Coincidence sets in quasilinear elliptic problems of monostable type *

Shingo Takeuchi †
Department of General Education, Kogakuin University
2665-1 Nakano, Hachioji, Tokyo 192-0015, JAPAN
E-mail: shingo@cc.kogakuin.ac.jp

Abstract

This paper concerns the formation of a coincidence set for the positive solution of the boundary value problem: $-\varepsilon\Delta_p u = u^{q-1} f(a(x)-u)$ in Ω with $u = 0$ on $\partial\Omega$, where ε is a positive parameter, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $1 < q \le p < \infty$, $f(s) \sim |s|^{\theta-1}s$ ($s \to 0$) for some $\theta > 0$ and $a(x)$ is a positive smooth function satisfying $\Delta_p a = 0$ in Ω with $\inf_\Omega |\nabla a| > 0$. It is proved in this paper that if $0 < \theta < 1$ the coincidence set $O_\varepsilon = \{x \in \Omega : u_\varepsilon(x) = a(x)\}$ has a positive measure and converges to Ω with order $O(\varepsilon^{1/p})$ as $\varepsilon \to 0$. Moreover, it is also shown that if $\theta \ge 1$, then O_ε is empty for any $\varepsilon > 0$. The proofs rely on comparison theorems and an energy method for obtaining local comparison functions.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \ge 2$) with smooth boundary $\partial\Omega$, and we consider the boundary value problem of quasilinear elliptic equations of monostable type:

$$\begin{cases} -\varepsilon\Delta_p u = u^{q-1} f(a(x) - u) & \text{in } \Omega, \\ u \ge 0,\ u \not\equiv 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where ε is a positive parameter, $\Delta_p u$ denotes the p-Laplacian $\mathrm{div}(\nabla_p u)$ with the p-gradient $\nabla_p u = |\nabla u|^{p-2}\nabla u$, $1 < q \le p < \infty$, $a : \Omega \to \mathbb{R}$ is a positive and smooth function and f is a function satisfying the following conditions.

(F1) $f \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ and $f(0) = 0$.

(F2) f is strictly increasing on $\mathbb{R}$.

(F3) There exists $\theta > 0$ such that $\lim_{s\to 0} \frac{f(s)}{|s|^{\theta-1}s} = C$ for some $C > 0$.

By a solution of (1.1) we mean a function $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfying (1.1) (for details, see Section 2). Applying the theorem of Díaz and Saá [4] and the regularity result of Lieberman [14], we see that if $\varepsilon < \varepsilon_a$ then (1.1) admits a unique positive solution $u_\varepsilon \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$; if $\varepsilon \ge \varepsilon_a$ then (1.1) has no solution. Here, $\varepsilon_a = \infty$ if $p > q$ and $\varepsilon_a = 1/\lambda_{f(a)}$ if $p = q$, where $\lambda_{f(a)}$ denotes the first eigenvalue of the definite weight eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda f(a(x))|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and it can be characterized by

$$\lambda_{f(a)} = \inf_{u\in W_0^{1,p}(\Omega),\ \neq 0} \frac{\displaystyle\int_\Omega |\nabla u(x)|^p \, dx}{\displaystyle\int_\Omega f(a(x))|u(x)|^p \, dx}.$$

We define the *coincidence set* of the positive solution u_ε of (1.1) with $a(x)$ as

$$\mathcal{O}_\varepsilon = \{x \in \Omega : u_\varepsilon(x) = a(x)\}.$$

In case $a(x)$ is constant, problem (1.1) has been already studied by several authors. Let $a(x) \equiv 1$ and $p = q > 2$. Then, Guedda and Véron [10] for $N = 1$ and Kamin and Véron [12] for $N \ge 2$ established that there exists a non-empty coincidence set $\mathcal{O}_\varepsilon$ (or a *flat core*, because the graph of u_ε is flat on $\mathcal{O}_\varepsilon$) for ε small enough (when Ω is a ball and $f(s) = s$, Kichenassamy and Smoller [13] had obtained the positive radial solution with a flat core). They and García-Melián and Sabina de Lis [9] proved that if $0 < \theta < p - 1$, then the flat core has a positive measure for small $\varepsilon \in (0, f(a)/\lambda_{f(a)})$ and it converges to Ω as $\mathrm{dist}(x, \mathcal{O}_\varepsilon) \sim \varepsilon^{1/p}$ $(\varepsilon \to 0)$ for any $x \in \partial\Omega$; while if $\theta \ge p - 1$, then the flat core is empty. These earlier results [9, 10, 12, 13] are substantially sharpened by Guo [11]. Moreover, even if $a(x)$ is constant on a plural subdomain of Ω, there exists a flat core in each subdomain (see [16]). General references for coincidence set are given in the monographs [3] of Díaz and [15] of Pucci and Serrin.

In this paper we shall investigate the case where $a(x)$ is variable. It is heuristic that if the coincidence set $\mathcal{O}_\varepsilon$ has an interior point, then $a(x)$ has to satisfy $\Delta_p a = 0$ on its neighborhood. Inversely, we shall assume $a(x)$ to be p-harmonic: $\Delta_p a = 0$ in Ω, and hence $a(x)$ satisfies the equation of (1.1). Then, our major finding is that the p-harmonicity of $a(x)$ is also a sufficient condition for an appearance of coincidence set.

Before stating the result, we give precise conditions to $a(x)$:

(A1) $\inf_{x\in\Omega} a(x) > 0$,

(A2) $a \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and $\Delta_p a = 0$ in Ω, and

(A3) $\inf_{x\in\Omega} |\nabla a(x)| > 0$.

We notice that by DiBenedetto [6] and Tolksdorf [19], (A2) follows from, e.g.,

(A2') there exists a domain $\Omega' \supset \overline{\Omega}$ such that $a \in W^{1,p}_{\mathrm{loc}}(\Omega')$ and $\Delta_p a = 0$ in Ω'.

The following theorem suggests that with regard to the coincidence set of positive solution, it is unnecessary to assume $a(x)$ to be constant as in the past studies.

Theorem 1.1. *Assume* (A1), (A2) *and* (A3). *Let* $0 < \theta < 1$. *Then, there exist* $L > 0$ *and* $\varepsilon_0 \in (0, \varepsilon_a)$ *such that for each* $\varepsilon \in (0, \varepsilon_0)$ *the solution* u_ε *of* (1.1) *satisfies*

$$u_\varepsilon(x) = a(x) \quad \textit{if } \operatorname{dist}(x, \partial\Omega) \geq L\varepsilon^{1/p}.$$

The corresponding theorem for $p = 2$ has been already proved in the author's paper [17]. As mentioned above, the condition $0 < \theta < p-1$ seems to be valid as a modification to the case $1 < p < \infty$, while the condition $0 < \theta < 1$ in the theorem is same as that in case $p = 2$. However, this is natural because the principal part of equation of (1.1) is neither degenerate nor singular in O_ε when $a(x)$ satisfies the non-degeneracy condition (A3).

The condition $0 < \theta < 1$ in Theorem 1.1 is optimal in the following sense.

Theorem 1.2. *Assume* $a(x)$ *to be same in Theorem* 1.1. *Let* $\theta \geq 1$. *Then, for every* $\varepsilon \in (0, \varepsilon_a)$, $u_\varepsilon < a$ *in* Ω, *and hence* $O_\varepsilon = \emptyset$.

In our approach, it is significant to study the translation $-\varepsilon\Delta_p(v - a)$ of the principal part $-\varepsilon\Delta_p v$. Putting $\Phi_p(\nabla v, \nabla a) = \nabla_p(v - a) + \nabla_p a$ and using (A2), we see that $\Phi_p(0, \nabla a) = 0$ and that the translation can be represented as the monotone operator $v \mapsto -\varepsilon \operatorname{div} \Phi_p(\nabla v, \nabla a)$. The vector-valued function $\Phi_p(\eta, \nabla a)$ has a different order at $\eta = 0$ from what $\Phi_p(\eta, 0)$ has if and only if $a(x)$ is non-degenerate. This is the reason why the conditions of θ in the theorems differ from those in case $a(x)$ is constant.

Theorems 1.1 and 1.2 are proved in Section 4. In order to show Theorem 1.1, letting the solution u_ε be close to $a(x)$ as $\varepsilon \to 0$ (the convergence will be shown in Section 2), we compare u_ε with a local comparison function which attains $a(x)$. Such a comparison function is obtained in Section 3 by means of the energy method developed by Díaz and Véron [5] (see also Díaz [3], and Antontsev, Díaz and Shmarev [1]). In proving Theorem 1.2, we give a Harnack type inequality by Trudinger [20] for an associated differential inequality. Finally, in Section 5, we apply our method to the known case where $a(x)$ is constant and realize the necessity of modifying the condition of θ to $0 < \theta < p - 1$.

The corresponding theorems for $N = 1$ to Theorems 1.1 and 1.2 have been already obtained in the author's paper [18].

Remark 1.1. If $\Omega = \mathbb{R}^N$, then the corresponding problem to (1.1)

$$-\varepsilon \Delta_p u = u^{q-1} f(a(x) - u) \quad \text{in } \mathbb{R}^N$$

is trivial. Indeed, since $a(x)$ is a positive and p-harmonic function in $\mathbb{R}^N$, it is constant by Liouville's theorem for p-Laplacian [15, Corollary 7.2.3] and any nonnegative solution of (1.1) must be the constant (see Du and Guo [7]).

Through the paper, we denote by C positive constants independent of ε and δ, unless otherwise noted.

2 Convergence to $a(x)$ as $\varepsilon \to 0$

In this section, we show that the solution of (1.1) converges to $a(x)$ uniformly in any compact set of Ω as $\varepsilon \to 0$.

A function $u = u_\varepsilon \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is called a *solution of* (1.1) if $u \geq 0$ a.e. in Ω, u does not vanish in a set of positive measure, and

$$\varepsilon \int_\Omega \nabla_p u \cdot \nabla \varphi \, dx = \int_\Omega u^{q-1} f(a(x) - u) \varphi \, dx$$

for all $\varphi \in W_0^{1,p}(\Omega)$. A function $u = u_\varepsilon \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is called a *supersolution* (resp. *subsolution*) *of* (1.1) if $u \geq 0$ (resp. $u \leq 0$) a.e. on $\partial\Omega$ and

$$\varepsilon \int_\Omega \nabla_p u \cdot \nabla \varphi \, dx \geq \text{ (resp. } \leq) \int_\Omega u^{q-1} f(a(x) - u) \varphi \, dx$$

for all $\varphi \in W_0^{1,p}(\Omega)$ satisfying $\varphi \geq 0$ a.e. in Ω. If a function u is not only a supersolution but also a subsolution, then u must be a solution of (1.1).

We denote by λ_1 the first eigenvalue to the following eigenvalue problem and by z the corresponding eigenfunction to λ_1 with $\|z\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} |z(x)| = 1$:

$$\begin{cases} -\Delta_p z = \lambda |z|^{p-2} z & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well-known that $\lambda_1 > 0$, $z \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and $z > 0$ in Ω. Let $B(x_0, r) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$, $\Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq \varepsilon\}$ and $d = \inf_{x \in \Omega} a(x)/2 > 0$.

Proposition 2.1. *Assume $a(x)$ to satisfy* (A1) *and* (A2)*. For each $\delta \in (0, 2d)$, there exist $K > 0$ and $\varepsilon_* \in (0, \varepsilon_a)$ such that if $\varepsilon \in (0, \varepsilon_*)$ then the solution u_ε of* (1.1) *satisfies*

$$a(x) - \delta \le u_\varepsilon(x) \le a(x) \quad \textit{for all } x \in \Omega_{K\varepsilon^{1/p}}.$$

Proof. It is clear from (A2) that $\overline{u} = a$ is a supersolution of (1.1) for every $\varepsilon > 0$.

We shall construct a subsolution of (1.1). From the uniform continuity of $a(x)$ in $\overline{\Omega}$, there exists $r > 0$ such that for every $x_0 \in \Omega$, $a(x) > a(x_0) - \delta/2$ for all $x \in B(x_0, r) \cap \Omega$, and hence for each $x \in B(x_0, r) \cap \Omega$, $a(x) - u > \delta/2$ for all $u \in [0, a(x_0) - \delta]$. Therefore, $f(a(x) - u) \ge \sigma = f(\delta/2)$ for all $x \in B(x_0, r) \cap \Omega$ and $u \in [0, a(x_0) - \delta]$. Let $K > 0$ be a constant satisfying $K^p > \lambda_1 \|a\|^{p-q}_{L^\infty(\Omega)}/\sigma$ and choose $\varepsilon_* \in (0, \varepsilon_a)$ such that $K\varepsilon_*^{1/p} < r$.

Take any $\varepsilon \in (0, \varepsilon_*)$ and $x_0 \in \Omega_{K\varepsilon^{1/p}}$. Changing scaling as $\underline{z}(x) = z((x - x_0)/(K\varepsilon^{1/p}))$, we have

$$\begin{cases} -\varepsilon \Delta_p \underline{z} = \dfrac{\lambda_1}{K^p} \underline{z}^{p-1} & \text{in } B(x_0, K\varepsilon^{1/p}), \\ \underline{z} = 0 & \text{on } \partial B(x_0, K\varepsilon^{1/p}). \end{cases}$$

Then the function

$$\underline{u}(x) = \begin{cases} (a(x_0) - \delta)\underline{z}(x), & x \in B(x_0, K\varepsilon^{1/p}), \\ 0, & x \in \Omega \setminus B(x_0, K\varepsilon^{1/p}) \end{cases}$$

is a nonnegative subsolution of (1.1). Indeed, $a(x_0) \ge 2d > \delta$, and for every $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \ge 0$

$$\begin{aligned} &\frac{1}{(a(x_0) - \delta)^{q-1}} \left(\varepsilon \int_\Omega \nabla_p \underline{u} \cdot \nabla \varphi \, dx - \int_\Omega \underline{u}^{q-1} f(a(x) - \underline{u}) \varphi \, dx \right) \\ &\quad \le -\varepsilon \int_{B(x_0, K\varepsilon^{1/p})} (a(x_0) - \delta)^{p-q} \Delta_p \underline{z} \, \varphi \, dx - \sigma \int_{B(x_0, K\varepsilon^{1/p})} \underline{z}^{q-1} \varphi \, dx \\ &\quad = \int_{B(x_0, K\varepsilon^{1/p})} \left(\frac{\lambda_1 (a(x_0) - \delta)^{p-q}}{K^p} \underline{z}^{p-q} - \sigma \right) \underline{z}^{q-1} \varphi \, dx \\ &\quad \le \left(\frac{\lambda_1 \|a\|^{p-q}_{L^\infty(\Omega)}}{K^p} - \sigma \right) \int_{B(x_0, K\varepsilon^{1/p})} \underline{z}^{q-1} \varphi \, dx \le 0. \end{aligned}$$

Since $\underline{u} < \overline{u}$ in Ω, there exists a solution u^* of (1.1) with $\underline{u} \le u^* \le \overline{u}$ in Ω (e.g., Deuel and Hess [2]). As mentioned in Section 1, the solution of (1.1) is unique. Therefore, $u^* = u_\varepsilon$, and hence $\underline{u} \le u_\varepsilon \le \overline{u}$ in Ω. In particular, $a(x_0) - \delta \le u_\varepsilon(x_0) \le a(x_0)$ for all $x_0 \in \Omega_{K\varepsilon^{1/p}}$ when $0 < \varepsilon < \varepsilon_*$. □

Remark 2.1. Even if (A2) is not assumed, then we can prove that $|u_\varepsilon - a| < \delta$. Indeed, we can construct a supersolution of (1.1) close to $a(x)$ from above. Let $p \geq 2$ for simplicity, and assume $\overline{u}$ to be an arbitrary *smooth* function satisfying $a + \delta/2 < \overline{u} < a + \delta$. Since

$$-\varepsilon\Delta_p\overline{u} - \overline{u}^{q-1} f(a(x) - \overline{u}) \geq -\varepsilon\Delta_p\overline{u} + C(\overline{u} - a(x))^\theta \geq -\varepsilon\Delta_p\overline{u} + C\left(\frac{\delta}{2}\right)^\theta$$

for all $x \in \Omega$ and $\Delta_p\overline{u}$ is continuous in $\overline{\Omega}$, the last expression can be positive provided ε is small enough. For the case $1 < p < 2$, we refer to [16].

3 Auxiliary problem near $a(x)$

In this section, we show that there exists a comparison function with dead core, which satisfies an equation having a subsolution $a - u_\varepsilon \geq 0$.

We define the vector-valued function $\Phi_p : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ as

$$\Phi_p(\eta, \xi) = |\eta - \xi|^{p-2}(\eta - \xi) + |\xi|^{p-2}\xi.$$

In particular, we note that $\Phi_p(\nabla u, \nabla v) = \nabla_p(u - v) + \nabla_p v$ for gradients.

The following lemma means that for each $\xi \neq 0$ the function $\Phi_p(\eta, \xi)$ is of order 1 at $\eta = 0$.

Lemma 3.1. *For all* $\eta,\ \xi \in \mathbb{R}^N$ *with* $|\eta - \xi| + |\xi| > 0$

$$\Phi_p(\eta, \xi) \cdot \eta \geq \min\{p - 1, 2^{2-p}\}(|\eta - \xi| + |\xi|)^{p-2}|\eta|^2, \tag{3.1}$$

$$|\Phi_p(\eta, \xi)| \leq \max\{p - 1, 2^{2-p}\}(|\eta - \xi| + |\xi|)^{p-2}|\eta|. \tag{3.2}$$

For all $\eta,\ \eta',\ \xi \in \mathbb{R}^N$ *with* $|\eta - \xi| + |\eta' - \xi| > 0$

$$(\Phi_p(\eta, \xi) - \Phi_p(\eta', \xi)) \cdot (\eta - \eta') \geq \min\{p - 1, 2^{2-p}\}(|\eta - \xi| + |\eta' - \xi|)^{p-2}|\eta - \eta'|^2, \tag{3.3}$$

$$|\Phi_p(\eta, \xi) - \Phi_p(\eta', \xi)| \leq \max\{p - 1, 2^{2-p}\}(|\eta - \xi| + |\eta' - \xi|)^{p-2}|\eta - \eta'|. \tag{3.4}$$

Proof. By the mean value theorem, we have

$$(\Phi_p(\eta, \xi), \eta) = (p - 1)|\eta|^2 \int_0^1 |t\eta - \xi|^{p-2}\, dt, \tag{3.5}$$

$$|\Phi_p(\eta, \xi)| = (p - 1)|\eta| \int_0^1 |t\eta - \xi|^{p-2}\, dt. \tag{3.6}$$

Since $|t\eta - \xi| = |t(\eta - \xi) - (1 - t)\xi| \leq |\eta - \xi| + |\xi|$ for all $t \in [0, 1]$, equation (3.5) yields (3.1) if $1 < p \leq 2$, while (3.6) yields (3.2) if $p \geq 2$.

Putting $t_0 = |\xi|/(|\eta - \xi| + |\xi|) \in (0, 1]$, we have

$$|t\eta - \xi| \geq |t|\eta - \xi| - (1 - t)|\xi|| = (|\eta - \xi| + |\xi|)|t - t_0|.$$

If $p > 2$ (resp. $1 < p < 2$), then for every $t_0 \in (0, 1]$ we have that $\int_0^1 |t - t_0|^{p-2}\, dt \geq$ (resp. $\leq$) $2\int_0^{1/2} z^{p-2}\, dz = 2^{2-p}/(p-1)$, thus (3.5) (resp. (3.6)) yields (3.1) (resp. (3.2)).

Since $\Phi_p(\eta, \xi) - \Phi_p(\eta', \xi) = \Phi_p(\eta - \eta', \xi - \eta')$, (3.3) and (3.4) follow from (3.1) and (3.2), respectively. □

Let Λ be a positive constant. Take $x_0 \in \Omega$, $\delta \in (0, 1)$ and $\varepsilon \in (0, 1)$ such that $B = B(x_0, \varepsilon^{1/p}) \subset \Omega$. Consider the boundary value problem

$$\begin{cases} -\varepsilon \operatorname{div} \Phi_p(\nabla w, \nabla a) + \Lambda |w|^{\theta-1} w = 0 & \text{in } B, \\ w = \delta & \text{on } \partial B. \end{cases} \tag{3.7}$$

For Propositions 3.1 and 3.2 below, we assume only $a \in W^{1,p}(B)$ without (A1), (A2) and (A3).

Proposition 3.1. *Let g be a non-decreasing function, and suppose that $u,\ v \in W^{1,p}(B) \cap L^{\sigma}(B)$, where $\sigma \in [1, \infty]$, satisfy $g(u),\ g(v) \in L^{\sigma^*}(B)$, where $\sigma^* = \frac{\sigma}{\sigma-1}$ ($\sigma^* = \infty$ if $\sigma = 1$ and $\sigma^* = 1$ if $\sigma = \infty$), and*

$$\begin{cases} -\operatorname{div} \Phi_p(\nabla u, \nabla a) + g(u) \leq -\operatorname{div} \Phi_p(\nabla v, \nabla a) + g(v) & \text{in } B, \\ u \leq v & \text{on } \partial B. \end{cases}$$

Then, $u \leq v$ a.e. in B.

Proof. Using $(u - v)^+ \in W_0^{1,p}(B) \cap L^{\sigma}(B)$ as a test function, we get

$$\int_D (\Phi_p(\nabla u, \nabla a) - \Phi_p(\nabla v, \nabla a)) \cdot (\nabla u - \nabla v)\, dx \leq -\int_D (g(u) - g(v))(u - v)\, dx \leq 0,$$

where $D = \{x \in B : u(x) > v(x)\}$. On the other hand, the integrand of the left-hand side is non-negative because of (3.3). Thus, we conclude $\nabla u = \nabla v$ a.e. in D, and hence $\nabla(u - v)^+ = 0$ a.e. in B, which means $(u - v)^+ = 0$ a.e. in B. Therefore, $u \leq v$ a.e. in B. □

Proposition 3.2. *For any $\varepsilon > 0$, there exists a unique solution $w \in W^{1,p}(B) \cap L^{\infty}(B)$ of* (3.7)*. Moreover,* $0 \leq w \leq \delta$ *a.e. in B.*

Proof. We set the C^1-energy functional J corresponding to (3.7) as

$$J(u) = \frac{\varepsilon}{p} \int_B |\nabla u - \nabla a|^p\, dx + \varepsilon \int_B \nabla_p a \cdot \nabla u\, dx + \Lambda \int_B |u|^{1+\theta}\, dx,$$

which is defined in

$$K = \{u \in W^{1,p}(B) \cap L^{1+\theta}(B) : u - \delta \in W^{1,p}_0(B)\}.$$

Since

$$|\nabla_p a \cdot \nabla u| \le |\nabla a|^{p-1}|\nabla u - \nabla a| + |\nabla a|^p \le \frac{1}{2p}|\nabla u - \nabla a|^p + C|\nabla a|^p,$$

we have

$$J(u) \ge \frac{\varepsilon}{2p}\int_B |\nabla u - \nabla a|^p\,dx + \Lambda \int_B |u|^{1+\theta}\,dx - C\varepsilon \int_B |\nabla a|^p\,dx. \tag{3.8}$$

Then we see that J is bounded from below and $J_0 = \inf_{u\in K} J(u)$ exists. It suffices to show that there exists $w \in K$ such that $J(w) = J_0$.

Let $\{u_n\}$ be a minimizing sequence such that $u_n \in K$ and $J(u_n) \to J_0$ as $n \to \infty$. Then, by (3.8) we obtain

$$\int_B |\nabla u_n - \nabla a|^p\,dx, \quad \int_B |u_n|^{1+\theta}\,dx \quad \le C,$$

so that $\{u_n - \delta\}$ and $\{u_n\}$ are bounded in the reflexive Banach spaces $W^{1,p}_0(B)$ and $L^{1+\theta}(B)$, respectively. Thus, we can choice a subsequence, which is denoted u_n again, and $w \in K$ such that $u_n \to w$ weakly in $W^{1,p}(B)$ and weakly in $L^{1+\theta}(B)$. Thus,

$$\liminf_{n\to\infty} \|u_n - a\|_{W^{1,p}(B)} \ge \|w - a\|_{W^{1,p}(B)}, \tag{3.9}$$

$$\lim_{n\to\infty} \int_B \nabla_p a \cdot \nabla u_n\,dx = \int_B \nabla_p a \cdot \nabla w\,dx, \tag{3.10}$$

$$\liminf_{n\to\infty} \|u_n\|_{L^{1+\theta}(B)} \ge \|w\|_{L^{1+\theta}(B)}. \tag{3.11}$$

Since $u_n \to w$ strongly in $L^p(B)$ by the Poincaré inequality, it follows from (3.9) that

$$\liminf_{n\to\infty} \|\nabla(u_n - a)\|_{L^p(B)} \ge \|\nabla(w - a)\|_{L^p(B)}. \tag{3.12}$$

Therefore, by (3.10), (3.11) and (3.12), we conclude that $J_0 = \liminf_{n\to\infty} J(u_n) \ge J(w) \ge J_0$, so that $J(w) = J_0$. The uniqueness and the boundedness of solutions follow from Proposition 3.1 with $g(s) = |s|^{\theta-1}s$ and $\sigma = 1 + \theta$. □

To show that the solution w of (3.7) has a dead core for any $\varepsilon > 0$, scaling is useful: setting $y = \varepsilon^{-1/p}(x - x_0)$, $\tilde{w}(y) = \tilde{w}(y; \varepsilon, x_0) = w(x + \varepsilon^{1/p}y)$ and $\tilde{a}(y) = \tilde{a}(y; \varepsilon, x_0) = a(x_0 + \varepsilon^{1/p}y)$ in (3.7), we have

$$\begin{cases} -\operatorname{div} \Phi_p(\nabla\tilde{w}, \nabla\tilde{a}) + \Lambda\tilde{w}^\theta = 0 & \text{in } B(0,1), \\ \tilde{w} = \delta & \text{on } \partial B(0,1). \end{cases} \tag{3.13}$$

We shall write B_ρ to represent $B(0, \rho)$.

Lemma 3.2. *Let* $a(x)$ *satisfy* (A2)*, and assume* $\tilde{w}$ *to be the unique solution of* (3.13)*. Then* $\tilde{w} \in C^{1,\alpha}(\overline{B_1})$ *for some* $\alpha \in (0,1)$ *and* $\|\nabla(\tilde{w} - \tilde{a})\|_{L^\infty(B_1)} \le C$*, where* C *is independent of* ε*,* δ *and* x_0*.*

Proof. Setting $v(y) = \tilde{w}(y) - \tilde{a}(y)$, we have

$$\begin{cases} -\Delta_p v + \Lambda(v + \tilde{a})^\theta = 0 & \text{in } B_1, \\ v = \delta + \tilde{a} & \text{on } \partial B_1. \end{cases}$$

Since $\|v + \tilde{a}\|_{L^\infty(B_1)} \le \delta \le 1$ by Proposition 3.1 and $\delta + \tilde{a}\ |_{\partial B_1} \in C^{1,\alpha}(\partial B_1)$ with $\|\delta + \tilde{a}\|_{C^{1,\alpha}(\partial B_1)} \le \|\delta + \tilde{a}\|_{C^{1,\alpha}(\overline{B_1})} \le 1 + \|a\|_{C^{1,\alpha}(\overline{\Omega})}$ (for the norm of $C^{1,\alpha}(\partial B_1)$, see Gilbarg and Trudinger [8, Section 6.2]), it follows from a regularity result of Lieberman [14] that $v \in C^{1,\alpha}(\overline{B_1})$ and $\|v\|_{C^{1,\alpha}(\overline{B_1})} \le C$ for some $\alpha \in (0,1)$ and $C > 0$ are independent of ε, δ and x_0. In particular, $\|\nabla v\|_{L^\infty(B_1)} \le C$. □

Proposition 3.3. *Let* $a(x)$ *satisfy* (A2) *and* (A3)*, and assume* w *to be the unique solution of* (3.7)*. If* $0 < \theta < 1$*, then there exists* $M > 0$ *independent of* ε*,* δ *and* x_0 *such that* $w(x) = 0$ *for all* $x \in B(x_0, (1 - M\delta^{(1+\theta)\gamma})^{1/\tau}\varepsilon^{1/p})$*, where*

$$\gamma = \frac{\frac{1}{1+\theta} - \frac{1}{2}}{N\left(\frac{1}{1+\theta} - \frac{1}{2}\right) + 1} \in \left(0, \frac{1}{N+2}\right),$$
$$\tau = 2N\left(\frac{1}{1+\theta} - \frac{1}{2}\right) + 2 \in (2, N+2)\,.$$

In particular, $w(x_0) = 0$ *for arbitrary* $\varepsilon > 0$ *if* $\delta^{(1+\theta)\gamma} < M^{-1}$*.*

Proof. It is sufficient to prove the existence of dead core for the solution of (3.13). To do this, we follow the energy method developed by Díaz and Véron [5] (see also Díaz [3], and Antontsev, Díaz and Shmarev [1]).

We define the diffusion and absorption energy functions $E_D(\rho)$ and $E_A(\rho)$ in $(0,1)$ as follows:

$$E_D(\rho) = \int_{B_\rho} \Phi_p(\nabla\tilde{w}(y), \nabla\tilde{a}(y)) \cdot \nabla\tilde{w}(y)\, dy,$$
$$E_A(\rho) = \int_{B_\rho} |\tilde{w}(y)|^{1+\theta}\, dy.$$

The total energy function $E_T(\rho)$ is defined as

$$E_T(\rho) = E_D(\rho) + \Lambda E_A(\rho).$$

The global total energy $E_T(1)$ is finite. Indeed, (we write w, a instead of $\tilde{w}$, $\tilde{a}$, respectively), multiplying the equation of (3.13) by the nonnegative function $\delta - w \in W_0^{1,p}(B_1)$ and integrating by parts in B_1, we have

$$E_T(1) \le \Lambda\delta^{1+\theta}|B_1| \le C\delta^{1+\theta}. \tag{3.14}$$

Multiplying the equation of (3.13) by w and integrating by parts in B_ρ, we have also (now we shall write S_ρ to represent ∂B_ρ)

$$E_T(\rho) = \int_{S_\rho} \Phi_p(\nabla w(y), \nabla a(y)) \cdot n\, w(y)\, ds, \tag{3.15}$$

where $n = n(s)$ is the outward normal vector at $y \in S_\rho$. By (3.15), Lemmas 3.1 and 3.2 with (A3)

$$\begin{aligned}
E_T(\rho) &= \int_{S_\rho} |\Phi_p(\nabla w, \nabla a)||w|\, ds \\
&\le \left(\int_{S_\rho} |\Phi_p(\nabla w, \nabla a)|^2\, ds\right)^{1/2} \left(\int_{S_\rho} |w|^2\, ds\right)^{1/2} \\
&\le \left(\int_{S_\rho} (|\nabla w - \nabla a| + |\nabla a|)^{2(p-2)} (\Phi_p(\nabla w, \nabla a) \cdot \nabla w)\, ds\right)^{1/2} \|w\|_{L^2(S_\rho)} \\
&\le C\left(\int_{S_\rho} \Phi_p(\nabla w, \nabla a) \cdot \nabla w\, ds\right)^{1/2} \|w\|_{L^2(S_\rho)}.
\end{aligned} \tag{3.16}$$

On the other hand, by using spherical coordinates (ω, r) with center x_0, we have

$$E_D(\rho) = \int_0^\rho \int_{S^{N-1}} \Phi_p(\nabla w(r\omega), \nabla a(r\omega)) \cdot \nabla w(r\omega)\, r^{N-1}\, d\omega\, dr.$$

Hence, E_D is almost everywhere differentiable and

$$\begin{aligned}
\frac{dE_D(\rho)}{d\rho} &= \int_{S^{N-1}} \Phi_p(\nabla w(\rho\omega), \nabla a(\rho\omega)) \cdot \nabla w(\rho\omega)\, \rho^{N-1}\, d\omega \\
&= \int_{S_\rho} \Phi_p(\nabla w, \nabla a) \cdot \nabla w\, ds.
\end{aligned} \tag{3.17}$$

Similarly,

$$\frac{dE_A(\rho)}{d\rho} = \int_{S_\rho} |w|^{1+\theta}\, ds. \tag{3.18}$$

Moreover, since $0 < \theta < 1$, we have the following inequality (see Díaz *et al.* [5, 3, 1]):

$$\|w\|_{L^2(S_\rho)} \le C\left(\|\nabla w\|_{L^2(B_\rho)} + \rho^{-\alpha}\|w\|_{L^{1+\theta}(B_\rho)}\right)^{\beta} \|w\|^{1-\beta}_{L^{1+\theta}(B_\rho)},$$

where $C = C(N, \theta)$ and

$$\alpha = \frac{N(1-\theta) + 2(1+\theta)}{2(1+\theta)} = N\left(\frac{1}{1+\theta} - \frac{1}{2}\right) + 1 \in \left(1, \frac{N}{2} + 1\right) \subset (1, \infty),$$

$$\beta = \frac{N(1-\theta) + 1 + \theta}{N(1-\theta) + 2(1+\theta)} = \frac{N\left(\frac{1}{1+\theta} - \frac{1}{2}\right) + \frac{1}{2}}{N\left(\frac{1}{1+\theta} - \frac{1}{2}\right) + 1} \in \left(\frac{1}{2}, \frac{N+1}{N+2}\right) \subset (0, 1).$$

Thus, from (3.1) and Lemma 3.2, we obtain $E_D(\rho) \ge C\|\nabla w\|^2_{L^2(B_\rho)}$, so that

$$\begin{aligned}
\|w\|^{1/\beta}_{L^2(S_\rho)} &\le C\left(\|\nabla w\|_{L^2(B_\rho)} + \rho^{-\alpha}\|w\|_{L^{1+\theta}(B_\rho)}\right)\|w\|^{\frac{1-\beta}{\beta}}_{L^{1+\theta}(B_\rho)} \\
&= C\left(\|\nabla w\|_{L^2(B_\rho)}\|w\|^{\frac{1-\beta}{\beta}}_{L^{1+\theta}(B_\rho)} + \rho^{-\alpha}\|w\|^{1/\beta}_{L^{1+\theta}(B_\rho)}\right) \\
&\le C\rho^{-\alpha}\left(\rho^{\alpha}E_D(\rho)^{\frac{1}{2}}E_A(\rho)^{\frac{1-\beta}{\beta(1+\theta)}} + E_A(\rho)^{\frac{1}{\beta(1+\theta)}}\right) \\
&\le C\rho^{-\alpha}\left(E_T(\rho)^{\frac{1}{2}+\frac{1-\beta}{\beta(1+\theta)}} + E_A(1)^{\frac{1}{1+\theta}-\frac{1}{2}}E_A(\rho)^{\frac{1}{2}+\frac{1-\beta}{\beta(1+\theta)}}\right) \\
&\le C\rho^{-\alpha}E_T(\rho)^{\frac{1}{2}+\frac{1-\beta}{\beta(1+\theta)}}.
\end{aligned} \tag{3.19}$$

Here we have used that $E_A(1) \le C\delta^{1+\theta} < C$ and $0 < \theta < 1$. Combining (3.16)–(3.18) and (3.19), we obtain

$$E_T(\rho) \le C\left(\frac{dE_T(\rho)}{d\rho}\right)^{1/2}\rho^{-\alpha\beta}E_T(\rho)^{\frac{\beta}{2}+\frac{1-\beta}{1+\theta}},$$

that is,

$$\frac{dE_T(\rho)}{d\rho} \ge C\rho^{\tau-1}E_T(\rho)^{1-\gamma},$$

where

$$\gamma = 2(1-\beta)\left(\frac{1}{1+\theta} - \frac{1}{2}\right) = \frac{\frac{1}{1+\theta} - \frac{1}{2}}{N\left(\frac{1}{1+\theta} - \frac{1}{2}\right) + 1} \in \left(0, \frac{1}{N+2}\right),$$

$$\tau = 1 + 2\alpha\beta = 2N\left(\frac{1}{1+\theta} - \frac{1}{2}\right) + 2 \in (2, N+2).$$

Integrating it on $[\rho, 1]$ and using (3.14), we have

$$E_T(\rho)^\gamma \le E_T(1)^\gamma - C(1-\rho^\tau) \le C(\rho^\tau - (1 - M\delta^{(1+\theta)\gamma}))$$

for some $M > 0$, thus $E_T((1 - M\delta^{(1+\theta)\gamma})^{1/\tau}) = 0$, i.e., $\tilde{w}(y) = 0$ for all $y \in B(0,(1 - M\delta^{(1+\theta)\gamma})^{1/\tau})$. Scaling back to x, we conclude the assertion. □

4 Proofs of Theorems

Now we are in a position to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Fix $\delta \in (0, d)$ such that $M\delta^{(1+\theta)\gamma} < 1$, where M and γ are the constants appearing in Proposition 3.3. Thanks to the p-harmonicity of $a(x)$, the function $v = a - u_\varepsilon$ satisfies that $-\varepsilon \operatorname{div} \Phi_p(\nabla v, \nabla a) = -(a(x) - v)^{q-1} f(v)$ in the distribution sense in Ω. Since

$$(a(x) - s)^{q-1} f(s) \ge d^{q-1} C s^\theta =: \Lambda_1 s^\theta \quad \text{for all } x \in \Omega \text{ and } s \in [0, \delta]$$

and by Proposition 2.1, $\max_{x \in \Omega_{K\varepsilon^{1/p}}} v_\varepsilon(x) \le \delta$ for every $\varepsilon \in (0, \varepsilon_*)$, we have

$$-\varepsilon \operatorname{div} \Phi_p(\nabla v, \nabla a) + \Lambda_1 v^\theta \le 0 \quad \text{in } \Omega_{K\varepsilon^{1/p}}. \tag{4.1}$$

Let $\varepsilon_0 \in (0, \varepsilon_*)$ be small such that $\Omega_{(K+1)\varepsilon_0^{1/p}} \ne \emptyset$. Take any $\varepsilon \in (0, \varepsilon_0)$ and $x_0 \in \Omega_{(K+1)\varepsilon^{1/p}}$. Letting w be the solution of (3.7), we can see

$$\begin{cases} -\varepsilon \operatorname{div} \Phi_p(\nabla w, \nabla a) + \Lambda_1 w^\theta = 0 & \text{in } B(x_0, \varepsilon^{1/p}), \\ w = \delta & \text{on } \partial B(x_0, \varepsilon^{1/p}). \end{cases} \tag{4.2}$$

Since $B(x_0, \varepsilon^{1/p}) \subset \Omega_{K\varepsilon^{1/p}}$ and $v \le \delta = w$ on $\partial B(x_0, \varepsilon^{1/p})$, it follows from (4.1) and (4.2) that v is a subsolution of (4.2). Therefore, Proposition 3.1 gives $v \le w$ in $B(x_0, \varepsilon^{1/p})$. Proposition 3.3 implies that $0 \le v_\varepsilon(x_0) \le w(x_0) = 0$, and hence $u(x_0) = a(x_0)$ for all $x_0 \in \Omega_{(K+1)\varepsilon^{1/p}}$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let u_ε be a solution of (1.1). The function $v = a - u_\varepsilon \ge 0, \not\equiv 0$, satisfies

$$-\varepsilon \operatorname{div} \Phi_p(\nabla v, \nabla a) + \Lambda_2 v^\theta \ge 0$$

for some $\Lambda_2 > 0$. Since $u_\varepsilon \in C^1(\overline{\Omega})$ by the regularity result of Lieberman [14], so is v, and there exists $k > 0$ such that $\|\nabla v\|_{L^\infty(\Omega)} \le k$. We define

$$M_{p,k} = \sup_{|\eta| \le k, x \in \Omega} (|\eta - \nabla a(x)| + |\nabla a(x)|)^{p-2},$$

$$m_{p,k} = \inf_{|\eta| \le k, x \in \Omega} (|\eta - \nabla a(x)| + |\nabla a(x)|)^{p-2},$$

which are both finite and positive for any $p > 1$ because of (A3). Then, v is also a nonnegative bounded function satisfying

$$-\varepsilon \operatorname{div} \tilde{\Phi}_p(\nabla v, \nabla a) + \Lambda_2 |v|^{\theta-1} v \geq 0,$$

where $\tilde{\Phi}_p(\eta, \nabla a)$ is a vector measurable function as

$$\tilde{\Phi}_p(\eta, \nabla a) = \begin{cases} \Phi_p(\eta, \nabla a) & \text{if } |\eta| \leq k, \\ M_{p,k}\eta & \text{if } |\eta| > k, \end{cases}$$

which satisfies (from (3.2) and (3.1) in Lemma 3.1)

$$|\tilde{\Phi}_p(\eta, \nabla a(x))| \leq M_{p,k} \max\{p-1, 2^{2-p}\}\, |\eta|,$$
$$\tilde{\Phi}_p(\eta, \nabla a(x)) \cdot \eta \geq m_{p,k} \min\{p-1, 2^{2-p}\}\, |\eta|^2.$$

Moreover, if $\theta \geq 1$, then there exists $C > 0$ such that $||s|^{\theta-1}s| \leq C|s|$ if $|s| \leq \|v\|_{L^\infty(\Omega)}$. Thus, the weak Harnack inequality by Trudinger [20, Theorem 1.2] (see also Pucci and Serrin [15, Theorem 7.1.2]) follows: for any $\overline{B(x_0, 4\rho)} \subset \Omega$ and $\gamma \in (0, \frac{N}{N-2})$ ($\gamma \in (0, \infty)$ if $N = 2$), there exists $C = C(N, \gamma, \Lambda_2/\varepsilon, \rho, p, k, M_{p,k}, m_{p,k})$ such that

$$\rho^{-\frac{N}{\gamma}} \|v\|_{L^\gamma(B(x_0, 2\rho))} \leq C \inf_{x \in B(x_0, 2\rho)} v(x). \tag{4.3}$$

Suppose $v(x_0) = 0$ with $x_0 \in \Omega$. Then the set $O = \{x \in \Omega : v(x) = 0\}$, which is closed relatively to Ω since v is continuous, is nonempty. Since v is continuous, if $x \in O$ and $\overline{B(x, 4\delta)} \subset \Omega$, then $\inf_{B(x,2\rho)} v = v(x) = 0$. From (4.3) we have that $\|v\|_{L^\gamma(B(x,2\rho))} = 0$ so that $v \equiv 0$ in $B(x, 2\rho)$.So O is also open and since Ω is connected it must be $O = \Omega$, i.e., $v \equiv 0$ in Ω, which is a contradiction. Therefore, v is strictly positive in Ω, i.e., $u_\varepsilon < a$ in Ω. □

5 Degenerate case

In this section, we consider the case where $a(x)$ is constant in Ω. As introduced in Section 1, this case has been already treated by several papers [9, 10, 11, 12, 13]. Our approach can be applied to the case.

Since $\nabla a \equiv 0$ in this case, we note $\Phi_p(\nabla w, \nabla a) = \nabla_p w$ and Propositions 3.1, 3.2 and Lemma 3.2 are all satisfied. However, Proposition 3.3 has to be changed as follows.

Proposition 3.3' . *Let $a(x)$ be a constant in Ω, and assume w to be the unique solution of* (3.7). *If $0 < \theta < p - 1$, then there exists $M > 0$ independent of ε, δ and x_0 such that*

$w(x) = 0$ for all $x \in B(x_0, (1 - M\delta^{(1+\theta)\gamma})^{1/\tau}\varepsilon^{1/p})$, where

$$\gamma = \frac{\frac{1}{1+\theta} - \frac{1}{p}}{N\left(\frac{1}{1+\theta} - \frac{1}{p}\right) + 1} \in \left(0, \frac{1}{N + p^*}\right),$$
$$\tau = Np^*\left(\frac{1}{1+\theta} - \frac{1}{p}\right) + p^* \in (p^*, N + p^*)\,,$$

where $p^* = \frac{p}{p-1}$. In particular, $w(x_0) = 0$ for arbitrary $\varepsilon > 0$ if $\delta^{(1+\theta)\gamma} < M^{-1}$.

Proof. It is sufficient to prove the existence of dead core of solution of (3.13). We define the diffusion and absorption energy functions $E_D(\rho)$ and $E_A(\rho)$ in $(0, 1)$ as follows:

$$E_D(\rho) = \int_{B_\rho} |\nabla\tilde{w}(y)|^p\, dy,$$
$$E_A(\rho) = \int_{B_\rho} |\tilde{w}(y)|^{1+\theta}\, dy.$$

The total energy function $E_T(\rho)$ is defined as

$$E_T(\rho) = E_D(\rho) + \Lambda E_A(\rho).$$

The global total energy $E_T(1)$ is finite. Indeed, (we write w instead of $\tilde{w}$), multiplying the equation of (3.13) by the nonnegative function $\delta - w \in W_0^{1,p}(B_1)$ and integrating by parts in B_1, we have

$$E_T(1) \le \Lambda\delta^{1+\theta}|B_1| \le C\delta^{1+\theta}. \tag{5.1}$$

Multiplying the equation of (3.13) by w and integrating by parts in B_ρ, we have also (now we shall write S_ρ to represent ∂B_ρ)

$$E_T(\rho) = \int_{S_\rho} \nabla_p w(y) \cdot n\, w(y)\, ds, \tag{5.2}$$

where $n = n(s)$ is the outward normal vector at $y \in S_\rho$. By (5.2)

$$E_T(\rho) = \int_{S_\rho} |\nabla_p w||w|\, ds \le \|\nabla w\|^{p-1}_{L^p(S_\rho)}\|w\|_{L^p(S_\rho)}. \tag{5.3}$$

On the other hand, by using spherical coordinates (ω, r) with center x_0, we have

$$E_D(\rho) = \int_0^\rho \int_{S^{N-1}} |\nabla w(r\omega)|^p\, r^{N-1}\, d\omega\, dr.$$

Hence, E_D is almost everywhere differentiable and

$$\frac{dE_D(\rho)}{d\rho} = \int_{S^{N-1}} |\nabla w(r\omega)|^p \rho^{N-1}\, d\omega = \int_{S_\rho} |\nabla w|^p\, ds. \tag{5.4}$$

Similarly,

$$\frac{dE_A(\rho)}{d\rho} = \int_{S_\rho} |w|^{1+\theta}\, ds. \tag{5.5}$$

Moreover, since $0 < \theta < p-1$, we have the following inequality (see Díaz *et al*. [5, 3, 1]):

$$\|w\|_{L^p(S_\rho)} \leq C \left(\|\nabla w\|_{L^p(B_\rho)} + \rho^{-\alpha} \|w\|_{L^{1+\theta}(B_\rho)} \right)^\beta \|w\|^{1-\beta}_{L^{1+\theta}(B_\rho)},$$

where $C = C(N, \theta)$ and

$$\alpha = \frac{N(p-1-\theta) + p(1+\theta)}{p(1+\theta)} = N\left(\frac{1}{1+\theta} - \frac{1}{p}\right) + 1 \in \left(1, \frac{N}{p^*} + 1\right) \subset (1, \infty),$$

$$\beta = \frac{N(p-1-\theta) + 1 + \theta}{N(p-1-\theta) + p(1+\theta)} = \frac{N\left(\frac{1}{1+\theta} - \frac{1}{p}\right) + \frac{1}{p}}{N\left(\frac{1}{1+\theta} - \frac{1}{p}\right) + 1} \in \left(\frac{1}{p}, \frac{N + \frac{1}{p-1}}{N + p^*}\right) \subset (0, 1).$$

Thus,

$$\begin{aligned}
\|w\|^{1/\beta}_{L^p(S_\rho)} &\leq C \left(\|\nabla w\|_{L^p(B_\rho)} + \rho^{-\alpha} \|w\|_{L^{1+\theta}(B_\rho)} \right) \|w\|^{\frac{1-\beta}{\beta}}_{L^{1+\theta}(B_\rho)} \\
&= C \left(\|\nabla w\|_{L^p(B_\rho)} \|w\|^{\frac{1-\beta}{\beta}}_{L^{1+\theta}(B_\rho)} + \rho^{-\alpha} \|w\|^{1/\beta}_{L^{1+\theta}(B_\rho)} \right) \\
&= C\rho^{-\alpha} \left(\rho^\alpha E_D(\rho)^{\frac{1}{p}} E_A(\rho)^{\frac{1-\beta}{\beta(1+\theta)}} + E_A(\rho)^{\frac{1}{\beta(1+\theta)}} \right) \\
&\leq C\rho^{-\alpha} \left(E_T(\rho)^{\frac{1}{p} + \frac{1-\beta}{\beta(1+\theta)}} + E_A(1)^{\frac{1}{1+\theta} - \frac{1}{p}} E_A(\rho)^{\frac{1}{p} + \frac{1-\beta}{\beta(1+\theta)}} \right) \\
&\leq C\rho^{-\alpha} E_T(\rho)^{\frac{1}{p} + \frac{1-\beta}{\beta(1+\theta)}}.
\end{aligned} \tag{5.6}$$

Here we have used that $E_A(1) \leq C\delta^{1+\theta} < C$ and $0 < \theta < p-1$. Combining (5.3)–(5.5) and (5.6), we obtain

$$E_T(\rho) \leq C \left(\frac{dE_T(\rho)}{d\rho} \right)^{(p-1)/p} \rho^{-\alpha\beta} E_T(\rho)^{\frac{\beta}{p} + \frac{1-\beta}{1+\theta}},$$

that is,

$$\frac{dE_T(\rho)}{d\rho} \geq C\rho^{\tau-1} E_T(\rho)^{1-\gamma},$$

where

$$\gamma = p^*(1-\beta)\left(\frac{1}{1+\theta} - \frac{1}{p}\right) = \frac{\frac{1}{1+\theta} - \frac{1}{p}}{N\left(\frac{1}{1+\theta} - \frac{1}{p}\right) + 1} \in \left(0, \frac{1}{N+p^*}\right),$$

$$\tau = 1 + p^*\alpha\beta = Np^*\left(\frac{1}{1+\theta} - \frac{1}{p}\right) + p^* \in (p^*, N+p^*)\,.$$

Integrating it on $[\rho, 1]$ and using (5.1), we have

$$E_T(\rho)^\gamma \le E_T(1)^\gamma - C(1-\rho^\tau) \le C(\rho^\tau - (1 - M\delta^{(1+\theta)\gamma}))$$

for some $M > 0$, thus $E_T((1 - M\delta^{(1+\theta)\gamma})^{1/\tau}) = 0$, i.e., $\tilde{w}(y) = 0$ for all $y \in B(0, (1 - M\delta^{(1+\theta)\gamma})^{1/\tau})$. Scaling back to x, we conclude the assertion. □

As in Section 4, we obtain the corresponding Theorems 5.1 and 5.2 below to Theorems 1.1 and 1.2, respectively, in the case when $a(x)$ is constant. For the proof of Theorem 5.2, we have only to use the weak Harnack inequality directly to $-\varepsilon\Delta_p v + \Lambda_2 v^\theta \ge 0$ with $0 < \theta < p-1$. We note again that these have been already obtained by [12].

Theorem 5.1. *Assume $a(x)$ to be a positive constant. Let $0 < \theta < p-1$. Then, there exist $L > 0$ and $\varepsilon_0 \in (0, \varepsilon_a)$ such that for each $\varepsilon \in (0, \varepsilon_0)$ the solution u_ε of (1.1) satisfies*

$$u_\varepsilon(x) = a(x) \quad \textit{if } \operatorname{dist}(x, \partial\Omega) \ge L\varepsilon^{1/p}.$$

Theorem 5.2. *Assume $a(x)$ to be a positive constant. Let $\theta \ge p-1$. Then, for every $\varepsilon \in (0, \varepsilon_a)$, $u_\varepsilon < a$ in Ω, and hence $O_\varepsilon = \emptyset$.*

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