Noether's problem for some 2-groups

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Abstract. Let G be a finite group and k be a field. Let G act on the rational function field $k(x_g : g \in G)$ by k-automorphisms defined by $g \cdot x_h = x_{gh}$ for any $g, h \in G$. Noether's problem asks whether the fixed field $k(G) = k(x_g : g \in G)^G$ is rational (i.e. purely transcendental) over k. We will prove that, if G is a group of order 2^n ($n \geq 4$) and of exponent 2^e such that (i) $e \geq n-2$ and (ii) $\zeta_{2^{e-1}} \in k$, then $k(G)$ is k-rational.

§1. Introduction

Let k be any field and G be a finite group. Let G act on the rational function field $k(x_g : g \in G)$ by k-automorphisms such that $g \cdot x_h = x_{gh}$ for any $g, h \in G$. Denote by $k(G)$ the fixed field $k(x_g : g \in G)^G$. Noether's problem asks whether $k(G)$ is rational (=purely transcendental) over k. It is related to the inverse Galois problem, to the existence of generic G-Galois extensions over k, and to the existence of versal G-torsors over k-rational field extensions [Sw; Sa; GMS, 33.1, p.86]. Noether's problem

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for abelian groups was studied extensively by Swan, Voskresenskii, Endo, Miyata and Lenstra, etc. The reader is referred to Swan's paper for a survey of this problem [Sw].

On the other hand, just a handful of results about Noether's problem are obtained when the groups are not abelian. It is the case even when the group G is a p-group. The reader is referred to [CK; Ka1; HuK; Ka4] for previous results of Noether's problem for p-groups. In the following we will list only those results relevant to the 2-groups which are the main subjects of this paper.

Theorem 1.1 (Chu, Hu and Kang [CHK; Ka2]) *Let k be any field. Suppose that G is a non-abelian group of order 8 or 16. Then $k(G)$ is rational over k, except when $\operatorname{char} k \neq 2$ and $G = Q_{16}$, the generalized quaternion group of order 16. When $\operatorname{char} k \neq 2$ and $G = Q_{16}$, then $k(G)$ is also rational over k provided that $k(\zeta_8)$ is a cyclic extension over k where ζ_8 is a primitive 8-th root of unity.*

Theorem 1.2 (Serre [GMS, Theorem 34.7]) *If $G = Q_{16}$, then $\mathbb{Q}(G)$ is not stably rational over $\mathbb{Q}$; in particular, it is not rational over $\mathbb{Q}$.*

We don't know the answer whether $k(G)$ is rational over k or not, if $G = Q_{16}$ and k is a field other than $\mathbb{Q}$ such that $k(\zeta_8)$ is not a cyclic extension of k. The reader is referred to [CHKP; CHKK] for groups of order 32 and 64. Now we turn to metacyclic p-groups.

Theorem 1.3 (Hu and Kang [HuK; Ka4]) *Let $n \geq 4$ and G be a non-abelian group of order 2^n. Assume that either* (i) $\operatorname{char} k = 2$, *or* (ii) $\operatorname{char} k \neq 2$ *and k contains a primitive 2^{n-2}-th root of unity. If G contains an element whose order $\geq 2^{n-2}$, then $k(G)$ is rational over k.*

The main result of this paper is the following theorem which strengthens parts of the above Theorem 1.3.

Theorem 1.4 *Let $n \geq 4$ and G be a group of order 2^n and of exponent 2^e where $e \geq n-2$. Assume that either* (i) $\operatorname{char} k = 2$, *or* (ii) $\operatorname{char} k \neq 2$ *and k contains a primitive 2^{e-1}-th root of unity. Then $k(G)$ is rational over k.*

We claim that in order to prove Theorem 1.4 we may assume the following extra conditions on G and k without loss of generality

(1.1) $n \geq 5$, $|G| = 2^n$, $\exp(G) = 2^{n-2}$, G is non-abelian, $\operatorname{char} k \neq 2$ and $\zeta_{2^{n-3}} \in k$.

For, it is not difficult to prove Theorem 1.4 when G is an abelian group by applying Lenstra's Theorem [Le]. Moreover, Kuniyoshi's Theorem asserts that, if $\operatorname{char} k = p > 0$ and G is a p-group, then $k(G)$ is rational over k [Ku; KP, Corollary 1.2]. Thus we may assume that G is non-abelian and $\operatorname{char} k \neq 2$. When G is a non-abelian group of order 2^n, the case of Theorem 1.4 when $n = 4$ is taken care by Theorem 1.1, and the case when $\exp(G) = 2^{n-1}$ is taken care by Theorem 1.3. Thus only the situation of (1.1) remains.

The key idea to prove Theorem 1.4 is, by applying Theorem 2.2, to find a low-dimensional faithful G-subspace $W = \bigoplus_{1\le i\le m} k\cdot y_i$ of the regular representation space $\bigoplus_{g\in G} k\cdot x(g)$ and to show that $k(y_i : 1 \le i \le m)^G$ is rational over k. The subspace W is obtained as an induced representation from some abelian subgroup of G. This method is reminiscent of some techniques exploited in [Ka4]. However, the proof of Theorem 1.4 is more subtle and requires elaboration. For examples, in [Ka4], the following two theorems were used to solve the rationality problem for many groups G_i in Theorem 2.1.

Theorem 1.5 ([Ka1]) *Let k be a field and G be a metacyclic p-group. Assume that* (i) $\operatorname{char} k = p > 0$, *or* (ii) $\operatorname{char} k \ne p$ *and* $\zeta_e \in k$ *where* $e = \exp(G)$. *Then $k(G)$ is rational over k.*

Theorem 1.6 ([Ka3, Theorem 1.4]) *Let k be a field and G be a finite group. Assume that* (i) *G contains an abelian normal subgroup H so that G/H is cyclic of order n,* (ii) *$\mathbb{Z}[\zeta_n]$ is a unique factorization domain, and* (iii) *$\zeta_e \in k$ where e is the exponent of G. If $G \to GL(V)$ is any finite-dimensional linear representation of G over k, then $k(V)^G$ is rational over k.*

Because we assume $\zeta_{2^{n-3}} \in k$ (instead of $\zeta_{2^{n-2}} \in k$) in (1.1), the above two theorems are not directly applicable in the present situation. This is the reason why we should find judiciously a faithful subspace W. Fortunately we can find these subspaces W in an almost unified way. In fact, the proof for the group G_8 in Theorem 2.1 is a typical case; the proof for other groups is either similar to that of G_8 or has appeared in [Ka4].

We organize this paper as follows. In Section 2 we recall Ninomiya's classification of non-abelian groups G with $|G| = 2^n$ and $\exp(G) = 2^{n-2}$ (where $n \ge 4$). We also recall some preliminaries which will be used in the proof of Theorem 1.4. The proof of Theorem 1.4 is given in Section 3.

Standing Notations. Throughout this article, $K(x_1,\ldots,x_n)$ or $K(x,y)$ will be rational function fields over K. ζ_n denotes a primitive n-th root of unity. A field extension L of K is called rational over K (or K-rational, for short) if $L \simeq K(x_1,\ldots,x_n)$ over K for some integer n. L is stably rational over K if $L(y_1,\ldots,y_m)$ is rational over K for some $y_1,\ldots,y_m$ which are algebraically independent over L. Recall that $K(G)$ denotes $K(x_g : g \in G)^G$ where $h\cdot x_g = x_{hg}$ for $h, g \in G$.

The exponent of a finite group G, denoted by $\exp(G)$, is $\operatorname{lcm}\{\operatorname{ord}(g) : g \in G\}$ where $\operatorname{ord}(g)$ is the order of g.

If G is a finite group acting on a rational function field $K(x_1,\ldots,x_n)$ by K-automorphisms, the actions of G are called purely monomial actions if, for any $\sigma \in G$, any $1 \le j \le n$, $\sigma\cdot x_j = \prod_{1\le i\le n} x_i^{a_{ij}}$ where $a_{ij} \in \mathbb{Z}$; similarly, the actions of G are called monomial actions if, for any $\sigma \in G$, any $1 \le j \le n$, $\sigma\cdot x_j = \lambda_j(\sigma)\cdot\prod_{1\le i\le n} x_i^{a_{ij}}$ where $a_{ij} \in \mathbb{Z}$ and $\lambda_j(\sigma) \in K\backslash\{0\}$. All the groups in this article are finite groups.

§2. Preliminaries

Theorem 2.1 (Ninomiya [Ni, Theorem 2]) *Let $n \geq 4$. The finite non-abelian groups of order 2^n which have a cyclic subgroup of index 4, but haven't a cyclic subgroup of index 2 are of the following types:*

(I) $n \geq 4$

$G_1 = \langle \sigma, \tau : \sigma^{2^{n-2}} = \tau^4 = 1, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-3}} \rangle$,
$G_2 = \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \lambda^2 = 1, \sigma^{2^{n-3}} = \tau^2, \tau^{-1}\sigma\tau = \sigma^{-1}, \sigma\lambda = \lambda\sigma, \tau\lambda = \lambda\tau \rangle$,
$G_3 = \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1}, \sigma\lambda = \lambda\sigma, \tau\lambda = \lambda\tau \rangle$,
$G_4 = \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \sigma\tau = \tau\sigma, \sigma\lambda = \lambda\sigma, \lambda^{-1}\tau\lambda = \sigma^{2^{n-3}}\tau \rangle$,
$G_5 = \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \sigma\tau = \tau\sigma, \lambda^{-1}\sigma\lambda = \sigma\tau, \tau\lambda = \lambda\tau \rangle$.

(II) $n \geq 5$

$G_6 = \langle \sigma, \tau : \sigma^{2^{n-2}} = \tau^4 = 1, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle$,
$G_7 = \langle \sigma, \tau : \sigma^{2^{n-2}} = \tau^4 = 1, \tau^{-1}\sigma\tau = \sigma^{-1+2^{n-3}} \rangle$,
$G_8 = \langle \sigma, \tau : \sigma^{2^{n-2}} = 1, \sigma^{2^{n-3}} = \tau^4, \tau^{-1}\sigma\tau = \sigma^{-1} \rangle$,
$G_9 = \langle \sigma, \tau : \sigma^{2^{n-2}} = \tau^4 = 1, \sigma^{-1}\tau\sigma = \tau^{-1} \rangle$,
$G_{10} = \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-3}}, \sigma\lambda = \lambda\sigma, \tau\lambda = \lambda\tau \rangle$,
$G_{11} = \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1+2^{n-3}}, \sigma\lambda = \lambda\sigma, \tau\lambda = \lambda\tau \rangle$,
$G_{12} = \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \sigma\tau = \tau\sigma, \lambda^{-1}\sigma\lambda = \sigma^{-1}, \lambda^{-1}\tau\lambda = \sigma^{2^{n-3}}\tau \rangle$,
$G_{13} = \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \sigma\tau = \tau\sigma, \lambda^{-1}\sigma\lambda = \sigma^{-1}\tau, \tau\lambda = \lambda\tau \rangle$,
$G_{14} = \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = 1, \sigma^{2^{n-3}} = \lambda^2, \sigma\tau = \tau\sigma, \lambda^{-1}\sigma\lambda = \sigma^{-1}\tau, \tau\lambda = \lambda\tau \rangle$,
$G_{15} = \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-3}}, \lambda^{-1}\sigma\lambda = \sigma^{-1+2^{n-3}}, \tau\lambda = \lambda\tau \rangle$,
$G_{16} = \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-3}}, \lambda^{-1}\sigma\lambda = \sigma^{-1+2^{n-3}}, \lambda^{-1}\tau\lambda = \sigma^{2^{n-3}}\tau \rangle$,
$G_{17} = \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-3}}, \lambda^{-1}\sigma\lambda = \sigma\tau, \tau\lambda = \lambda\tau \rangle$,
$G_{18} = \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = 1, \lambda^2 = \tau, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-3}}, \lambda^{-1}\sigma\lambda = \sigma^{-1}\tau \rangle$.

(III) $n \geq 6$

$G_{19} = \langle \sigma, \tau : \sigma^{2^{n-2}} = \tau^4 = 1, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-4}} \rangle$,
$G_{20} = \langle \sigma, \tau : \sigma^{2^{n-2}} = \tau^4 = 1, \tau^{-1}\sigma\tau = \sigma^{-1+2^{n-4}} \rangle$,
$G_{21} = \langle \sigma, \tau : \sigma^{2^{n-2}} = 1, \sigma^{2^{n-3}} = \tau^4, \sigma^{-1}\tau\sigma = \tau^{-1} \rangle$,
$G_{22} = \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \sigma\tau = \tau\sigma, \lambda^{-1}\sigma\lambda = \sigma^{1+2^{n-4}}\tau, \lambda^{-1}\tau\lambda = \sigma^{2^{n-3}}\tau \rangle$,
$G_{23} = \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \sigma\tau = \tau\sigma, \lambda^{-1}\sigma\lambda = \sigma^{-1+2^{n-4}}\tau, \lambda^{-1}\tau\lambda = \sigma^{2^{n-3}}\tau \rangle$,
$G_{24} = \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = \lambda^2 = 1, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-3}}, \lambda^{-1}\sigma\lambda = \sigma^{-1+2^{n-4}}, \tau\lambda = \lambda\tau \rangle$,
$G_{25} = \langle \sigma, \tau, \lambda : \sigma^{2^{n-2}} = \tau^2 = 1, \sigma^{2^{n-3}} = \lambda^2, \tau^{-1}\sigma\tau = \sigma^{1+2^{n-3}}, \lambda^{-1}\sigma\lambda = \sigma^{-1+2^{n-4}}, \tau\lambda = \lambda\tau \rangle$,

(IV) $n = 5$

$G_{26} = \langle \sigma, \tau, \lambda : \sigma^8 = \tau^2 = 1, \sigma^4 = \lambda^2, \tau^{-1}\sigma\tau = \sigma^5, \lambda^{-1}\sigma\lambda = \sigma\tau, \tau\lambda = \lambda\tau \rangle$.

Theorem 2.2 ([HK, Theorem 1]) *Let G be a finite group acting on $L(x_1, \ldots, x_n)$, the rational function field of n variables over a field L. Suppose that*

(i) *for any* $\sigma \in G$, $\sigma(L) \subset L$*;*

(ii) *the restriction of the action of* G *to* L *is faithful;*

(iii) *for any* $\sigma \in G$*,*

$$\begin{pmatrix} \sigma(x_1) \\ \sigma(x_2) \\ \vdots \\ \sigma(x_n) \end{pmatrix} = A(\sigma) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + B(\sigma)$$

where $A(\sigma) \in GL_n(L)$ *and* $B(\sigma)$ *is an* $n \times 1$ *matrix over* L*.*

Then there exist elements $z_1, \ldots, z_n \in L(x_1, \ldots, x_n)$ *such that* $L(x_1, \ldots, x_n) = L(z_1, \ldots, z_n)$ *and* $\sigma(z_i) = z_i$ *for any* $\sigma \in G$*, any* $1 \le i \le n$*.*

Theorem 2.3 ([AHK, Theorem 3.1]) *Let* L *be any field,* $L(x)$ *the rational function field of one variable over* L*, and* G *a finite group acting on* $L(x)$*. Suppose that, for any* $\sigma \in G$*,* $\sigma(L) \subset L$ *and* $\sigma(x) = a_\sigma \cdot x + b_\sigma$ *where* $a_\sigma, b_\sigma \in L$ *and* $a_\sigma \neq 0$*. Then* $L(x)^G = L^G(f)$ *for some polynomial* $f \in L[x]$*. In fact, if* $m = \min\{\deg g(x) : g(x) \in L[x]^G \backslash L\}$*, any polynomial* $f \in L[x]^G$ *with* $\deg f = m$ *satisfies the property* $L(x)^G = L^G(f)$*.*

Theorem 2.4 (Hoshi, Kitayama and Yamasaki [HKY, 5.4]) *Let* k *be a field with* $\operatorname{char} k \neq 2$*,* $\varepsilon \in \{1, -1\}$ *and* $a, b \in k\backslash\{0\}$*. Let* $G = \langle \sigma, \tau \rangle$ *act on* $k(x, y, z)$ *by* k*-automorphisms defined by*

$$\begin{aligned} &\sigma : x \mapsto a/x,\ y \mapsto a/y,\ z \mapsto \varepsilon z, \\ &\tau : x \mapsto y \mapsto x,\ z \mapsto b/z. \end{aligned}$$

Then $k(x, y, z)^G$ *is rational over* k*.*

Theorem 2.5 (Hajja [Ha]) *Let* G *be a finite group acting on the rational function field* $k(x, y)$ *be monomial* k*-automorphisms. Then* $k(x, y)^G$ *is rational over* k*.*

Theorem 2.6 (Kang and Plans [KP, Theorem 1.3]) *Let* k *be any field,* G_1 *and* G_2 *two finite groups. If both* $k(G_1)$ *and* $k(G_2)$ *are rational over* k*, then so is* $k(G_1 \times G_2)$ *over* k*.*

§3. The proof of Theorem 1.4

We will prove Theorem 1.4 in this section.

By the discussion of Section 1, it suffices to consider those groups G in Theorem 2.1 (with $n \ge 5$) under the assumptions of (1.1), i.e. $\operatorname{char} k \neq 2$ and $\zeta_{2^{n-3}} \in k$. These assumptions will remain in force throughout this section.

Write $\zeta = \zeta_{2^{n-3}} \in k$ from now on. Since $n \geq 5$, $\zeta^{2^{n-5}} \in k$ and $\zeta^{2^{n-5}}$ is a primitive 4-th root of unity. We write $\zeta^{2^{n-5}} = \sqrt{-1}$.

Case 1. $G = G_1$ where G_1 is the group in Theorem 2.1.

G is a metacyclic group. But we cannot apply Theorem 1.5 because $\zeta_{2^{n-2}} \notin k$.

Let V be a k-vector space whose dual space V^* is defined as $V^* = \bigoplus_{g \in G} k \cdot x(g)$ and $h \cdot x(g) = x(hg)$ for any $g, h \in G$. Note that $k(G) = k(x(g) : g \in G)^G = k(V)^G$. We will find a faithful G-subspace W of V^*.

Note that $\langle \sigma^2, \tau \rangle$ is an abelian subgroup of G and $\mathrm{ord}(\sigma^2) = 2^{n-3}$. Define

$$
\begin{aligned}
X &= \sum_{0 \leq i \leq 2^{n-3}-1} \zeta^{-i} \left[x(\sigma^{2i}) + x(\sigma^{2i}\tau) + x(\sigma^{2i}\tau^2) + x(\sigma^{2i}\tau^3) \right], \\
Y &= \sum_{\substack{0 \leq i \leq 2^{n-3}-1 \\ 0 \leq j \leq 3}} \left(\sqrt{-1}\right)^{-j} x(\sigma^{2i}\tau^j).
\end{aligned} \tag{3.1}
$$

We find that

$$
\begin{aligned}
\sigma^2 &: X \mapsto \zeta X,\ Y \mapsto Y, \\
\tau &: X \mapsto X,\ Y \mapsto \sqrt{-1} Y.
\end{aligned}
$$

Define $x_0 = X$, $x_1 = \sigma X$, $y_0 = Y$, $y_1 = \sigma Y$. The actions of σ, τ are given by

$$
\begin{aligned}
\sigma &: x_0 \mapsto x_1 \mapsto \zeta x_0,\ y_0 \mapsto y_1 \mapsto y_0, \\
\tau &: x_0 \mapsto x_0,\ x_1 \mapsto -x_1,\ y_0 \mapsto \sqrt{-1} y_0,\ y_1 \mapsto \sqrt{-1} y_1.
\end{aligned}
$$

It follows that $W = k \cdot x_0 \oplus k \cdot x_1 \oplus k \cdot y_0 \oplus k \cdot y_1$ is a faithful G-subspace of V^*. By Theorem 2.2, $k(G)$ is rational over $k(x_0, x_1, y_0, y_1)^G$. It remains to show that $k(x_0, x_1, y_0, y_1)^{\langle \sigma, \tau \rangle}$ is rational over k.

Define $z_1 = x_1/x_0$, $z_2 = y_1/y_0$. Then $k(x_0, x_1, y_0, y_1) = k(z_1, z_2, x_0, y_0)$ and

$$
\begin{aligned}
\sigma &: x_0 \mapsto z_1 x_0,\ y_0 \mapsto z_2 y_0,\ z_1 \mapsto \zeta/z_1,\ z_2 \mapsto 1/z_2, \\
\tau &: x_0 \mapsto x_0,\ y_0 \mapsto \sqrt{-1} y_0,\ z_1 \mapsto -z_1,\ z_2 \mapsto z_2.
\end{aligned}
$$

By Theorem 2.3, $k(z_1, z_2, x_0, y_0)^{\langle \sigma, \tau \rangle} = k(z_1, z_2)^{\langle \sigma, \tau \rangle}(z_3, z_4)$ for some z_3, z_4 with $\sigma(z_j) = \tau(z_j) = z_j$ for $j = 3, 4$.

The actions of σ and τ on z_1, z_2 are monomial automorphisms. By Theorem 2.5, $k(z_1, z_2)^{\langle \sigma, \tau \rangle}$ is rational. Thus $k(x_0, x_1, y_0, y_1)^{\langle \sigma, \tau \rangle}$ is also rational over k.

Case 2. $G = G_2$, G_3, G_{10} or G_{11}.

These four groups are direct products of subgroups $\langle \sigma, \tau \rangle$ and $\langle \lambda \rangle$. We may apply Theorem 1.6 to study $k(G)$ since $H := \langle \sigma, \tau \rangle$ is a group of order 2^{n-1}, $\mathrm{ord}(\sigma) = 2^{n-2}$ and $\zeta_{2^{n-3}} \in k$. By Theorem 1.3 we find that $k(H)$ is rational over k.

Case 3. $G = G_4$.

As in the proof of Case 1. $G = G_1$, we will find a faithful G-subspace W in $V^* = \bigoplus_{g \in G} k \cdot x(g)$. The construction of W is similar to that in Case 1, but some modification should be made.

Although $\langle \sigma^2, \tau \rangle$ is an abelian subgroup of G, we will consider $\langle \sigma^2 \rangle$ instead. Explicitly, define

$$X = \sum_{0 \le i \le 2^{n-3}-1} \zeta^{-i} \left[x(\sigma^{2i}) + x(\sigma^{2i}\tau) \right].$$

It follows that $\sigma^2(X) = \zeta X$ and $\tau(X) = X$.

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \lambda X$, $x_3 = \lambda \sigma X$. We find that

$$\begin{aligned}
&\sigma : x_0 \mapsto x_1 \mapsto \zeta x_0, \ x_2 \mapsto x_3 \mapsto \zeta x_2, \\
&\tau : x_0 \mapsto x_0, \ x_1 \mapsto x_1, \ x_2 \mapsto -x_2, \ x_3 \mapsto -x_3, \\
&\lambda : x_0 \mapsto x_2 \mapsto x_0, \ x_1 \mapsto x_3 \mapsto x_1.
\end{aligned}$$

Note that G acts faithfully on $k(x_i : 0 \le i \le 3)$. Hence $k(G)$ is rational over $k(x_i : 0 \le i \le 3)^G$ by Theorem 2.2.

Define $y_0 = x_0^{2^{n-3}}$, $y_1 = x_1/x_0$, $y_2 = x_2/x_1$, $y_3 = x_3/x_2$. Then $k(x_i : 0 \le i \le 3)^{\langle \sigma^2 \rangle} = k(y_i : 0 \le i \le 3)$ and

$$\begin{aligned}
&\sigma : y_0 \mapsto y_1^{2^{n-3}} y_0, \ y_1 \mapsto \zeta/y_1, \ y_2 \mapsto \zeta^{-1} y_1 y_2 y_3, \ y_3 \mapsto \zeta/y_3, \\
&\tau : y_0 \mapsto y_0, \ y_1 \mapsto y_1, \ y_2 \mapsto -y_2, \ y_3 \mapsto y_3, \\
&\lambda : y_0 \mapsto y_1^{2^{n-3}} y_2^{2^{n-3}} y_0, \ y_1 \mapsto y_3 \mapsto y_1, \ y_2 \mapsto 1/(y_1 y_2 y_3).
\end{aligned}$$

By Theorem 2.3, we find that $k(y_i : 0 \le i \le 3)^{\langle \sigma, \tau, \lambda \rangle} = k(y_i : 1 \le i \le 3)^{\langle \sigma, \tau, \lambda \rangle}(y_4)$ for some y_4 with $\sigma(y_4) = \tau(y_4) = \lambda(y_4) = y_4$.

It is clear that $k(y_i : 1 \le i \le 3)^{\langle \tau \rangle} = k(y_1, y_2^2, y_3)$.

Define $z_1 = y_1$, $z_2 = y_3$, $z_3 = y_1 y_3 y_2^2$. Then $k(y_1, y_2^2, y_3) = k(z_i : 1 \le i \le 3)$ and

$$\begin{aligned}
&\sigma : z_1 \mapsto \zeta/z_1, \ z_2 \mapsto \zeta/z_2, \ z_3 \mapsto z_3, \\
&\lambda : z_1 \mapsto z_2 \mapsto z_1, \ z_3 \mapsto 1/z_3.
\end{aligned}$$

By Theorem 2.4, $k(z_i : 1 \le i \le 3)^{\langle \sigma, \lambda \rangle}$ is rational over k.

Case 4. $G = G_5$.

The proof is similar to Case 3. $G = G_4$. We define X such that $\sigma^2(X) = \zeta X$, $\lambda(X) = X$ (note that in the present case we require $\lambda(X) = X$ instead of $\tau(X) = X$).

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \tau X$, $x_3 = \tau \sigma X$. It follows that

$$\begin{aligned}
&\sigma : x_0 \mapsto x_1 \mapsto \zeta x_0, \ x_2 \mapsto x_3 \mapsto \zeta x_2, \\
&\tau : x_0 \mapsto x_2 \mapsto x_0, \ x_1 \mapsto x_3 \mapsto x_1, \\
&\lambda : x_0 \mapsto x_0, \ x_1 \mapsto x_3 \mapsto x_1, \ x_2 \mapsto x_2.
\end{aligned}$$

It follows that G acts faithfully on $k(x_i : 0 \le i \le 3)$. By Theorem 2.2 it suffices to show that $k(x_i : 0 \le i \le 3)^G$ is rational over k.

Define $y_0 = x_0 - x_2$, $y_1 = x_1 - x_3$, $y_2 = x_0 + x_2$, $y_3 = x_1 + x_3$. It follows that $k(x_i : 0 \le i \le 3) = k(y_0 : 0 \le i \le 3)$ and

$$\begin{aligned}
&\sigma : y_0 \mapsto y_1 \mapsto \zeta y_0,\ y_2 \mapsto y_3 \mapsto \zeta y_2,\\
&\tau : y_0 \mapsto -y_0,\ y_1 \mapsto -y_1,\ y_2 \mapsto y_2,\ y_3 \mapsto y_3,\\
&\lambda : y_0 \mapsto y_0,\ y_1 \mapsto -y_1,\ y_2 \mapsto y_2,\ y_3 \mapsto y_3.
\end{aligned}$$

By Theorem 2.2 $k(y_i : 0 \le i \le 3)^G = k(y_0, y_1)^G(y_4, y_5)$ for some y_4, y_5 with $g(y_4) = y_4$, $g(y_5) = y_5$ for any $g \in G$. Note the the actions of G on y_0, y_1 are monomial automorphisms. By Theorem 2.5 $k(y_0, y_1)^G$ is rational over k.

Case 5. $G = G_6, G_7$.

Consider the case $G = G_6$ first.

Note that $\langle \sigma^2, \tau^2 \rangle$ is an abelian subgroup of G. As in the proof of Case 1. $G = G_1$ we define X and Y in $V^* = \bigoplus_{g \in G} k \cdot x(g)$ by

$$\begin{aligned}
X &= \sum_{0 \le i \le 2^{n-3}-1} \zeta^{-i} \left[x(\sigma^{2i}) + x(\sigma^{2i}\tau^2) \right], \\
Y &= \sum_{\substack{0 \le i \le 2^{n-3}-1 \\ 0 \le j \le 3}} \left(\sqrt{-1}\right)^{-j} x(\sigma^{2i}\tau^j).
\end{aligned} \tag{3.2}$$

It follows that $\sigma^2(X) = \zeta X$, $\tau^2(X) = X$, $\sigma^2(Y) = Y$, $\tau(Y) = \sqrt{-1} Y$.

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \tau X$, $x_3 = \tau\sigma X$, $y_0 = Y$, $y_1 = \sigma Y$. We get

$$\begin{aligned}
&\sigma : x_0 \mapsto x_1 \mapsto \zeta x_0,\ x_2 \mapsto \zeta^{-1} x_3,\ x_3 \mapsto x_2,\ y_0 \mapsto y_1 \mapsto y_0,\\
&\tau : x_0 \mapsto x_2 \mapsto x_0,\ x_1 \mapsto x_3 \mapsto x_1,\ y_0 \mapsto \sqrt{-1} y_0,\ y_1 \mapsto \sqrt{-1} y_1.
\end{aligned}$$

Note that G acts faithfully on $k(x_i, y_0, y_1 : 0 \le i \le 3)$. We will show that $k(x_i, y_0, y_1 : 0 \le i \le 3)^{\langle \sigma, \tau \rangle}$ is rational over k.

Define $y_2 = y_1/y_0$. It follows that $\sigma(y_2) = 1/y_2$, $\sigma(y_0) = y_2 y_0$, $\tau(y_2) = y_2$, $\tau(y_0) = \sqrt{-1} y_0$. By Theorem 2.3 $k(x_i, y_0, y_1 : 0 \le i \le 3)^{\langle \sigma, \tau \rangle} = k(x_i, y_2, y_0 : 0 \le i \le 3)^{\langle \sigma, \tau \rangle} = k(x_i, y_2 : 0 \le i \le 3)^{\langle \sigma, \tau \rangle}(y_3)$ for some y_3 with $\sigma(y_3) = \tau(y_3) = y_3$.

Define $y_4 = (1 - y_2)/(1 + y_2)$. Then $\sigma(y_4) = -y_4$, $\tau(y_4) = y_4$. By Theorem 2.3 $k(x_i, y_2 : 0 \le i \le 3)^{\langle \sigma, \tau \rangle} = k(x_i : 0 \le i \le 3)^{\langle \sigma, \tau \rangle}(y_5)$ for some y_5 with $\sigma(y_5) = \tau(y_5) = y_5$.

Define $z_0 = x_0$, $z_1 = x_1/x_0$, $z_2 = x_3/x_2$, $z_3 = x_2/x_1$. We find that

$$\begin{aligned}
&\sigma : z_0 \mapsto z_1 z_0,\ z_1 \mapsto \zeta / z_1,\ z_2 \mapsto \zeta / z_2,\ z_3 \mapsto \zeta^{-2} z_1 z_2 z_3,\\
&\tau : z_0 \mapsto z_1 z_3 z_0,\ z_1 \mapsto z_2 \mapsto z_1,\ z_3 \mapsto 1/(z_1 z_2 z_3).
\end{aligned}$$

By Theorem 2.3 $k(x_i : 0 \le i \le 3)^{\langle \sigma, \tau \rangle} = k(z_i : 0 \le i \le 3)^{\langle \sigma, \tau \rangle} = k(z_i : 1 \le i \le 3)^{\langle \sigma, \tau \rangle}(z_4)$ for some z_4 with $\sigma(z_4) = \tau(z_4) = z_4$.

Define $u_1 = z_3^{2^{n-4}}$. Then $k(z_i : 1 \le i \le 3)^{\langle \sigma^2 \rangle} = k(z_1, z_2, u_1)$ and

$$\sigma : z_1 \mapsto \zeta/z_1,\ z_2 \mapsto \zeta/z_2,\ u_1 \mapsto (z_1 z_2)^{2^{n-4}} u_1,$$
$$\tau : z_1 \mapsto z_2 \mapsto z_1,\ u_1 \mapsto \left((z_1 z_2)^{2^{n-4}} \cdot u_1\right)^{-1}.$$

Define $u_2 = (z_1 z_2)^{2^{n-5}} u_1$. Then $k(z_1, z_2, u_1) = k(z_1, z_2, u_2)$ and

$$\sigma : z_1 \mapsto \zeta/z_1,\ z_2 \mapsto \zeta/z_2,\ u_2 \mapsto -u_2,$$
$$\tau : z_1 \mapsto z_2 \mapsto z_1,\ u_2 \mapsto 1/u_2.$$

By Theorem 2.4 $k(z_1, z_2, u_2)^{\langle \sigma, \tau \rangle}$ is rational over k. This solves the case $G = G_6$.

When $G = G_7$, we use the same X and Y in (3.2). Define x_0, x_1, x_2, x_3, y_0, y_1 by the same formula. The proof is almost the same as $G = G_6$. Done.

Case 6. $G = G_8$.

Note that $\tau^8 = 1$ and $\sigma\tau^2 = \tau^2\sigma$.

Define X and Y in $V^* = \bigoplus_{g \in G} k \cdot x(g)$ by

$$X = \sum_{\substack{0 \le i \le 2^{n-3}-1 \\ 0 \le j \le 3}} \zeta^{-i} x(\sigma^{2i}\tau^{2j}), \quad Y = \sum_{\substack{0 \le i \le 2^{n-3}-1 \\ 0 \le j \le 3}} \left(\sqrt{-1}\right)^{-j} x(\sigma^{2i}\tau^{2j}).$$

It follows that $\sigma^2(X) = \zeta X$, $\sigma^2(Y) = Y$, $\tau^2(X) = X$, $\tau^2(Y) = \sqrt{-1}Y$.

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \tau X$, $x_3 = \tau\sigma X$, $y_0 = Y$, $y_1 = \sigma Y$, $y_2 = \tau Y$, $y_3 = \tau\sigma Y$. We find that

$$\sigma : x_0 \mapsto x_1 \mapsto \zeta x_0,\ x_2 \mapsto \zeta^{-1} x_3,\ x_3 \mapsto x_2,\ y_0 \leftrightarrow y_1,\ y_2 \leftrightarrow y_3,$$
$$\tau : x_0 \leftrightarrow x_2,\ x_1 \leftrightarrow x_3,\ y_0 \mapsto y_2 \mapsto \sqrt{-1} y_0,\ y_1 \mapsto y_3 \mapsto \sqrt{-1} y_1.$$

Since $G = \langle \sigma, \tau \rangle$ acts faithfully on $k(x_i, y_i : 0 \le i \le 3)$, it remains to show that $k(x_i, y_i : 0 \le i \le 3)^{\langle \sigma, \tau \rangle}$ is rational over k.

Define $z_i = x_i y_i$ for $0 \le i \le 3$. We get

$$\sigma : z_0 \mapsto z_1 \mapsto \zeta z_0,\ z_2 \mapsto \zeta^{-1} z_3,\ z_3 \mapsto z_2,$$
$$\tau : z_0 \mapsto z_2 \mapsto \sqrt{-1} z_0,\ z_1 \mapsto z_3 \mapsto \sqrt{-1} z_1. \tag{3.3}$$

Note that $k(x_i, y_i : 0 \le i \le 3) = k(x_i, z_i : 0 \le i \le 3)$ and G acts faithfully on $k(z_i : 0 \le i \le 3)$. By Theorem 2.2 $k(x_i, z_i : 0 \le i \le 3)^{\langle \sigma, \tau \rangle} = k(z_i : 0 \le i \le 3)^{\langle \sigma, \tau \rangle}$ (X_0, X_1, X_2, X_3) for some X_i $(0 \le i \le 3)$ with $\sigma(X_i) = \tau(X_i) = X_i$.

Define $u_0 = z_0$, $u_1 = z_1/z_0$, $u_2 = z_3/z_2$, $u_3 = z_2/z_1$. The actions are given by

$$\sigma : u_0 \mapsto u_1 u_0,\ u_1 \mapsto \zeta/u_1,\ u_2 \mapsto \zeta/u_2,\ u_3 \mapsto \zeta^{-2} u_1 u_2 u_3,$$
$$\tau : u_0 \mapsto u_1 u_3 u_0,\ u_1 \leftrightarrow u_2,\ u_3 \mapsto \sqrt{-1}/(u_1 u_2 u_3). \tag{3.4}$$

By Theorem 2.3 $k(z_i : 0 \le i \le 3)^{\langle\sigma,\tau\rangle} = k(u_i : 0 \le i \le 3)^{\langle\sigma,\tau\rangle} = k(u_i : 1 \le i \le 3)^{\langle\sigma,\tau\rangle}(u_4)$ for some u_4 with $\sigma(u_4) = \tau(u_4) = u_4$.

Define $v_1 = u_3^{2^{n-4}}$. Then $k(u_i : 1 \le i \le 3)^{\langle\sigma^2\rangle} = k(u_1, u_2, v_1)$ and $\sigma(v_1) = (u_1u_2)^{2^{n-4}} v_1$, $\tau(v_1) = \varepsilon/\big((u_1u_2)^{2^{n-4}}u_4\big)$ where $\varepsilon = 1$ if $n \ge 6$, and $\varepsilon = -1$ if $n = 5$.

Define $v_2 = (u_1u_2)^{2^{n-5}}v_1$. Then $\sigma(v_2) = -v_2$, $\tau(v_2) = \varepsilon/v_2$. Since $k(u_1, u_2, v_1)^{\langle\sigma,\tau\rangle} = k(u_1, u_2, v_2)^{\langle\sigma,\tau\rangle}$ is rational over k by Theorem 2.4, the proof is finished.

Case 7. $G = G_9$.

Note that $\sigma^2\tau = \tau\sigma^2$.

Define X and Y in $V^* = \bigoplus_{g\in G} k \cdot x(g)$ by

$$X = \sum_{\substack{0\le i\le 2^{n-3}-1\\ 0\le j\le 3}} \zeta^{-i}x(\sigma^{2i}\tau^j), \quad Y = \sum_{\substack{0\le i\le 2^{n-3}-1\\ 0\le j\le 3}} \big(\sqrt{-1}\big)^{-j} x(\sigma^{2i}\tau^j).$$

It follows that $\sigma^2(X) = \zeta X$, $\sigma^2(Y) = Y$, $\tau(X) = X$, $\tau(Y) = \sqrt{-1}Y$.

Define $x_0 = X$, $x_1 = \sigma X$, $y_0 = Y$, $y_1 = \sigma Y$. We get

$$\begin{aligned}
&\sigma : x_0 \mapsto x_1 \mapsto \zeta x_0,\ y_0 \mapsto y_1 \mapsto y_0,\\
&\tau : x_0 \mapsto x_0,\ x_1 \mapsto x_1,\ y_0 \mapsto \sqrt{-1}y_0,\ y_1 \mapsto -\sqrt{-1}y_1.
\end{aligned}$$

It remains to prove $k(x_0, x_1, y_0, y_1)^{\langle\sigma,\tau\rangle}$ is rational over k. The proof is almost the same as Case 1. $G = G_1$. Done.

Case 8. $G = G_{12}$.

Define $X \in V^* = \bigoplus_{g\in G} h \cdot x(g)$ by

$$X = \sum_{0\le i\le 2^{n-3}-1} \zeta^{-i}\left[x(\sigma^{2i}) + x(\sigma^{2i}\tau)\right].$$

Then $\sigma^2 X = \zeta X$, $\tau X = X$.

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \lambda X$, $x_3 = \lambda\sigma X$. We find that

$$\begin{aligned}
&\sigma : x_0 \mapsto x_1 \mapsto \zeta x_0,\ x_2 \mapsto \zeta^{-1}x_3,\ x_3 \mapsto x_2,\\
&\tau : x_0 \mapsto x_0,\ x_1 \mapsto x_1,\ x_2 \mapsto -x_2,\ x_3 \mapsto -x_3,\\
&\lambda : x_0 \leftrightarrow x_2,\ x_1 \leftrightarrow x_3.
\end{aligned}$$

Since $G = \langle\sigma, \tau, \lambda\rangle$ is faithful on $k(x_i : 0 \le i \le 3)$, it remains to show that $k(x_i : 0 \le i \le 3)^{\langle\sigma,\tau,\lambda\rangle}$ is rational over k.

Define $y_0 = x_0$, $y_1 = x_1/x_0$, $y_2 = x_3/x_2$, $y_3 = x_2/x_1$. We get

$$\begin{aligned}
&\sigma : y_0 \mapsto y_1y_0,\ y_1 \mapsto \zeta/y_1,\ y_2 \mapsto \zeta/y_2,\ y_3 \mapsto \zeta^{-2}y_1y_2y_3,\\
&\tau : y_0 \mapsto y_0,\ y_1 \mapsto y_1,\ y_2 \mapsto y_2,\ y_3 \mapsto -y_3,\\
&\lambda : y_0 \mapsto y_1y_3y_0,\ y_1 \leftrightarrow y_2,\ y_3 \mapsto 1/(y_1y_2y_3).
\end{aligned} \tag{3.5}$$

By Theorem 2.3 $k(y_i : 0 \le i \le 3)^{\langle\sigma,\tau,\lambda\rangle} = k(y_i : 1 \le i \le 3)^{\langle\sigma,\tau,\lambda\rangle}(y_4)$ for some y_4 with $\sigma(y_4) = \tau(y_4) = \lambda(y_4) = y_4$.

Define $z_1 = y_3^2$. Then $k(y_i : 1 \le i \le 3)^{\langle\tau\rangle} = k(y_1, y_2, z_1)$ and $\sigma(z_1) = \zeta^{-4} y_1^2 y_2^2 z_1$, $\lambda(z_1) = 1/(y_1^2 y_2^2 z_1)$.

Define $z_2 = z_1^{2^{n-5}}$. Then $k(y_1, y_2, z_1)^{\langle\sigma^2\rangle} = k(y_1, y_2, z_2)$ and $\sigma(z_2) = (y_1y_2)^{2^{n-4}} z_2$, $\lambda(z_2) = 1/\big((y_1y_2)^{2^{n-4}} z_2\big)$.

Define $z_3 = (y_1y_2)^{2^{n-5}} z_2$. We find that $k(y_1, y_2, z_2) = k(y_1, y_2, z_3)$ and $\sigma(z_3) = -z_3$, $\lambda(z_3) = 1/z_3$. By Theorem 2.4, $k(y_1, y_2, y_3)^{\langle\sigma,\tau\rangle}$ is rational over k. Done.

Case 9. $G = G_{13}$, G_{14}.

We consider the case $G = G_{13}$ only, because the proof for $G = G_{14}$ is almost the same (with the same way of changing the variables).

Define X and Y in $V^* = \bigoplus_{g\in G} k \cdot x(g)$ by

$$X = \sum_{0\le i\le 2^{n-3}-1} \zeta^{-i}\left[x(\sigma^{2i}) + x(\sigma^{2i}\tau)\right],$$
$$Y = \sum_{0\le i\le 2^{n-3}-1} x(\sigma^{2i}) - \sum_{0\le i\le 2^{n-3}-1} x(\sigma^{2i}\tau).$$

We find that $\sigma^2(X) = \zeta X$, $\sigma^2(Y) = Y$, $\tau(X) = X$, $\tau(Y) = -Y$.

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \lambda X$, $x_3 = \lambda\sigma X$, $y_0 = Y$, $y_1 = \sigma Y$, $y_2 = \lambda Y$, $y_3 = \lambda\sigma Y$. It follows that

$$\begin{aligned}
&\sigma : x_0 \mapsto x_1 \mapsto \zeta x_0,\ x_2 \mapsto \zeta^{-1} x_3,\ x_3 \mapsto x_2,\ y_0 \leftrightarrow y_1,\ y_2 \leftrightarrow -y_3,\\
&\tau : x_i \mapsto x_i,\ y_i \mapsto -y_i,\\
&\lambda : x_0 \leftrightarrow x_2,\ x_1 \leftrightarrow x_3,\ y_0 \leftrightarrow y_2,\ y_1 \leftrightarrow y_3.
\end{aligned}$$

Note that G acts faithfully on $k(x_i, y_i : 0 \le i \le 3)$. Thus it remains to show that $k(x_i, y_i : 0 \le i \le 3)^{\langle\sigma,\tau,\lambda\rangle}$ is rational over k.

Define $x_4 = y_0 + y_1, x_5 = y_2 + y_3, x_6 = y_0 - y_1, x_7 = y_2 - y_3$. Then $k(x_i, y_i : 0 \le i \le 3) = k(x_i : 0 \le i \le 7)$, and $\sigma(x_i) = x_i$ for $i = 4, 7$, $\sigma(x_i) = -x_i$ for $i = 5, 6$, $\tau(x_i) = -x_i$ for $4 \le i \le 7$, $\lambda : x_4 \leftrightarrow x_5,\ x_6 \leftrightarrow x_7$.

Apply Theorem 2.2 to $k(x_i : 0 \le i \le 7)$. It suffices to prove that $k(x_i : 0 \le i \le 5)^{\langle\sigma,\tau,\lambda\rangle}$ is rational over k.

Define $Z = x_5/x_4$. Then $k(x_i : 0 \le i \le 5) = k(x_i, Z : 0 \le i \le 4)$ and $\sigma(Z) = -Z, \tau(Z) = Z, \lambda(Z) = 1/Z$. Apply Theorem 2.3 to $k(x_i : 0 \le i \le 5)$. It remains to prove that $k(x_i, Z : 0 \le i \le 3)^{\langle\sigma,\tau,\lambda\rangle}$ is rational over k. Note that the action of τ becomes trivial on $k(x_i, Z : 0 \le i \le 3)$.

Define $u_0 = x_0, u_1 = x_1/x_0, u_2 = x_3/x_2, u_3 = x_2/x_1, u_4 = Z$. By Theorem 2.3 $k(x_i, Z : 0 \le i \le 3)^{\langle\sigma,\lambda\rangle} = k(u_i : 1 \le i \le 4)^{\langle\sigma,\lambda\rangle}(U)$ for some element U fixed by the action of G. The actions of σ and λ are given by

$$\begin{aligned}
&\sigma : u_1 \mapsto \zeta/u_1,\ u_2 \mapsto \zeta/u_2,\ u_3 \mapsto \zeta^{-2} u_1u_2u_3,\ u_4 \mapsto -u_4,\\
&\lambda : u_1 \leftrightarrow u_2,\ u_3 \mapsto 1/(u_1u_2u_3), u_4 \mapsto 1/u_4.
\end{aligned}$$

Note that σ^2 fixes u_1, u_2, u_4 and $\sigma^2(u_3) = \zeta^{-2}u_3$. Define $u_5 = u_3^{2^{n-4}}$. Then $k(u_i : 1 \le i \le 4)^{\langle\sigma^2\rangle} = k(u_1, u_2, u_4, u_5)$ and $\sigma(u_5) = (u_1u_2)^{2^{n-4}}u_5, \lambda(u_5) = 1/((u_1u_2)^{2^{n-4}}u_5)$.

Define $u_6 = (u_1u_2)^{2^{n-5}}u_5$. Then $k(u_1, u_2, u_4, u_5) = k(u_1, u_2, u_4, u_6)$ and we get

$$\begin{aligned}&\sigma : u_1 \mapsto \zeta/u_1,\ u_2 \mapsto \zeta/u_2,\ u_6 \mapsto -u_6,\ u_4 \mapsto -u_4,\\ &\lambda :\ u_1 \leftrightarrow u_2,\ u_6 \mapsto 1/u_6,\ u_4 \mapsto 1/u_4.\end{aligned}$$

Define $u_7 = u_4u_6$. Then $\sigma(u_7) = u_7, \lambda(u_7) = 1/u_7$. Define $u_8 = (1 - u_7)/(1 + u_7)$. Then $\sigma(u_8) = u_8, \lambda(u_8) = -u_8$. Since $k(u_1, u_2, u_4, u_6) = k(u_1, u_2, u_6, u_8)$, we may apply Theorem 2.3. Thus it suffices to prove that $k(u_1, u_2, u_6)^{\langle\sigma,\lambda\rangle}$ is rational over k. By Theorem 2.4 $k(u_1, u_2, u_6)^{\langle\sigma,\lambda\rangle}$ is rational over k. Done.

Case 10. $G = G_{15}$, G_{16}, G_{17}, G_{18}, G_{24}, G_{25}.

These cases were proved in [Ka4, Section 5]. Note that in Cases 5 $\sim$ 8 of [Ka4, Section 5], only $\zeta_{2^{n-3}} \in k$ was used. Hence the result.

Case 11. $G = G_{19}$, G_{20}.

We consider the case $G = G_{19}$ only, because the proof for $G = G_{20}$ is almost the same.

Define $X \in V^* = \bigoplus_{g\in G} k \cdot x(g)$ by

$$X = \sum_{0\le i\le 2^{n-3}-1} \zeta^{-i}\left[x(\sigma^{2i}) + x(\sigma^{2i}\tau^2)\right].$$

Then $\sigma^2(X) = \zeta X$ and $\tau^2(X) = X$.

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \tau X$, $x_3 = \tau\sigma X$. We find that

$$\begin{aligned}&\sigma : x_0 \mapsto x_1 \mapsto \zeta x_0,\ x_2 \mapsto \sqrt{-1}x_3,\ x_3 \mapsto \sqrt{-1}\zeta x_2,\\ &\tau : x_0 \leftrightarrow x_2,\ x_1 \mapsto x_3 \mapsto -x_1.\end{aligned}$$

Thus G acts faithfully on $k(x_i : 0 \le i \le 3)$. It remains to prove $k(x_i : 0 \le i \le 3)^{\langle\sigma,\tau\rangle}$ is rational over k.

Define $u_0 = x_0$, $u_1 = x_1/x_0$, $u_2 = x_3/x_2$, $u_3 = x_2/x_1$. We find that

$$\begin{aligned}&\sigma : u_0 \mapsto u_1u_0,\ u_1 \mapsto \zeta/u_1,\ u_2 \mapsto \zeta/u_2,\ u_3 \mapsto \sqrt{-1}\zeta^{-1}u_1u_2u_3,\\ &\tau : u_0 \mapsto u_1u_3u_0,\ u_1 \mapsto u_2 \mapsto -u_1,\ u_3 \mapsto 1/(u_1u_2u_3).\end{aligned} \tag{3.6}$$

Compare the formula (3.6) with the formula (3.4) in the proof of Case 6. $G = G_8$. It is not difficult to see that the proof is almost the same as that of Case 6. $G = G_8$ (by taking the fixed field of the subgroup $<\sigma^2>$ first, and then making similar changes of variables). Done.

Case 12. $G = G_{21}$.

Note that $\tau^8 = 1$ and $\sigma^2\tau = \tau\sigma^2$.

Define X and Y in $V^* = \bigoplus_{g\in G} k \cdot x(g)$ by

$$X = \sum_{\substack{0\le i\le 2^{n-3}-1\\ 0\le j\le 3}} \zeta^{-i} x(\sigma^{2i}\tau^{2j}), \quad Y = \sum_{\substack{0\le i\le 2^{n-3}-1\\ 0\le j\le 2}} \left(\sqrt{-1}\right)^{-j} x(\sigma^{2i}\tau^{2j}).$$

Then $\sigma^2(X) = \zeta X$, $\sigma^2(Y) = Y$, $\tau^2(X) = X$, $\tau^2(Y) = \sqrt{-1}Y$.

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \tau X$, $x_3 = \tau\sigma X$, $y_0 = Y$, $y_1 = \sigma Y$, $y_2 = \tau Y$, $y_3 = \tau\sigma Y$. We find that

$$\begin{aligned}
&\sigma : x_0 \mapsto x_1 \mapsto \zeta x_0,\ x_2 \mapsto x_3 \mapsto \zeta x_2,\ y_0 \leftrightarrow y_1,\ y_2 \leftrightarrow \sqrt{-1}y_3,\\
&\tau : x_0 \leftrightarrow x_2,\ x_1 \leftrightarrow x_3,\ y_0 \mapsto y_2 \mapsto \sqrt{-1}y_0,\ y_1 \mapsto y_3 \mapsto -\sqrt{-1}y_1.
\end{aligned}$$

Since G is faithful on $k(x_i, y_i : 0 \le i \le 3)$, it remains to show that $k(x_i, y_i : 0 \le i \le 3)^{\langle\sigma,\tau\rangle}$ is rational over k.

Define $z_i = x_i y_i$ for $0 \le i \le 3$. It follows that

$$\begin{aligned}
&\sigma : z_0 \mapsto z_1 \mapsto \zeta z_0,\ z_2 \mapsto \sqrt{-1}z_3,\ z_3 \mapsto -\sqrt{-1}\zeta z_2,\\
&\tau : z_0 \mapsto z_2 \mapsto \sqrt{-1}z_0,\ z_1 \mapsto z_3 \mapsto -\sqrt{-1}z_1.
\end{aligned} \tag{3.7}$$

Compare the formulae (3.7) and (3.3). They are almost the same. Thus it is obvious that $k(x_i, y_i : 0 \le i \le 3)^{\langle\sigma,\tau\rangle}$ is rational over k.

Case 13. $G = G_{22}, G_{23}$.

We consider the case $G = G_{23}$, because the proof for $G = G_{22}$ is almost the same.

Define $X \in V^* = \bigoplus_{g\in G} k \cdot x(g)$ by

$$X = \sum_{0\le i\le 2^{n-3}-1} \zeta^{-i} \left[x(\sigma^{2i}) + x(\sigma^{2i}\tau)\right].$$

Then $\sigma^2(X) = \zeta X$, $\tau(X) = X$.

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \lambda X$, $x_3 = \lambda\sigma X$. We find that

$$\begin{aligned}
&\sigma : x_0 \mapsto x_1 \mapsto \zeta x_0,\ x_2 \mapsto \sqrt{-1}\zeta^{-1}x_3,\ x_3 \mapsto \sqrt{-1}x_2,\\
&\tau : x_0 \mapsto x_0, x_1 \mapsto x_1,\ x_2 \mapsto -x_2,\ x_3 \mapsto -x_3,\\
&\lambda : x_0 \leftrightarrow x_2,\ x_1 \leftrightarrow x_3.
\end{aligned}$$

Note that G acts faithfully on $k(x_i : 0 \le i \le 3)$. It remains to show that $k(x_i : 0 \le i \le 3)^{\langle\sigma,\tau,\lambda\rangle}$ is rational over k.

Define $y_0 = x_0$, $y_1 = x_1/x_0$, $y_2 = x_3/x_2$, $y_3 = x_2/x_1$. We get

$$\begin{aligned}
&\sigma : y_0 \mapsto y_1 y_0,\ y_1 \mapsto \zeta/y_1,\ y_2 \mapsto \zeta/y_2,\ y_3 \mapsto \sqrt{-1}\zeta^{-2}y_1y_2y_3,\\
&\tau : y_0 \mapsto y_0,\ y_1 \mapsto y_1,\ y_2 \mapsto y_2,\ y_3 \mapsto -y_3,\\
&\lambda : y_0 \mapsto y_1y_3y_0,\ y_1 \leftrightarrow y_2,\ y_3 \leftrightarrow 1/(y_1y_2y_3).
\end{aligned} \tag{3.8}$$

Compare the formula (3.8) with the formula (3.5) in the proof of Case 8. $G = G_{12}$. It is not difficult to show that $k(x_i : 0 \le i \le 3)^{\langle\sigma,\tau,\lambda\rangle}$ is rational over k in the present case.

Case 14. $G = G_{26}$.

Note that $\lambda^4 = 1$ and $\sigma^2\tau = \tau\sigma^2$.

Define $X \in V^* = \bigoplus_{g\in G} k \cdot x(g)$ by

$$X = \sum_{0\le i\le 3} \left(\sqrt{-1}\right)^{-i} \left[x(\sigma^{2i}) + x(\sigma^{2i}\tau)\right].$$

Then $\sigma^2(X) = \sqrt{-1}X$, $\tau(X) = X$.

Define $x_0 = X$, $x_1 = \sigma X$, $x_2 = \lambda X$, $x_3 = \lambda\sigma X$. We find that

$$\begin{aligned}
&\sigma : x_0 \mapsto x_1 \mapsto \sqrt{-1}x_0,\ x_2 \mapsto x_3 \mapsto -\sqrt{-1}x_2,\\
&\tau : x_0 \mapsto x_0,\ x_1 \mapsto -x_1,\ x_2 \mapsto x_2,\ x_3 \mapsto -x_3,\\
&\lambda : x_0 \mapsto x_2 \mapsto -x_0,\ x_1 \mapsto x_3 \mapsto -x_1.
\end{aligned}$$

Since G is faithful on $k(x_i : 0 \le i \le 3)$, it remains to show that $k(x_i : 0 \le i \le 3)^{\langle\sigma,\tau,\lambda\rangle}$ is rational over k.

Define $y_0 = x_0$, $y_1 = x_1/x_0$, $y_2 = x_3/x_2$, $y_3 = x_2/x_1$. We get

$$\begin{aligned}
&\sigma : y_0 \mapsto y_1y_0,\ y_1 \mapsto \sqrt{-1}/y_1,\ y_2 \mapsto -\sqrt{-1}/y_2,\ y_3 \mapsto -\sqrt{-1}y_1y_2y_3,\\
&\tau : y_0 \mapsto y_0,\ y_1 \mapsto -y_1,\ y_2 \mapsto -y_2,\ y_3 \mapsto -y_3,\\
&\lambda : y_0 \mapsto y_1y_3y_0,\ y_1 \leftrightarrow y_2,\ y_3 \mapsto -1/(y_1y_2y_3).
\end{aligned}$$

By Theorem 2.3 $k(y_i : 0 \le i \le 3)^{\langle\sigma,\tau,\lambda\rangle} = k(y_i : 1 \le i \le 3)^{\langle\sigma,\tau,\lambda\rangle}(y_4)$ for some y_4 with $\sigma(y_4) = \tau(y_4) = \lambda(y_4) = y_4$.

Define $v_0 = y_3^2$. Then $k(y_i : 1 \le i \le 3)^{\langle\sigma^2\rangle} = k(v_0, y_1, y_2)$ and

$$\sigma(v_0) = -(y_1y_2)^2v_0, \quad \tau(v_0) = v_0, \quad \lambda(v_0) = 1/(y_1^2y_2^2v_0).$$

Define $v_1 = y_1y_2$, $v_2 = y_1/y_2$. Then $k(v_0, y_1, y_2)^{\langle\tau\rangle} = k(v_i : 0 \le i \le 3)$ and

$$\begin{aligned}
&\sigma : v_1 \mapsto 1/v_1,\ v_2 \mapsto -1/v_2,\ v_0 \mapsto -v_1^2v_0,\\
&\lambda : v_1 \mapsto v_1,\ v_2 \mapsto 1/v_2,\ v_0 \mapsto 1/(v_1^2v_0).
\end{aligned}$$

Define $u_1 = v_1v_0$, $u_2 = v_2$, $u_3 = (1 - v_1)/(1 + v_1)$. Then $k(v_i : 0 \le i \le 2) = k(u_i : 1 \le i \le 3)$ and

$$\begin{aligned}
&\sigma : u_1 \mapsto -u_1,\ u_2 \mapsto -1/u_2,\ u_3 \mapsto -u_3,\\
&\lambda : u_1 \mapsto 1/u_1,\ u_2 \mapsto 1/u_2,\ u_3 \mapsto u_3.
\end{aligned}$$

By Theorem 2.2 $k(u_i : 1 \le i \le 3)^{\langle\sigma,\tau\rangle} = k(u_1, u_2)^{\langle\sigma,\tau\rangle}(u_4)$ for some u_4 with $\sigma(u_4) = \tau(u_4) = u_4$. By Theorem 2.5 $k(u_1, u_2)^{\langle\sigma,\tau\rangle}$ is rational over k. Hence $k(u_i : 1 \le i \le 3)^{\langle\sigma,\tau\rangle}$ is rational over k. □

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