

GENERALIZED LANDAU-LIFSHITZ EQUATION INTO  $S^n$ 

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ABSTRACT. In this paper, a type of integrable evolution equation—the generalized Landau-Lifshitz equation into  $S^n$  is considered. We deal with this equation from a geometric point of view by rewriting it in a geometric form. Through the geometric energy method, we show the global well-posedness of the corresponding Cauchy problem.

## 1. INTRODUCTION

Let  $u$  be a map from  $S^1 \times \mathbb{R}$  to the  $n$ -dimensional sphere  $S^n$ , which is embedded into  $\mathbb{R}^{n+1}$ . In this paper the following equation:

$$(1.1) \quad u_t = (u_{xx} + \frac{3}{2}|u_x|^2u)_x + \frac{3}{2}(u, Au)u_x, \quad |u|^2 = 1$$

is considered. Here  $A$  is a constant symmetric matrix and  $(\cdot, \cdot)$  denotes the standard inner product on  $\mathbb{R}^{n+1}$ . Without loss of generality we may assume that  $A$  is a diagonal matrix, i.e.  $A = \text{diag}(r_1, \dots, r_n)$ . We will also use  $\langle \cdot, \cdot \rangle$  to denote the metric on  $S^n$ . It is compatible with the inner product on  $\mathbb{R}^{n+1}$  in the sense that  $\langle X, Y \rangle = (X, Y)$  for any tangent vector fields  $X, Y \in TS^n$ .

For  $n = 1$ , with the trigonometric parameterizations of a circle, equation (1.1) becomes a well-known model in the theory of exactly integrable systems [3]. For  $n = 2$ , equation (1.1) defines an infinitesimal symmetry for the well-known Noemann system [15] describing the dynamics of a particle on the sphere  $S^2$  under the influence of field with the quadratic potential  $U = \frac{1}{2}(u, Au)$ . Besides, equation (1.1) coincides with the higher symmetry of third order for the famous Landau-Lifshitz equation

$$(1.2) \quad u_t = u \times u_{xx} + (Au) \times u, \quad |u|^2 = 1.$$

Here the symbol ‘ $\times$ ’ denotes a vector product. Thus system (1.1) is a generalized Landau-Lifshitz equation into  $n$ -dimensional sphere  $S^n$ .

In [6], I. Z. Golubchik and V. V. Sokolov showed that for any dimension  $n$  and matrix  $A$  the system (1.1) is exactly integrable by the inverse scattering method, and has an infinite number of symmetries. After that, Meshkov and Sokolov [9] used the symmetry approach to give a complete classification of integrable vector evolution equations similar to equation (1.1). In 2008, S. Igonin, J. Van De Leur, G. Manno, and V. Trushkov [7] successfully applied the Wahlquist-Estabrook method to recover the infinite-dimensional Lie algebra related to this system. For more reference on this topic, see also [1, 2, 11].

However, from the perspective of partial differential equations, one may naturally propose the problem of well-posedness of the corresponding Cauchy problem of equation (1.1) in appropriate Sobolev spaces. In this paper, we intend to provide such results. Let  $\nabla_x$  denote the covariant derivative  $\nabla_{\frac{\partial}{\partial x}}$  on the pull-back bundle  $u^*TS^n$  induced from the Levi-Civita connection  $\nabla$  on  $S^n$ , then

$$\nabla_x^2 u_x = u_{xxx} + 3(u_x, u_{xx})u + \langle u_x, u_x \rangle u_x.$$

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Therefore, equation (1.1) can be rewritten as a geometric flow on  $S^n$ :

$$(1.3) \quad u_t = \nabla_x^2 u_x + \frac{1}{2}|u_x|^2 u_x + \frac{3}{2}(u, Au)u_x.$$

When  $n = 2$  and  $A \equiv 0$ , equation (1.3) coincides with the geometric KdV Flow defined by Sun and Wang [13]. In this sense, equation (1.1) is a type of generalization of the so-called KdV Flow.

The Cauchy problem corresponding to equation (1.1), i.e. (1.3) is

$$(1.4) \quad \begin{cases} u_t = \nabla_x^2 u_x + \frac{1}{2}|u_x|^2 u_x + \frac{3}{2}(u, Au)u_x, \\ u(0) = u_0, \end{cases}$$

where  $u_0$  is an initial map from  $S^1$  into  $S^n$ . Now it is natural for us to treat this problem as a geometric evolution equation and implement the so-called geometric energy method (see Section 2). This method relies heavily on the geometric structure of equation (1.3) and seems more intrinsic. With this powerful tool, we prove the global well-posedness of Cauchy problem (1.4). Our main result is the following theorem.

**Theorem 1.1.** *Suppose  $u_0 \in W^{k,2}(S^1, S^n)$  for  $k \geq 3$ , then the Cauchy problem (1.4) admits a unique global solution  $u \in L^\infty(\mathbb{R}^+, W^{k,2}(S^1, S^n))$ .*

We sketch our strategy as follows:

First, we prove the local existence of solution to Cauchy problem (1.4) by perturbing the system with a 4th order term  $-\epsilon \nabla_x^3 u_x$ , where  $\epsilon > 0$  is a small positive number. Namely, we consider the following perturbed system:

$$(1.5) \quad \begin{cases} u_t = -\epsilon \nabla_x^3 u_x + \nabla_x^2 u_x + \frac{1}{2}|u_x|^2 u_x + \frac{3}{2}(u, Au)u_x, \\ u(0) = u_0. \end{cases}$$

This is a 4th order parabolic system and it is well-known that there exists a unique local solution  $u_\epsilon$  of (1.5) with smooth initial data. Then we use this solution to approximate the desired solution of (1.4) by vanishing the perturbing term, i.e. by letting  $\epsilon$  go to 0. The key step is to establish an uniform estimate of the solution  $u_\epsilon$  for  $\epsilon > 0$ , see Lemma 3.4. With this estimate, we are able to show that  $u_\epsilon$  converges to a limit map  $u$  which is a local solution to Cauchy problem (1.4). Furthermore, a careful calculation yields the uniqueness of the solution.

Next, instead of computing conservation laws of the integrable system, we define the following ‘energy’ integrals:

$$\begin{aligned} E_2(u) &= \int_{S^1} |\nabla_x u_x|^2 - \frac{1}{4} \int_{S^1} |u_x|^4 - \frac{9}{4} \int_{S^1} (u, Au)|u_x|^2 + \int_{S^1} (u_x, Au_x), \\ E_3(u) &= \int_{S^1} |\nabla_x^2 u_x|^2 - \int_{S^1} \langle u_x, \nabla_x u_x \rangle^2 - \frac{3}{2} \int_{S^1} |u_x|^2 |\nabla_x u_x|^2. \end{aligned}$$

Then we show that these geometric energies satisfy semi-conservation laws under the flow (1.3), i.e.

$$\begin{aligned} \frac{d}{dt} E_2(u) &\leq C(E_2(u) + 1), \\ \frac{d}{dt} E_3(u) &\leq C(E_3(u) + 1), \end{aligned}$$

where  $C$  is a constant depending only on the  $W^{3,2}$ -norm of the initial data  $u_0$ , the matrix  $A$  and the existing time of the solution. Applying Gronwall’s inequality, we get the bound of these energies, which implies a global bound of the  $W^{3,2}$ -norm of the solution. A standard argument then yields the global existence of solution to Cauchy problem (1.4).

Though we only discuss the situation when  $u$  is a map from  $S^1$  to  $S^n$ , the same result also holds for  $u$  mapping from the real line  $\mathbb{R}^1$ . Actually, one may check this by following the argument in [5]. The crucial fact is that the interpolation inequality in Theorem 2.1 is scaling invariant, and hence the main estimate in Lemma 3.4 does not depend on the diameter of the domain. Thus we have the following

**Theorem 1.2.** *Suppose  $u_0 \in W^{k,2}(\mathbb{R}^1, S^n)$  for  $k \geq 3$ , then the Cauchy problem (1.4) admits a unique global solution  $u \in L^\infty(\mathbb{R}^+, W^{k,2}(\mathbb{R}^1, S^n))$ .*

The rest of this paper is arranged as follows: we first recall the geometric energy method in Section 2. Then we apply this method to show the local existence and uniqueness of solution to the Cauchy problem (1.4) in Section 3 and Section 4 respectively. Next we compute the semi-conservation laws we need in Section 5. At last, we finish the proof of global existence in Section 6.

## 2. GEOMETRIC ENERGY METHOD

In this section, we recall the geometric energy method. This method was first introduced by Ding and Wang in their seminar paper [5] to show the local well-posedness of the Schrödinger flow. Then similar methods were employed to treat different kinds of problems in geometric analysis. It is especially powerful when applied to non-linear evolution equations. For example, Kenig, etc. [8] showed the same method works efficiently for a difference scheme to approach the Schrödinger flow equation. Song and Wang [12] used a similar method to prove the local well-posedness of the wave map with potential, which implies the existence of Schrödinger solitons on Lorentzian manifolds. Moser [10] also defined a geometric energy to deal with the biharmonic map. Recently, Sun and Wang [13] applied this method to investigate the geometric KdV flow.

The geometric energy method starts with a kind of geometric Sobolev-type norms defined on Riemannian vector bundles. Its main idea is to derive a priori estimates of the geometric energies, i.e. the geometric Sobolev norms of the solution. Since the geometric norm naturally involves with the geometry of the underlying manifolds, it seems more intrinsic to investigate these norms instead of the classical Sobolev norms when dealing with specific equations with geometric backgrounds. However, these norms are non-linear in general and harder to be dealt with than the normal ones, because the Sobolev embedding theorems fail to hold. Fortunately, Ding and Wang [5] discovered a generalized Gagliardo-Nirenberg inequality for the geometric Sobolev norms which plays the key role in the geometric energy method. Moreover, they found these geometric norms are in some sense equivalent to the normal Sobolev norms. We summarize their results with two theorems in the rest part of this section.

Let  $\pi : E \rightarrow M$  be a Riemannian vector bundle over an  $m$ -dimensional closed Riemannian manifold  $M$  and  $D$  denote the covariant derivative on  $E$  induced by the Riemannian metric. Then we can define a Sobolev norm which we denote by  $H^{k,p}$  for any  $k \geq 1$  and  $p > 0$  via the bundle metric for every section  $s \in \Gamma(E)$  by

$$\|s\|_{H^{k,q}} = \sum_{l=0}^k \|D^l s\|_{L^q}.$$

**Theorem 2.1.** ([5]) *Suppose  $s \in C^\infty(E)$  is a section where  $E$  is a vector bundle on  $M$ . Then we have*

$$\|D^j s\|_{L^p} \leq C \|s\|_{H^{k,q}}^a \|s\|_{L^r}^{1-a},$$

where  $1 \leq p, q, r \leq \infty$ , and  $j/k \leq a \leq 1$  ( $j/k \leq a < 1$  if  $q = m/(k-j) \neq 1$ ) are numbers such that

$$\frac{1}{p} = \frac{j}{m} + \frac{1}{r} + a\left(\frac{1}{q} - \frac{1}{r} - \frac{k}{m}\right).$$

The constant  $C$  only depends on  $M$  and the numbers  $j, k, q, r, a$ .

**Corollary 2.2.** *Suppose  $s \in C^\infty(E)$  is a section where  $E$  is a vector bundle on  $S^1$ . Then we have*

$$(2.1) \quad \|s\|_{L^\infty} \leq C \|s\|_{H^{1,2}}^{\frac{1}{2}} \|s\|_{L^2}^{\frac{1}{2}}$$

*Proof.* Since  $m = 1$ , just let  $j = 0, p = \infty, a = 1/2, k = 1, q = 2, r = 2$  and apply Theorem 2.1.  $\square$

Especially, for a map  $u \in C^\infty(S^1, N)$ , the pull-back bundle  $u^*(TN)$  is a Riemannian vector bundle on 1-dimensional manifold  $M = S^1$ . So the above inequality (2.1) applies for section  $s = \nabla_x^l u_x \in \Gamma(u^*(TN))$  with  $l \geq 0$ , which yields

$$(2.2) \quad \|\nabla_x^l u_x\|_{L^\infty} \leq C \|u_x\|_{H^{l+1,2}}^{\frac{1}{2}} \|u_x\|_{L^2}^{\frac{1}{2}} \leq C \|u_x\|_{H^{l+1,2}}.$$

For any map  $u$  from a  $m$ -dimensional Riemannian manifold  $M$  to a compact Riemannian manifold  $N$  which can be embedded into a Euclidean space  $\mathbb{R}^K$ , we have two kinds of Sobolev norms—namely, the above  $H^{k,p}$  norms of section  $Du = \nabla u \in \Gamma(u^*TN)$  and the normal  $W^{k,p}$  Sobolev norms of function  $u : M \rightarrow \mathbb{R}^K$ , i.e.

$$\|u\|_{W^{k,p}} = \sum_{l=0}^k \|\nabla^l u\|_{L^p},$$

where  $\nabla$  denotes the covariant derivative of functions on  $M$ . Ding and Wang showed that for  $k > m/2$ , the  $H^{k,p}$  norm of  $Du$  is equivalent to the  $W^{k+1,p}$  norm of  $u$ . Precisely, we have

**Theorem 2.3.** ([5]) *Assume that  $k > m/2$ . Then there exists a constant  $C = C(N, k)$  such that for all  $u \in C^\infty(M, N)$ ,*

$$\|\nabla u\|_{W^{k-1,2}} \leq C \sum_{i=1}^k \|Du\|_{H^{k-1,2}}^i$$

and

$$\|Du\|_{H^{k-1,2}} \leq C \sum_{i=1}^k \|\nabla u\|_{W^{k-1,2}}^i$$

In our case,  $m = 1$  and  $k$  can be any positive integer. That means, for all  $k \geq 1$ , the Sobolev norms  $\|u_x\|_{W^{k,2}}$  are equivalent to the nonlinear norms  $\|u_x\|_{H^{k,2}}$  of the same order.

### 3. LOCAL EXISTENCE

A basic property of the flow (1.3) is that it preserves the perturbed energy

$$(3.1) \quad E_1(u) = \frac{1}{2} \int_{S^1} |u_x|^2 dx + \frac{1}{2} \int_{S^1} (u, Au) dx.$$

**Lemma 3.1.**  *$E_1$  is conserved under the flow (1.3).*

*Proof.* A direct computation yields:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{S^1} |u_x|^2 \\
&= \int_{S^1} \langle \nabla_t u_x, u_x \rangle = \int_{S^1} \langle \nabla_x u_t, u_x \rangle \\
&= - \int_{S^1} \langle \nabla_x^2 u_x + \frac{1}{2} |u_x|^2 u_x + \frac{3}{2} (u, Au) u_x, \nabla_x u_x \rangle \\
&= - \int_{S^1} \langle \nabla_x^2 u_x, \nabla_x u_x \rangle - \frac{1}{2} \int_{S^1} |u_x|^2 \langle u_x, \nabla_x u_x \rangle - \frac{3}{2} \int_{S^1} (u, Au) \langle u_x, \nabla_x u_x \rangle.
\end{aligned}$$

The first two terms vanishes, when integrating by parts. Thus,

$$(3.2) \quad \frac{1}{2} \frac{d}{dt} \int_{S^1} |u_x|^2 = \frac{3}{2} \int_{S^1} |u_x|^2 (u, Au_x).$$

On the other hand, using equation (1.1), we have

$$\begin{aligned}
(3.3) \quad & \frac{1}{2} \frac{d}{dt} \int_{S^1} (u, Au) = \int_{S^1} (u_t, Au) \\
&= - \int_{S^1} (u_{xx} + \frac{3}{2} |u_x|^2 u, Au_x) + \frac{3}{2} \int_{S^1} (u, Au) (u_x, Au) \\
&= - \frac{3}{2} \int_{S^1} |u_x|^2 (u, Au_x).
\end{aligned}$$

From (3.2) and (3.3), we get

$$\frac{d}{dt} E_1(u(t)) = 0.$$

Lemma 3.1 follows.  $\square$

Since the matrix  $A$  is constant and  $|u| = 1$  on the sphere  $S^n$ , we have

$$(3.4) \quad \int_{S^1} (u, Au) dx \leq C.$$

Thus we get the following easy corollary:

**Lemma 3.2.**  $\|u_x\|_{L^2}$  is bounded under flow (1.3). Moreover, the following inequality holds:

$$(3.5) \quad \|u_x\|_{L^\infty} \leq C \|u_x\|_{H^{1,2}}^{\frac{1}{2}}.$$

*Proof.* The first statement follows directly from Lemma 3.1 and (3.4). Combining this and the interpolation inequality (2.1), we get

$$\|u_x\|_{L^\infty} \leq C \|u_x\|_{H^{1,2}}^{\frac{1}{2}} \|u_x\|_{L^2}^{\frac{1}{2}} \leq C \|u_x\|_{H^{1,2}}^{\frac{1}{2}}.$$

$\square$

To attain the local existence, we approximate equation (1.3) by a 4th-order parabolic system:

$$(3.6) \quad \begin{cases} u_t = -\epsilon \nabla_x^3 u_x + \nabla_x^2 u_x + \frac{1}{2} |u_x|^2 u_x + \frac{3}{2} (u, Au) u_x, \\ u(0) = u_0. \end{cases}$$

where  $\epsilon > 0$  is a small number.

Since  $N = S^n$  is a submanifold of  $\mathbb{R}^{n+1}$ ,  $u$  could be considered as a mapping from  $S^1$  into  $\mathbb{R}^{n+1}$ . The equation (3.6) then becomes a fourth order parabolic equation in  $\mathbb{R}^{n+1}$  which is analogous to the heat flow of biharmonic map. By the standard parabolic theory (See [14], for example), equation (3.6) admits a local solution  $u_\epsilon \in C^\infty([0, T_\epsilon) \times S^1, S^n)$ , if the initial map  $u_0$  is smooth. Moreover, one

can verify that  $u(t)$  lies on the sphere  $S^n$  for any  $t \in [0, T_\epsilon)$ , if the initial map does [16].

Similarly to Lemma 3.1, one can prove the following lemma through direct computation.

**Lemma 3.3.** *If  $u_\epsilon : [0, T_\epsilon) \times S^1 \rightarrow S^n$  is a solution to the Cauchy problem (3.6), then*

$$E_1(u_\epsilon(t)) \leq E_1(u_0), \quad \forall t \in [0, T].$$

Next we approximate the solution of the original system (1.4) by vanishing the perturbing term, i.e. letting  $\epsilon$  go to 0. To achieve this goal, we need the following lemma which provides an uniform estimate on the solutions of (3.6) for  $\epsilon > 0$ .

**Lemma 3.4.** *Let  $u_0 \in C^\infty(S^1, S^n)$  and  $u \in C^\infty([0, T_\epsilon) \times S^1, S^n)$  be a solution of (3.6) with  $\epsilon \in (0, 1]$ . Then for any integer  $k \geq 2$ , there exists a  $T_k > 0$  which is independent of  $\epsilon$ , such that*

$$(3.7) \quad \|\nabla_x u(t)\|_{H^{k,2}} \leq C(k, \|\nabla_x u_0\|_{H^{k,2}}), \quad t \in [0, T_k].$$

*Proof.* For any  $l \geq 2$ , we compute

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{S^1} |\nabla_x^l u_x|^2 = \int_{S^1} \langle \nabla_t \nabla_x^l u_x, \nabla_x^l u_x \rangle \\ &= \int_{S^1} \langle \nabla_x^l \nabla_t u_x + Q_l(u_t, u_x), \nabla_x^l u_x \rangle \\ (3.8) \quad &= \int_{S^1} \langle \nabla_x^{l+1} (-\epsilon \nabla_x^3 u_x + \nabla_x^2 u_x), \nabla_x^l u_x \rangle \\ & \quad + \frac{1}{2} \int_{S^1} \langle \nabla_x^{l+1} (|u_x|^2 u_x + 3(u, Au)u_x), \nabla_x^l u_x \rangle + \int_{S^1} \langle Q_l(u_t, u_x), \nabla_x^l u_x \rangle \\ & := I_1 + I_2 + I_3, \end{aligned}$$

where  $Q_l(u_t, u_x)$  denotes the curvature terms, and will be treated later.

For the first term in (3.8), it is easy to see

$$(3.9) \quad I_1 = -\epsilon \int_{S^1} \langle \nabla_x^{l+4} u_x, \nabla_x^l u_x \rangle + \int_{S^1} \langle \nabla_x^{l+3} u_x, \nabla_x^l u_x \rangle = -\epsilon \int_{S^1} |\nabla_x^{l+2} u_x|^2.$$

For the second term, we have

$$(3.10) \quad \begin{aligned} I_2 &= \frac{1}{2} \int_{S^1} \langle \nabla_x^{l+1} (|u_x|^2 u_x), \nabla_x^l u_x \rangle + \frac{3}{2} \int_{S^1} \langle \nabla_x^{l+1} ((u, Au)u_x), \nabla_x^l u_x \rangle \\ & := J_1 + J_2 \end{aligned}$$

We first estimate  $J_1$ . After differentiating, we get

$$(3.11) \quad \begin{aligned} J_1 &\leq \left| \int_{S^1} \langle \nabla_x^{l+1} u_x, u_x \rangle \langle u_x, \nabla_x^l u_x \rangle \right| + \frac{1}{2} \left| \int_{S^1} \langle u_x, u_x \rangle \langle \nabla_x^{l+1} u_x, \nabla_x^l u_x \rangle \right| \\ & \quad + C \sum_{a,b,c} \int_{S^1} |\nabla_x^a u_x| |\nabla_x^b u_x| |\nabla_x^c u_x| |\nabla_x^l u_x|, \end{aligned}$$

where the sum is taken over all integers  $a, b, c$  satisfying

$$(3.12) \quad a + b + c = l + 1, \quad \text{and} \quad l \geq a, b, c \geq 0.$$

For the first two terms in (3.11), integrating by parts, we have

$$(3.13) \quad \int_{S^1} \langle \nabla_x^{l+1} u_x, u_x \rangle \langle u_x, \nabla_x^l u_x \rangle = - \int_{S^1} \langle \nabla_x^l u_x, \nabla_x u_x \rangle \langle u_x, \nabla_x^l u_x \rangle,$$

and

$$\int_{S^1} \langle u_x, u_x \rangle \langle \nabla_x^{l+1} u_x, \nabla_x^l u_x \rangle = - \int_{S^1} \langle u_x, \nabla_x u_x \rangle \langle \nabla_x^l u_x, \nabla_x^l u_x \rangle,$$

Thus these two terms are of the same form as the summation term in (3.11), only with  $a = l$  and  $b = 1, c = 0$ . Now we may recall inequality (2.2) to estimate

$$\begin{aligned} \int_{S^1} |\nabla_x^l u_x| |\nabla_x u_x| |u_x| |\nabla_x^l u_x| &\leq \|\nabla_x u_x\|_{L^\infty} \|u_x\|_{L^\infty} \|\nabla_x^l u_x\|_{L^2}^2 \\ &\leq C \|u_x\|_{H^{l,2}}^4. \end{aligned}$$

Note that we used the assumption  $l \geq 2$  here. However for  $l = 1$ , this term is also bounded by

$$(3.14) \quad \int_{S^1} |\nabla_x u_x|^3 |u_x| \leq C \|u_x\|_{H^{2,2}}^4.$$

For the other terms of the summation where  $a, b, c \leq l - 1$ , we have

$$\begin{aligned} \int_{S^1} |\nabla_x^a u_x| |\nabla_x^b u_x| |\nabla_x^c u_x| |\nabla_x^l u_x| &\leq \|\nabla_x^a u_x\|_{L^\infty} \|\nabla_x^b u_x\|_{L^\infty} \|\nabla_x^c u_x\|_{L^\infty} \|\nabla_x^l u_x\|_{L^1} \\ &\leq C \|u_x\|_{H^{l,2}}^4. \end{aligned}$$

Hence, we find  $J_1$  bounded by

$$(3.15) \quad J_1 \leq C \|u_x\|_{H^{l,2}}^4.$$

Similarly, the second term  $J_2$  satisfies

$$\begin{aligned} J_2 &\leq 3 \left| \int_{S^1} (D_x^{l+1} u, Au) \langle u_x, \nabla_x^l u_x \rangle \right| + \frac{3}{2} \left| \int_{S^1} (u, Au) \langle \nabla_x^{l+1} u_x, \nabla_x^l u_x \rangle \right| \\ &\quad + C \sum_{a,b,c} \int_{S^1} |D_x^a u| |D_x^b u| |\nabla_x^c u_x| |\nabla_x^l u_x|, \end{aligned}$$

where  $D_x$  denotes the derivative of functions, and the sum is taken over all integers  $a, b, c$  satisfying (3.12). Now we may treat  $J_2$  in almost the same way as  $J_1$ , except that we shift to the classical Sobolev inequalities for the terms in the Euclidean inner product  $(\cdot, \cdot)$  this time. Still, we can obtain

$$(3.16) \quad J_2 \leq C \|u_x\|_{W^{l,2}}^2 \|u_x\|_{H^{l,2}}^2 \leq C \|u_x\|_{H^{l,2}}^4,$$

since the  $W^{l,2}$  and  $H^{l,2}$  Sobolev norms are equivalent by Theorem 2.3.

Combining (3.10), (3.15), (3.16), we get

$$(3.17) \quad I_2 \leq C \|u_x\|_{H^{l,2}}^4.$$

Finally we turn to the third term, i.e. the curvature term

$$I_3 = \int_{S^1} \langle Ql(u_t, u_x), \nabla_x^l u_x \rangle = \sum_{a,b,c} C_{a,b,c} \int_{S^1} \langle R(\nabla_x^a u_t, \nabla_x^b u_x) \nabla_x^c u_x, \nabla_x^l u_x \rangle,$$

where the sum is taken over all integers  $a, b, c$  satisfying

$$a + b + c = l - 1, \text{ and } l - 1 \geq a, b, c \geq 0,$$

and  $C_{a,b,c}$  are combination numbers bounded by a constant  $C_l$  only depending on  $l$ . Substituting  $u_t$  by equation (3.6), we get

$$\begin{aligned} I_3 &\leq C_l \sum_{a,b,c} \left\{ -\epsilon \int_{S^1} \langle R(\nabla_x^{a+3} u_x, \nabla_x^b u_x) \nabla_x^c u_x, \nabla_x^l u_x \rangle \right. \\ &\quad + \int_{S^1} \langle R(\nabla_x^{a+2} u_x, \nabla_x^b u_x) \nabla_x^c u_x, \nabla_x^l u_x \rangle \\ &\quad \left. + \frac{1}{2} \int_{S^1} |\nabla_x^a (|u_x|^2 u_x + 3(u, Au) u_x)| |\nabla_x^b u_x| |\nabla_x^c u_x| |\nabla_x^l u_x| \right\} \\ &:= C_l (J_4 + J_5), \end{aligned}$$

where  $J_4$  denotes the higher-order terms with  $\nabla_x^d u_x$ ,  $d \geq l+1$ , while  $J_5$  denotes the summation of the rest terms. We only need to deal with  $J_4$  here, since all the lower-order terms in  $J_5$  can be bounded. Namely, after a similar argument as the estimate of  $I_2$ , which we omit here, we can obtain

$$J_5 \leq C \|u_x\|_{H^{l,2}}^6.$$

On the other hand, there are four terms in  $J_4$ , i.e.

$$\begin{aligned} J_4 = & -\epsilon \left( \int_{S^1} \langle R(\nabla_x^{l+2} u_x, u_x) u_x, \nabla_x^l u_x \rangle + \int_{S^1} \langle R(\nabla_x^{l+1} u_x, \nabla_x u_x) u_x, \nabla_x^l u_x \rangle \right. \\ & \left. + \int_{S^1} \langle R(\nabla_x^{l+1} u_x, u_x) \nabla_x u_x, \nabla_x^l u_x \rangle + \int_{S^1} \langle R(\nabla_x^{l+1} u_x, u_x) u_x, \nabla_x^l u_x \rangle \right). \end{aligned}$$

The last term can be handled as we have done in (3.13). For the first term, it follows from Young's inequality that for any  $0 < \delta < 1$

$$\begin{aligned} |\epsilon \int_{S^1} \langle R(\nabla_x^{l+2} u_x, u_x) u_x, \nabla_x^l u_x \rangle| & \leq \epsilon \int_{S^1} |\nabla_x^{l+2} u_x| |\nabla_x^l u_x| |u_x|^2 \\ & \leq \epsilon \delta \int_{S^1} |\nabla_x^{l+2} u_x|^2 + \frac{\epsilon}{\delta} \int_{S^1} |\nabla_x^l u_x|^2 |u_x|^4 \\ & \leq \epsilon \delta \int_{S^1} |\nabla_x^{l+2} u_x|^2 + \frac{C}{\delta} \|u_x\|_{H^{l,2}}^6. \end{aligned}$$

Similarly, for the rest two terms in  $J_4$ ,

$$\begin{aligned} & \epsilon \left( \int_{S^1} \langle R(\nabla_x^{l+1} u_x, \nabla_x u_x) u_x, \nabla_x^l u_x \rangle + \int_{S^1} \langle R(\nabla_x^{l+1} u_x, u_x) \nabla_x u_x, \nabla_x^l u_x \rangle \right) \\ & \leq \epsilon \delta \int_{S^1} |\nabla_x^{l+1} u_x|^2 + \frac{4\epsilon}{\delta} \int_{S^1} |\nabla_x^l u_x|^2 |\nabla_x u_x|^2 |u_x|^2 \\ & \leq \epsilon \delta \int_{S^1} |\nabla_x^{l+1} u_x|^2 + \frac{C}{\delta} \|u_x\|_{H^{l,2}}^6. \end{aligned}$$

Note that we assume  $l \geq 2$  again in the above. As for  $l = 1$ , it is obviously bounded by  $C \|u_x\|_{H^{2,2}}^6$ . Hence, if we choose  $\delta = 1/C_l$ , we get

$$J_4 \leq \frac{\epsilon}{C_l} \int_{S^1} |\nabla_x^{l+2} u_x|^2 + \frac{\epsilon}{C_l} \int_{S^1} |\nabla_x^{l+1} u_x|^2 + C \|u_x\|_{H^{l,2}}^6,$$

which now implies

$$(3.18) \quad I_3 \leq \frac{\epsilon}{2} \int_{S^1} |\nabla_x^{l+2} u_x|^2 + \frac{\epsilon}{2} \int_{S^1} |\nabla_x^{l+1} u_x|^2 + C \|u_x\|_{H^{l,2}}^6.$$

Correspondingly, for  $l = 1$  we have

$$(3.19) \quad I_3 \leq \frac{\epsilon}{2} \int_{S^1} |\nabla_x^3 u_x|^2 + C \|u_x\|_{H^{2,2}}^6.$$

Combining (3.8),(3.9),(3.17) and (3.18), we arrive at

$$(3.20) \quad \frac{1}{2} \frac{d}{dt} |\nabla_x^l u_x|^2 \leq -\frac{\epsilon}{2} \int_{S^1} |\nabla_x^{l+2} u_x|^2 + \frac{\epsilon}{2} \int_{S^1} |\nabla_x^{l+1} u_x|^2 + C \|u_x\|_{H^{l,2}}^4 + C \|u_x\|_{H^{l,2}}^6,$$

for any  $l \geq 2$ . Also for the case  $l = 1$ , the above arguments together with (3.14),(3.19) shows that

$$(3.21) \quad \frac{1}{2} \frac{d}{dt} \int_{S^1} |\nabla_x u_x|^2 \leq -\frac{\epsilon}{2} \int_{S^1} |\nabla_x^3 u_x|^2 + C \|u_x\|_{H^{2,2}}^4 + C \|u_x\|_{H^{2,2}}^6.$$

Summing inequality (3.20) from  $l = 2$  to  $k$ , and putting (3.21) and Lemma 3.3 into account, we conclude that there exists a constant  $C_k$  depending only on  $k$  such that

$$\frac{d}{dt} \|u_x\|_{H^{k,2}}^2 \leq C_k (1 + \|u_x\|_{H^{k,2}}^2)^3$$



for any  $k \geq 2$ .

For this ordinary differential inequality with respect to  $\|u_x\|_{H^{k,2}}^2$ , whose initial data is  $u(0) = u_0$ , there exist  $T_k = T(\|\nabla_x u_0\|_{H^{k,2}})$  for all  $k \geq 2$  such that

$$\|u_x(t)\|_{H^{k,2}}^2 \leq C(k, \|\nabla_x u_0\|_{H^{k,2}}), \quad \forall t \in (0, T_k].$$

Thus we complete the proof of the lemma.  $\square$

**Remark 3.5.** *Actually, we can say more for  $k > 2$ . With the same procedure but more careful treatment with the interpolations (See [5, 13]), one can prove that*

$$(3.22) \quad \frac{d}{dt} \|u_x\|_{H^{k,2}}^2 \leq C(k, \|u_x\|_{H^{k-1,2}})(1 + \|u_x\|_{H^{k,2}}^2).$$

*One important fact is that the expression (3.22) is a linear differential inequality for  $\|u_x\|_{H^{k,2}}^2$ , therefore the existing time of the solution only depends on the existing time of  $\|u_x\|_{H^{k-1,2}}$ , which in turn equals to  $T_0 = T(\|\nabla_x u_0\|_{H^{2,2}}) = T(\|u_0\|_{W^{3,2}})$  by induction.*

Now we are ready to prove the local existence of the solution to Cauchy problem (1.4). We state this result in a separate theorem.

**Theorem 3.6.** *Suppose  $u_0 \in W^{k,2}(S^1, S^n)$  for  $k \geq 3$ , then the Cauchy problem (1.4) admits a local solution  $u \in L^\infty([0, T], W^{k,2}(S^1, S^n))$  for some positive number  $T > 0$ . Moreover, if the initial data  $u_0$  is smooth, so is the solution  $u$ .*

*Proof.* We first assume the initial map  $u_0$  is smooth. By Lemma 3.4, we know that the Cauchy problem (3.6) admits a unique smooth solution  $u_\epsilon \in C^\infty([0, T] \times S^1, S^n)$  which satisfies the estimates (3.7) with  $T$  only depending on  $\|u_0\|_{W^{3,2}}$ . Then by Theorem 2.3, we have for any integer  $p > 0$  and  $\epsilon \in (0, 1]$ :

$$\sup_{t \in [0, T]} \|u_\epsilon\|_{W^{p,2}(N)} \leq C_p(N, u_0),$$

where  $C_p(N, u_0)$  does not depend on  $\epsilon$ . Thus, by sending  $\epsilon \rightarrow 0$  and applying the embedding theorem of Sobolev spaces to  $u$ , we have  $u_\epsilon \rightarrow u \in C^p(S^1 \times [0, T])$  for any  $p$ . It is easy to check that  $u$  is a solution to the Cauchy problem (1.4).

Next, if  $u_0 \in W^{k,2}(S^1, S^n)$  for  $k \geq 3$ , then we can always choose a sequence  $u_0^i \in C^\infty(S^1, S^n)$ , such that  $u_0^i$  converges to  $u_0$  in  $W^{k,2}(S^1, S^n)$ . Now for each initial data  $u_0^i$ , we have a solution  $u^i \in C^\infty([0, T^i] \times S^1, S^n)$  to Cauchy problem (1.4). They all satisfy estimate (3.7), i.e.

$$\|\nabla_x u^i(t)\|_{H^{k-1,2}} \leq C(k, \|\nabla_x u_0^i\|_{H^{k-1,2}}) \leq C(k, \|\nabla_x u_0\|_{H^{k-1,2}}), \quad t \in [0, T^i]$$

Moreover, we have an uniform lower bound on the existing time  $T^i$ , i.e.

$$T^i = T(\|u_0^i\|_{W^{3,2}}) \geq T = T(\|u_0\|_{W^{3,2}}).$$

Again by Theorem 2.3, we have

$$\sup_{t \in [0, T]} \|u^i(t)\|_{W^{k,2}} \leq C(k, \|u_0\|_{W^{k,2}}).$$

Therefore there exists a subsequence  $\{u^j\}$  and a map  $u \in L^\infty([0, T], W^{k,2}(S^1, S^n))$  such that

$$u^j \rightarrow u \quad [\text{weakly}^*] \quad \text{in} \quad L^\infty([0, T], W^{k,2}(S^1, S^n)).$$

It is easy to verify that the limit map  $u$  we get above is indeed a strong solution of Cauchy problem (1.4).  $\square$

## 4. UNIQUENESS

This section is devoted to prove the following uniqueness theorem. This result relies heavily on the geometric structure of equation (1.3).

**Theorem 4.1.** *Suppose  $u_0 \in W^{k,2}(S^1, S^n)$  for  $k \geq 3$ , then the solution of Cauchy problem (1.4) is unique.*

*Proof.* Suppose  $u$  and  $v$  are two solutions of Cauchy problem (1.4) with same initial data  $u_0$ . Let  $w = u - v$ , then  $w$  satisfies the following equation:

$$\begin{aligned}
(4.1) \quad w_t &= w_{xxx} + 3[(u_x, u_{xx})u - (v_x, v_{xx})v] \\
&\quad + \frac{3}{2}[|u_x|^2 u_x - |v_x|^2 v_x] + \frac{3}{2}[(u, Au)u_x - (v, Av)v_x] \\
&=: w_{xxx} + 3I_1 + \frac{3}{2}I_2 + \frac{3}{2}I_3.
\end{aligned}$$

By inserting intermediate terms, we have

$$\begin{aligned}
I_1 &= (u_x, u_{xx})u - (u_x, v_{xx})u + (u_x, v_{xx})u - (v_x, v_{xx})u + (v_x, v_{xx})u - (v_x, v_{xx})v \\
&= (u_x, w_{xx})u + (w_x, v_{xx})u + (v_x, v_{xx})w.
\end{aligned}$$

Similarly, for the last two terms in (4.1), we have

$$I_2 = (w_x, u_x)u_x + (w_x, v_x)u_x + (v_x, v_x)w_x,$$

and

$$I_3 = (w, Au)u_x + (w, Av)u_x + (v, Av)w_x.$$

Now, we are going to compute  $\frac{d}{dt}\|w\|_{W^{1,2}}$ . First, it's easy to see

$$\begin{aligned}
(4.2) \quad \frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 &= \int_{S^1} (w_t, w) \\
&= \int_{S^1} (w_{xxx}, w) + 3 \int_{S^1} (u_x, w_{xx})(u, w) \\
&\quad + 3 \int_{S^1} (w_x, v_{xx})(u, w) + 3 \int_{S^1} (v_x, v_{xx})|w|^2 \\
&\quad + \frac{3}{2} \int_{S^1} (w_x, u_x)(u_x, w) + (w_x, v_x)(u_x, w) + (v_x, v_x)(w_x, w) \\
&\quad + \frac{3}{2} \int_{S^1} (w, Au)(u_x, w) + (w, Av)(u_x, w) + (v, Av)(w_x, w) \\
&\leq C(\|u\|_{W^{3,2}} + \|v\|_{W^{3,2}})\|w\|_{W^{1,2}}^2.
\end{aligned}$$

Next, we claim that

$$\begin{aligned}
(4.3) \quad & - \frac{1}{2} \frac{d}{dt} \|w_x\|_{L^2}^2 \\
&= \int_{S^1} (w_t, w_{xx}) \\
&= \int_{S^1} (w_{xxx}, w_{xx}) + 3 \int_{S^1} (u_x, w_{xx})(u, w_{xx}) \\
&\quad + 3 \int_{S^1} (w_x, v_{xx})(u, w_{xx}) + 3 \int_{S^1} (v_x, v_{xx})(w, w_{xx}) \\
&\quad + \frac{3}{2} \int_{S^1} (w_x, u_x)(u_x, w_{xx}) + (w_x, v_x)(u_x, w_{xx}) + (v_x, v_x)(w_x, w_{xx}) \\
&\quad + \frac{3}{2} \int_{S^1} (w, Au)(u_x, w_{xx}) + (w, Av)(u_x, w_{xx}) + (v, Av)(w_x, w_{xx}) \\
&\leq C(\|u\|_{W^{3,2}} + \|v\|_{W^{3,2}})\|w\|_{W^{1,2}}^2.
\end{aligned}$$

We shall examine this term by term carefully. First of all, it's obvious

$$\int_{S^1} (w_{xxx}, w_{xx}) = 0.$$

Also, it's easy to check

$$\begin{aligned} 3 \int_{S^1} (v_x, v_{xx})(w, w_{xx}) + \frac{3}{2} \int_{S^1} (w, Au)(u_x, w_{xx}) + (w, Av)(u_x, w_{xx}) \\ \leq C(\|u\|_{W^{3,2}} + \|v\|_{W^{3,2}})\|w\|_{W^{1,2}}^2. \end{aligned}$$

Furthermore, by integrating by parts, we have

$$\begin{aligned} & \frac{3}{2} \int_{S^1} (w_x, u_x)(u_x, w_{xx}) + (v_x, v_x)(w_x, w_{xx}) + (v, Av)(w_x, w_{xx}) \\ = & -\frac{3}{2} \int_{S^1} (w_x, u_x)(u_{xx}, w_x) + (v_x, v_{xx})(w_x, w_x) + (v, Av_x)(w_x, w_x) \\ \leq & C(\|u\|_{W^{3,2}} + \|v\|_{W^{3,2}})\|w_x\|_{L^2}^2. \end{aligned}$$

Similarly,

$$\begin{aligned} & \frac{3}{2} \int_{S^1} (w_x, v_x)(u_x, w_{xx}) \\ = & \frac{3}{2} \int_{S^1} (w_x, v_x)(w_x, w_{xx}) + (w_x, v_x)(v_x, w_{xx}) \\ \leq & \frac{3}{2} \|v_x\|_{L^\infty} \|w_{xx}\|_{L^\infty} \|w_x\|_{L^2}^2 - \frac{3}{2} \int_{S^1} (w_x, v_x)(v_{xx}, w_x) \\ \leq & C(\|u\|_{W^{3,2}} + \|v\|_{W^{3,2}})\|w_x\|_{L^2}^2. \end{aligned}$$

Thus there are only two terms left, i.e.

$$(4.4) \quad 3 \int_{S^1} (u_x, w_{xx})(u, w_{xx}) \quad \text{and} \quad 3 \int_{S^1} (w_x, v_{xx})(u, w_{xx}).$$

To treat them, we observe that  $|u|^2 = 1$  implies  $(u, u_x) = 0$ , hence  $(u, u_{xx}) + (u_x, u_x) = 0$ . Therefore,

$$\begin{aligned} (u, w_{xx}) &= (u, u_{xx} - v_{xx}) = -(u_x, u_x) - (u, v_{xx}) \\ &= -(u_x, u_x) + (u_x, v_x) - (u_x, v_x) + (v_x, v_x) + (v, v_{xx}) - (u, v_{xx}) \\ &= -(u_x, w_x) - (w_x, v_x) - (w, v_{xx}). \end{aligned}$$

Taking this into account, we can bound (4.4) in the same way as above. Namely, we have

$$3 \int_{S^1} (u_x, w_{xx})(u, w_{xx}) + 3 \int_{S^1} (w_x, v_{xx})(u, w_{xx}) \leq C(\|u\|_{W^{3,2}} + \|v\|_{W^{3,2}})\|w\|_{W^{1,2}}^2.$$

So we proved the claim and finally get from (4.2) and (4.3) that

$$\frac{d}{dt} \|w\|_{W^{1,2}}^2 \leq C(\|u\|_{W^{3,2}} + \|v\|_{W^{3,2}})\|w\|_{W^{1,2}}^2.$$

Thus Lemma 3.4 implies

$$\|w(t)\|_{W^{1,2}}^2 \leq C\|w(0)\|_{W^{1,2}}^2.$$

Since  $u$  and  $v$  share the same initial data, we know  $w(0) = 0$ . Hence we conclude that  $w(t) = 0$ , i.e. the solution is unique.  $\square$

## 5. SEMI-CONSERVATION LAWS

After getting a local solution  $u \in L^\infty([0, T], W^{k,2}(S^1, S^n))$  of Cauchy problem (1.4), what we need to do next is to derive an uniform estimate of  $\|u\|_{W^{3,2}}$  for all  $t \in [0, T]$ . Then Theorem 1.1 ensures that the local solution can be extended to a global solution. In geometric evolution problems, this is usually done by finding some energy conservation laws, see [4, 13, 17] for example. However, in the current situation we fail to find conservation quantities except the energy  $E_1(u)$ . Nevertheless, we do find semi-conservation laws for two higher order energies which is sufficient to prove the global existence.

**Remark 5.1.** *Our goal here is to bound the norm  $\|u_x\|_{H^{2,2}}$ , which is equivalent to the Sobolev norm  $\|u\|_{W^{3,2}}$ , for all existing time  $t \in [0, T]$ . However, there are some unexpected terms emerging, when we calculate  $\frac{d}{dt}\|u_x\|_{H^{2,2}}$  directly. These ‘bad’ terms can’t be controlled by the linear form of  $\|u_x\|_{H^{2,2}}$ , which is necessary when carrying out Gronwall’s inequality. Luckily, we find some other energies which satisfy semi-conservative laws. More importantly, these semi-conservative laws implies a global bound of  $\|u\|_{W^{3,2}}$ .*

Through out this section, we let  $u \in L^\infty([0, T], W^{k,2}(S^1, S^n))$  be a local solution of Cauchy problem (1.4) on the time interval  $[0, T]$ , and use  $C$  to denote constants which may depend on the initial data, the matrix  $A$  and the maximal time  $T$ . First we define a second order ‘energy’

$$E_2(u) = \int_{S^1} |\nabla_x u_x|^2 - \frac{1}{4} \int_{S^1} |u_x|^4 - \frac{9}{4} \int_{S^1} (u, Au)|u_x|^2 + \int_{S^1} (u_x, Au_x).$$

To derive the semi-conservation law, we take the time derivative

$$(5.1) \quad \frac{d}{dt} E_2(u) = \frac{d}{dt} \int_{S^1} |\nabla_x u_x|^2 - \frac{1}{4} \frac{d}{dt} \int_{S^1} |u_x|^4 - \frac{9}{4} \frac{d}{dt} \int_{S^1} (u, Au)|u_x|^2 + \frac{d}{dt} \int_{S^1} (u_x, Au_x)$$

and compute the four terms in (5.1) one by one.

Using the equation (1.3) and changing the order of derivatives, we have

$$\begin{aligned} & \frac{d}{dt} \int_{S^1} |\nabla_x u_x|^2 \\ &= 2 \int_{S^1} \langle \nabla_t \nabla_x u_x, \nabla_x u_x \rangle \\ &= 2 \int_{S^1} \langle \nabla_x \nabla_x u_t, \nabla_x u_x \rangle + 2 \int_{S^1} \langle R(u_x, u_t)u_x, \nabla_x u_x \rangle \\ &= 2 \int_{S^1} \langle u_t, \nabla_x^3 u_x \rangle + 2 \int_{S^1} \langle R(u_x, \nabla_x^2 u_x)u_x, \nabla_x u_x \rangle \\ & \quad + \int_{S^1} \langle R(u_x, u_x)u_x, \nabla_x u_x \rangle |u_x|^2 + 3 \int_{S^1} \langle R(u_x, u_x)u_x, \nabla_x u_x \rangle (u, Au) \\ &= 2 \int_{S^1} \langle u_t, \nabla_x^3 u_x \rangle + 2 \int_{S^1} \langle R(u_x, \nabla_x^2 u_x)u_x, \nabla_x u_x \rangle \\ &:= I_1 + I_2 \end{aligned}$$

Here we noticed that the curvature tensor  $R$  on  $S^n$  is constant, hence  $\nabla_x R \equiv 0$ . We first compute the term  $I_1$ .

$$\begin{aligned}
I_1 &= 2 \int_{S^1} \langle \nabla_x^2 u_x + \frac{1}{2} \langle u_x, u_x \rangle u_x + \frac{3}{2} (u, Au) u_x, \nabla_x^3 u_x \rangle \\
&= 2 \int_{S^1} \langle \nabla_x^2 u_x, \nabla_x^3 u_x \rangle + \int_{S^1} \langle u_x, u_x \rangle \langle u_x, \nabla_x^3 u_x \rangle + 3 \int_{S^1} (u, Au) \langle u_x, \nabla_x^3 u_x \rangle \\
&= -2 \int_{S^1} \langle \nabla_x u_x, u_x \rangle \langle u_x, \nabla_x^2 u_x \rangle - \int_{S^1} \langle u_x, u_x \rangle \langle \nabla_x u_x, \nabla_x^2 u_x \rangle \\
&\quad - 6 \int_{S^1} (u_x, Au) \langle u_x, \nabla_x^2 u_x \rangle - 3 \int_{S^1} (u, Au) \langle \nabla_x u_x, \nabla_x^2 u_x \rangle \\
&= 2 \int_{S^1} \langle \nabla_x u_x, u_x \rangle \langle \nabla_x u_x, \nabla_x u_x \rangle + \int_{S^1} \langle \nabla_x u_x, u_x \rangle \langle \nabla_x u_x, \nabla_x u_x \rangle \\
&\quad + 6 \int_{S^1} (u_{xx}, Au) \langle u_x, \nabla_x u_x \rangle + 6 \int_{S^1} (u_x, Au_x) \langle u_x, \nabla_x u_x \rangle \\
&\quad + 6 \int_{S^1} (u_x, Au) \langle \nabla_x u_x, \nabla_x u_x \rangle + 3 \int_{S^1} (u_x, Au) \langle \nabla_x u_x, \nabla_x u_x \rangle \\
&= 3 \int_{S^1} \langle \nabla_x u_x, u_x \rangle \langle \nabla_x u_x, \nabla_x u_x \rangle + 6 \int_{S^1} (u_{xx}, Au) \langle u_x, \nabla_x u_x \rangle \\
&\quad + 6 \int_{S^1} (u_x, Au_x) \langle u_x, \nabla_x u_x \rangle + 9 \int_{S^1} (u_x, Au) \langle \nabla_x u_x, \nabla_x u_x \rangle
\end{aligned}$$

For  $I_2$ , since  $\nabla_x R \equiv 0$ , we have

$$\begin{aligned}
I_2 &= 2 \int_{S^1} \langle R(u_x, \nabla_x^2 u_x) u_x, \nabla_x u_x \rangle \\
&= \int_{S^1} \langle R(u_x, \nabla_x^2 u_x) u_x, \nabla_x u_x \rangle - \int_{S^1} \langle R(\nabla_x u_x, \nabla_x u_x) u_x, \nabla_x u_x \rangle \\
&\quad - \int_{S^1} \langle (\nabla_x R)(u_x, \nabla_x u_x) u_x, \nabla_x u_x \rangle - \int_{S^1} \langle R(u_x, \nabla_x u_x) \nabla_x u_x, \nabla_x u_x \rangle \\
&\quad - \int_{S^1} \langle R(u_x, \nabla_x u_x) u_x, \nabla_x^2 u_x \rangle \\
&= 0.
\end{aligned}$$

Thus for the first term in (5.1), we get

$$\begin{aligned}
(5.2) \quad & \frac{d}{dt} \int_{S^1} |\nabla_x u_x|^2 \\
&= 3 \int_{S^1} \langle \nabla_x u_x, u_x \rangle \langle \nabla_x u_x, \nabla_x u_x \rangle + 6 \int_{S^1} (u_{xx}, Au) \langle u_x, \nabla_x u_x \rangle \\
&\quad + 6 \int_{S^1} (u_x, Au_x) \langle u_x, \nabla_x u_x \rangle + 9 \int_{S^1} (u_x, Au) \langle \nabla_x u_x, \nabla_x u_x \rangle.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \frac{d}{dt} \int_{S^1} |u_x|^4 \\
&= 4 \int_{S^1} \langle \nabla_t u_x, u_x \rangle \langle u_x, u_x \rangle \\
&= -4 \int_{S^1} \langle u_t, \nabla_x u_x \rangle \langle u_x, u_x \rangle - 8 \int_{S^1} \langle u_t, u_x \rangle \langle u_x, \nabla_x u_x \rangle \\
&= -4 \int_{S^1} \langle \nabla_x^2 u_x, \nabla_x u_x \rangle \langle u_x, u_x \rangle - 2 \int_{S^1} \langle u_x, u_x \rangle \langle u_x, u_x \rangle \langle u_x, \nabla_x u_x \rangle \\
(5.3) \quad & - 6 \int_{S^1} (u, Au) \langle u_x, \nabla_x u_x \rangle \langle u_x, u_x \rangle - 8 \int_{S^1} \langle \nabla_x^2 u_x, u_x \rangle \langle u_x, \nabla_x u_x \rangle \\
& - 4 \int_{S^1} \langle u_x, u_x \rangle \langle u_x, u_x \rangle \langle u_x, \nabla_x u_x \rangle - 12 \int_{S^1} (u, Au) \langle u_x, u_x \rangle \langle u_x, \nabla_x u_x \rangle \\
&= 4 \int_{S^1} \langle \nabla_x u_x, \nabla_x u_x \rangle \langle u_x, \nabla_x u_x \rangle - 6 \int_{S^1} (u, Au) \langle u_x, \nabla_x u_x \rangle \langle u_x, u_x \rangle \\
& + 8 \int_{S^1} \langle \nabla_x u_x, u_x \rangle \langle \nabla_x u_x, \nabla_x u_x \rangle - 12 \int_{S^1} (u, Au) \langle u_x, u_x \rangle \langle u_x, \nabla_x u_x \rangle \\
&= 12 \int_{S^1} \langle \nabla_x u_x, \nabla_x u_x \rangle \langle u_x, \nabla_x u_x \rangle + 9 \int_{S^1} (u_x, Au) \langle u_x, u_x \rangle \langle u_x, u_x \rangle.
\end{aligned}$$

Besides,

$$\begin{aligned}
& \frac{d}{dt} \int_{S^1} (u, Au) \langle u_x, u_x \rangle \\
&= 2 \int_{S^1} (u_t, Au) \langle u_x, u_x \rangle + 2 \int_{S^1} (u, Au) \langle \nabla_t u_x, u_x \rangle \\
&= 2 \int_{S^1} (u_t, Au) \langle u_x, u_x \rangle - 2 \int_{S^1} (u, Au) \langle u_t, \nabla_x u_x \rangle - 4 \int_{S^1} (u_x, Au) \langle u_t, u_x \rangle \\
&= 2 \int_{S^1} (\nabla_x^2 u_x, Au) \langle u_x, u_x \rangle + \int_{S^1} (u_x, Au) \langle u_x, u_x \rangle^2 \\
& + 3 \int_{S^1} (u, Au) (u_x, Au) \langle u_x, u_x \rangle - 2 \int_{S^1} (u, Au) \langle \nabla_x^2 u_x, \nabla_x u_x \rangle \\
(5.4) \quad & - \int_{S^1} (u, Au) \langle u_x, u_x \rangle \langle u_x, \nabla_x u_x \rangle - 3 \int_{S^1} (u, Au)^2 \langle u_x, \nabla_x u_x \rangle \\
& - 4 \int_{S^1} (u_x, Au) \langle \nabla_x^2 u_x, u_x \rangle - 2 \int_{S^1} (u_x, Au) \langle u_x, u_x \rangle^2 \\
& - 6 \int_{S^1} (u_x, Au) (u, Au) \langle u_x, u_x \rangle \\
&= 2 \int_{S^1} (\nabla_x^2 u_x, Au) \langle u_x, u_x \rangle - \frac{1}{2} \int_{S^1} (u_x, Au) \langle u_x, u_x \rangle^2 \\
& + 3 \int_{S^1} (u, Au) (u_x, Au) \langle u_x, u_x \rangle + 4 \int_{S^1} (u_x, Au) \langle \nabla_x u_x, \nabla_x u_x \rangle \\
& + 4 \int_{S^1} (u_{xx}, Au) \langle \nabla_x u_x, u_x \rangle + 4 \int_{S^1} (u_x, Au_x) \langle \nabla_x u_x, u_x \rangle
\end{aligned}$$

To proceed, we recall that

$$\nabla_x^2 u_x = u_{xxx} + 3(u_x, u_{xx})u + \langle u_x, u_x \rangle u_x.$$

So for the first term in the last equality of (5.4), we have

$$\begin{aligned}
& 2 \int_{S^1} (\nabla_x^2 u_x, Au) \langle u_x, u_x \rangle \\
&= 2 \int_{S^1} (u_{xxx}, Au) \langle u_x, u_x \rangle + 6 \int_{S^1} \langle u_x, u_{xx} \rangle (u, Au) \langle u_x, u_x \rangle \\
(5.5) \quad &+ 2 \int_{S^1} \langle u_x, u_x \rangle^2 (u_x, Ax) \\
&= 2 \int_{S^1} (u_x, Au_x) \langle \nabla_x u_x, u_x \rangle - 4 \int_{S^1} (u_{xxx}, Au) \langle \nabla_x u_x, u_x \rangle \\
&\quad - \int_{S^1} (u_x, Au) \langle u_x, u_x \rangle^2
\end{aligned}$$

(5.4) and (5.5) yields

$$\begin{aligned}
& \frac{d}{dt} \int_{S^1} (u, Au) \langle u_x, u_x \rangle \\
(5.6) \quad &= 4 \int_{S^1} (u_x, Au) \langle \nabla_x u_x, \nabla_x u_x \rangle + 6 \int_{S^1} (u_x, Au_x) \langle \nabla_x u_x, u_x \rangle \\
&\quad + 3 \int_{S^1} (u, Au) (u_x, Au) \langle u_x, u_x \rangle - \frac{3}{2} \int_{S^1} (u_x, Au) \langle u_x, u_x \rangle^2
\end{aligned}$$

For the last term in (5.1), we have

$$\begin{aligned}
(5.7) \quad & \frac{d}{dt} \int_{S^1} (u_x, Au_x) \\
&= 2 \int_{S^1} (u_{xt}, Au_x) = -2 \int_{S^1} (u_t, Au_{xx}) \\
&= -2 \int_{S^1} \langle \nabla_x^2 u_x + \frac{1}{2} \langle u_x, u_x \rangle u_x + \frac{3}{2} (u, Au) u_x, Au_{xx} \rangle \\
&= -2 \int_{S^1} (\nabla_x^2 u_x, Au_{xx}) - \int_{S^1} \langle u_x, u_x \rangle (u_x, Au_{xx}) - 3 \int_{S^1} (u, Au) (u_x, Au_{xx}) \\
&= -2 \int_{S^1} (u_{xxx}, Au_{xx}) - 6 \int_{S^1} (u, Au_{xx}) \langle u_x, \nabla_x u_x \rangle - 2 \int_{S^1} (u_x, Au_{xx}) \langle u_x, u_x \rangle \\
&\quad - \int_{S^1} \langle u_x, u_x \rangle (u_x, Au_{xx}) + 3 \int_{S^1} (u_x, Au) (u_x, Au_x) \\
&= -6 \int_{S^1} (u, Au_{xx}) \langle u_x, \nabla_x u_x \rangle + 3 \int_{S^1} (u_x, Au_x) \langle u_x, \nabla_x u_x \rangle \\
&\quad + 3 \int_{S^1} (u_x, Au) (u_x, Au_x)
\end{aligned}$$

Here we notice that  $\nabla_x u_x = u_{xx} + \langle u_x, u_x \rangle u$  and  $\nabla_x^2 u_x = u_{xxx} + 3 \langle u_x, u_{xx} \rangle u + \langle u_x, u_x \rangle u_x$ .

Combining (5.2),(5.3),(5.6) and(5.7), we finally get

$$\begin{aligned}
(5.8) \quad & \frac{d}{dt} E_2(u) = \frac{9}{4} \int_{S^1} (u_x, Au) |u_x|^4 + 3 \int_{S^1} (u_x, Au) (u_x, Au_x) \\
&\quad - \frac{9}{2} \int_{S^1} (u_x, Au_x) \langle \nabla_x u_x, u_x \rangle - \frac{27}{4} \int_{S^1} (u, Au) (u_x, Au) \langle u_x, u_x \rangle.
\end{aligned}$$

Now we can derive the desired Gronwall-type inequality. Since  $A$  is a constant matrix and  $|u| = 1$ , (5.8) yields

$$(5.9) \quad \begin{aligned} \frac{d}{dt} E_2(u) &\leq C \int_{S^1} |u_x|^5 + C \int_{S^1} |u_x|^3 + C \int_{S^1} |u_x|^3 |\nabla_x u_x| \\ &\leq C \int_{S^1} |u_x|^5 + C \int_{S^1} |u_x|^3 |\nabla_x u_x|. \end{aligned}$$

At this point, we may recall that Lemma 3.2 provides the desired bounds for both  $\|u_x\|_{L^2}$  and  $\|u_x\|_{L^\infty}$ . Therefore,

$$(5.10) \quad \begin{aligned} \int_{S^1} |u_x|^5 &\leq \|u_x\|_{L^\infty}^3 \int_{S^1} |u_x|^2 \\ &\leq C \|u_x\|_{H^{1,2}}^{3/2} \|u_x\|_{L^2}^2 \\ &\leq C \int_{S^1} |\nabla_x u_x|^2 + C. \end{aligned}$$

Furthermore,

$$(5.11) \quad \begin{aligned} \int_{S^1} |u_x|^3 |\nabla_x u_x| &\leq \|u_x\|_{L^\infty}^2 \int_{S^1} |u_x| |\nabla_x u_x| \\ &\leq C \|u_x\|_{H^{1,2}} \cdot \|u_x\|_{L^2} \|\nabla_x u_x\|_{L^2} \\ &\leq C \int_{S^1} |\nabla_x u_x|^2 + C. \end{aligned}$$

So we arrive at

$$(5.12) \quad \frac{d}{dt} E_2(u) \leq C \int_{S^1} |\nabla_x u_x|^2 + C.$$

Next, we claim that the integral  $\int_{S^1} |\nabla_x u_x|^2$  is controlled by  $E_2(u)$ . Indeed,

$$(5.13) \quad \begin{aligned} \int_{S^1} |\nabla_x u_x|^2 &= E_2(u) + \frac{1}{4} \int_{S^1} |u_x|^4 + \frac{9}{4} \int_{S^1} (u, Au) |u_x|^2 - \int_{S^1} (u_x, Au_x) \\ &\leq E_2(u) + C \int_{S^1} |u_x|^4 + C. \end{aligned}$$

By the same virtual of the estimate (5.10), we have

$$(5.14) \quad \begin{aligned} \int_{S^1} |u_x|^4 &\leq \|u_x\|_{L^\infty}^2 \int_{S^1} |u_x|^2 \\ &\leq C \|u_x\|_{H^{1,2}} \|u_x\|_{L^2}^2 \\ &\leq C(\epsilon \|u_x\|_{H^{1,2}}^2 + \frac{1}{\epsilon} \|u_x\|_{L^2}^4) \\ &\leq C\epsilon \int_{S^1} |\nabla_x u_x|^2 + C. \end{aligned}$$

Here we employed Young's inequality with  $\epsilon$ , i.e.

$$ab \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}, \text{ for } a, b > 0.$$

Thus if we choose  $\epsilon$  sufficiently small, we proved the claim from (5.13) and (5.14) that

$$(5.15) \quad \int_{S^1} |\nabla_x u_x|^2 \leq C E_2(u) + C.$$

Consequently, we conclude from (5.12) and (5.15) that

$$(5.16) \quad \frac{d}{dt} E_2(u) \leq C E_2(u) + C.$$



By Gronwall's inequality, we finally arrive at

**Lemma 5.2.** *Suppose  $u : S^1 \times [0, T] \rightarrow S^n$  is a solution to the cauchy problem (1.4), then for all  $t \in [0, T]$*

$$E_2(u(t)) \leq C(T), \int_{S^1} |\nabla_x u_x(t)|^2 dx \leq C(T),$$

where  $C(T)$  is a constant depending on  $T$  and the initial data  $u_0$ .

Our last ingredient in proving the global existence is the semi-conservation law for a third order energy, which is given by

$$(5.17) \quad E_3(u) = \int_{S^1} |\nabla_x^2 u_x|^2 - \int_{S^1} \langle u_x, \nabla_x u_x \rangle^2 - \frac{3}{2} \int_{S^1} |u_x|^2 |\nabla_x u_x|^2.$$

Similarly, we have the following lemma.

**Lemma 5.3.** *Suppose  $u : S^1 \times [0, T] \rightarrow S^n$  is a solution to the cauchy problem (1.4), then for all  $t \in [0, T]$*

$$E_3(u(t)) \leq C(T), \int_{S^1} |\nabla_x^2 u_x(t)|^2 dx \leq C(T),$$

where  $C(t)$  is a constant depending on  $T$  and the initial data  $u_0$ .

It takes a lot of efforts to find the energy functional  $E_3$  which satisfies the semi-conservation law and therefore bounded under the flow. The spirit is all the same as that of  $E_2$  demonstrated in the proof of Lemma 5.2. So we are going to omit the lengthy computation which mainly involves integration by parts and changing orders of derivatives, and only give the key steps instead.

The time-derivative of  $E_3(u)$  consists of four terms. A long computation yields

$$\begin{aligned} & \frac{d}{dt} \int_{S^1} |\nabla_x^2 u_x|^2 \\ &= 2 \int_{S^1} \langle \nabla_x^3 u_t, \nabla_x^2 u_x \rangle + 2 \int_{S^1} \langle \nabla_x (R(u_t, u_x) u_x), \nabla_x^2 u_x \rangle \\ & \quad + 2 \int_{S^1} \langle R(u_t, u_x) \nabla_x u_x, \nabla_x^2 u_x \rangle \\ (5.18) &= 6 \int_{S^1} \langle u_x, \nabla_x^2 u_x \rangle \langle \nabla_x u_x, \nabla_x^2 u_x \rangle + 9 \int_{S^1} \langle u_x, \nabla_x u_x \rangle |\nabla_x^2 u_x|^2 \\ & \quad + 6 \int_{S^1} (u_{xxx}, Au) \langle u_x, \nabla_x^2 u_x \rangle + 15 \int_{S^1} (u_x, Au) |\nabla_x^2 u_x|^2 \\ & \quad + 18 \int_{S^1} (u_{xx}, Au) \langle \nabla_x u_x, \nabla_x^2 u_x \rangle + 18 \int_{S^1} (u_x, Au_x) \langle \nabla_x u_x, \nabla_x^2 u_x \rangle \\ & \quad + 18 \int_{S^1} (u_{xx}, Au_x) \langle u_x, \nabla_x^2 u_x \rangle. \end{aligned}$$

For the second term, we have

$$\begin{aligned}
& \frac{d}{dt} \int_{S^1} \langle u_x, \nabla_x u_x \rangle^2 \\
&= 2 \int_{S^1} \langle \nabla_x u_t, \nabla_x u_x \rangle \langle u_x, \nabla_x u_x \rangle + 2 \int_{S^1} \langle u_x, \nabla_x \nabla_x u_x \rangle \langle u_x, \nabla_x u_x \rangle \\
&\quad + 2 \int_{S^1} \langle u_x, R(u_t, u_x) u_x \rangle \langle u_x, \nabla_x u_x \rangle \\
(5.19) \quad &= 6 \int_{S^1} \langle u_x, \nabla_x^2 u_x \rangle \langle \nabla_x u_x, \nabla_x^2 u_x \rangle + 3 \int_{S^1} \langle u_x, \nabla_x u_x \rangle^3 \\
&\quad + 15 \int_{S^1} (u_x, Au) \langle u_x, \nabla_x u_x \rangle^2 + 6 \int_{S^1} |u_x|^2 (u_{xx}, Au) \langle u_x, \nabla_x u_x \rangle \\
&\quad + 6 \int_{S^1} |u_x|^2 (u_x, Au_x) \langle u_x, \nabla_x u_x \rangle
\end{aligned}$$

Besides, for the third term,

$$\begin{aligned}
& \frac{d}{dt} \int_{S^1} |u_x|^2 |\nabla_x u_x|^2 \\
&= 2 \int_{S^1} \langle \nabla_x u_t, u_x \rangle |\nabla_x u_x|^2 + 2 \int_{S^1} |u_x|^2 \langle \nabla_x \nabla_x u_t, \nabla_x u_x \rangle \\
&\quad + 2 \int_{S^1} |u_x|^2 \langle R(u_t, u_x) u_x, \nabla_x u_x \rangle \\
(5.20) \quad &= 6 \int_{S^1} \langle u_x, \nabla_x u_x \rangle |\nabla_x^2 u_x|^2 + 15 \int_{S^1} (u_x, Au) |u_x|^2 |\nabla_x u_x|^2 \\
&\quad + 3 \int_{S^1} |u_x|^2 \langle u_x, \nabla_x u_x \rangle |\nabla_x u_x|^2 + 6 \int_{S^1} |u_x|^2 (u_{xx}, Au) \langle u_x, \nabla_x u_x \rangle \\
&\quad + 6 \int_{S^1} |u_x|^2 (u_x, Au_x) \langle u_x, \nabla_x u_x \rangle
\end{aligned}$$

Putting these three terms together, we get

$$\begin{aligned}
(5.21) \quad \frac{d}{dt} E_3(u) &= 6 \int_{S^1} (u_{xxx}, Au) \langle u_x, \nabla_x^2 u_x \rangle + 15 \int_{S^1} (u_x, Au) \langle \nabla_x^2 u_x, \nabla_x^2 u_x \rangle \\
&\quad - 3 \int_{S^1} \langle u_x, \nabla_x u_x \rangle^3 + \{\text{lower order terms}\}.
\end{aligned}$$

The key point in the above is that all the ‘bad’ terms, i.e. the higher order terms including

$$\int_{S^1} \langle u_x, \nabla_x^2 u_x \rangle \langle \nabla_x u_x, \nabla_x^2 u_x \rangle, \int_{S^1} \langle u_x, \nabla_x u_x \rangle |\nabla_x^2 u_x|^2$$

vanish in the summation. Because we have already shown that  $\|u_x\|_{H^{1,2}}$  and  $\|u_x\|_{L^\infty}$  are bounded by Lemma 3.2 and Lemma 5.2. The other terms left, though seems a lot, are all controllable. Here we only take the term  $\int_{S^1} \langle u_x, \nabla_x u_x \rangle^3$  for example to demonstrate this. Similar to the estimate of  $E_2$ , the interpolation inequality—namely, Corollary 2.2—plays an important role here.

$$\begin{aligned}
(5.22) \quad \int_{S^1} \langle u_x, \nabla_x u_x \rangle^3 &\leq C \|u_x\|_{L^\infty}^3 \|\nabla_x u_x\|_{L^\infty}^3 \\
&\leq C \|u_x\|_{H^{1,2}}^{\frac{3}{2}} \cdot \|\nabla_x u_x\|_{H^{1,2}}^{\frac{3}{2}} \|\nabla_x u_x\|_{L^2}^{\frac{3}{2}} \\
&\leq C \int_{S^1} |\nabla_x^2 u_x|^2 + C.
\end{aligned}$$

Finally, we can get

$$(5.23) \quad \frac{d}{dt} E_3(u) \leq C \int_{S^1} |\nabla_x^2 u_x|^2 + C \leq C E_3(u) + C.$$

Here, the constant  $C$  depends on  $T$  and  $\|u_0\|_{W^{2,2}}$ . Thus,  $E_3$  is bounded in  $[0, T)$  by Gronwall's inequality. Moreover, we also have

$$(5.24) \quad \int_{S^1} |\nabla_x^2 u_x|^2 \leq C(T).$$

## 6. GLOBAL EXISTENCE

In this section we finish the proof of Theorem 1.1.

Let  $u$  be the local smooth solution of (1.4) which exists on the maximal time interval  $[0, T)$ . If  $T = \infty$ , then Theorem 1.1 holds true. Thus we only need to consider the case where  $T < \infty$ .

From Lemma 5.2 and Lemma 5.3, we have

$$\sup_{t \in [0, T)} \|u_x\|_{H^{2,2}} \leq C,$$

where  $C$  depends on  $T$  and the  $W^{3,3}$ -norm of the initial data  $u_0$ . By Theorem 2.3, the  $W^{3,2}$ -Sobolev norm of  $u$  is bounded by

$$(6.1) \quad \sup_{t \in [0, T)} \|u\|_{W^{3,2}} \leq C \sup_{t \in [0, T)} \|u_x\|_{H^{2,2}}^3 \leq C(T, \|u_0\|_{W^{3,2}}).$$

Thus by Theorem 3.6, if  $T$  is finite, we can find a local solution  $u_1$  of (1.4) satisfying the initial value condition

$$u_1(x, 0) = u(x, T - \epsilon),$$

where  $0 < \epsilon < T$  is a small number. By uniqueness Theorem 4.1, we know that  $u$  and  $u_1$  coincides on the overlapped time interval. Then from Lemma 3.4, one can see that  $u_1$  exists on the time interval  $[0, T_1)$  with  $T_1 > 0$  only depending on the Sobolev norm  $\|u(x, T - \epsilon)\|_{W^{3,2}}$ . However, this norm is in turn decided by the initial data  $\|u_0\|_{W^{3,2}}$ . The uniform bound (6.1) implies  $T_1$  is independent of  $\epsilon$ . Thus, by choosing  $\epsilon$  sufficiently small, we can glue  $u$  and  $u_1$  together to obtain a solution of the Cauchy problem (1.4) on a larger time interval  $[0, T - \epsilon + T_1)$ , where  $T - \epsilon + T_1 > T$ . This contradicts to the maximality of  $T$ . Hence  $T = \infty$  and the proof is done.

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