# A SUFFICIENCY CLASS FOR GLOBAL (IN TIME) SOLUTIONS TO THE 3D NAVIERSTOKES EQUATIONS II

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ABSTRACT. In this paper, we simplify and extend the results of [GZ] to include the case in which $\Omega = \mathbb{R}^3$. Let $[L^2(\mathbb{R}^3)]^3$ be the Hilbert space of square integrable functions on $\mathbb{R}^3$ and let $\mathbb{H}[\mathbb{R}^3]^3 =: \mathbb{H}$ be the completion of the set, $\left\{\mathbf{u} \in (\mathbb{C}_0^\infty[\mathbb{R}^3])^3 \mid \nabla \cdot \mathbf{u} = 0\right\}$, with respect to the inner product of $[L^2(\mathbb{R}^3)]^3$. In this paper, we consider sufficiency conditions on a class of functions in $\mathbb{H}$ which allow global-in-time strong solutions to the three-dimensional Navier-Stokes equations on $\mathbb{R}^3$. These equations describe the time evolution of the fluid velocity and pressure of an incompressible viscous homogeneous Newtonian fluid in terms of a given initial velocity and given external body forces. Our approach uses the analytic nature of the Stokes semigroup to construct an equivalent norm for $\mathbb{H}$ which allows us to prove a reverse of the Poincaré inequality. This result allows us to provide strong bounds on the nonlinear term. We then prove that, under appropriate conditions, there exists a positive constant $u_+$, depending only on the domain, the viscosity and the body forces such that, for all functions in a dense set $\mathbb{D}$ contained in the closed ball $\mathbb{B}(\mathbb{R}^3) =: \mathbb{B}$ of radius $(1/2)u_+$ in $\mathbb{H}$, the Navier-Stokes equations have unique strong solutions in $\mathbb{C}^1\left((0,\infty),\mathbb{H}\right)$.

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## Introduction

Let $[L^2(\mathbb{R}^3)]^3$ be the Hilbert space of square integrable functions on $\mathbb{R}^3$ and let $\mathbb{H}_0[\mathbb{R}^3]$ be the completion of the set of functions in $\left\{\mathbf{u} \in \mathbb{C}_0^\infty[\mathbb{R}^3]^3 \mid \nabla \cdot \mathbf{u} = 0\right\}$ which vanish at infinity with respect to the inner product of $[L^2(\mathbb{R}^3)]^3$, and let $\mathbb{V}_0[\mathbb{R}^3]$ be the completion of the above functions which vanish at infinity with respect to the inner product of $\mathbb{H}_0^1[\mathbb{R}^3]$, the functions in $\mathbb{H}_0[\mathbb{R}^3]$ with weak derivatives in $[L^2(\mathbb{R}^3)]^3$. The global-in-time classical Navier-Stokes initial-value problem (on $\mathbb{R}^3$ and all $T > 0$) is to find functions $\mathbf{u} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ and $p : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ such that

$$
\begin{aligned}
&\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}(t) \text{ in } (0,T) \times \mathbb{R}^3, \\
&\nabla \cdot \mathbf{u} = 0 \text{ in } (0,T) \times \mathbb{R}^3 \text{ (in the weak sense)}, \\
&\lim_{\|\mathbf{x}\| \to \infty} \mathbf{u}(t,\mathbf{x}) = 0 \text{ on } (0,T) \times \mathbb{R}^3, \\
&\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \mathbb{R}^3.
\end{aligned}
\tag{1}
$$

The equations describe the time evolution of the fluid velocity $\mathbf{u}(\mathbf{x},t)$ and the pressure $p$ of an incompressible viscous homogeneous Newtonian fluid with constant viscosity coefficient $\nu$ in terms of a given initial velocity $\mathbf{u}_0(\mathbf{x})$ and given external body forces $\mathbf{f}(\mathbf{x},t)$. (Note that our third condition, $\lim_{\|\mathbf{x}\| \to \infty} \mathbf{u}(t,\mathbf{x}) = 0$ on $(0,T) \times \mathbb{R}^3$, is natural in this case since it is well-known that $\mathbb{H}_0^k[\mathbb{R}^3]^3 = \mathbb{H}^k[\mathbb{R}^3]^3$ (see Stein [S] or [SY].)

## Purpose

Let $\mathbb{P}$ be the (Leray) orthogonal projection of $(L^2[\mathbb{R}^3])^3$ onto $\mathbb{H}_0[\mathbb{R}^3]$ and define the Stokes operator by: $\mathbf{A}\mathbf{u} =: -\mathbb{P}\Delta\mathbf{u}$, for $\mathbf{u} \in D(\mathbf{A}) \subset \mathbb{H}_0^2[\mathbb{R}^3]$, the domain of $\mathbf{A}$. The purpose of this paper is to prove that there exists a number $u_+$, depending only on $\mathbf{A}$, $f$ and $\nu$ such that, for all functions in a certain subset (defined in the paper) of $\mathbb{D} = D(\mathbf{A}) \cap \mathbb{B}$, where $\mathbb{B}$ is the closed ball of radius $\frac{1}{2}u_+$ in $\mathbb{H}_0(\mathbb{R}^3)$, the Navier-Stokes equations have unique strong solutions in $\mathbf{u} \in L^\infty_{\text{loc}}[[0,\infty); \mathbb{V}_0(\mathbb{R}^3)] \cap \mathbb{C}^1[(0,\infty); \mathbb{H}_0(\mathbb{R}^3)]$.

## Preliminaries

Applying the Leray projection to equation (1), with $C(\mathbf{u},\mathbf{u}) = \mathbb{P}(\mathbf{u}\cdot\nabla)\mathbf{u}$, we can recast equation (1) in the standard form:

$$
\begin{aligned}
&\partial_t \mathbf{u} = -\nu \mathbf{A}\mathbf{u} - C(\mathbf{u},\mathbf{u}) + \mathbb{P}\mathbf{f}(t) \text{ in } (0,T)\times\mathbb{R}^3,\\
&\lim_{\|\mathbf{x}\|\to\infty} \mathbf{u}(t,\mathbf{x}) = 0 \text{ on } (0,T)\times\mathbb{R}^3,\\
&\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \mathbb{R}^3,
\end{aligned}
\tag{2}
$$

where we have used the fact that the orthogonal complement of $\mathbb{H}_0$ relative to $\{L^2(\mathbb{R}^3)\}^3$ is $\{\mathbf{v} : \mathbf{v} = \nabla q, \ q \in (H^1)^3\}$ to eliminate the pressure term (see Galdi [GA] or [SY, T1,T2]).

**Definition 1.** *We say that the operator $\mathcal{A}(\cdot,t)$ is (for each $t$)*

(1) *0-Dissipative if* $\langle \mathcal{A}(\mathbf{u},t), \mathbf{u}\rangle_{\mathbb{H}} \le 0$.

(2) *Dissipative if* $\langle \mathcal{A}(\mathbf{u},t) - \mathcal{A}(\mathbf{v},t), \mathbf{u}-\mathbf{v}\rangle_{\mathbb{H}} \le 0$.

(3) *Strongly dissipative if there exists an $\delta > 0$ such that*

$$\langle \mathcal{A}(\mathbf{u},t) - \mathcal{A}(\mathbf{v},t), \mathbf{u}-\mathbf{v}\rangle_{\mathbb{H}} \le -\delta \|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{H}}.$$

Note that, if $\mathcal{A}(\cdot,t)$ is a linear operator, definitions (1) and (2) coincide. Theorem 2 below is essentially due to Browder [B], see Zeidler [Z, Corollary 32.27, page 868 and Corollary 32.35 page 887, in Vol. IIB], while Theorem 3 is from Miyadera [M,

p. 185, Theorem 6.20], and is a modification of the Crandall-Liggett Theorem [CL] (see the appendix to the first section of [CL]) .

**Theorem 2.** *Let* $\mathbb{B}$ *be a closed, bounded, convex subset of* $\mathbb{H}$*. If* $\mathcal{A}(\cdot,t):\mathbb{B}\to\mathbb{H}$ *is a strongly dissipative mapping for each fixed* $t\geq 0$*, then for each* $\mathbf{b}\in\mathbb{B}$*, there is a* $\mathbf{u}\in\mathbb{B}$ *with* $\mathcal{A}(\mathbf{u},t)=\mathbf{b}$ *(i.e., the range, Ran[*$\mathcal{A}(\cdot,t)$*]* $\supset\mathbb{B}$*).*

**Theorem 3.** *Let* $\mathcal{A}(\cdot,t), t\in I=[0,\infty)\}$ *be a family of operators defined on* $\mathbb{H}$ *with domains* $D(\mathcal{A}(\cdot,t))=D$*, independent of* $t$*. We assume that* $\mathbb{D}=D(A)\cap\mathbb{B}$ *is a closed convex set (in an appropriate topology):*

(1) *The operator* $\mathcal{A}(\cdot,t)$ *is the generator of a contraction semigroup for each* $t\in I$*.*
(2) *The function* $\mathcal{A}(\mathbf{u},t)$ *is continuous in both variables on* $\mathbb{D}\times I$*.*

*Then, for every* $\mathbf{u}_0\in\mathbb{D}$*, the problem* $\partial_t\mathbf{u}(t,\mathbf{x})=\mathcal{A}(\mathbf{u}(t,\mathbf{x}),t)$*,* $\mathbf{u}(0,\mathbf{x})=\mathbf{u}_0(\mathbf{x})$*, has a unique solution* $\mathbf{u}(t,\mathbf{x})\in\mathbb{C}^1(I;\mathbb{D})$*.*

**Stokes Equation.** The difficulty in proving the existence of global-in-time strong solutions for equation (2) is directly linked to the problem of getting good estimates for the nonlinear term $C(\mathbf{u},\mathbf{u})$. In [GZ], we obtained an extension of the important result due to Constantin and Foias [CF]). This result, see below, is one of the major estimates used to study this equation. In what follows, we assume that $\mathbf{u}$, $\mathbf{v}\in D(\mathbf{A})$.

**Theorem 4.** *Let* $0\leq\alpha_i,\ 1\leq i\leq 3$*, satisfy* $\alpha_1+\alpha_2+\alpha_3=3/2$ *and*

$$(\alpha_1,\alpha_2,\alpha_3)\notin\{(3/2,0,0),(0,3/2,0),(0,0,3/2)\}\,.$$

*Then there is a positive constant $c = c(\alpha_i, \mathbb{R}^3)$ such that*

$$|\langle C(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{\mathbb{H}}| \le c \left\| \mathbf{A}^{\alpha_1/2}\mathbf{u} \right\|_{\mathbb{H}} \left\| \mathbf{A}^{(1+\alpha_2)/2}\mathbf{v} \right\|_{\mathbb{H}} \left\| \mathbf{A}^{\alpha_3/2}\mathbf{w} \right\|_{\mathbb{H}}. \tag{3}$$

In [GZ] we showed that, by renorming $\mathbb{H}$, we could prove a very strong inequality for equation (3). Since this result is not well-known and important for this paper, we give a proof. First we need to review the Stokes equation.

If we drop the nonlinear term, we get the well-known Stokes equation ($\mathbb{P}\mathbf{f}(t) = \mathbf{0}$):

$$\partial_t \mathbf{u} = -\nu \mathbf{A}\mathbf{u} \text{ in } (0,T) \times \mathbb{R}^3,$$

$$\lim_{\|\mathbf{x}\|\to\infty} \mathbf{u}(t,\mathbf{x}) = 0 \text{ on } (0,T) \times \mathbb{R}^3,$$

$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \mathbb{R}^3.$$

A proof of the next theorem may be found in Sell and You [SY] (page 114):

**Theorem 5.** *Let $\mathbf{A}$ be the Stokes operator on $\mathbb{R}^3$. Then the following holds:*

(1) *The operator $\mathbf{A}$ is a positive selfadjoint generator of a contraction semigroup $T(t)$.*
(2) *The operator $\mathbf{A}$ is sectorial and $T(t)$ is analytic.*

**Equivalent Norms.** Recall that an equivalent norm, $\|\cdot\|_{\mathcal{H},1}$, on a Hilbert space $\mathcal{H}$, with norm $\|\cdot\|_{\mathcal{H}}$, is one that satisfies: (for positive constants $M,\ M_1$)

$$\|u\|_{\mathcal{H}} \leqslant M \|u\|_{\mathcal{H},1} \leqslant M_1 \|u\|_{\mathcal{H}},\ u \in \mathcal{H}.$$

It is easy to show that any equivalent norm on $\mathcal{H}$ can be identified with a transformation of $\mathcal{H}$ which preserves the topology. In order to see how an equivalent norm can help us, let $T(t) = exp\{-t\mathbf{A}\}$ be the analytic contraction semigroup generated by the Stokes operator $\mathbf{A}$, with $\|T(t)\mathbf{u}\|_{\mathbb{H}} \leqslant e^{-\omega t}\|\mathbf{u}\|_{\mathbb{H}}$. Let $S(t) = e^{\omega t}T(t)$ and choose $M$ so that $\|\mathbf{u}\|_{\mathbb{H},1} = \|S(r)\mathbf{u}\|_{\mathbb{H}}$ is an equivalent norm, where $r$ is a *fixed value*, to be determined. Since $\mathbf{A}$ is analytic, there is a constant $c_z$ such that, for $\mathbf{u} \in D(\mathbf{A}^z)$,

$$\|\mathbf{A}^z\mathbf{u}\|_{\mathbb{H},1} = e^{\omega r} \|\mathbf{A}^z T(r)\mathbf{u}\|_{\mathbb{H}} \leqslant e^{\omega r} e^{-\omega r} \frac{c_z}{(r)^z} \|\mathbf{u}\|_{\mathbb{H}} \leqslant \frac{M c_z}{(r)^z} \|\mathbf{u}\|_{\mathbb{H},1}.$$

Since the norms are equivalent, we also have that

$$\|\mathbf{A}^z\mathbf{u}\|_{\mathbb{H}} \leq M \|\mathbf{A}^z\mathbf{u}\|_{\mathbb{H},1} \leqslant \frac{M^2 c_z}{(r)^z} \|\mathbf{u}\|_{\mathbb{H},1}. \tag{4}$$

From Theorem 4, we have the following result:

**Theorem 6.** *Let* $\mathbf{u} \in D(\mathbf{A})$*, set* $\mathbf{S} = S(r)$ *and renorm* $\mathbb{H}$ *so that* $\|\mathbf{u}\|_{\mathbb{H},1} = \|\mathbf{S}\mathbf{u}\|_{\mathbb{H}}$*. We define* $\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w})_{\mathbb{H},1} = \langle \mathbf{S}C(\mathbf{u},\mathbf{v}), \mathbf{S}\mathbf{w}\rangle_{\mathbb{H}}$*. Then:*

(1) *If we let* $\alpha_1 = 0$*,* $\alpha_2 = 1$ *and* $\alpha_3 = 1/2$*, there are positive constants* $c = c(\alpha_i, \mathbb{R}^3)$*,* $c_1$ *and* $c_2$ *such that*

$$\begin{aligned} \left|\langle C(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{\mathbb{H},1}\right| &\leq \frac{M^4 c c_1 c_2}{r^{5/4}} \|\mathbf{u}\|_{\mathbb{H},1} \|\mathbf{w}\|_{\mathbb{H},1} \|\mathbf{v}\|_{\mathbb{H},1} \text{ and} \\ \left|\langle C(\mathbf{v},\mathbf{u}),\mathbf{w}\rangle_{\mathbb{H},1}\right| &\leq \frac{M^4 c c_1 c_2}{r^{5/4}} \|\mathbf{u}\|_{\mathbb{H},1} \|\mathbf{w}\|_{\mathbb{H},1} \|\mathbf{v}\|_{\mathbb{H},1}. \end{aligned} \tag{5}$$

(2)

$$max\{\|C(\mathbf{u},\mathbf{v})\|_{\mathbb{H},1},\ \|C(\mathbf{v},\mathbf{u})\|_{\mathbb{H},1}\} \leqslant \frac{M^4 c c_1 c_2}{r^{5/4}} \|\mathbf{u}\|_{\mathbb{H},1} \|\mathbf{v}\|_{\mathbb{H},1}. \tag{6}$$

*Proof.* We prove the first equation of (5), the proof of the second is similar. Set $S(r) = \mathbf{S}$ and $\mathbf{S}^2\mathbf{w} = \mathbf{w}_1$, then we have:

$$\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w})_{\mathbb{H},1} = \langle \mathbf{S}C(\mathbf{u},\mathbf{v}), \mathbf{S}\mathbf{w}\rangle_{\mathbb{H}} = \mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w}_1)_{\mathbb{H}}.$$

Using the selfadjoint property of $\mathbf{A}$, and integration by parts, we have

$$\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w}_1)_{\mathbb{H}} = -\mathbf{b}(\mathbf{u},\mathbf{w_1},\mathbf{v})_{\mathbb{H}}.$$

It follows that:

$$\left|\langle C(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{\mathbb{H},1}\right| \le c\left\|\mathbf{A}^{\alpha_1/2}\mathbf{u}\right\|_{\mathbb{H}}\left\|\mathbf{A}^{(1+\alpha_2)/2}\mathbf{w}_1\right\|_{\mathbb{H}}\left\|\mathbf{A}^{\alpha_3/2}\mathbf{v}\right\|_{\mathbb{H}}.$$

Setting $\alpha_1 = 0$, $\alpha_2 = 1$ $\alpha_3 = 1/2$ and, using equation (4), we have:

$$\begin{aligned}
&\left|\langle C(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle_{\mathbb{H},1}\right| \le c\left\|\mathbf{u}\right\|_{\mathbb{H}}\left\|\mathbf{A}\mathbf{w}_1\right\|_{\mathbb{H}}\left\|\mathbf{A}^{1/4}\mathbf{v}\right\|_{\mathbb{H}}\\
&\le \frac{Mcc_1c_2}{r^{5/4}}\left\|\mathbf{u}\right\|_{\mathbb{H}}\left\|\mathbf{w}\right\|_{\mathbb{H}}\left\|\mathbf{v}\right\|_{\mathbb{H}}\\
&\le \frac{M^4cc_1c_2}{r^{5/4}}\left\|\mathbf{u}\right\|_{\mathbb{H},1}\left\|\mathbf{w}\right\|_{\mathbb{H},1}\left\|\mathbf{v}\right\|_{\mathbb{H},1}.
\end{aligned}$$

The proof of (6) is clear. □

The following extension of the Poincaré inequality will also prove useful.

**Lemma 7.** *Let* $\mathbf{A}^{1/2}$ *generate an analytic contraction semigroup* $S(t)$*. If* $r > 0$ *is any fixed number, then there exists an* $\alpha = \alpha(r) > 0$ *such that for any* $\mathbf{u} \in D(\mathbf{A}^{1/2})$*,*

$$\alpha^{1/2}\left\|\mathbf{u}\right\|_{H,1} \leqslant r^{1/2}\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{H,1}.$$

*Proof.* First observe that

$$\int_0^{r^{1/2}}\mathbf{A}^{1/2}S(t)\mathbf{u}dt = \int_0^{r^{1/2}}\frac{d}{dt}S(t)\mathbf{u}dt = S(r^{1/2})\mathbf{u}-\mathbf{u}.$$

Now choose $\alpha^{1/2}$ so that $\left\|S(r^{1/2})\mathbf{u}-\mathbf{u}\right\|_{H,1} \ge \alpha^{1/2}\left\|\mathbf{u}\right\|_{H,1}$. It follows that

$$\alpha^{1/2}\left\|\mathbf{u}\right\|_{H,1} \leqslant \int_0^{r^{1/2}}\left\|S(t)\mathbf{A}^{1/2}\mathbf{u}\right\|_{H,1}dt \leqslant r^{1/2}\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|_{H,1}.$$

□

## M-Dissipative Conditions

Let us assume that $\mathbf{f}(t) \in L^{\infty}[[0,\infty);\mathbb{H}]$ and is Hölder continuous in $t$, with $\left\|\mathbf{f}(t)-\mathbf{f}(\tau)\right\|_{\mathbb{H},1} \leq d\left|t-\tau\right|^{\theta},\ d>0,\ 0<\theta<1$. We can now rewrite equation (2) in the form:

$$\begin{gathered} \partial_t \mathbf{u} = \mathcal{A}(\mathbf{u},t) \text{ in } (0,T)\times\mathbb{R}^3, \\ \mathcal{A}(\mathbf{u},t) = -\nu\mathbf{A}\mathbf{u} - C(\mathbf{u},\mathbf{u}) + \mathbb{P}\mathbf{f}(t). \end{gathered} \tag{7}$$

We begin with a study of the operator $\mathcal{A}(\cdot,t)$, for fixed $t$, and seek conditions depending on $\mathbf{A}$, $\nu$, and $\mathbf{f}(t)$ which guarantee that $\mathcal{A}(\cdot,t)$ is m-dissipative for each $t$. Clearly $\mathcal{A}(\cdot,t)$ is defined on $D(\mathbf{A})$ and, since $\nu\mathbf{A}$ is a closed positive (m-accretive) operator, $-\nu(\mathbf{A})$ generates a linear contraction semigroup. Thus, we need to ensure that $\nu\mathcal{A}(\cdot,t)$ will be m-dissipative for each $t$. The following Lemma follows from the properties of $\mathbf{f}(t)$.

**Lemma 8.** *For $t \in I = [0,\infty)$ and, for each fixed $\mathbf{u} \in D(\mathbf{A})$, $\mathcal{A}(\mathbf{u},t)$ is Hölder continuous, with $\left\|\mathcal{A}(\mathbf{u},t)-\mathcal{A}(\mathbf{u},\tau)\right\|_{\mathbb{H},1} \leq d\left|t-\tau\right|^{\theta}$, where $d$ is the Hölder constant for the function $\mathbf{f}(t)$.*

## Main Results

**Theorem 9.** *Let $f = \sup_{t\in\mathbf{R}^+}\left\|\mathbb{P}\mathbf{f}(t)\right\|_{\mathbb{H},1} < \infty$, then there exists a positive constants $u_+$, $u_-$, depending only on $f$, $\mathbf{A}$ and $\nu$ such that, for all $\mathbf{u}$ with $0 \leq u_- \leq \left\|\mathbf{u}\right\|_{\mathbb{H},1} \leq u_+$, $\mathcal{A}(\cdot,t)$ is strongly dissipative.*

*Proof.* The proof of our first assertion has two parts. First, we require that the nonlinear operator $\mathcal{A}(\cdot,t)$ be 0-dissipative, which gives us an upper bound $u_+$ and lower bound $u_-$ in terms of the norm (i.e., $\left\|\mathbf{u}\right\|_{\mathbb{H},1} \leqslant u_+$ ). We then use this part to show that $\mathcal{A}(\cdot,t)$ is strongly dissipative on any closed convex ball, $\mathbb{D}$ inside the annulus defined by $\left\{\mathbf{u}\in D(\mathbf{A}): 0\le u_-\le \|\mathbf{u}\|_{\mathbb{H},1}\leqslant \frac{1}{2}u_+\right\}$.

Part 1) From equation (7), we consider the expression

$$\langle \mathcal{A}(\mathbf{u},t),\mathbf{u}\rangle_{\mathbb{H},1} = -\nu\langle \mathbf{A}\mathbf{u},\mathbf{u}\rangle_{\mathbb{H},1} + \langle[-C(\mathbf{u},\mathbf{u})+\mathbb{P}\mathbf{f}],\mathbf{u}\rangle_{\mathbb{H},1}$$

$$= -\nu\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|^2_{\mathbb{H},1} - \langle C(\mathbf{u},\mathbf{u}),\mathbf{u}\rangle_{\mathbb{H},1} + \langle \mathbb{P}\mathbf{f},\mathbf{u}\rangle_{\mathbb{H},1}.$$

We now use the fact that

$$\tfrac{\alpha}{r}\|\mathbf{u}\|^2_{\mathbb{H},1}\leqslant \left\|\mathbf{A}^{1/2}\mathbf{u}\right\|^2_{\mathbb{H},1} \Rightarrow -\nu\tfrac{\alpha}{r}\|\mathbf{u}\|^2_{\mathbb{H},1}\geqslant -\nu\left\|\mathbf{A}^{1/2}\mathbf{u}\right\|^2_{\mathbb{H},1}$$

to get that

$$\langle \mathcal{A}(\mathbf{u},t),\mathbf{u}\rangle_{\mathbb{H},1}\le -\nu\tfrac{\alpha}{r}\|\mathbf{u}\|^2_{\mathbb{H},1} + \frac{M^4cc_1c_2}{r^{5/4}}\|\mathbf{u}\|^3_{\mathbb{H},1} + f\|\mathbf{u}\|_{\mathbb{H},1}. \tag{8}$$

In the last line, we used our estimate from Theorem 6.

Since $\|\mathbf{u}\|_{\mathbb{H},1}>0$, we have that $\mathcal{A}(\cdot,t)$ is 0-dissipative if

$$-\frac{\nu\alpha}{r}\|\mathbf{u}\|_{H,1} + \frac{M^4cc_1c_2}{r^{5/4}}\|\mathbf{u}\|^2_{H,1} + f\leqslant 0$$

Solving the equality, we get that

$$u_\pm = \tfrac{\nu\alpha r^{1/4}}{2M^4cc_1c_2}\left\{1\pm\sqrt{1-(4fM^4cc_1c_2)\big/(\nu^2r^{1/2}\alpha^2)}\right\} = \tfrac{\nu\alpha r^{1/4}}{2M^4cc_1c_2}\left\{1\pm\sqrt{1-\gamma}\right\},$$

where $\gamma=(4r^{3/4}fM^4cc_1c_2)\big/(\nu^2\alpha^2)$. Since we want real distinct solutions, we must require that

$$\gamma = \frac{4r^{3/4}fM^4cc_1c_2}{\nu^2\alpha^2}<1 \quad\Rightarrow \frac{2M^2r^{3/8}}{\alpha}\left[fcc_1c_2\right]^{1/2}<\nu.$$

It follows that, if $\mathbb{P}\mathbf{f}\neq\mathbf{0}$, then $u_-<u_+$, and our requirement that $\mathcal{A}(\mathbf{u},t)$ is 0-dissipative implies that, since our solution factors as $(\|\mathbf{u}\|_{\mathbb{H},1}-u_+)(\|\mathbf{u}\|_{\mathbb{H},1}-u_-)\le 0$,

we must have that:

$$\|\mathbf{u}\|_{\mathbb{H},1} - u_+ \leq 0, \ \ \|\mathbf{u}\|_{\mathbb{H},1} - u_- \geq 0.$$

It follows that, for $u_- \leq \|\mathbf{u}\|_{\mathbb{H},1} \leq u_+$, $\langle \mathcal{A}(\mathbf{u},t), \mathbf{u}\rangle_{\mathbb{H},1} \leq 0$. (It is clear that, when $\mathbb{P}\mathbf{f}(t) = \mathbf{0}$, $u_- = 0$, and $u_+ = \frac{\nu\alpha r^{1/4}}{M^4 c c_1 c_2}$.)

Part 2): Now, for any $\mathbf{u}, \mathbf{v} \in D(\mathbf{A})$ with $\mathbf{u} - \mathbf{v} \in D(\mathbf{A})$ and

$$u_- \leq \min( \ \|\mathbf{u}\|_{\mathbb{H},1}, \|\mathbf{v}\|_{\mathbb{H},1}) \leq \max( \ \|\mathbf{u}\|_{\mathbb{H},1}, \|\mathbf{v}\|_{\mathbb{H},1}) \leq (1/2)u_+,$$

we have that

$$\begin{aligned}
&\langle \mathcal{A}(\mathbf{u},t) - \mathcal{A}(\mathbf{v},t), (\mathbf{u}-\mathbf{v})\rangle_{\mathbb{H},1} = -\nu \left\| \mathbf{A}^{1/2}(\mathbf{u}-\mathbf{v}) \right\|_{\mathbb{H},1}^2 \\
&\quad - \langle [C(\mathbf{u}, \mathbf{u}-\mathbf{v}) + C(\mathbf{v}, \mathbf{u}-\mathbf{v})], (\mathbf{u}-\mathbf{v})\rangle_{\mathbb{H},1} \\
&\quad \leqslant -\tfrac{\nu\alpha}{r} \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H},1}^2 + [1/(r^{5/4})] M^4 c c_1 c_2 \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H},1}^2 \left( \|\mathbf{u}\|_{\mathbb{H},1} + \|\mathbf{v}\|_{\mathbb{H},1} \right) \\
&\quad \leq -\tfrac{\nu\alpha}{r} \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H},1}^2 + [1/(r^{5/4})] M^4 c c_1 c_2 \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H},1}^2 u_+ \\
&\quad = -\tfrac{\nu\alpha}{r} \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H},1}^2 + [1/(r^{5/4})] M^4 c c_1 c_2 \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H},1}^2 \left( \tfrac{\nu\alpha r^{1/4}}{2M^4 c c_1 c_2} \left\{ 1 + \sqrt{1-\gamma} \right\} \right) \\
&\quad = -\tfrac{\nu\alpha}{2r} \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H},1}^2 \left\{ 1 - \sqrt{1-\gamma} \right\} \\
&\quad = -\delta \|\mathbf{u}-\mathbf{v}\|_{\mathbb{H},1}^2, \ \delta = \tfrac{\nu\alpha}{2r} \left\{ 1 - \sqrt{1-\gamma} \right\}.
\end{aligned}$$

□

Let $\mathbb{D}$ be any closed convex set (in the graph norm of $\mathbf{A}$) inside the annulus bounded by $\frac{1}{2}u_+$ and $u_-$.

**Theorem 10.** *The operator $\mathcal{A}(\cdot, t)$ is closed, strongly dissipative and jointly continuous in $\mathbf{u}$ and $t$. Furthermore, for each $t \in \mathbf{R}^+$ and $\beta > 0$, $Ran[I - \beta\mathcal{A}(t)] \supset \mathbb{D}$, so that $\mathcal{A}(t)$ is m-dissipative on $\mathbb{D}$.*

*Proof.* It is easy to see that $\mathcal{A}(\cdot,t)$ is closed. Since $\mathcal{A}(\cdot,t)$ is strongly dissipative, it is maximal dissipative, so that $Ran[I-\beta\mathcal{A}(\cdot,t)] \supset \mathbb{D}$. It follows that $\mathcal{A}(\cdot,t)$ is m-dissipative on $\mathbb{D}$ for each $t \in \mathbf{R}^+$ (since $\mathbb{H}$ is a Hilbert space). To see that $\mathcal{A}(\mathbf{u},t)$ is continuous in both variables, let $\mathbf{u}_n, \mathbf{u} \in \mathbb{B}$, $\|(\mathbf{u}_n-\mathbf{u})\|_{\mathbb{H},1} \to 0$, with $t_n, t \in I$ and $t_n \to t$. Then, if $\|\mathbf{A}\mathbf{u}\|_{\mathbb{H},1} \leqslant \frac{Mc_3}{r}\|\mathbf{u}\|_{\mathbb{H},1}$, we have

$$\|\mathcal{A}(\mathbf{u}_n,t_n)-\mathcal{A}(\mathbf{u},t)\|_{\mathbb{H},1} \leqslant \|\mathcal{A}(\mathbf{u},t_n)-\mathcal{A}(\mathbf{u},t)\|_{\mathbb{H},1} + \|\mathcal{A}(\mathbf{u}_n,t_n)-\mathcal{A}(\mathbf{u},t_n)\|_{\mathbb{H},1}$$

$$= \|[\mathbb{P}\mathbf{f}(t_n)-\mathbb{P}\mathbf{f}(t)]\|_{\mathbb{H},1} + \|\nu\mathbf{A}(\mathbf{u}_n-\mathbf{u}) + [C(\mathbf{u}_n-\mathbf{u},\mathbf{u}_n)+C(\mathbf{u},\mathbf{u}_n-\mathbf{u})]\|_{\mathbb{H},1}$$

$$\leqslant d\,|t_n-t|^\theta + \nu\,\|\mathbf{A}(\mathbf{u}_n-\mathbf{u})\|_{\mathbb{H},1} + \|C(\mathbf{u}_n-\mathbf{u},\mathbf{u}_n)+C(\mathbf{u},\mathbf{u}_n-\mathbf{u})\|_{\mathbb{H},1}$$

$$\leqslant d\,|t_n-t|^\theta + \nu\tfrac{Mc_3}{r}\,\|(\mathbf{u}_n-\mathbf{u})\|_{\mathbb{H},1} + \tfrac{M^4cc_1c_2}{r^{5/4}}\,\|(\mathbf{u}_n-\mathbf{u})\|_{\mathbb{H},1}\left\{\|\mathbf{u}_n\|_{\mathbb{H},1}+\|\mathbf{u}\|_{\mathbb{H},1}\right\}$$

$$\leqslant d\,|t_n-t|^\theta + \nu\tfrac{Mc_3}{r}\,\|(\mathbf{u}_n-\mathbf{u})\|_{\mathbb{H},1} + \tfrac{M^4cc_1c_2}{r^{5/4}}\,\|(\mathbf{u}_n-\mathbf{u})\|_{\mathbb{H},1}\,u_+.$$

It follows that $\mathcal{A}(\mathbf{u},t)$ is continuous in both variables. □

When $\mathbf{f}=\mathbf{0}$, $\mathbb{D}$ is the graph closure of $D(\mathbf{A})\cap\mathbb{B}_+$ in the $\mathbb{H}$ norm, where $\mathbb{B}_+$ is the ball of radius $\frac{1}{2}u_+$. In this case, it follows that $\mathbb{D}$ is a closed, bounded, convex set. We now have:

**Theorem 11.** *For each $T \in \mathbf{R}^+$, $t \in (0,T)$ and $\mathbf{u}_0 \in \mathbb{D}$, the global-in-time Navier-Stokes initial-value problem in $\mathbb{R}^3$ :*

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{0} \text{ in } (0,T)\times\mathbb{R}^3,$$

$$\nabla\cdot\mathbf{u} = 0 \text{ in } (0,T)\times\mathbb{R}^3,$$

$$\lim_{\|\mathbf{x}\|\to\infty}\mathbf{u}(t,\mathbf{x}) = \mathbf{0} \text{ on } (0,T)\times\mathbb{R}^3, \tag{9}$$

$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \text{ in } \mathbb{R}^3,$$

*has a unique strong solution $\mathbf{u}(t,\mathbf{x})$, which is in $L^2_{loc}[[0,\infty);\mathbb{H}]$ and in $L^\infty_{loc}[[0,\infty);\mathbb{V}]\cap\mathbb{C}^1[(0,\infty);\mathbb{H}]$.*

*Proof.* Theorem 3 allows us to conclude that, when $\mathbf{u}_0 \in \mathbb{D}$, the initial value problem is solved and the solution $\mathbf{u}(t,\mathbf{x})$ is in $\mathbb{C}^1[(0,\infty);\mathbb{D}]$. Since $\mathbb{D} \subset \mathbb{H}^2$, it follows that $\mathbf{u}(t,\mathbf{x})$ is also in $\mathbb{V}$ for each $t > 0$. It is now clear that, for any $T > 0$,

$$\int_0^T \|\mathbf{u}(t,\mathbf{x})\|_{\mathbb{H}}^2 \, dt < \infty, \text{ and } \sup_{0<t<T} \|\mathbf{u}(t,\mathbf{x})\|_{\mathbb{V}}^2 < \infty.$$

This gives our conclusion. □

When $\mathbf{f} \neq \mathbf{0}$, $u_- \neq 0$. Let $\mathbb{k} = \left\{ \mathbf{u} \, : \, \|\mathbf{u}\|_{\mathbb{H},1} < u_- \, \&, \, \|\mathbf{u}\|_{\mathbb{H},1} > \frac{1}{2} u_+ \right\}$ and set $\mathbb{B}_- = \mathbb{B} \cap \mathbb{k}^c$, where $\mathbb{k}^c$ is the complement of $\mathbb{k}$. We can now take the graph closure of $\mathbb{B}_- \cap D(\mathbf{A})$ and use the largest closed convex set containing the initial data inside this set.

## Discussion

It is known that, if $\mathbf{u}_0 \in \mathbb{V}$ and $\mathbf{f}(t) \in L^\infty[(0,\infty),\mathbb{H}]$, then there is a time $T > 0$ such that a weak solution with this data is uniquely determined on any subinterval of $[0,T)$ (see Sell and You, page 396, [SY]). Thus, we also have that:

**Corollary 12.** *For each* $t \in \mathbf{R}^+$ *and* $\mathbf{u}_0 \in \mathbb{D}$ *the Navier-Stokes initial-value problem on* $\mathbb{R}^3$ :

$$\begin{aligned} &\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}(t) \; \textit{in} \; (0,T) \times \mathbb{R}^3, \\ &\nabla \cdot \mathbf{u} = 0 \; \textit{in} \; (0,T) \times \mathbb{R}^3, \\ &\lim_{\|\mathbf{x}\| \to \infty} \mathbf{u}(t,\mathbf{x}) = \mathbf{0} \; \textit{on} \; (0,T) \times \mathbb{R}^3, \\ &\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \; \textit{in} \; \mathbb{R}^3, \end{aligned} \tag{10}$$

*has a unique weak solution* $\mathbf{u}(t,\mathbf{x})$ *which is in* $L^2_{loc}[[0,\infty);\mathbb{H}^2]$ *and in* $L^\infty_{loc}[[0,\infty);\mathbb{V}] \cap \mathbb{C}^1[(0,\infty);\mathbb{H}]$.

As in [GZ], our results show that the Leray-Hopf weak solutions do not develop singularities if $\mathbf{u}_0(\mathbf{x}) \in \mathbb{H}^2$ (see Giga [G] and references therein).

We should note that the constant $\alpha$ in Lemma 7 depends on $r$ so we can't change $r$ without affecting $\alpha$. This means that the size of $u_+$ need not increase with large values of $r$.

A close review of the results of this paper show that all theorems hold for the bounded domain case. This provides an improvement of the results in [GZ]. Furthermore, in that case, we can take $\alpha = \lambda_1$, the first eigenvalue of $\mathbf{A}$, which is independent of $r$. This means that choosing larger values for $r$ could increase the possible size of $\mathbb{D}$ for bounded domains. However, the inequality for $\nu$ must be maintained, so that increasing $\mathbb{D}$ is not certain.

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