, \tau]$. First of all, note that k can be decomposed into $k = k_s + k_a$ where k_s is symmetric positive semidefinite, and k_a is anti symmetric. We can use $D = k_a$ in the lossless approximation (1) and need only to consider the symmetric matrix k_s next.

First, choose the time interval of interest, $[0, \tau]$, and rewrite $y(t) = k_s u(t)$ as the convolution

$$y(t) = \int_{-\infty}^{\infty} \kappa(t-s) u(s) ds, \quad \kappa(t) := k_s \delta(t), \tag{5}$$

where $u(t)$ is at least continuous and has support in the interval $[0, \tau]$,

$$u(t) = 0, \quad t \in (-\infty, 0] \cup [\tau, \infty),$$

and $\delta(t)$ is the Dirac distribution. The time interval $[0, \tau]$ should contain all the time instants where we perform input-output experiments on the system (4)–(5). The impulse response $\kappa(t)$ can be formally expanded in a Fourier series over the interval $[-\tau, \tau]$,

$$\kappa(t) \sim \frac{k_s}{2\tau} + \sum_{l=1}^{\infty} \frac{k_s}{\tau} \cos l\omega_0 t, \quad \omega_0 := \pi/\tau. \tag{6}$$

To be precise, the Fourier series (6) converges to $k_s \delta(t)$ in the sense of distributions. Define the truncated Fourier series by $\kappa_N(t) := k_s/(2\tau) + \sum_{l=1}^{N-1} (k_s/\tau) \cos l\omega_0 t$ and split $\kappa_N(t)$ into a causal and an anti-causal part:

$$\kappa_N(t) =: \kappa_N^c(t) + \kappa_N^{ac}(t)$$
$$\kappa_N^c(t) = 0 \quad (t < 0), \quad \kappa_N^{ac}(t) = 0 \quad (t \geq 0).$$

The causal part $\kappa_N^c(t)$ can be realized as the impulse response of a lossless linear system (1) of order $(2N-1)r$ using the matrices

$$J = J_N := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \Omega_N \\ 0 & -\Omega_N & 0 \end{bmatrix},$$

$$\Omega_N := \mathrm{diag}\{\omega_0 I_r, 2\omega_0 I_r, \ldots, (N-1)\omega_0 I_r\}, \tag{7}$$

$$B = B_N := \sqrt{\frac{1}{\tau}} \left(\frac{k_f^T}{\sqrt{2}} \quad k_f^T \quad \ldots \quad k_f^T \quad 0 \quad \ldots \quad 0 \right)^T,$$

where $r = \mathrm{rank}\, k_s$ and $k_f \in \mathbb{R}^{r \times p}$ satisfies $k_s = k_f^T k_f$. That the series (6) converges in the sense of distributions means that for all smooth $u(t)$ of support in $[0, \tau]$ we have that

$$k_s u(t) = \lim_{N \to \infty} \int_{-\infty}^{\infty} \left(\kappa_N^{ac}(t-s) + \kappa_N^c(t-s) \right) u(s) ds.$$

A closer study of the two terms under the integral reveals that

$$\lim_{N \to \infty} \int_{-\infty}^{\infty} \kappa_N^{ac}(t-s) u(s) ds = \frac{1}{2} k_s u(t+),$$

$$\lim_{N \to \infty} \int_{-\infty}^{\infty} \kappa_N^c(t-s) u(s) ds = \frac{1}{2} k_s u(t-),$$

because of the anti-causal/causal decomposition and $\kappa_N^c(t) = \kappa_N^{ac}(-t)$, $t > 0$. Thus since $u(t)$ is smooth, we can also model $y(t) = k_s u(t)$ using only the causal part $\kappa_n^c(t)$ if it is scaled by a factor of two. This leads to a linear lossless approximation of $y(t) = k_s u(t)$ that we denote by the linear operator $K_N : \mathcal{C}^2(0, \tau) \to \mathcal{C}^2(0, \tau)$ defined by

$$
\begin{aligned}
y_N(t) = (K_N u)(t) &= \int_{-\infty}^{\infty} 2\kappa_N^c(t-s) u(s) ds \\
&= \int_0^t 2\kappa_N^c(t-s) u(s) ds.
\end{aligned}
\tag{8}
$$

Here $\mathcal{C}^2(0, \tau)$ denotes the space of twice continuously differentiable functions on the interval $[0, \tau]$. The linear operator K_N is realized by the triple $(J_N, \sqrt{2} B_N, \sqrt{2} B_N^T)$. We can bound the approximation error as seen in the following theorem.

Theorem 1. *Assume that $u \in \mathcal{C}^2(0, \tau)$ and $u(0) = 0$. Let $y(t) = ku(t) = k_s u(t) + k_a u(t)$ with k_s symmetric positive semidefinite and k_a anti symmetric. Define a lossless approximation with realization $(J_N, \sqrt{2}B_N, \sqrt{2}B_N^T, k_a)$, $y_N(t) = K_N u(t) + k_a u(t)$. Then the approximation error is bounded as*

$$\|y(t) - y_N(t)\|_2 \leq \frac{2\bar{\sigma}(k_s)\tau}{\pi^2(N-1)} \left(\|\dot{u}(t)\|_2 + \|\dot{u}(0)\|_2 + \|\ddot{u}\|_{L_1[0,t]} \right),$$

for t in $[0, \tau]$.

Proof. We have that $y(t) - y_N(t) = \sum_{l=N}^{\infty} (2k_s/\tau) \int_0^t \cos l\omega_0(t - s)u(s)ds$, $t \in [0, \tau]$. The order of summation and integration has changed because this is how the value of the series is defined in distribution sense. We proceed by using repeated integration by parts on each term in the series. It holds that $\int_0^t \cos l\omega_0(t - s)u(s)ds = [\int_0^t \sin l\omega_0(t - s)\dot{u}(s)ds]/(l\omega_0) = [\dot{u}(t) - \dot{u}(0)\cos l\omega_0 t - \int_0^t \cos l\omega_0(t - s)\ddot{u}(s)ds]/(l^2\omega_0^2)$. Hence, we have the bound

$$\|y(t) - y_N(t)\|_2 \leq \frac{2\bar{\sigma}(k_s)}{\tau} \sum_{l=N}^{\infty} \frac{1}{l^2\omega_0^2}(\|\dot{u}(t)\|_2$$

$$+ \|\dot{u}(0)\|_2 + \int_0^t \|\ddot{u}(s)\|_1 ds).$$

Since $\sum_{l=N}^{\infty} 1/l^2 \leq 1/(N-1)$, we can establish the bound in the theorem. $\square$

The theorem shows that by choosing the truncation order N sufficiently large, the memoryless model (4) can be approximated as well as we like with a lossless linear system, if inputs are smooth. Hence we cannot then distinguish between the systems $y = ku$ and $y_N = K_N u + k_a u$ using finite-time input-output experiments. On physical grounds one may prefer the model $K_N + k_a$ even though it is more complex, since it assumes the form (1) of a lossless system (and is time reversible if k is reciprocal, see Theorem 3). Additional support for this idea is given in Section 3. Note that the lossless approximation K_N is far from unique: The time interval $[0, \tau]$ is arbitrary, and other Fourier expansions than (6) are possible to consider. The point is, however, that it is always possible to approximate the dissipative behavior using a lossless model.

It is often a reasonable assumption that inputs $u(t)$, for example voltages, are smooth if we look at a sufficiently fine time scale. This is because we usually cannot change inputs arbitrarily fast due to physical limitations. Physically, we can think of the approximation order $(2N-1)r$ as the number

of degrees of freedom in a physical system, usually of the order of Avogadro's number, $N \approx 10^{23}$. It is then clear that the interval length τ can be very large without making the approximation error bound in Theorem 1 large. This explains how the dissipative system (4) is consistent with a physics based on energy conserving systems.

Remark 3. *Note that it is well known that a dissipative memoryless system can be modeled by an* infinite-dimensional *lossless system. We can model an electrical resistor by a semi-infinite lossless transmission line using the telegraphists's equation (the wave equation), see [38], for example. If the inductance and capacitance per unit length of the line are L and C, respectively, then the characteristic impedance of the line, $\sqrt{L/C}$, is purely resistive. One possible interpretation of K_N is as a finite-length lossless transmission line where only the N lowest modes of the telegraphists's equation are retained. Also in the physics literature lossless (or Hamiltonian) approximations of dissipative memoryless systems can be found. In [10–12], a so-called Ohmic bath is used, for example. Note that it is not shown in these papers when, and how fast, the approximation converges to the dissipative system. This is in contrast to the analysis presented herein, and the error bound in Theorem 1.*

2.3   Lossless approximation of dissipative systems with memory

In this section, we generalize the procedure from Section 2.2 to dissipative systems that have memory. We consider asymptotically stable time-invariant linear causal systems G with impulse response $g(t) \in \mathbb{R}^{p \times p}$. Their input-output relation is given by

$$y(t) = (Gu)(t) = \int_0^t g(t-s)u(s)ds. \tag{9}$$

Possible direct terms in G can be approximated separately as shown in Section 2.2. The system (9) is dissipative with respect to the work rate $w(t) = y(t)^T u(t)$ if and only if $\int_0^\tau y(t)^T u(t)dt \geq 0$, for all $\tau \geq 0$ and admissible $u(t)$. An equivalent condition, see [28], is that the transfer function satisfies

$$\hat{g}(j\omega) + \hat{g}(-j\omega)^T \geq 0 \quad \text{for all} \quad \omega. \tag{10}$$

Here $\hat{g}(j\omega)$ is the Fourier transform of $g(t)$.

We will next consider the problem of how well, and when, a system (9) can be approximated using a linear lossless system (1) (call it G_N) with

fixed initial state x_0,

$$y_N(t) = B^T e^{Jt} x_0 + \int_0^t B^T e^{J(t-s)} Bu(s)ds, \qquad (11)$$

for a set of input signals. Let us formalize the problem.

Problem 1. *For any fixed time horizon $[0, \tau]$ and arbitrarily small $\epsilon > 0$, when is it possible to find a lossless system with fixed initial state x_0 and output y_N such that*

$$\|y(t) - y_N(t)\|_2 \leq \epsilon \|u\|_{L_2[0,t]}, \qquad (12)$$

for all input signals $u \in L_2[0, t]$ and $0 \leq t \leq \tau$?

Note that we require x_0 to be fixed in Problem 1, so that it is independent of the applied input $u(t)$. This means the approximation should work even if the applied input is not known beforehand. Let us next state a necessary condition for linear lossless approximations.

Proposition 1. *Assume there is a linear lossless system G_N that solves Problem 1. Then it holds that*

(i) If $x_0 \neq 0$, then x_0 is an unobservable state;

(ii) If $x_0 \neq 0$, then x_0 is an uncontrollable state; and

(iii) If the realization of G_N is minimal, then $x_0 = 0$.

Proof. (i): The inequality (12) holds for $u = 0$ when $y = 0$. Then (12) reduces to $\|y_N(t)\|_2 \leq 0$, for $t \in [0, \tau]$, which implies $y_N(t) = B^T e^{Jt} x_0 = 0$. Thus a nonzero x_0 must be unobservable. (ii): For the lossless realizations it holds that $\mathcal{N}(\mathcal{O}) = \mathcal{R}(\mathcal{O}^T)^\perp = \mathcal{R}(\mathcal{C})^\perp$, where $\mathcal{O}$ and $\mathcal{C}$ are the observability and controllability matrices for the realization (J, B, B^T). Thus if x_0 is unobservable, it is also uncontrollable. (iii): Both (i) and (ii) imply (iii). $\square$

Proposition 1 significantly restricts the classes of systems G we can approximate using linear lossless approximations. Intuitively, to approximate active systems there must be energy stored in the initial state of G_N. But Proposition 1 says that such initial energy is not available for the inputs and outputs of G_N. The next theorem shows that we can approximate G using G_N if, and only if, G is dissipative.

Theorem 2. *Suppose G is a linear time-invariant causal system (9), where $\|g(t)\|_2$ is uniformly bounded, $g(t) \in L_1 \cap L_2(0, \infty)$, and $\dot{g}(t) \in L_1(0, \infty)$. Then Problem 1 is solvable using a linear lossless G_N if, and only if, G is dissipative.*

Proof. See Appendix 6.1. □

The proof of Theorem 2 shows that the number of states needed in G_N is proportional to τ/ϵ^2, and again the required state space is large. The result shows that for finite-time input-output experiments with finite-energy inputs it is not possible to distinguish between the dissipative system and its lossless approximations. Theorem 2 illustrates that a very large class of dissipative systems (macroscopic systems) can be approximated by the lossless linear systems we introduced in (1). The lossless systems are dense in the dissipative systems, in the introduced topology. Again this shows how dissipative systems are consistent with a physics based on energy-conserving systems.

In [28, Theorem 8], necessary and sufficient conditions for time reversible systems are given. We can now use this result together with Theorem 2 to prove a result reminiscent to the *Onsager reciprocal relations* which say physical systems tend to be reciprocal, see for example [6]. Before stating the result, we properly define what is meant by reciprocal and time reversible systems. These definitions are slight reformulations of those found in [28].

A *signature matrix* Σ_e is a diagonal matrix with entries either $+1$ and -1.

Definition 1. *A linear time-invariant system G with impulse response $g(t)$ is* reciprocal *with respect to the signature matrix Σ_e if $\Sigma_e g(t) = g(t)^T \Sigma_e$.*

Definition 2. *Consider a finite-dimensional linear time-invariant system G and assume that $x(0) = 0$. Let u_1, u_2 be admissible inputs to G, and y_1, y_2 be the corresponding outputs. Then G is* time reversible *with respect to the signature matrix Σ_e if $y_2(t) = \Sigma_e y_1(-t)$ whenever $u_2(t) = -\Sigma_e u_1(-t)$.*

Theorem 3. *Suppose G satisfies the assumptions in Theorem 2. Then G is dissipative and reciprocal with respect to Σ_e if, and only if, there exists a time-reversible (with respect to Σ_e) arbitrarily good linear lossless approximation G_N of G.*

Proof. See Appendix 6.2. □

Hence, one can understand that macroscopic physical systems close to equilibrium usually are reciprocal because their underlying dynamics are lossless *and* time reversible.

Remark 4. *There is a long-standing debate in physics about how macroscopic time-irreversible dynamics can result from microscopic time-reversible dynamics. The debate goes back to* Loschmidt's paradox *and the* Poincaré recurrence theorem. *The Poincaré recurrence theorem says that bounded trajectories of volume-preserving systems (such as lossless systems) will return arbitrarily close to their initial conditions if we wait long enough (the Poincaré recurrence time). This seems counter-intuitive for real physical systems. One common argument is that the Poincaré recurrence time for macroscopic physical systems is so long that we will never experience a recurrence. But this argument is not universally accepted and other explanations exist. The debate still goes on, see for example [13]. In this paper we construct lossless and time-reversible systems with arbitrarily large Poincaré recurrence times, that are consistent with observations of all linear dissipative (time-irreversible) systems, as long as those observations take place before the recurrence time. For a control-oriented related discussion about the arrow of time, see [34].*

2.4 Nonlinear lossless approximations

In Section 2.2, it was shown that a dissipative memoryless system can be approximated using a lossless linear system. Later in Section 2.3 it was also shown that the approximation procedure can be applied to any dissipative (linear) system. Because of Proposition 1 and Theorem 2, it is clear that it is not possible to approximate a linear active system using a *linear* lossless system with fixed initial state. Next we will show that it is possible to solve Problem 1 for active systems if we use *nonlinear* lossless approximations.

Consider the simplest possible active system,

$$y(t) = ku(t), \tag{13}$$

where $k \in \mathbb{R}^{p \times p}$ is negative definite. This can be a model of a negative resistor, for example. More general active systems are considered below. The reason a linear lossless approximation of (13) cannot exist is that the active device has an internal infinite energy supply, but we cannot store any energy in the initial state of a linear lossless system and simultaneously track a set of outputs, see Proposition 1. However, if we allow for lossless nonlinear approximations, (13) can be arbitrarily well approximated. This is shown next by means of an example.

Consider the nonlinear system

$$\dot{x}_E(t) = \frac{1}{\sqrt{2E_0}} u(t)^T k u(t), \quad x_E(0) = \sqrt{2E_0}, \ E_0 > 0,$$

$$y_E(t) = \frac{x_E(t)}{\sqrt{2E_0}} k u(t), \tag{14}$$

with a scalar energy-supply state $x_E(t)$, and total energy $E(x_E) = \frac{1}{2} x_E^2$. The system (14) has initial total energy $\frac{1}{2} x_E(0)^2 =: E_0$, and is a lossless system with respect to the work rate $w(t) = y_E(t) u(t)$, since

$$\frac{d}{dt} E(x_E(t)) = x_E(t) \dot{x}_E(t) = y_E(t)^T u(t).$$

The input-output relation of (14) is given by

$$x_E(t) = \sqrt{2E_0} + \frac{1}{\sqrt{2E_0}} \int_0^t u(s)^T k u(s) ds,$$

$$y_E(t) = k u(t) + \frac{1}{2E_0} k u(t) \int_0^t u(s)^T k u(s) ds. \tag{15}$$

We have the following approximation result.

Theorem 4. *For uniformly bounded inputs, $\|u(t)\|_2 \leq \bar{u}$, $t \in [0, \tau]$, the error between the active system (13) and the nonlinear lossless approximation (14) can be bounded as*

$$\|y_E(t) - y(t)\|_2 \leq \epsilon \|u\|_{L_2[0,t]},$$

for $t \in [0, \tau]$, where $\epsilon = \bar{\sigma}(k)^2 \bar{u}^2 \sqrt{\tau}/(2E_0)$.

Proof. A simple bound on $y_E(t) - k u(t)$ from (15) gives $\|y_E(t) - y(t)\|_2 \leq \frac{\bar{\sigma}(k)^2 \|u(t)\|_2}{2E_0} \int_0^t \|u(s)\|_2^2 ds$. Then using $\|u(t)\|_2 \leq \bar{u}$, $t \in [0, \tau]$, gives the result. $\square$

The error bound in Theorem 4 can be made arbitrarily small for finite time intervals if the initial total energy E_0 is large enough. This example shows that active systems can also be approximated by lossless systems, if the lossless systems are allowed to be nonlinear and are charged with initial energy.

The above approximation method can in fact be applied to much more general systems. Consider the ordinary differential equation

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0,$$

$$y(t) = g(x(t), u(t)), \tag{16}$$

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where $x(t) \in \mathbb{R}^n$, and $u(t), y(t) \in \mathbb{R}^p$. In general, this is not a lossless system with respect to the supply rate $w(t) = y(t)^T u(t)$. A nonlinear lossless approximation of (16) is given by

$$\dot{\hat{x}}(t) = \frac{x_E(t)}{\sqrt{2E_0}} f(\hat{x}(t), u(t)), \quad \hat{x}(0) = x_0,$$

$$\dot{x}_E(t) = \frac{1}{\sqrt{2E_0}} g(\hat{x}(t), u(t))^T u(t) - \frac{1}{\sqrt{2E_0}} \hat{x}(t)^T f(\hat{x}(t), u(t)), \qquad (17)$$

$$y_E(t) = \frac{x_E(t)}{\sqrt{2E_0}} g(\hat{x}(t), u(t)), \quad x_E(0) = \sqrt{2E_0},$$

where again $x_E(t)$ is a scalar energy-supply state, and $\hat{x}(t) \in \mathbb{R}^n$ can be interpreted as an approximation of $x(t)$ in (16). That (17) is lossless can be verified using the storage function

$$E = \frac{1}{2}\hat{x}(t)^T \hat{x}(t) + \frac{1}{2}x_E(t)^2,$$

since

$$\dot{E} = (x_E/\sqrt{2E_0})(\hat{x}^T f(\hat{x}, u) + g(\hat{x}, u)^T u - \hat{x}^T f(\hat{x}, u))$$
$$= (x_E/\sqrt{2E_0})g(\hat{x}, u)^T u = y_E^T u = w.$$

Since $x_E(t)/\sqrt{2E_0} \approx 1$ for small t, it is intuitively clear that $\hat{x}(t)$ in (17) will be close to $x(t)$ in (16), at least for small t and large initial energy E_0. We have the following theorem.

Theorem 5. *Assume that $\partial f/\partial x$ is continuous with respect to x and t, and that (16) has a unique solution $x(t)$ for $0 \le t \le \tau$. Then there exist positive constants C_1 and E_1 such that for all $E_0 \ge E_1$ (17) has a unique solution $\hat{x}(t)$ which satisfies $\|x(t) - \hat{x}(t)\|_2 \le C_1/\sqrt{2E_0}$ for all $0 \le t \le \tau$.*

Proof. Introduce the new coordinate $\Delta x_E = x_E - \sqrt{2E_0}$ and define $\epsilon_0 := 1/\sqrt{2E_0}$. The system (17) then takes the form

$$\dot{\hat{x}} = (1 + \epsilon_0 \Delta x_E)f(\hat{x}, u), \quad \hat{x}(0) = x_0,$$
$$\Delta \dot{x}_E = \epsilon_0 g(\hat{x}, u)^T u - \epsilon_0 \hat{x}^T f(\hat{x}, u), \quad \Delta x_E(0) = 0.$$

Perturbation analysis [39, Section 10.1] in the parameter ϵ_0 as $\epsilon_0 \to 0$ yields that there are positive constants ϵ_1 and C_1 such that $\|x - \hat{x}\|_2 \le C_1|\epsilon_0|$ for all $|\epsilon_0| \le \epsilon_1$. The result then follows with $E_1 = 1/(2\epsilon_1^2)$. $\qquad \square$

Just as in Section 2.3, the introduced lossless approximations are not unique. The one introduced here, (17), is very simple since only one extra state x_E is added. Its accuracy (C_1, E_0) of course depends on the particular system (f, g) and the time horizon τ. An interesting topic for future work is to develop a theory for "optimal" lossless approximations using a fixed amount of energy and a fixed number of states.

2.5 Summary

In Section 2, we have seen that a large range of systems, both dissipative and active, can be approximated by lossless systems. Lossless systems account for the total energy, and we claim these models are more physical. It was shown that linear lossless systems are dense in the set of linear dissipative systems. It was also shown that time reversibility of the lossless approximation is equivalent to a reciprocal dissipative system. To approximate active systems nonlinearity is needed. The introduced nonlinear lossless approximation has to be initialized at a precise state with a large total energy (E_0). The nonlinear approximation achieves better accuracy (smaller ϵ) by increasing initial energy (increasing E_0). This is in sharp contrast to the linear lossless approximations of dissipative systems that are initialized with zero energy ($E_0 = 0$). These achieve better accuracy (smaller ϵ) by increasing the number of states (increasing N). The next section deals with uncertainties in the initial state of the lossless approximations.

3 The Fluctuation-Dissipation Theorem

As discussed in the introduction, the fluctuation-dissipation theorem plays a major role in close-to-equilibrium statistical mechanics. The theorem has been stated in many different settings and for different models. See for example [17, 20], where it is stated for Hamiltonian systems and Langevin equations. In [18, 19], it is stated for electrical circuits. A fairly general form of the fluctuation-dissipation theorem is given in [6, p. 500]. We re-state this version of the theorem here.

Suppose that y_i and u_i, $i = 1, \ldots, p$, are conjugate external variables (inputs and outputs) for a dissipative system in thermal equilibrium of temperature T [Kelvin] (as defined in Section 3.1). We can interpret y_i as a generalized velocity and u_i as the corresponding generalized force, such that $y_i u_i$ is a work rate [Watt]. Although the system is generally nonlinear, we only consider small variations of the state around a fixpoint of the dynamics,

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which allows us to assume the system to be linear. Assume first that the system has no direct term (no memoryless element). If we make a perturbation in the forces u, the velocities y respond according to

$$y(t) = \int_0^t g(t-s)u(s)ds,$$

where $g(t) \in \mathbb{R}^{p \times p}$ is the impulse response matrix by definition. The following fluctuation-dissipation theorem now says that the velocities y actually also fluctuates around the equilibrium.

Proposition 2. *The total response of a linear dissipative system G with no memoryless element and in thermal equilibrium of temperature T is given by*

$$y(t) = n(t) + \int_0^t g(t-s)u(s)ds, \tag{18}$$

for perturbations u. The fluctuations $n(t) \in \mathbb{R}^p$ is a stationary Gaussian stochastic process, where

$$\begin{aligned}
\mathbf{E}n(t) &= 0, \\
R_n(t,s) &:= \mathbf{E}n(t)n(s)^T \\
&= \begin{cases} k_B T g(t-s), \, t-s \geq 0 \\ k_B T g(s-t)^T, \, t-s < 0, \end{cases}
\end{aligned} \tag{19}$$

where k_B is Boltzmann's constant.

Proof. See Section 3.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

The covariance function of the noise n is determined by the impulse response g, and vice versa. The result has found wide-spread use in for example fluid mechanics: By empirical estimation of the covariance function we can estimate how the system responds to external forces. In circuit theory, the result is often used in the other direction: The forced response determines the color of the inherent thermal noise. One way of understanding the fluctuation-dissipation theorem is by using linear lossless approximations of dissipative models, as seen in the next subsection.

We may also express (18) in state space form in the following way. A dissipative system with no direct term can always be written as [28, Theorem 3]:

$$\begin{aligned}
\dot{x}(t) &= (J-K)x(t) + Bu(t), \\
y(t) &= B^T x(t),
\end{aligned} \tag{20}$$

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where $K = K^T$ is positive semidefinite and J anti symmetric. To account for (18)–(19), it suffices to introduce a white noise term $v(t)$ in (20) in the following way,

$$\dot{x}(t) = (J - K)x(t) + Bu(t) + \sqrt{2k_B T} Lv(t),$$
$$y(t) = B^T x(t), \tag{21}$$

where the matrix L is chosen such that $LL^T = K$. Equation (21) is the called the Langevin equation of the dissipative system.

Dissipative systems with memoryless elements are of great practical significance. Proposition 2 needs to be slightly modified for such systems.

Proposition 3. *The total response of a linear dissipative memoryless system in thermal equilibrium of temperature T and for perturbations u is given by*

$$y(t) = n(t) + ku(t) = n(t) + k_s u(t) + k_a u(t), \tag{22}$$

where $k_s \geq 0$ is symmetric positive semidefinite, and k_a anti symmetric. The fluctuations $n(t) \in \mathbb{R}^p$ is a white Gaussian stochastic process, where

$$\mathbf{E}n(t) = 0,$$
$$R_n(t,s) := \mathbf{E}n(t)n(s)^T = 2k_B T k_s \delta(t - s).$$

Proposition 3 follows from Proposition 2 if one extracts the dissipative term $k_s u(t)$ from the memoryless model $ku(t)$ and puts $g(t) = k_s \delta(t)$. However, the integral in (18) runs up to $s = t$ and cuts the impulse $\delta(t)$ in half. The re-normalized impulse response of the dissipative term is therefore given by $g(t) = 2k_s \delta(t)$ (see also Section 2.2). The result then follows using this $g(t)$ by application of Proposition 2. One explanation for why the anti symmetric term k_a can be removed from $g(t)$ is that it can be realized exactly using the direct term D in linear lossless approximation (1). An application of Proposition 3 gives the Johnson-Nyquist noise of a resistor.

Example 2. *As first shown theoretically in [15] and experimentally in [14], a resistor R of temperature T generates white noise. The total voltage over the resistor, $v(t)$, satisfies $v(t) = Ri(t) + n(t)$, $\mathbf{E}n(t)n(s) = 2k_B T R \delta(t - s)$, where $i(t)$ is the current.*

3.1 Derivation using linear lossless approximations

Let us first consider systems without memoryless elements. The general solution to the linear lossless system (1) is then

$$y(t) = B^T e^{Jt} x_0 + \int_0^t B^T e^{J(t-s)} Bu(s)ds, \tag{23}$$

where x_0 is the initial state. It is the second term, the convolution, that approximates the dissipative $(Gu)(t)$ in the previous section. In Proposition 1, we showed that the first transient term is not desired in the approximation. Theorems 1 and 2 suggest that we will need a system of extremely high order to approximate a linear dissipative system on a reasonably long time horizon. When dealing with systems of such high dimensions, it is reasonable to assume that the exact initial state x_0 is not known, and it can be hard to enforce $x_0 = 0$. Therefore, let us take a statistical approach to study its influence. We have that

$$\mathbf{E}y(t) = B^T e^{Jt} \mathbf{E}x_0 + \int_0^t B^T e^{J(t-s)} Bu(s)ds, \quad t \geq 0,$$

if the input $u(t)$ is deterministic and $\mathbf{E}$ is the expectation operator. The autocovariance function R_y for $y(t)$ is then

$$
\begin{aligned}
R_y(t,s) &:= \mathbf{E}[y(t) - \mathbf{E}y(t)][y(s) - \mathbf{E}y(s)]^T \\
&= B^T e^{Jt} X_0 e^{-Js} B,
\end{aligned}
\tag{24}
$$

where X_0 is the covariance of the initial state,

$$X_0 := \mathbf{E}\Delta x_0 \Delta x_0^T, \tag{25}$$

where $\Delta x_0 := x_0 - \mathbf{E}x_0$ is the stochastic uncertain component of the initial state, which evolves as $\Delta x(t) = e^{Jt}\Delta x_0$. The positive semidefinite matrix X_0 can be interpreted as a measure of how well the initial state is known. For a lossless system with total energy $E(x) = \frac{1}{2}x^T x$ we define the *internal energy* as

$$U(x) := \frac{1}{2}\Delta x^T \Delta x, \quad \Delta x := x - \mathbf{E}x. \tag{26}$$

The expected total energy of the system equals $\mathbf{E}E(x) = \frac{1}{2}(\mathbf{E}x)^T\mathbf{E}x + \mathbf{E}U(x)$. Hence the internal energy captures the stochastic part of the total energy, see also [25, 30]. In statistical mechanics, see [6–8], the temperature of a system is defined using the internal energy.

Definition 3 (Temperature). *A system with internal energy $U(x)$ [Joule] has temperature T [Kelvin] if, and only if, its state x belongs to Gibbs's distribution with probability density function*

$$p(x) = \frac{1}{Z}\exp[-U(x)/k_B T], \tag{27}$$

where k_B is Boltzmann's constant and Z is the normalizing constant called the partition function. A system with temperature is said to be at thermal equilibrium.

When the internal energy function is quadratic and the system is at thermal equilibrium, it is well known that the uncertain energy is equipartitioned between the states, see [6, Sec. 4-5].

Proposition 4. *Suppose a lossless system with internal energy function $U(x) = \frac{1}{2}\Delta x^T \Delta x$ has temperature T at time $t = 0$. Then the initial state x_0 belongs to a Gaussian distribution with covariance matrix $X_0 = k_B T I_n$, and $\mathbf{E}U(x_0) = \frac{n}{2}k_B T$.*

Hence, the temperature T is proportional to how much uncertain equipartitioned energy there is per degree of freedom in the lossless system. There are many arguments in the physics and information theory literature for adopting the above definition of temperature. For example, Gibbs's distribution maximizes the Shannon continuous entropy (principle of maximum entropy [40, 41]). In this paper, we will simply accept this common definition of temperature, although it is interesting to investigate more general definitions of temperature of dynamical systems.

Remark 5. *Note that lossless systems may have a temperature at any time instant, not only at $t = 0$. For instance, a lossless linear system (23) of temperature T at $t = 0$ that is driven by a deterministic input remains at the same temperature and has constant internal energy at all times, since $\Delta x(t)$ is independent of $u(t)$. To change the internal energy using deterministic inputs, nonlinear systems are needed as explained in [23, 24]. For the related issue of entropy for dynamical systems, see [23, 25].*

If a lossless linear system (23) has temperature T at $t = 0$ as defined in Definition 3 and Proposition 4, then the autocovariance function (24) takes the form

$$R_y(t, s) = k_B T \cdot B^T e^{J(t-s)} B = k_B T \cdot [B^T e^{J(s-t)} B]^T,$$

since $J^T = -J$. It is seen that linear lossless systems satisfy the fluctuation-dissipation theorem (Proposition 2) if we identify the stochastic transient in (23) with the fluctuation, i.e. $n(t) = B^T e^{Jt} x_0$ (assuming $\mathbf{E}x_0 = 0$), and the impulse response as $g(t) = B^T e^{Jt} B$. In particular, $n(t)$ is a Gaussian process of mean zero because x_0 is Gaussian and has mean zero.

Theorem 2 showed that dissipative systems with memory can be arbitrarily well approximated by lossless systems. Hence we cannot distinguish between the two using only input-output experiments. One reason for preferring the lossless model is that its transient also explains the thermal noise that is predicted by the fluctuation-dissipation theorem. To explain

the fluctuation-dissipation theorem for systems without memory (Proposition 3), one can repeat the above arguments by making a lossless approximation of k_s (see Theorem 1). The anti symmetric part k_a does not need to be approximated but can be included directly in the lossless system by using the anti symmetric direct term D in (12).

Proposition 3 captures the notion of a *heat bath*, modelling it (as described in Theorem 1) with a lossless system so large that for moderate inputs and within the chosen time horizon, the interaction with its environment is not significantly affected.

That the Langevin equation (21) is a valid state-space model for (18) is shown by a direct calculation. If we assume that (20) is a low-order approximation for a high-order linear lossless system (23), in the sense of Theorem 2, it is enough to require that both systems are at thermal equilibrium with the same temperature T in order to be described by the same stochastic equation (18), at least in the time interval in which the approximation is valid.

3.2 Nonlinear lossless approximations and thermal noise

Lossless approximations are not unique. We showed in Section 2.4 that low-order nonlinear lossless approximations can be constructed. As seen next, these do *not* satisfy the fluctuation-dissipation theorem. This is not surprising since they can also model active systems. If they are used to implement linear dissipative systems, the linearized form is not in the form (1). By studying the thermal noise of a system, it could in principle be possible to determine what type of lossless approximation that is used.

Consider the nonlinear lossless approximation (14) of $y(t) = ku(t)$, where k is scalar and can be either positive or negative. The approximation only works well when the initial total energy E_0 is large. To study the effect of thermal noise, we add a random Gaussian perturbation Δx_0 to the initial state so that the system has temperature T at $t = 0$ according to Definition 3 and Proposition 4. This gives the system

$$\dot{x}_E(t) = \frac{k}{\sqrt{2E_0}}u(t)^2, \quad x_E(0) = \sqrt{2E_0} + \Delta x_0, \ \mathbf{E}\Delta x_0 = 0,$$
$$y_E(t) = \frac{k}{\sqrt{2E_0}}x_E(t)u(t), \quad \mathbf{E}\Delta x_0^2 = k_B T. \tag{28}$$

The solution to the lossless approximation (28) is given by

$$y_E(t) = ku(t) + n_s(t) + n_d(t), \tag{29}$$

where
$$n_d(t) = \frac{k^2}{2E_0} u(t) \int_0^t u(s)^2 ds, \quad n_s(t) = \frac{k\Delta x_0}{\sqrt{2E_0}} u(t). \qquad (30)$$

We call $n_d(t)$ the deterministic implementation noise and $n_s(t)$ the stochastic thermal noise. The ratio between the deterministic and stochastic noise is

$$\frac{n_d(t)}{n_s(t)} = \frac{k}{\sqrt{2E_0}\Delta x_0} \int_0^t u(s)^2 ds = \frac{ku(0)^2}{\sqrt{2E_0}\Delta x_0} t + O(t^2),$$

as $t \to 0$, if $u(t)$ is continuous. Hence, for sufficiently small times t and if $\Delta x_0 \neq 0$, the stochastic noise $n_s(t)$ is the dominating noise in the lossless approximation (28). Since Δx_0 belongs to a Gaussian distribution, there is zero probability that $\Delta x_0 = 0$. Hence, the solution $y_E(t)$ can be written

$$y_E(t) = ku(t) + n_s(t) + O(t),$$
$$\mathbf{E} n_s(t) = 0, \quad \mathbf{E} n_s(t)^2 = \frac{k^2 k_B T}{2E_0} u(t)^2. \qquad (31)$$

Just as in Proposition 3, the noise variance is proportional to the temperature T. Notice, however, that the noise is significantly smaller in (31) than in Proposition 3. There the noise is white and unbounded for each t. The expression (31) is further used in Section 4.

3.3 Summary

In Section 3, we have seen that uncertainty in the initial state of a linear lossless approximation gives a simple explanation for the fluctuation-dissipation theorem. We have also seen seen that uncertainty in the initial state of a nonlinear lossless approximation gives rise to noise which does not satisfy the fluctuation-dissipation theorem. In all cases, the variance of the noise is proportional to the temperature of the system. Only when the initial state is perfectly know, that is when the system has temperature zero, perfect approximation using lossless systems can be achieved.

4 Limits on Measurements and Back Action

In this section, we study measurement strategies and devices using the developed theory. In quantum mechanics, the problem of measurements and their interpretation have been much studied and debated. Also in classical physics there have been studies on limits on measurement accuracy. Two

examples are [42, 43], where thermal noise in measurement devices is analyzed and bounds on possible measurement accuracy derived. Nevertheless, the effect of the measurement device on the measured system, the "back action", is usually neglected in classical physics. That such effects exist also in classical physics is well known, however, and is called the "observer effect". Also in control engineering these effects are usually neglected: The sensor is normally modeled to interact with the controlled plant only through the feedback controller.

Using the theory developed in this paper, we will quantify and give limits on observer effects in a fairly general setting. These limitations should be of practical importance for control systems on the small physical scale, such as for MEMS and in systems biology.

4.1 Measurement problem formulation

Assume that the problem is to estimate the scalar potential $y(t_m)$ (an output) of a linear dissipative dynamical system $\mathcal{S}$ at some time $t_m > 0$. Furthermore, assume that the conjugate variable of y is u (the "flow" variable). Then the product $y(t)u(t)$ is a work rate. As has been shown in Section 2.3, all single-input–single-output linear dissipative systems can be arbitrarily well approximated by a dynamical system in the form,

$$
\mathcal{S} : \quad \begin{cases} \dot{x}(t) = Jx(t) + Bu(t), & x(0) = x_0, \\ y(t) = B^T x(t), & y(0) = y_0 = B^T x_0, \end{cases} \tag{32}
$$

for a fixed initial state x_0. Note that this system evolves deterministically since x_0 is fixed. Let us also define the parameter C by $B^T B =: 1/C$. Then $1/C$ is the first Markov parameter of the transfer function of $\mathcal{S}$. If $\mathcal{S}$ is an electrical capacitor and the measured quantity a voltage, C coincides with the capacitance.

To estimate the potential $y(t_m)$, an idealized measurement device called $\mathcal{M}$ is connected to $\mathcal{S}$ in the time interval $[0, t_m]$, see Fig. 2. The validity of Kirchoff's laws is assumed in the interconnection. That is, the flow out of $\mathcal{S}$ goes into $\mathcal{M}$, and the potential difference $y(t)$ over the devices is the same (a lossless interconnection). The device $\mathcal{M}$ has an ideal flow meter that gives the scalar value $u_m(t) = -u(t)$. Therefore the problem is to estimate the potential of $\mathcal{S}$ given knowledge of the flow $u(t)$. For this problem, two related effects are studied next, the *back action* $b(t_m)$, and the *estimation error* $e(t_m)$. By back action we mean how the interconnection with $\mathcal{M}$ effects the state of $\mathcal{S}$. It quantifies how much the state of $\mathcal{S}$ deviates from its
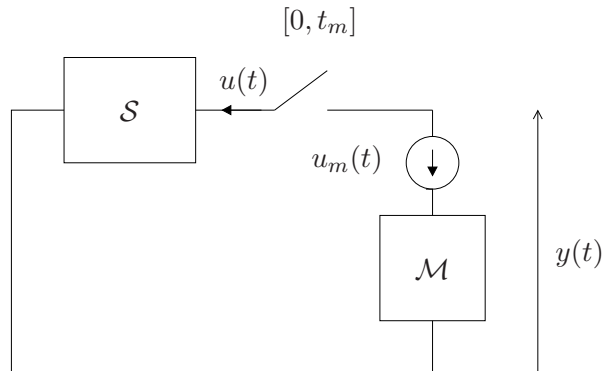
Figure 2: Circuit diagram of an idealized measurement device $\mathcal{M}$ and the measured system $\mathcal{S}$. The measurement is performed in the time interval $[0, t_m]$. The problem is to estimate the potential $y(t_m)$ as well as possible, given the flow measurement $u_m = -u$.
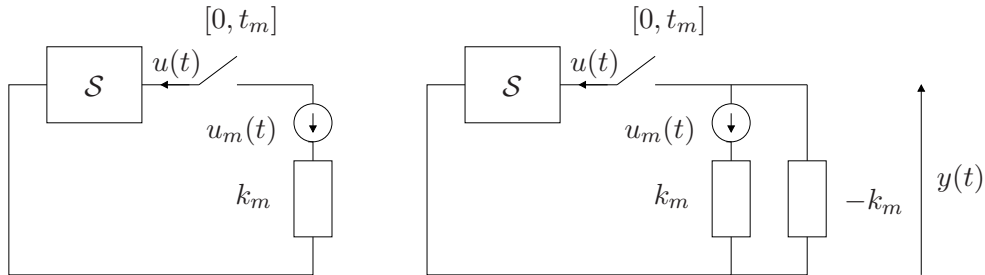

Figure 3: Circuit diagrams of the memoryless dissipative measurement device $\mathcal{M}_1$ (left) and the memoryless active measurement device $\mathcal{M}_2$ (right).

natural trajectory after the measurement. Estimation error is the difference between the actual potential and the estimated potential. Next we consider two measurement strategies and their lossless approximations in order to study the impact of physical implementation.

Remark 6. *The reason the initial state x_0 in $\mathcal{S}$ is fixed is that we want to compare how different measurement strategies succeed when used on exactly the same system. We also assume that $y_0 = B^T x_0$ is completely unknown to the measurement device before the measurement starts.*

4.2 Memoryless dissipative measurement device

Consider the measurement device $\mathcal{M}_1$ to the left in Fig. 3. This measurement device connected to $\mathcal{S}$ is modeled by a memoryless system with (a known) admittance $k_m > 0$,

$$\mathcal{M}_1 : \begin{cases} u_m(t) = -u(t) = k_m y(t) \\ y_m(t) = \dfrac{u_m(t)}{k_m} = y(t). \end{cases}$$

The signal $y_m(t)$ is the measurement signal produced by $\mathcal{M}_1$. The dynamics of the interconnected measured system becomes

$$\mathcal{SM}_1 : \begin{cases} \dot{x}_1(t) = (J - k_m B B^T) x_1(t), \quad x_1(0) = x_0, \\ y_1(t) = B^T x_1(t), \end{cases}$$

where $x_1(t)$ is the state of $\mathcal{S}$ when it is interconnected to $\mathcal{M}_1$. If the measurement circuit is closed in the time interval $[0, t_m]$, then the state of the system $\mathcal{S}$ gets perturbed from its natural trajectory by a quantity

$$b(t_m) := x_1(t_m) - x(t_m) = e^{(J - k_m BB^T)t_m} x_0 - e^{Jt_m} x_0$$
$$= -k_m y_0 B t_m + O(t_m^2),$$

where $x(t)$ satisfies (32) with $u(t) = 0$, and $b(t_m)$ is the back action. By making the measurement time t_m small, the back action can be made arbitrarily small.

In this situation, a good estimation policy for the potential $y_1(t_m)$ is to choose $\hat{y}(t_m) = y_m(t_m)$, since the estimation error $e(t_m)$ is identically zero in this case,

$$e(t_m) := \hat{y}(t_m) - y_1(t_m) = 0.$$

The signal $\hat{y}(t_m)$ should here, and in the following, be interpreted as the best possible estimate of the potential of $\mathcal{S}$ for someone who has access to the measurement signal $y_m(t)$, $0 \leq t \leq t_m$. Note that the estimation error e is defined with respect to the perturbed system $\mathcal{SM}_1$. Given that we already have defined back action it is easy to give a relation to the unperturbed system $\mathcal{S}$ by

$$y(t_m) = \hat{y}(t_m) - e(t_m) - B^T b(t_m), \tag{33}$$

which is valid for non-zero estimation errors also.

Remark 7. *Whether one is interested in the perturbed potential $y_1(t_m)$ or the unperturbed potential $y(t_m)$ of $\mathcal{S}$ depends on the reason for the measurement. For a control engineer who wants to act on the measured system, $y_1(t_m)$ is likely to be of most interest. A physicist, on the other hand, who is curious about the uncontrolled system may be more interested in $y(t_m)$. Either way, knowing the back action b, one can always get $y(t_m)$ from $y_1(t_m)$ using (33).*

4.2.1 Lossless realization $\hat{\mathcal{M}}_1$

Next we make a linear lossless realization of the admittance $k_m > 0$ in $\mathcal{M}_1$, using Proposition 3, so that it satisfies the fluctuation-dissipation theorem. Linear physical implementations of $\mathcal{M}_1$ inevitably exhibit this type of Johnson-Nyquist noise. We obtain

$$
\hat{\mathcal{M}}_1 : \begin{cases} u_m(t) = -u(t) = k_m y(t) + \sqrt{2k_m k_B T_m}\, n(t), \\[2mm] y_m(t) = \dfrac{u_m(t)}{k_m} = y(t) + \sqrt{\dfrac{2k_B T_m}{k_m}}\, n(t), \end{cases}
$$

where T_m is the temperature of the measurement device, and $n(t)$ is unit-intensity white noise. As shown before, the noise can be interpreted as due to our ignorance of the exact initial state of the measurement device. The interconnected measured system $\mathcal{S}\hat{\mathcal{M}}_1$ satisfies a Langevin-type equation,

$$
\mathcal{S}\hat{\mathcal{M}}_1 : \begin{cases} \dot{x}_1(t) = (J - k_m BB^T)x_1(t) - \sqrt{2k_m k_B T_m}\, Bn(t), \\ x_1(0) = x_0, \\ y_1(t) = B^T x_1(t). \end{cases}
$$

The solution for $\mathcal{S}\hat{\mathcal{M}}_1$ is

$$
x_1(t) = e^{(J-k_m BB^T)t} x_0
$$

$$
- \int_0^t e^{(J-k_m BB^T)(t-s)} B \sqrt{2k_m k_B T_m}\, n(s)ds.
$$

The back action can be calculated as

$$b(t_m) = x_1(t_m) - x(t_m) = b_d(t_m) + b_s(t_m),$$
$$b_d(t_m) := \mathbf{E}x_1(t_m) - x(t_m) = e^{(J - k_m BB^T)t_m}x_0 - e^{Jt_m}x_0$$
$$= -k_m y_0 B t_m + O(t_m^2),$$
$$b_s(t_m) := x_1(t_m) - \mathbf{E}x_1(t_m)$$
$$= -\int_0^{t_m} e^{(J - k_m BB^T)(t_m - s)} B\sqrt{2k_m k_B T_m}n(s)ds,$$

where we have split the back action into deterministic and stochastic parts. The deterministic back action coincides with the back action for $\mathcal{M}_1$. The stochastic back action comes from the uncertainty in the lossless realization of the measurement device. The measurement device $\hat{\mathcal{M}}_1$ injects a stochastic perturbation into the measured system $\mathcal{S}$.

The covariance P of the back action b at time t_m is

$$P(t_m) := \mathbf{E}[b(t_m) - \mathbf{E}b(t_m)][b(t_m) - \mathbf{E}b(t_m)]^T$$
$$= \mathbf{E}b_s(t_m)b_s(t_m)^T = 2k_m k_B T_m \int_0^{t_m} e^{(J - k_m BB^T)(t_m - s)}B$$
$$\times B^T(e^{(J - k_m BB^T)(t_m - s)})^T ds = 2BB^T k_m k_B T_m t_m + O(t_m^2). \quad (34)$$

It holds that $P(t_m) \to k_B T_m I_n$ and $\mathbf{E}x_1(t) \to 0$ as $t_m \to \infty$, see [30, Propositions 1 and 2], and the measured system attains temperature T_m after an infinitely long measurement. It is therefore reasonable to keep t_m small if one wants to have a small back action.

Next we analyze and bound the estimation error. The measurement equation is given by

$$y_m(t) = \frac{u_m(t)}{k_m} = y_1(t) + \sqrt{\frac{2k_B T_m}{k_m}}n(t).$$

Note that $\hat{y}(t_m) = y_m(t_m)$ is now a poor estimator of $y_1(t_m)$, since the variance of the estimation error $e(t) = \hat{y}(t) - y_1(t)$ is infinite due to the white noise $n(t)$. Using filtering theory, we can construct an optimal estimator that achieves a fundamental lower bound on the possible accuracy (minimum variance) given $y_m(t)$ in the interval $0 \le t \le t_m$. The solution is the Kalman filter,

$$\dot{\hat{x}}_1(t) = (J - k_m BB^T)\hat{x}_1(t) + K(t)[y_m(t) - B^T\hat{x}_1(t)],$$
$$\hat{y}(t) = B^T\hat{x}_1(t), \quad (35)$$

where $K(t)$ is the Kalman gain (e.g. [44]). The minimum possible variance of the estimation error, $M^*(t_m) = \min \mathbf{E}(\hat{y}(t_m) - y_1(t_m))^2$ (* denotes optimal) can be computed from the differential Riccati equation

$$\dot{X}(t) = J_{k_m}X(t) + X(t)J_{k_m}^T + 2k_m k_B T_m BB^T$$
$$- \frac{k_m}{2k_B T_m}(X(t) - 2k_B T_m I_n)B$$
$$\times B^T(X(t) - 2k_B T_m I_n)^T, \qquad (36)$$
$$M^*(t_m) = B^T X(t_m)B, \quad J_{k_m} := J - k_m BB^T.$$

A series expansion $X(t) = \frac{1}{t}X_{-1} + X_0 + tX_1 + \ldots$ of the solution to (36) yields that the coefficient X_{-1} should satisfy $X_{-1} = \frac{k_m}{2k_B T_m}X_{-1}BB^T X_{-1}$. Note that X_{-1} is independent on J_{k_m}. From the X_1 equation, we yield that

$$M^*(t_m) = \frac{2k_B T_m}{k_m t_m} + O(1),$$

since $M^*(t) = \frac{1}{t}B^T X_{-1}B + B^T X_0 B + tB^T X_1 B + \ldots$ Here the boundary condition $M^*(0) = +\infty$ has been used, since it is assumed that y_0 is completely unknown, see Remark 6. It is easy to verify that $M^*(t_m) \to 0$ as $t_m \to \infty$, and given an infinitely long measurement a perfect estimate is obtained. This comes at the expense of a large back action.

To implement the Kalman filter (35) requires a complete model (J, B, k_m, T_m) which is not always reasonable to assume. Nevertheless, the Kalman filter is optimal and the variance of the estimation error, $M(t) := \mathbf{E}e(t)^2$, of any other estimator, in particular those that do not require complete model knowledge, must satisfy

$$M(t_m) \geq M^*(t_m) = \frac{2k_B T_m}{k_m t_m} + O(1). \qquad (37)$$

4.2.2 Back action and estimation error trade-off

Define the root mean square back action and the root mean square estimation error of the potential y by

$$|\Delta y(t_m)| := \sqrt{B^T P(t_m)B}, \quad |\Delta \hat{y}(t_m)| := \sqrt{M(t_m)}.$$

This is the typical magnitude of the change of the potential y and the estimation error after a measurement. Using (34) and (37), the appealing relation

$$|\Delta y(t_m)||\Delta \hat{y}(t_m)| \geq 2k_B T_m/C + O(t_m), \qquad (38)$$

Table 1: Summary of back action and estimation error after a measurement in the time interval $[0, t_m]$. $b_d(t_m)$ - deterministic back action, $P(t_m)$ - covariance of back action, $|\Delta y|^2$ - variance of potential, and $M^*(t_m)$ - lower bound on estimation error.

Device	$b_d(t_m)$	$P(t_m) = \mathbf{E}b_s(t_m)b_s(t_m)^T$	$|\Delta y(t_m)|^2 = B^T P(t_m)B$	$M^*(t_m) = \min|\Delta\hat{y}|^2$
$\mathcal{M}_1$	$-k_m y_0 B t_m + O(t_m^2)$	0	0	0
$\hat{\mathcal{M}}_1$	$-k_m y_0 B t_m + O(t_m^2)$	$2k_m k_B T_m BB^T t_m + O(t_m^2)$	$\frac{2k_m k_B T_m}{C^2}t_m + O(t_m^2)$	$\frac{2k_B T_m}{k_m}t_m^{-1} + O(1)$
$\mathcal{M}_2$	0	0	0	0
$\hat{\mathcal{M}}_2$	$\frac{y_0^3 k_m}{4E_m}B t_m^2 + O(t_m^3)$	$2k_m k_B T_m BB^T t_m + O(t_m^2)$	$\frac{2k_m k_B T_m}{C^2}t_m + O(t_m^2)$	$\frac{2k_B T_m}{k_m}t_m^{-1} + O(1)$

where $1/C = B^T B$, is obtained. Hence, there is a direct trade-off between the accuracy of estimation and the perturbation in the potential, independently on (small) t_m and admittance k_m. It is seen that the more "capacitance" (C) $\mathcal{S}$ has, the less important the trade-off is. One can interpret C as a measure of the physical size or inertia of the system. The trade-off is more important for "small" system in "hot" environments. Using an optimal filter, the trade-off is satisfied with equality.

4.3 Memoryless active measurement device

A problem with the device $\mathcal{M}_1$ is that it causes back action b even in the most ideal situation. If active elements are allowed in the measurement device, this perturbation can apparently be easily eliminated, but of course with the inherent costs of an active device. Consider the measurement device $\mathcal{M}_2$ to the right in Fig. 3. It is modeled by

$$\mathcal{M}_2: \begin{cases} u_m(t) = k_m y(t), \\ u(t) = u_m(t) - k_m y(t) = 0, \\ y_m(t) = \dfrac{u_m(t)}{k_m} = y(t), \end{cases}$$

where an active element $-k_m$ exactly compensates for the back action in $\mathcal{M}_1$. It is clear that there is no back action and no estimation error using this device,

$$b(t_m) = 0, \quad e(t_m) = 0,$$

for all t_m. Next, a lossless approximation of $\mathcal{M}_2$ is performed.

4.3.1 Lossless realization $\hat{\mathcal{M}}_2$

Let the dissipative element k_m in $\mathcal{M}_2$ be implemented with a linear lossless system, see Proposition 3, and the active element $-k_m$ be implemented using the nonlinear lossless system in (28). This approximation of $\mathcal{M}_2$ captures the reasonable assumption that the measurement device must be charged with energy to behave like an active device, and that its linear dissipative element satisfies the fluctuation-dissipation theorem.

Assume that the temperature of the measurement device $\hat{\mathcal{M}}_2$ is T_m and the deterministic part of the total energy of the active element is E_m. Then the interconnected system becomes

$$
\mathcal{S}\hat{\mathcal{M}}_2 : \begin{cases}
\dot{x}_2(t) = (J - k_m B B^T) x_2(t) \\
\qquad + \dfrac{k_m}{\sqrt{2E_m}} x_r(t) B B^T x_2(t) \\
\qquad - B\sqrt{2k_m k_B T_m}\, n(t), \quad x_2(0) = x_0, \\
\dot{x}_r(t) = \dfrac{k_m}{\sqrt{2E_m}} (B^T x_2(t))^2, \\
x_r(0) = \sqrt{2E_m} + \Delta x_{r0}, \\
\mathbf{E}\Delta x_{r0} = 0, \quad \mathbf{E}\Delta x_{r0}^2 = k_B T_m, \\
y_m(t) = \dfrac{u_m(t)}{k_m} = B^T x_2(t) + \sqrt{\dfrac{2k_B T_m}{k_m}}\, n(t),
\end{cases}
$$

where x_2 is the state of $\mathcal{S}$, and x_r is the state of the active element. Using the closed-form solution (29)–(30) to eliminate x_r, we can also write the equations as

$$
\mathcal{S}\hat{\mathcal{M}}_2 : \begin{cases}
\dot{x}_2(t) = \left(J + \dfrac{k_m \Delta x_{r0}}{\sqrt{2E_m}} B B^T \right) x_2(t) \\
\qquad + B w_d(t) - B\sqrt{2k_m k_B T_m}\, n(t), \; x_2(0) = x_0, \\
y_m(t) = \dfrac{u_m(t)}{k_m} = B^T x_2(t) + \sqrt{\dfrac{2k_B T_m}{k_m}}\, n(t),
\end{cases} \tag{39}
$$

with the deterministic perturbation $w_d(t) = \frac{k_m^2 y_0^3}{2E_m} t + O(t^2)$. The solution to

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(39) can be expanded as

$$x_2(t) = x_0 - \sqrt{2k_m k_B T_m} BN(t)$$
$$+ \left(J + \frac{k_m \Delta x_{r0}}{\sqrt{2E_m}} BB^T \right) x_0 t$$
$$- \sqrt{2k_m k_B T_m} \left(J + \frac{k_m \Delta x_{r0}}{\sqrt{2E_m}} BB^T \right)$$
$$\times B \int_0^t N(s)ds + B \frac{k_m^2 y_0^3}{4E_m} t^2 + o(t^2), \quad (40)$$

where $N(t) = \int_0^t n(s)ds = O(\sqrt{t})$ is integrated white noise (a Brownian motion). It can be seen that the white noise disturbance n is much more important than the deterministic disturbance w_d. The back action becomes

$$b(t_m) = x_2(t_m) - x(t_m) = b_d(t_m) + b_s(t_m)$$
$$b_d(t_m) := \mathbf{E}x_2(t_m) - x(t_m) = \frac{k_m^2 y_0^3}{4E_m} Bt_m^2 + O(t_m^3),$$
$$b_s(t_m) := x_2(t_m) - \mathbf{E}x_2(t_m)$$
$$= -\sqrt{2k_m k_B T_m} BN(t_m) + \frac{k_m \Delta x_{r0}}{\sqrt{2E_m}} By_0 t_m$$
$$+ O(t_m \sqrt{t_m}),$$

where we used that the covariance between Δx_{r0} and N is zero. The covariance of the back action becomes

$$P(t_m) := \mathbf{E}b_s(t_m)b_s(t_m)^T = 2k_m k_B T_m BB^T t_m + O(t_m^2). \quad (41)$$

It is seen that the dominant term in the stochastic back action is the same as for $\hat{\mathcal{M}}_1$, but the deterministic back action b_d is much smaller.

Remark 8. *Using a nonlinear lossless approximation of* $-k_m$ *of order larger than one, we can make the deterministic back action smaller for fixed* E_m, *at the expense of model complexity.*

The measurement noise in $\mathcal{S}\hat{\mathcal{M}}_2$ is the same as in $\mathcal{S}\hat{\mathcal{M}}_1$, and we can essentially repeat the argument from Section 4.2.1. The difference between $\mathcal{S}\hat{\mathcal{M}}_2$ and $\mathcal{S}\hat{\mathcal{M}}_1$ lies in the dynamics. In $\mathcal{S}\hat{\mathcal{M}}_2$, the system matrix is $J + \frac{k_m \Delta x_{r0}}{\sqrt{2E_m}} BB^T$ and there is a deterministic perturbation $w_d(t)$. To make an estimate $\hat{y}(t_m)$, knowledge of $y_m(t)$ in the interval $[0, t_m]$ is assumed. If we assume that the model (J, B, k_m, T_m) is known plus that the observer

somehow knows $w_d(t)$ and Δx_{r0}, then the optimal estimate again has the error covariance $M^*(t_m) = \frac{2k_B T_m}{k_m t_m} + O(1)$. Any other estimator that has less information available must be worse, so that

$$M(t) \geq M^*(t_m) = \frac{2k_B T_m}{k_m t_m} + O(1).$$

Again, we have the trade-off (38)

$$|\Delta y(t_m)||\Delta \hat{y}(t_m)| \geq 2k_B T_m/C + O(t_m),$$

which holds even though we have inserted an active element in device. The only effect of the active element is to eliminate the deterministic back action.

4.4   Summary and Discussion

The back action and estimation error of the measurement devices are summarized in Table 1. For the ideal devices $\mathcal{M}_1$ and $\mathcal{M}_2$ no real trade-offs exist. However, if we realize them with lossless elements very reasonable trade-offs appear. It is only in the limit of infinite available energy and zero temperature that the trade-offs disappear. The deterministic back action can be made small with large E_m, charging the measurement device with much energy. However, the effect of stochastic back action is inescapable for both $\hat{\mathcal{M}}_1$ and $\hat{\mathcal{M}}_2$, and the trade-off

$$|\Delta y||\Delta \hat{y}| \geq 2k_B T_m/C \quad \text{for small } t_m, \tag{42}$$

holds in both cases. The reason for having short measurements is to minimize the effect of the back action. The lower bound on the estimation error $M^*(t_m)$ tends to zero for large t_m, but at the same time the measured system $\mathcal{S}$ tends to a thermodynamic equilibrium with the measurement device.

It is possible to increase the estimation accuracy by making the admittance k_m of the measurement device large, but only at the expense of making a large stochastic perturbation of the measured system. Hence, we have quantified a limit for the observer effect discussed in the introduction of this section. We conjecture that inequalities like (42) hold for very general measurement devices as soon as the dissipative elements satisfy the fluctuation-dissipation theorem. Note, for example, that if a lossless transmission cable of admittance k_m and of temperature T_m is used to interconnect the system $\mathcal{S}$ to an arbitrary measurement device $\mathcal{M}$, then the trade-off (42) holds. The deterministic back action, on the other hand, is possible to make smaller by using more elaborate nonlinear lossless implementations.

5   Conclusions

In this paper, we constructed lossless approximations of both dissipative and active systems. We obtained an if-and-only-if characterization of linear dissipative systems (linear lossless systems are dense in the linear dissipative systems) and gave explicit approximation error bounds that depend on the time horizon, the order, and the available energy of the approximations. We showed that the fluctuation-dissipation theorem, that quantifies macroscopic thermal noise, can be explained by uncertainty in the initial state of a linear lossless approximation of very high order. We also saw that using these techniques, it was relatively easy to quantify limitations on the back action of measurement devices. This gave rise to a trade-off between process and measurement noise.

6   Appendices

6.1   Proof of Theorem 2

We first show the 'only if' direction. Assume the opposite: There is a lossless approximation G_N that satisfy (12) for arbitrarily small $\epsilon > 0$ even though G is not dissipative. From Proposition 1 it is seen that we can without loss of generality assume G_N has a minimal realization and $x_0 = 0$. If G is not dissipative, we can find an input $u(t)$ over the interval $[0, \tau]$ such that $\int_0^\tau y(t)^T u(t)dt = -K_1 < 0$, i.e., we extract energy from G even though its initial state is zero. Call $\|u\|_{L_1[0,\tau]} = K_2$. We have $\int_0^\tau (y_N(t) - y(t))^T u(t)dt \leq \epsilon K_2$, by the assumption that a lossless approximation G_N exists and using the Cauchy-Schwarz inequality. But the lossless approximation satisfies $\int_0^\tau y_N(t)^T u(t)dt = \frac{1}{2}x(\tau)^T x(\tau)$, since $x_0 = 0$. Hence, $-\int_0^\tau y(t)^T u(t)dt = K_1 \leq \epsilon K_2 - \frac{1}{2}x_N(\tau)^T x_N(\tau) \leq \epsilon K_2$. But since ϵ can be made arbitrarily small, this leads to a contradiction.

To prove the 'if' direction we explicitly construct a G_N that satisfies (12), when G is dissipative. It turns out that we can fix the model parameters $D = 0$ in G_N. Furthermore, we must choose $x_0 = 0$ since otherwise the zero trajectory $y = 0$ cannot be tracked (see above). We thus need to construct a lossless system with impulse response $g_N(t)$ such that $\|g - g_N\|_{L_2[0,\tau_0]} \leq \epsilon$, where we have denoted the time interval given in the theorem statement by $[0, \tau_0]$. Note that we can increase this time interval without loss of generality, since if we prove $\|g - g_N\|_{L_2[0,\tau]} \leq \epsilon$ then $\|g - g_N\|_{L_2[0,\tau_0]} \leq \epsilon$, if $\tau \geq \tau_0$.

Let us define the constants

$$C_1 \geq \|g(t)\|_2, \quad t \geq 0; \qquad\qquad C_2 = \int_0^\infty \|\dot{g}(t)\|_1 dt;$$

$$C_3 = \int_0^\infty \|g(t)\|_1 dt; \qquad\qquad C = \frac{4C_1 + 2C_2}{\pi} + \frac{4C_3}{\tau_0},$$

which are all finite by the assumptions of the theorem. It will become clear later why the constants are defined this way.

Next let us fix the approximation time interval $[0, \tau]$ such that

$$\delta(\tau) := \int_\tau^\infty \|g(t)\|_1 dt \leq \frac{\epsilon^2}{2C\sqrt{p}}, \tag{43}$$

where $\tau \geq \tau_0$. Such a τ always exists since $\delta(\tau)$ is a continuously decreasing function that converges to zero. The lossless approximation is achieved by truncating a Fourier series keeping N terms. Let us choose the integer N such that

$$N \leq \frac{\tau C^2}{\epsilon^2} \leq N + 1, \tag{44}$$

where τ is fixed in (43). We proceed by constructing an appropriate Fourier series.

6.1.1 Fourier expansion

The extended function $\tilde{g}(t) \in L_2(-\infty, \infty)$ of $g(t)$ is given by

$$\tilde{g}(t) = \begin{cases} g(t), & t \geq 0, \\ g(-t)^T, & t < 0. \end{cases}$$

Let us make a Fourier expansion of $\tilde{g}(t)$ on the interval $[-\tau, \tau]$,

$$\tilde{g}_\tau(t) := \frac{1}{2} A_0 + \sum_{k=1}^\infty A_k \cos \frac{k\pi t}{\tau} + B_k \sin \frac{k\pi t}{\tau},$$

with convergence in $L_2[-\tau, \tau]$. For the restriction to $[0, \tau]$ it holds that $\|g - \tilde{g}_\tau\|_{L_2[0,\tau]} = 0$. The expressions for the (matrix) Fourier coefficients are

$$\begin{aligned} A_k &= \frac{1}{\tau} \int_0^\tau (g(t) + g(t)^T) \cos \frac{k\pi t}{\tau} dt \\ B_k &= \frac{1}{\tau} \int_0^\tau (g(t) - g(t)^T) \sin \frac{k\pi t}{\tau} dt. \end{aligned} \tag{45}$$

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Note that $A_k, B_k \in \mathbb{R}^{p \times p}$, and A_k are symmetric $(A_k = A_k^T)$ and B_k are anti-symmetric $(B_k = -B_k^T)$. Parseval's formula becomes

$$\|\tilde{g}_\tau\|_{L_2[0,\tau]}^2 = \int_0^\tau \text{Tr } g(t)g(t)^T dt$$

$$= \frac{\tau}{4} \text{Tr } A_0^T A_0 + \frac{\tau}{2} \sum_{k=1}^\infty \text{Tr } A_k^T A_k + \text{Tr } B_k^T B_k. \quad (46)$$

We also need to bound $\|A_k - jB_k\|_2^2 = \text{Tr } A_k^T A_k + \text{Tr } B_k^T B_k$. It holds

$$A_k - jB_k = \frac{1}{\tau} \int_{-\tau}^\tau \tilde{g}(t) e^{-j\pi k t/\tau} dt$$

$$= \frac{(-1)^k}{jk\pi} (g(\tau)^T - g(\tau)) + \frac{1}{jk\pi} (g(0) - g(0)^T)$$

$$+ \frac{1}{jk\pi} \int_0^\tau e^{-j\pi k t/\tau} \dot{g}(t) - e^{j\pi k t/\tau} \dot{g}(t)^T dt,$$

using integration by parts. Then

$$\|A_k - jB_k\|_2$$

$$\leq \frac{4C_1}{k\pi} + \frac{1}{k\pi} \left\| \int_0^\tau e^{-j\pi k t/\tau} \dot{g}(t) - e^{j\pi k t/\tau} \dot{g}(t)^T dt \right\|_2$$

$$\leq \frac{4C_1}{k\pi} + \frac{2}{k\pi} \int_0^\tau \|\dot{g}(t)\|_1 dt \leq \frac{1}{k} \frac{4C_1 + 2C_2}{\pi}.$$

Furthermore,

$$\|A_k - jB_k\|_2 = \left\| \frac{1}{\tau} \int_{-\tau}^\tau \tilde{g}(t) e^{-j\pi k t/\tau} dt \right\|_2 \leq \frac{2C_3}{\tau_0},$$

since $\tau \geq \tau_0$. If the former bound is multiplied by k and the latter is multiplied by two and they are added together, we obtain

$$\|A_k - jB_k\|_2 \leq \frac{C}{2+k}, \quad k \geq 0, \quad (47)$$

where C was defined above.

6.1.2 Lossless approximation G_N

Let us now truncate the series $\tilde{g}_\tau(t)$ and keep the terms with Fourier coefficients $A_0, \ldots, A_{N-1}$ and $B_1, \ldots, B_{N-1}$. The truncated impulse response

can be realized exactly by a finite-dimensional lossless system iff $A_0 \geq 0$ and $A_k - jB_k \geq 0$, $k = 1, \ldots, N - 1$, see [28, Theorem 5]. But these inequalities are not necessarily true. We will thus perturb the coefficients to ensure the system becomes lossless and yet ensure that the L_2-approximation error is less than ϵ.

We quantify a number $\xi \geq 0$ that ensures that $A_k - jB_k + \xi I_p \geq 0$ for all k. Note that by the assumption of G being dissipative, it holds that

$$\hat{g}(j\omega) + \hat{g}(-j\omega)^T = \int_{-\infty}^{\infty} \tilde{g}(t)e^{-j\omega t}dt \geq 0.$$

Remember that $\int_{-\tau}^{\tau} \tilde{g}(t)e^{-j\pi kt/\tau}dt = \tau A_k - j\tau B_k$, and therefore

$$A_k - jB_k + \Delta_k \geq 0$$

where $\Delta_k := \frac{1}{\tau}\int_{\tau}^{\infty} g(t)e^{-j\pi kt/\tau} + g(t)^T e^{j\pi kt/\tau}dt$. The size of Δ_k can be bounded and we have

$$\|\Delta_k\|_2 = \sqrt{\mathrm{Tr}\ \Delta_k^* \Delta_k} \leq \frac{2}{\tau}\int_{\tau}^{\infty} \|g(t)\|_1 dt \leq \frac{\epsilon^2}{\tau C\sqrt{p}},$$

using (43). Thus we can choose

$$\xi = \frac{\epsilon^2}{\tau C\sqrt{p}},$$

and $A_k - jB_k + \xi I_p \geq 0$ for all k, since $\rho(\Delta_k) \leq \|\Delta_k\|_2$.

Next we verify that a system G_N with impulse response

$$g_N(t) := \frac{1}{2}(A_0 + \xi I_p)$$

$$+ \sum_{k=1}^{N-1} (A_k + \xi I_p)\cos\frac{k\pi t}{\tau} + B_k \sin\frac{k\pi t}{\tau}, \quad (48)$$

where τ, N, ξ are fixed above satisfies the statement of the theorem. By the construction of ξ, G_N is lossless. It remains to show that the approximation

error $\|g - g_N\|_{L_2[0,\tau]}$ is less than ϵ. Using Parseval's formula (46), it holds

$$
\|g - g_N\|^2_{L_2[0,\tau]} = \|\tilde{g}_\tau - g_N\|^2_{L_2[0,\tau]}
$$

$$
= \left\| \frac{1}{2}\xi I + \sum_{k=1}^{N-1} \xi I \cos \frac{k\pi t}{\tau} + \sum_{k=N}^{\infty} A_k \cos \frac{k\pi t}{\tau} + B_k \sin \frac{k\pi t}{\tau} \right\|^2_2
$$

$$
\leq \frac{\tau}{2} N \xi^2 p + \frac{\tau}{2} \sum_{k=N}^{\infty} \|A_k - jB_k\|^2_2 \leq \frac{\tau}{2} N \xi^2 p + \frac{\tau}{2} \sum_{k=N}^{\infty} \frac{C^2}{(2+k)^2}
$$

$$
\leq \frac{\tau}{2} \frac{\tau C^2}{\epsilon^2} \frac{\epsilon^4}{\tau^2 C^2 p} p + \frac{\tau}{2} \frac{C^2}{N+1} \leq \frac{\epsilon^2}{2} + \frac{\tau}{2} \frac{C^2 \epsilon^2}{\tau C^2} = \epsilon^2,
$$

where the bounds (44) and (47) are used. The result has been proved.

6.2 Proof of Theorem 3

We first show the 'if' direction. Then there exists a lossless and time-reversible (with respect to Σ_e, see Definition 2) approximation G_N of G. Theorem 2 shows that G is dissipative. Theorem 8 in [28] shows that G_N necessarily is reciprocal with respect to Σ_e. Since G_N is an arbitrarily good approximation it follows that G also is reciprocal, which concludes the 'if' direction of the proof.

Next we show the 'only if' direction. Then G is dissipative and reciprocal with respect to Σ_e. Theorem 2 shows that there exists an arbitrarily good lossless approximation G_N, and we will use the approximation (48). That G is reciprocal with respect to Σ_e means that $\Sigma_e g(t) = g(t)^T \Sigma_e$, see Definition 1. Using this and the definition of A_k and B_k in (45), it is seen that

$$
\Sigma_e (A_k + \xi I_p) = (A_k + \xi I_p)^T \Sigma_e, \quad \Sigma_e B_k = B_k^T \Sigma_e.
$$

Thus the chosen G_N is also reciprocal, $\Sigma_e g_N(t) = g_N(t)^T \Sigma_e$, and Theorem 8 in [28] shows G_N is time reversible with respect to Σ_e. This concludes the proof.

Acknowledgment

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