Wheeler-Feynman Equations for Rigid Charges

CLASSICAL ABSORBER ELECTRODYNAMICS PART II

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Abstract. This is the second part of our mathematical survey on the equations of motion of classical *absorber electrodynamics*. Here we study the equations of Wheeler-Feynman (WF) electrodynamics, which describe the interaction of finitely many charges by both the advanced and retarded Liénard-Wiechert fields. These equations are non-linear and involve retarded as well as advanced arguments and belong to the class of delay (or functional) differential equations. Such delayed arguments do not permit a direct application of standard PDE techniques. We introduce a general strategy to handle existence and uniqueness questions for such functional differential equations. We observe that any WF solution gives rise to a solution to the Maxwell-Lorentz equations without self-interaction (ML-SI), which are a set of non-linear PDEs without delay that have been studied in Part I. In other words, WF solutions are special solutions among all solutions of the ML-SI equations. Hence, WF solutions arise as solutions to the ML-SI equations for special initial conditions. We employ this observation to prove existence of strong solutions to the WF equations on finite but arbitrarily large time intervals for any given Newtonian Cauchy data (i.e. initial positions and momenta of all charges at one time instant). As a byproduct we also prove existence and uniqueness of strong solutions to the Synge equations on the time half-line for a given history of charge trajectories. The latter equations are just like the WF equations except that they involve only interaction via the retarded Liénard-Wiechert fields, i.e. only retarded arguments.

Contents

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1. Introduction

In this second part of this survey we study the Wheeler-Feynman (WF) electrodynamics [WF49] which describes the classical, electrodynamic interaction of N point-like charges. The idea for this kind of electrodynamics ranges back to [Gau45] and was then picked up by [Sch03, Tet22, Fok29]. Later on Wheeler and Feynman used it to circumvent the self-energy problem (UV divergence) of the Maxwell-Lorentz equations of classical electrodynamics and with its help gave a derivation of the Lorentz-Dirac equations without the need of mass renormalization to describe the radiation reaction of the charges [WF45]; as briefly described in Part I. Furthermore, Wheeler and Feynman also demonstrate that unlike orthodox classical electrodynamics WF electrodynamics is capable of explaining the irreversible effect of radiation. In this work we study the basic equations of WF electrodynamics and, as a byproduct, also the Synge equations [Pau21, Syn40] which are close relatives to the WF equations.

In WF electrodynamics the charges are represented by $\mathbb{R}$-parametrized world lines $\tau \mapsto z_i^\mu(\tau)$, $1 \le i \le N$, with values in $3+1$ dimensional Minkowski space $\mathbb{M} := (\mathbb{R} \times \mathbb{R}^3, g)$ for which we use the metric tensor $g = \operatorname{diag}(1, -1, -1, -1)$. These world lines $1 \le i \le N$ obey the WF equations

$$m_i \ddot{z}_i^\mu(\tau) = e_i \sum_{k \neq i} \frac{1}{2} \left(F_{k,+}^{\mu\nu}(z_i(\tau)) + F_{k,-}^{\mu\nu}(z_i(\tau)) \right) \dot{z}_{i,\nu}(\tau) \tag{1}$$

which describe their interaction via the advanced and retarded Liénard-Wiechert fields $F_{k,+}, F_{k,-}$, of the kth charge, respectively, which are antisymmetric second rank tensor fields on $\mathbb{M}$. The parameters $m_i \neq 0$ denote the masses of the charges and e_i the coupling constants (their charges). The overset dot denotes a differentiation with respect to the parametrization τ of the world line. The Liénard-Wiechert fields $F_{k,\pm}$ are functionals of the world line of the kth charge and are given explicitly by

$$F_{k,\pm}^{\mu\nu} = \partial^\mu A_{k,\pm}^\nu - \partial^\nu A_{k,\pm}^\mu \qquad \text{and} \qquad A_{k,\pm}^\mu(x) = e_k \frac{\dot{z}_k^\mu(\tau_{k,\pm}(x))}{(x - z_k(\tau_{k,\pm}(x)))_\nu \, \dot{z}_k^\nu(\tau_{i,\pm}(x))}. \tag{2}$$

The world line parameters $\tau_{k,\pm} : \mathbb{M} \to \mathbb{R}$ are defined implicitly through

$$z_k^0(\tau_{k,+}(x)) = x^0 + \|\mathbf{x} - \mathbf{z}_k(\tau_{k,+}(x))\| \qquad \text{and} \qquad z_k^0(\tau_{k,-}(x)) = x^0 - \|\mathbf{x} - \mathbf{z}_k(\tau_{k,-}(x))\| \tag{3}$$

where we have used the notation $x = (x^0, \mathbf{x})$ for an $x \in \mathbb{M}$ in order to distinguish the time component x^0 from the spatial components $\mathbf{x} \in \mathbb{R}^3$. Furthermore, $\|\cdot\|$ denotes the euclidean norm. Given an $x \in \mathbb{M}$ and a time-like world line (i.e. $\dot{z}_{k,\mu}\dot{z}_k^\mu > 0$) the solutions $\tau_{k,+}(x)$, $\tau_{k,-}(x)$, are unique and given by the intersection of the forward, backward, light-cone of space-time point x and the world-line z_k, respectively. Therefore, for sufficiently regular, time-like world lines the fields (2) are well-defined everywhere on $\mathbb{M}$ except on the world line z_k where they diverge, so that, as long as two trajectories do not cross, we can expect the WF equations (1) to be well-defined. Furthermore, we infer that the acceleration on the left-hand side of the WF equations depends through (3) on advanced as well as retarded data (with respect to τ) of all the other world lines. This type of equations commonly goes by the name of delay (or functional) differential equations.

Note that the advance and retardation depends on the state of motion (since $F_{k,\pm}$ on the right-hand side of (1) is evaluated at $z_i(\tau)$) and is even unbounded, which does not permit the usual PDE notion of local solutions, and hence, makes it very difficult to study existence and uniqueness of solutions. In fact, the WF equations as well as other delay differential equations with advanced and retarded arguments appear only very sparsely in the literature. [Dri77, DWLvG95] provide great overviews to the topic of delay differential equations. The two basic but unsolved questions connected to the type of delay differential equations studied here are:

(1) Do solutions exist?
E.g., do solutions exist for any given Newtonian Cauchy data (i.e. positions and momenta of all charges at one time instant)?

(2) How can we speak about solutions?
E.g., what kind of data of the solutions is necessary and/or sufficient to characterize them uniquely?

So far only partly answers have been given: While some special but explicit solutions to the WF equations of motions were found [Sch63], general existence of solutions to these equations has only been settled in the case of restricted motion of two point particles with equal charge on a straight line in three dimensional space [Bau97]. There it is shown that all solutions can be characterized by asymptotic positions and momenta. For the Synge equations, existence in three dimensions has been studied in [Ang90]. In a recent work [Luc09] the Fokker variational principle for two charges in three dimensions is discussed mathematically, which can be used to yield WF solutions on finite time intervals by specifying starting and ending points of the two world lines and giving in addition a part of the future of the first charge and a part of the past of the other charge. For all cases uniqueness remains open. Only conjectures about uniqueness of WF solutions can be found, e.g. [WF49, DW65, And67, Syn76]. The two main conjectures are that the WF solutions are uniquely characterized either by Newtonian Cauchy data or strips of world lines such that every light-cone of the ending points of one strip has intersections with all other strips. For the one dimensional WF equations in the case of two equal charges sufficiently far apart existence and uniqueness was shown for Newtonian Cauchy data with relative initial velocities equal zero in [Dri79]. In [Dec10] a WF toy model in three dimensions was given in which two equal charges interact only by the advanced as well as delayed Coulomb forces. For it existence and uniqueness of solutions, which are characterized by world line strips as described above, can be shown. Further literature is about special analytic solutions [Ste92], numerical approximation [DW65], and a special case of existence and uniqueness of solutions to the Synge equations in one dimension[Dri69].

2. Main Results

It is convenient to write the WF equations ($1_{\text{p.2}}$) together with ($2_{\text{p.2}}$) for smeared out charges as a dynamical system in non-relativistic notation and in a special coordinate frame. In order to avoid a highly technical study to exclude the improbable (maybe even impossible) cases of crossing of world lines we employ, as in Part I [BDD10], rigid charges instead of point-like charges for which the WF equations stay well-defined even in the case of two crossing world lines. It should be understood that the point-particle limit of the WF equations for solutions with non-crossing world lines bare no obstacles as the charges do not acquire electrodynamic masses. We define the electric and magnetic field of each charge to be $\mathbf{E}_{i,t} := (F^{i0}(t,\cdot))_{1\le i\le 3}$, $\mathbf{B}_{i,t} := (F_i^{32}(t,\cdot), F_i^{13}(t,\cdot), F_i^{21}(t,\cdot))$, respectively. In this notation the WF equations take the form

$$\begin{aligned} \partial_t \mathbf{q}_{i,t} &= \mathbf{v}(\mathbf{p}_{i,t}) := \frac{\mathbf{p}_{i,t}}{\sqrt{m_i^2 + \mathbf{p}_{i,t}^2}} \\ \partial_t \mathbf{p}_{i,t} &= \sum_{j\neq i} \int d^3x\, \varrho_i(\mathbf{x}-\mathbf{q}_{i,t}) \left(\mathbf{E}_{j,t}^{\text{WF}}(\mathbf{x}) + \mathbf{v}(\mathbf{q}_{i,t}) \wedge \mathbf{B}_{j,t}^{\text{WF}}(\mathbf{x})\right) \end{aligned} \tag{4}$$

for $1 \le i \le N$ and the *WF fields* given by one half the sum of the advanced and retarded Liénard-Wiechert fields (compare ($1_{\text{p.2}}$)):

$$\begin{pmatrix} \mathbf{E}_{i,t}^{\text{WF}} \\ \mathbf{B}_{i,t}^{\text{WF}} \end{pmatrix} = \frac{1}{2} \sum_{\pm} 4\pi e_{\pm} \int ds \int d^3y\, K_{t-s}^{\pm}(\mathbf{x}-\mathbf{y}) \begin{pmatrix} -\nabla \varrho_i(\mathbf{y}-\mathbf{q}_{i,s}) - \partial_s\left(\mathbf{v}(\mathbf{p}_{i,s})\varrho_i(\mathbf{y}-\mathbf{q}_{i,s})\right) \\ \nabla \wedge \left(\mathbf{v}(\mathbf{p}_{i,s})\varrho_i(\mathbf{y}-\mathbf{q}_{i,s})\right) \end{pmatrix}. \tag{5}$$

Here, $K_t^{\pm}(\mathbf{x}) := \triangle^{\pm}(t,\mathbf{x}) = \frac{\delta(\|\mathbf{x}\|\pm t)}{4\pi\|\mathbf{x}\|}$ are the advanced and retarded Green's functions of the d'Alembert operator. That the Liénard-Wiechert fields ($2_{\text{p.2}}$) take the form (5) will be shown in Section $2.1_{\text{p.8}}$. The partial derivative with respect to time t is denoted by ∂_t, the gradient by ∇, the divergence by $\nabla\cdot$, and the curl by $\nabla\wedge$. We shall use the same notation and terminology as for the ML-SI equations in Part I [BDD10], i.e. at time t the ith charge for $1 \le i \le N$ is situated at position $\mathbf{q}_{i,t}$ in space $\mathbb{R}^3$, momentum $\mathbf{p}_{i,t} \in \mathbb{R}^3$ and carries the classical mass $m_i \in \mathbb{R}\setminus\{0\}$. The geometry of the rigid charge is given by the charge distribution $\varrho_i \in C_c^\infty(\mathbb{R}^3,\mathbb{R})$ for $1 \le i \le N$. Because at one place in this work we speak about the Synge equations we introduced the coefficients $e_\pm$ which are used to switch from the WF equations with $e_+ = 1 = e_-$ (which we shall always use if not otherwise noted) to the Synge equations with $e_+ = 0$, $e_- = 1$. For $\varrho_i = \delta^{(3)}$ one retrieves the corresponding equations for point charges, i.e. for in the WF case ($1_{\text{p.2}}$)-($2_{\text{p.2}}$).

Central Observation: Our study is based on the observation that there is an intimate connection between WF and ML-SI dynamics. To see it, let us consider charge trajectories $t \mapsto (\mathbf{q}_{i,t},\mathbf{p}_{i,t})_{1\le i\le N}$ that constitute a solution to the WF equations, i.e. assume they fulfill the Lorentz force law ($4_{\text{p.3}}$) for the WF fields $t \mapsto (\mathbf{E}^{\text{WF}}_{i,t},\mathbf{B}^{\text{WF}}_{i,t})_{1\le i\le N}$ given by ($5_{\text{p.3}}$) which are functionals of these charge trajectories. By definition of the Liénard-Wiechert fields, the WF fields fulfill the Maxwell equations. Hence, the map $t \mapsto (\mathbf{q}_{i,t},\mathbf{p}_{i,t},\mathbf{E}_{i,t},\mathbf{B}_{i,t})_{1\le i\le N} := (\mathbf{q}_{i,t},\mathbf{p}_{i,t},\mathbf{E}^{\text{WF}}_{i,t},\mathbf{B}^{\text{WF}}_{i,t})_{1\le i\le N}$ gives rise to a solution to the ML-SI equations (ML-SI stands for Maxwell-Lorentz without self-interaction) which have been studied in Part I [BDD10], i.e. the Maxwell equations plus the Maxwell constraints

$$
\begin{aligned}
&\partial_t\mathbf{E}_{i,t} = \nabla\wedge\mathbf{B}_{i,t} - 4\pi\mathbf{v}(\mathbf{p}_{i,t})\varrho_i(\cdot-\mathbf{q}_{i,t}) && \nabla\cdot\mathbf{E}_{i,t} = 4\pi\varrho_i(\cdot-\mathbf{q}_{t,i})\\
&\partial_t\mathbf{B}_{i,t} = -\nabla\wedge\mathbf{E}_{i,t} && \nabla\cdot\mathbf{B}_{i,t} = 0.
\end{aligned}
\tag{6}
$$

together with the *Lorentz* equations

$$
\begin{aligned}
\partial_t\mathbf{q}_{i,t} &= \mathbf{v}(\mathbf{p}_{i,t}) := \frac{\mathbf{p}_{i,t}}{\sqrt{m_i^2+\mathbf{p}_{i,t}^2}}\\
\partial_t\mathbf{p}_{i,t} &= \sum_{k\ne i}^{N}\int d^3x\,\varrho_i(\mathbf{x}-\mathbf{q}_{i,t})\left[\mathbf{E}_{k,t}(\mathbf{x})+\mathbf{v}_{i,t}\wedge\mathbf{B}_{k,t}(\mathbf{x})\right].
\end{aligned}
\tag{7}
$$

On the other hand, clearly not all solutions to the ML-SI equations give rise to charge trajectories that obey the WF equations. However, we know from Part I [BDD10] that the initial value problem of the ML-SI equations is well-defined for initial values $(p,F) \in D_w(A)$, i.e. in the admitted domain of initial conditions given by Definition $\text{B.4}_{\text{p.43}}$. Hence, given Newtonian Cauchy data $p := (\mathbf{q}_i^0,\mathbf{p}_i^0)_{1\le i\le N}$ and sufficiently regular initial fields $F := (\mathbf{E}_i^0,\mathbf{B}_i^0)_{1\le i\le N}$, e.g. at time $t_0 \in \mathbb{R}$, there exists a unique solution $t \mapsto (\mathbf{q}_{i,t},\mathbf{p}_{i,t},\mathbf{E}_{i,t},\mathbf{B}_{i,t})_{1\le i\le N} =: M_L[p,F](t,t_0)$ to the ML-SI equations; cf. Theorems $\text{B.6}_{\text{p.44}}$, $\text{B.7}_{\text{p.44}}$ and Definition $\text{B.10}_{\text{p.45}}$. Hence, for fixed Newtonian Cauchy data p one only needs to find special initial fields F such that $(p,F) \in D_w(A)$ in order to yield charge trajectories by the ML-SI time evolution that also solve the WF equations. Such special initial fields can be identified naturally by the following condition:

$$
F = (\mathbf{E}^{\text{WF}}_{i,t},\mathbf{B}^{\text{WF}}_{i,t})_{1\le i\le N}\big|_{t=t_0}.
\tag{8}
$$

This condition states that the initial fields equal the WF fields at initial time t_0. It ensures that the time-evolved fields $t \mapsto (\mathbf{E}_{i,t},\mathbf{B}_{i,t})_{1\le i\le N}$ of the ML-SI solution equal the WF fields $t \mapsto (\mathbf{E}^{\text{WF}}_{i,t},\mathbf{B}^{\text{WF}}_{i,t})_{1\le i\le N}$ for all times because their difference is a solution to the free Maxwell equations which are a set of linear time evolution equations. Having the equality for all times, (7) turns into the WF equations ($4_{\text{p.3}}$), and hence, the charge trajectories of the ML-SI solution solve the WF equations. We may therefore turn the question around and ask: Are there initial conditions for the ML-SI equations that fulfill (8) and thus give rise to

WF solutions? Since the ML-SI equations are well under control we shall employ an iterative procedure on ML-SI solutions the fix points of which will indeed be WF solutions.

For the further discussion we resort on the notation and definitions of Part I [BDD10] as summarized in the Appendix Bp.43. The first result aims at understanding question (2): As discussed in Part I [BDD10], the "worst" behaving WF trajectories $t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1\le i\le N}$ we expect and which we want to also include in our discussion are the Schild solutions [Sch63] (i.e. charges that revolve each other with circular orbits), and those are once differentiable, strictly time-like, and have uniformly bounded accelerations and momenta. The collection of such WF solutions shall be denoted by the set $\mathcal{T}_{\mathrm{WF}}$; cf. Definition 2.23p.19. We say "worst" since for such charge trajectories the acceleration dependent term depending in the Liénard-Wiechert fields does not decay fast enough for the Liénard-Wiechert fields to be square integrable - the scattering solutions on the straight line behave better [Bau97]. The missing decay is modulated by the weight function w. To emphasize that the WF fields $(\mathbf{E}^{\mathrm{WF}}_{i,t}, B^{\mathrm{WF}}_{i,t})$ defined in (5p.3) are functionals of a charge trajectory $t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})$ we employ the notation $(\mathbf{E}^{\mathrm{WF}}_{i,t}, \mathbf{B}^{\mathrm{WF}}_{i,t}) := \frac{1}{2}\sum_\pm M_{\varrho_i,m_i}[(\mathbf{q}_i, \mathbf{p}_i)](t, \pm\infty)$, i.e. one half the sum of the retarded and advanced Liénard-Wiechert fields; the M stands for Maxwell time evolution, cf. 2.17p.14. Similarly, we use the notation $\varphi_t = M_L[\varphi_{t_0}](t, t_0)$ for the strong ML-SI time evolution of the initial data $\varphi^0 \in D_w(A^\infty)$ for the set of sufficiently regular initial data; cf. Definition B.10p.45 and Definition B.4p.43. We prove:

Theorem 2.1 (Characterization of WF Solutions)**.** *There exists a $w \in \mathcal{W}^\infty$ such that the following is true:*

(i) Let $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N}$ be in $\mathcal{T}_{\mathrm{WF}}$ and define

$$(\mathbf{E}^{\mathrm{WF}}_{i,t}, \mathbf{B}^{\mathrm{WF}}_{i,t}) := \frac{1}{2}\sum_\pm e_\pm M_{\varrho_i,m_i}[(\mathbf{q}_i, \mathbf{p}_i)](t, \pm\infty) \qquad \text{and} \qquad \mathbb{R} \ni t \mapsto \varphi_t := (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}^{\mathrm{WF}}_{i,t}, \mathbf{B}^{\mathrm{WF}}_{i,t})_{1\le i\le N}$$

Then, for any $t_0 \in \mathbb{R}$ it holds that $\varphi_{t_0} \in D_w(A^\infty)$ and $t \mapsto \varphi_t$ is a strong solution to the ML-SI equations with initial value φ_{t_0} at $t = t_0$, i.e. $\varphi_t = M_L[\varphi_{t_0}](t, t_0)$ for all $t \in \mathbb{R}$.

(ii) For each $t_0 \in \mathbb{R}$ the following map is injective:

$$i_{t_0} : \mathcal{T}_{\mathrm{WF}} \to D_w(A^\infty),\ (\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N} \mapsto (\mathbf{q}_{i,t_0}, \mathbf{p}_{i,t_0}, \mathbf{E}^{\mathrm{WF}}_{i,t_0}, \mathbf{B}^{\mathrm{WF}}_{i,t_0})_{1\le i\le N}$$

where, again, the WF fields are given by $(\mathbf{E}^{\mathrm{WF}}_{i,t}, \mathbf{B}^{\mathrm{WF}}_{i,t}) := \frac{1}{2}\sum_\pm M_{\varrho_i,m_i}[(\mathbf{q}_i, \mathbf{p}_i)](t, \pm\infty)$ for all $t \in \mathbb{R}$.

This theorem guarantees that all considered WF solutions give rise to sufficient regular initial values for the ML-SI equations, and that each WF solution is uniquely characterized by their corresponding WF fields at one time instant. With respect to question (2) it states that we can use special initial data for the ML-SI initial value problem to speak about the WF solutions. For example, regularity of the WF solution can now simply be inferred by studying the ML-SI equations; cf. Theorem B.7p.44. In order to prove the theorem above a detailed study of the strong Maxwell solutions is required which permits to show that $\varphi_{t_0} \in D_w(A^\infty)$, and which is presented in Section 2.1p.8. Since the charge trajectories of $t \mapsto \varphi_t$ are actually the WF trajectories (using the discussed connection of the WF and ML-SI equations) we conclude the injectivity of the map i_{t_0} by the uniqueness assertion of the ML-SI equations. The formal proof of the above theorem is given in Section 2.2p.19.

This first result, however, does neither touch the question what minimal data is necessary to speak about WF solutions, nor the question of existence of WF solutions. For both one needs to study the range of i_{t_0} which turns out to be very difficult. Before we get to the existence of WF solutions let us explain that the situation is better for the Synge equations on the time half-line. Given past charge trajectories on the half-line $(-\infty, t_0]$ for any $t_0 \in \mathbb{R}$ we can compute the retarded Liénard-Wiechert fields at time t_0 and use them together with the positions and momenta at time t_0 as initial data for the ML-SI dynamics. This way we get existence and uniqueness of solutions to the Synge equations on the half-line $[t_0, \infty)$. The reason why this scenario behaves much better than the one in the case of the WF equations is that the notion of local solutions from PDE theory again makes sense as there is no interaction from the future. The given past charge trajectories on $(-\infty, t_0]$ simply act as external fields. Hence, as a byproduct of the WF analysis one gets:

Theorem 2.2 (Existence and Uniqueness of Synge Solutions)**.** *Let $\mathcal{T}_{\mathrm{SY}}$ be defined as $\mathcal{T}_{\mathrm{WF}}$ except that its elements fulfill the Synge equations instead of the WF equations. Furthermore, for any time interval $I \subset \mathbb{R}$ let $\mathcal{T}_{\triangledown}(I)$ be the collection of all families of once differentiable and strictly time-like charge trajectories.*

(i) Theorem 2.1p.5 also holds for the case of the Synge equations, i.e. for the choice of $e_+ = 0, e_- = 1$ and $\mathcal{T}_{\mathrm{WF}}$ replaced by $\mathcal{T}_{\mathrm{SY}}$.

(ii) For any $t_0 \in \mathbb{R}$ and any family of charge trajectories $(\mathbf{q}^-, \mathbf{p}^-) \in \mathcal{T}_{\triangledown}((-\infty, t_0])$ that fulfills the Synge equations at time t_0 there exist a unique extension $(\mathbf{q}^+, \mathbf{p}^+) \in \mathcal{T}_{\triangledown}([t_0, \infty))$ such that the concatenation

$$(\mathbf{q}, \mathbf{p})(t) = \begin{cases} (\mathbf{q}^-, \mathbf{p}^-)(t) \text{ for } t \le t_0 \\ (\mathbf{q}^+, \mathbf{p}^+)(t) \text{ for } t > t_0 \end{cases} \tag{9}$$

is a once differentiable on $\mathbb{R}$ and solves the Synge equations for $t \ge t_0$.

Clearly, in the case of the Synge equations one only needs past trajectory strips such that all backward light-cones of the charges at time t_0 have intersection points with all other trajectory strips in order to maintain uniqueness. If we ask for solutions to the Synge equations on whole $\mathbb{R}$ we again face the problem that the notion of local solutions makes no sense as the delay is unbounded. A reasonable way around this is to give initial conditions for $t_0 \to \infty$ as in [Bau97]. It seems that this way one should even be able to maintain uniqueness.

For the WF equations question (1), i.e. getting an existence result, turns out to be much more difficult. Having the above characterization of WF solutions $\mathcal{T}_{\mathrm{WF}}$ by the map i_{t_0} in mind, we ask the following: for any $T > 0$ and for any given but strictly time-like *shape of charge trajectories* with uniformly bounded acceleration and momenta in the future $[T, \infty)$ and in the past $(-\infty, T]$, as well as given Newtonian Cauchy data $p = (\mathbf{q}_i^0, \mathbf{p}_i^0)_{1 \le i \le N} \in \mathbb{R}^{6N}$ at time zero, do WF solutions on $[-T, T]$ exist? With "shape of charge trajectories" we mean a prescription to smoothly (or even differentiably) continue the WF solution on $[-T, T]$ to whole $\mathbb{R}$. A simple example of such a prescription is the straight line from the positions of each charge at time $\pm T$ to $\pm\infty$ with constant velocity equal the one of the respective one a time $\pm T$. The shape of future and past charge trajectories is encoded in form of their advanced and retarded Liénard-Wiechert fields $X_{i,T}^+$ and $X_{i,-T}^-$, respectively, which depend on the WF solution on $[-T, T]$ in order to be able to connect smoothly. We shall refer to these fields as boundary fields and express them as functions of the initial values (p, F) of the ML-SI equations that corresponds to the WF solution on $[-T, T]$, i.e. $(p, F) \mapsto X_{i,\pm T}^{\pm}[p, F]$.

In view of the preliminary discussion of this section the task is to find ML-SI solutions $t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1 \le i \le N} = M_L[p, F](t, 0)$ whose initial fields F fulfill the discussed condition (8p.4) adapted to the given boundary fields $X_{i,\pm T}^{\pm}$:

$$(\mathbf{E}_{i,t}, \mathbf{B}_{i,t})|_{t=0} = \frac{1}{2} \sum_{\pm} M_{\varrho_i, m_i}[X_{i,\pm T}^{\pm}, (\mathbf{q}_i, \mathbf{p}_i)](0, \pm T) \tag{10}$$

where the notation $t \mapsto (\mathbf{E}_t, \mathbf{B}_t) = M_{\varrho_i, m_i}[F, (\mathbf{q}, \mathbf{p})](t, t_0)$ denotes the Maxwell solution subject to the charge trajectory $t \mapsto (\mathbf{q}_t, \mathbf{p}_t)$ having initial value $(\mathbf{E}_{i,t}, \mathbf{B}_{i,t})|_{t=t_0} = F$ at time t_0; cf. Definition 2.16p.14 and Theorem 2.14p.12. Note that for $t_0 \to \infty$ the Maxwell time evolution forgets its initial values X_i as we show in Theorem 2.18p.14, so that in the limit we have $M_{\varrho_i, m_i}[(\mathbf{q}_i, \mathbf{p}_i)](t, \pm\infty) = \lim_{t_0 \to \pm\infty} M_{\varrho_i, m_i}[X_i, (\mathbf{q}_i, \mathbf{p}_i)](t, t_0)$. Hence in the limit $T \to \infty$ the condition (10) turns into the discussed condition (8p.4).

Using the discussed connection of between the WF and ML-SI equations, the question of existence of WF solutions on $[-T, T]$ can be rephrased in the question of existence of special initial fields for the ML-SI initial value problem. It seems natural to construct such initial fields by iteration of the map S_T^p:

INPUT: $F = (\mathbf{E}_i^0, \mathbf{B}_i^0)_{1\le i\le N}$ for any fields such that $(p, F) \in D_w(A^\infty)$.

(i) Compute the ML-SI solution $[-T, T] \ni t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N} := M_L[p, F](t, 0)$.

(ii) Compute the advanced and retarded fields

$$(\widetilde{\mathbf{E}}_{i,t}, \widetilde{\mathbf{B}}_{i,t}) = \frac{1}{2}\sum_{\pm} M_{\varrho_i,m_i}[X^\pm_{i,\pm T}, (\mathbf{q}_i, \mathbf{p}_i)](t, \pm T)$$

given by the Maxwell time-evolved boundary fields $X^\pm_{i,\pm T}$ subject to the just computed charge trajectories $t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})$ for $1 \le i \le N$

OUTPUT: $S_T^{p,X^\pm}[F] := (\widetilde{\mathbf{E}}_{i,t}, \widetilde{\mathbf{B}}_{i,t})_{1\le i\le N}|_{t=0}$.

Clearly any fixed point $(\mathbf{E}^{\mathrm{WF}}_{i,t}, \mathbf{B}^{\mathrm{WF}}_{i,t})_{1\le i\le N}$ of this map fulfills the equations (4p.3) together with

$$(\mathbf{E}^{\mathrm{WF}}_{i,t}, \mathbf{B}^{\mathrm{WF}}_{i,t}) = \frac{1}{2}\sum_{\pm} M_{\varrho_i,m_i}[X^\pm_{i,\pm T}, (\mathbf{q}_i, \mathbf{p}_i)](t, \pm T), \text{ for } 1 \le i \le N \tag{11}$$

which turn into (5p.3), and thus $t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1\le i\le N}$ into a WF solutions with Newtonian Cauchy data p either for $T \to \infty$, or if by chance the boundary fields $X^\pm_{i,\pm T}$ are already the correct WF fields. For finite T the WF equations are only satisfied for on time interval $[-T, T]$ where the future and past ends of these strips interact with the prescribes future and past charge trajectories corresponding to the boundary fields $X^\pm_{i,\pm T}$, respectively. We shall prove:

Theorem 2.3 (Existence of WF Solution for Finite Times)**.** *Let Newtonian Cauchy data $p \in \mathcal{P}$ be given. For the maps $S_T^{p,X^\pm}$ for finite $T > 0$ as defined in Definition 2.40*p.25 *the following is true:*

(i) For any boundary fields $X^\pm \in \mathcal{A}_w^{\mathrm{Lip}}$ and T sufficiently small, $S_T^{p,X^\pm}$ has a unique fixed point.

(ii) For any boundary fields $X^\pm \in \mathcal{A}_w^3$ and finite $T > 0$, the map $S_T^{p,X^\pm}$ has a fixed point.

where the classes of boundary fields $\mathcal{A}_w^3$ and $\mathcal{A}_w^{\mathrm{Lip}}$ are given in Definition 2.36p.24 which both include the case of the discussed example of the straight lines as shown in Lemma 2.43p.27. In fact, these classes have been chosen general enough to also include advanced or retarded Liénard-Wiechert fields corresponding to any strictly time-like future and past charge trajectories shapes with uniformly bound acceleration and momenta which at least connect smoothly to the position trajectory $t \mapsto q_{i,t}$ of the respective charges; thanks to to the introduction of the weight w as explained in Part I [BDD10]. The strategy behind the proof is (i) an application of Banach's fixed point Theorem for small times T and (ii) an application of Schauder's fixed point Theorem [Eva98, Chapter 9, Theorem 3, p.502] for all finite times T. The basic ingredient in the proof is that the range of $S_T^{p,X^\pm}$ on the Hilbert space $\mathcal{F}_w := \bigoplus_{i=1}^N L^2_w \oplus L^2_w$ (cf. Definition B.3p.43) can be bounded by a uniform constant depending only on T and p as shown in Lemma 2.46p.30. By the Banach-Alaoglu Theorem this already gives weak compactness. Furthermore, the maximal support in space-time of the fields produced by the charge trajectories on $[-T, T]$ is compact by the finiteness of the speed of light. Within this set of space-time we have good control over the fields as well as their spacial derivative so that the weak compactness already implies strong compactness. How the fields behave on the complementary set of space-time depends on the regularity of the boundary fields only which are therefore assumed to be as good as needed (while not ruling out reasonable cases like the future and past straight lines). We shall in Section 2.3p.21 show that $S_T^{p,X^\pm}$ restricted to the convex hull of its range is a well-defined and continues self-map, which by Schauder's Theorem guarantees the existence of a fixed point F and therewith the existence of a WF solution on $[-T, T]$ for given Newtonian Cauchy data p and given boundary fields $X^\pm$.

Recall that the Synge solutions on the time half-line $[t_0, \infty]$ for times sufficiently close to t_0 give only rise to interactions with the given past trajectories on $[-\infty, t_0]$. In a sense for such small times one solves an external field problem only. Not until larger times the interaction becomes truly retarded. In the worst case, if a charge approaches the speed of light too fast it could even happen that the time coordinate of the intersection of its backward light-cone with another charge trajectory is bounded. That would mean this charge will never interact with the part of the other charges trajectory beyond that maximal time. If this maximal time is already smaller equal t_0 we would again only solve an external field problem and would not see any truly retarded interaction. Such a scenario is of course so special that one would generally not expect it (especially since we have existence and uniqueness of the Synge solutions for as large times bigger than t_0 as we want). For the WF equations, however, we want to be more careful and study these situations in order to appreciate Theorem 2.3. For the WF solutions we give the shape of the charge trajectories for $[-\infty, -T]$ as well as $[T, \infty]$. If the WF solution on $[-T, T]$ does anything crazy like described above we might end up solving only an external field problem as the charge trajectories on $[-T, T]$ only "see" the given past and future shapes of charge trajectories. The following result makes sure that at least for some solutions this is not the case since on an interval $[-L, L]$ with $0 < L \leq T$ they interact exclusively with the charge trajectories on $[-T, T]$ and not with the given boundary fields. We prove:

Theorem 2.4. *Choose $a, b, T > 0$ and Coulomb boundary fields $X^\pm = C$ as defined in Definition 2.42**p.27**. Let further $R > 0$ be the smallest radius such that $\operatorname{supp} \varrho_i \subseteq B_R(0)$. Then, the velocities of all charges of any $ML-SI$ solution with any initial data Newtonian Cauchy data p and any initial fields F such that*

$$\|p\| \leq a, \|\varrho_i\|_{L^2_w} + \|w^{-1/2}\varrho_i\|_{L^2} \leq b, F \in \operatorname{Range} S_T^{p,X^\pm}$$

have an upper bound $\mathbf{v}_T^{a,b}$ with $\|\mathbf{v}_T^{a,b}\| < 1$. For Newtonian Cauchy data $p = (\mathbf{q}_i^0, \mathbf{p}_i^0)_{1\leq i\leq N} = p$ one defines the maximal distance between the initial positions of the charges $\triangle q_{max}(p) := \max_{1\leq i,j\leq N} \|\mathbf{q}_i^0 - \mathbf{q}_j^0\|$. Hence, we can arrange Newtonian Cauchy data p and a maximal charge radius R such that

$$L := \max\left\{\frac{(1 - v_T^{a,b})T - \triangle q_{max} - 2R}{1 + v_T^{a,b}}, T\right\}$$

*is positive. Any fixed point F^* of $S_T^{p,X^\pm}$ gives rise to ML-SI solutions that do not only solve the WF equations (4**p.3**)-(11**p.7**) with boundary fields $X^\pm$ but also the WF equations (4**p.3**)-(5**p.3**) without boundary fields.*

For the assumed Coulomb boundary fields (for times $|t| > T$ the charges are at rest) this result is shown by direct computation using harmonic analysis and a very rough Grönwall estimate from the ML-SI dynamics. Its conditions are therefore quite restrictive but merely technical. Any uniform velocity estimate makes this result redundant as then T can just be chosen arbitrarily large to ensure positivity of L. For two charge of equal sign and restricted to a straight line such an estimate is given for point-like charges in [Bau97]. We expect such a bound also without the restriction to a straight line. However, without such a uniform velocity bound this result already ensures that by Theorem 2.3p.7 we really see true advanced and retarded interaction between the charges of the WF solutions at least for some choices of Newtonian Cauchy data and charge densities.

2.1. **Strong Solutions to the Maxwell Equations.** In this section we give the explicit representation formulas for strong solutions $t \mapsto (\mathbf{E}_t, \mathbf{B}_t)$ to the Maxwell equations given a charge trajectory or charge-current density which will be frequently used in both of the following sections.

Definition 2.5 (Charge Trajectories)**.** *We shall call any map*

$$(\mathbf{q}, \mathbf{p}) \in C^1(\mathbb{R}, \mathbb{R}^3 \times \mathbb{R}^3),\ t \mapsto (\mathbf{q}_t, \mathbf{p}_t)$$

a charge trajectory where $\mathbf{q}_t$ *denotes the position and* $\mathbf{p}_t$ *the momentum of the charge with mass* $m \neq 0$. *We collect all time-like trajectories in the set*

$$\mathcal{T}^1_\vee := \left\{(\mathbf{q},\mathbf{p}) \in C^1(\mathbb{R},\mathbb{R}^3\times\mathbb{R}^3) \;\middle|\; \|\mathbf{v}(\mathbf{p}_t)\| < 1 \text{ for all } t \in \mathbb{R}\right\},$$

and all strictly time-like trajectories in the set

$$\mathcal{T}^1_\triangledown := \left\{(\mathbf{q},\mathbf{p}) \in \mathcal{T}^1_\vee \;\middle|\; \exists v_{max} < 1 \text{ such that } \sup_{t\in\mathbb{R}} \|\mathbf{v}(\mathbf{p}_t)\| \leq v_{max}\right\}$$

where $\mathbf{v}(\mathbf{p}) := \frac{\mathbf{p}}{\sqrt{m^2+\mathbf{p}^2}}$. *We shall also use the notation* $\mathcal{T}_\# := \times_{i=1}^N \mathcal{T}^1_\#$ *for the N-fold Cartesian product where* $\#$ *is a placeholder for* $\vee$ *or* $\triangledown$. *Furthermore, two charge trajectories are equal if and only if their positions and momenta are equal for all times.*

Definition 2.6 (Charge-Current Densities)**.** *We shall call any pair of maps* $\rho : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}, (t,\mathbf{x}) \mapsto \rho_t(\mathbf{x})$ *and* $\mathbf{j} : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^3, (t,\mathbf{x}) \mapsto \mathbf{j}_t(\mathbf{x})$ *a charge-current density whenever:*

(i) For all $\mathbf{x} \in \mathbb{R}^3$*:* $\rho_{(\cdot)}(\mathbf{x}) \in C^1(\mathbb{R},\mathbb{R})$ *and* $\mathbf{j}_{(\cdot)}(\mathbf{x}) \in C^1(\mathbb{R},\mathbb{R}^3)$.

(ii) For all $t \in \mathbb{R}$*:* $\rho_t, \partial_t\rho_t \in C^\infty(\mathbb{R}^3,\mathbb{R})$ *and* $\mathbf{j}_t, \partial_t\mathbf{j}_t \in C^\infty(\mathbb{R}^3,\mathbb{R}^3)$.

(iii) For all $(t,\mathbf{x}) \in \mathbb{R}\times\mathbb{R}^3$*:* $\partial_t\rho_t(\mathbf{x}) + \nabla\cdot\mathbf{j}_t(\mathbf{x}) = 0$ *which we call continuity equation.*

We denote the set of such pairs $(\rho,\mathbf{j})$ *by* $\mathcal{D}$.

We shall also need the following connection between charge trajectories and charge-current densities:

Definition 2.7 (Induced Charge-Current Densities)**.** *For* $\varrho \in C^\infty_c(\mathbb{R}^3,\mathbb{R})$ *and* $(\mathbf{q},\mathbf{p}) \in \mathcal{T}^1_\vee$ *we call* $(\rho,\mathbf{j}) \in \mathcal{D}$ *defined by*

$$\rho_t(\mathbf{x}) := \varrho(\mathbf{x}-\mathbf{q}_t) \qquad \text{and} \qquad \mathbf{j}_t(\mathbf{x}) := \frac{\mathbf{p}_t}{\sqrt{m^2+\mathbf{p}_t^2}}\varrho(\mathbf{x}-\mathbf{q}_t)$$

for all $(t,\mathbf{x}) \in \mathbb{R}\times\mathbb{R}^3$ *the* ϱ *induced charge-current density of* $(\mathbf{q},\mathbf{p})$ *with mass m.*

The Maxwell equations including the Maxwell constraints for a given charge-current density $(\rho,\mathbf{j}) \in \mathcal{D}$ read:

$$\begin{aligned} \dot{\mathbf{E}}_t &= \nabla\wedge\mathbf{B}_t - 4\pi\mathbf{j}_t & \qquad \nabla\cdot\mathbf{E}_t &= 4\pi\rho_t \\ \dot{\mathbf{B}}_t &= -\nabla\wedge\mathbf{E}_t. & \qquad \nabla\cdot\mathbf{B}_t &= 0. \end{aligned} \tag{12}$$

The class of fields $(\mathbf{E}_t,\mathbf{B}_t)$ we are interested in is:

Definition 2.8 (Space of the Fields)**.** $\mathcal{F}^1 := C^\infty(\mathbb{R}^3,\mathbb{R}^3) \oplus C^\infty(\mathbb{R}^3,\mathbb{R}^3)$.

The class of solutions to these Maxwell equations we want to study is characterized by:

Definition 2.9 (Maxwell Solutions)**.** *Let* $t_0 \in \mathbb{R}$ *and* $F^0 \in \mathcal{F}^1$. *Then any mapping* $F : \mathbb{R} \to \mathcal{F}^1, t \mapsto F_t := (\mathbf{E}_t,\mathbf{B}_t)$ *that solves (12) for initial value* $F_t|_{t=t_0} = F^0$ *is called a solution to the Maxwell equations with* t_0 *initial value* F^0.

The explicit representation formulas are constructed with the help of:

Definition 2.10 (Green's Functions of the d'Alembert)**.** *We set*

$$K_t^\pm(\mathbf{x}) := \frac{\delta(\|\mathbf{x}\| \pm t)}{4\pi\|\mathbf{x}\|}$$

where δ denotes the one-dimensional Dirac delta distribution. Furthermore, for every $f \in C^\infty(\mathbb{R}^3)$ we define

$$K_t^\pm * F(\mathbf{x}) = \begin{cases} 0 & \text{for } \pm t > 0 \\ t \fint_{\partial B_{|t|}(\mathbf{x})} d\sigma(y) F(\mathbf{y}) := t \int_{\partial B_{|t|}(\mathbf{x})} d\sigma(y) \frac{F(\mathbf{y})}{4\pi t^2} & \text{otherwise} \end{cases}$$

In the next lemma we collect useful properties of these Green's functions.

Lemma 2.11 (Green's Functions Properties)**.** *The distributions $K_t^\pm$ introduced in Definition 2.10**p.9* *have the following properties:*

(i) For any $f \in C^\infty(\mathbb{R}^3)$ the mapping $(t, \mathbf{x}) \mapsto [K_t^\pm * f](\mathbf{x})$ is in $C^\infty((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^3)$, $\square K_t^\pm * f = 0$ for $t \neq 0$ and for any $n \in \mathbb{N}$

$$\lim_{t \to 0\mp} \begin{pmatrix} \partial_t^{2n} K_t^\pm * f \\ \partial_t^{2n+1} K_t^\pm * f \end{pmatrix} = \begin{pmatrix} 0 \\ \mp \triangle^n f \end{pmatrix}. \tag{13}$$

(ii) For any $f \in C^\infty(\mathbb{R}^3)$ and $K_t = \sum_\pm \mp K_t^\pm$ the mapping $(\mathbb{R} \setminus \{0\}) \times \mathbb{R} \ni (t, \mathbf{x}) \mapsto [K_t^\pm * f](\mathbf{x})$ is continuously extendable to a $C^\infty(\mathbb{R} \times \mathbb{R}^3)$ function. Furthermore, $\square K_t * f = 0$ for all $t \in \mathbb{R}$.

(iii) Let $\mathbb{R}^3 \times \mathbb{R} \ni (\mathbf{x}, t) \mapsto f_t(\mathbf{x})$ be a map that is for each fixed $\mathbf{x} \in \mathbb{R}^3$ an once continuously differentiable function and for each fixed $t \in \mathbb{R}$ infinitely often differentiable then the following estimates hold for an $R \geq |t|$:

$$\|[K_t * f_t](\mathbf{x})\| \leq R \sup_{\mathbf{y} \in \partial B_R(\mathbf{x})} \|f_t(\mathbf{y})\| \qquad \text{and} \qquad \|[K_t * f_t](\mathbf{x})\| \leq \sup_{\mathbf{y} \in \partial B_R(\mathbf{x})} \left(\|f_t(\mathbf{y})\| + \frac{R^2}{3} \|\triangle f(\mathbf{y})\| \right)$$

Furthermore, for all $n \in \mathbb{N}$ it is true that

$$\lim_{t \to 0} K_t * f_t = 0 \qquad \text{and} \qquad \lim_{t \to 0} \partial_t K_t * f_t = f_0.$$

Proof. A straightforward computation yields

$$K_t^\pm * f = \mp t \fint_{\partial B_{\mp t}(0)} d\sigma(y) f(\cdot - \mathbf{y}) \tag{14}$$

$$\partial_t K_t^\pm * f = \mp \fint_{\partial B_{\mp t}(0)} d\sigma(y)\ f(\cdot - \mathbf{y}) \mp \frac{t^2}{3} \fint_{B_{\mp t}(0)} d^3y\ \triangle f(\cdot - \mathbf{y}) \tag{15}$$

$$\partial_t^2 K_t^\pm * f = K_t^\pm * \triangle f = \triangle K_t^\pm * f. \tag{16}$$

(i) Therefore, the first and second derivatives exist with respect to t, while the second derivative can be written as a spacial derivative on f. By induction one easily computes all combinations of $\mathbf{x}$ and t derivatives and finds that the mapping $(t, \mathbf{x}) \mapsto [K_t^\pm * f](\mathbf{x})$ is in $C^\infty\left((\mathbb{R} \setminus \{0\}) \times \mathbb{R}^3\right)$. With (14), (15), (16) and induction in $\mathbb{N}$ together with Lebesgue's differentiation theorem one finds (13). (ii) With (i) we need to show that for any $f \in C^\infty(\mathbb{R}^3)$ the limits of $K_t * f$ and $\partial_t K_t * f$ from the right and from the left exist and agree at $t = 0$. The former case is clear because the limit is zero. Regarding the latter we observe

$$\lim_{t \to 0+} \partial_t K_t * f = \lim_{t \to 0+} \partial_t K_t^- * f = f = -\lim_{t \to 0-} \partial_t K_t^+ * f = \lim_{t \to 0-} \partial_t K_t * f.$$

$\lim_{t \to 0} \square K_t * f = 0$ is a special case of the above. (iii) The estimates are the immediate consequence of (14) and (15). The limits can be computed by

$$\lim_{t \to 0} \left\|[K_t * f_t](\mathbf{x})\right\| \leq \lim_{t \to 0} \left\|[K_t * (f_t - f_0)](\mathbf{x})\right\| + \lim_{t \to 0} \left\|[K_t * f_0](\mathbf{x})\right\|$$

where the second term is zero by (i). For every $\mathbf{x} \in \mathbb{R}^3$, $f_t(\mathbf{x})$ is continuous in t, therefore choosing t small enough and $R > |t|$ we obtain

$$\lim_{t\to 0} \left\| [K_t * (f_t - f_0)](\mathbf{x}) \right\| \leq R \lim_{t\to 0} \sup_{y\in B_\delta(\mathbf{x})} \| f_t(\mathbf{y}) - f_0(\mathbf{y}) \| = 0.$$

Similarly, we find

$$\lim_{t\to 0} \left\| [\partial_t K_t * f_t](\mathbf{x}) - f_0(\mathbf{x}) \right\| \leq \lim_{t\to 0} \left\| [\partial_t K_t * (f_t - f_0)](\mathbf{x}) \right\| + \lim_{t\to 0} \left\| [\partial_t K_t * f_0](\mathbf{x}) - f_0(\mathbf{x}) \right\|$$

while, again, the second term is zero by (i$_{\text{p.10}}$). The same continuity argument as above gives

$$\lim_{t\to 0} \left\| [\partial_t K_t * (f_t - f_0)](\mathbf{x}) \right\| \leq \lim_{t\to 0} \sup_{y\in B_\delta(\mathbf{x})} \left(\| f_t(\mathbf{y}) - f_0(\mathbf{y}) \| + \frac{R^2}{3} \| \triangle f_t(\mathbf{y}) - \triangle f_0(\mathbf{y}) \| \right) = 0$$

which concludes the proof. □

REMARK 2.12. *In the future we will always denote this continuous extension by the same symbol K_t. It is often called the* propagator *of the homogeneous wave equation.*

A simply consequence of this lemma is:

Corollary 2.13 (Kirchoff's formula)**.** *A solution $t \mapsto A_t$ of the homogeneous wave equation $\Box A_t = 0$ for initial value $A_t|_{t=0} = A^0$ and $\partial_t A_t|_{t=0} = \dot{A}^0$, for $A^0, \dot{A}^0 \in C^\infty(\mathbb{R}^3)$, is given by*

$$A_t = \partial_t K_t * A^0 + K_t * \dot{A}^0 \tag{17}$$

The next result gives explicit representation formulas of the Maxwell equations (12$_{\text{p.9}}$). These formulas can be constructed by the following line of thought: In the distribution sense every solution to the Maxwell equations (12$_{\text{p.9}}$) is also a solution to

$$\Box \begin{pmatrix} \mathbf{E}_t \\ \mathbf{B}_t \end{pmatrix} = 4\pi \begin{pmatrix} -\nabla\rho_t - \partial_t \mathbf{j}_t \\ \nabla \wedge \mathbf{j}_t \end{pmatrix}$$

for initial values

$$(\mathbf{E}_t, \mathbf{B}_t)\Big|_{t=t_0} = (\mathbf{E}^0, \mathbf{B}^0) \qquad \text{as well as} \qquad \partial_t(\mathbf{E}_t, \mathbf{B}_t)\Big|_{t=t_0} = (\nabla \wedge \mathbf{B}^0 - 4\pi \mathbf{j}_{t_0}, -\nabla \wedge \mathbf{E}^0). \tag{18}$$

Using the abbreviation $F_t^\# = (\mathbf{E}_t^\#, \mathbf{B}_t^\#)$, using # as placeholder for future superscripts, and with the help of the Green's functions from Definition 2.10$_{\text{p.9}}$ we can easily guess the general form of any solution to these equations which is given by:

$$F_t = F_t^{hom} + \int_{-\infty}^{\infty} ds\, K_{t-t_0-s}^{\pm} * \begin{pmatrix} -\nabla\rho_{t_0+s} - \partial_s \mathbf{j}_{t_0+s} \\ \nabla \wedge \mathbf{j}_{t_0+s} \end{pmatrix} \tag{19}$$

where any homogeneous solution F_t^{hom} fulfills $\Box F_t^{hom} = 0$. Considering the forward as well as backward time evolution we regard two different kinds of initial value problems:

(i) Initial fields F^0 are given at some time $t_0 \in \mathbb{R} \cup \{-\infty\}$ and propagated to a time $t > t_0$.
(ii) Initial fields F^0 are given at some time $t_0 \in \mathbb{R} \cup \{+\infty\}$ and propagated to a time $t < t_0$.

The kind of initial value problem posed will then determine F_t^{hom} and the corresponding Green's function $K_t^\pm$. For (i) we shall use K_t^- and for (ii) K_t^+ which are uniquely determined by $\Box K_t^\pm = \delta(t)\delta^3$ and $K_t^\pm = 0$ for $\pm t > 0$. Without a proof we note at least for time-like charge trajectories and $\mp(t - t_0) > 0$

$$\Box \int_{\pm\infty}^{0} ds\, K_{t-t_0-s}^{\pm} * \begin{pmatrix} -\nabla\rho_{t_0+s} - \partial_s \mathbf{j}_{t_0+s} \\ \nabla \wedge \mathbf{j}_{t_0+s} \end{pmatrix} = \int_{\pm\infty}^{0} ds\, \Box K_{t-t_0-s}^{\pm} * \begin{pmatrix} -\nabla\rho_{t_0+s} - \partial_s \mathbf{j}_{t_0+s} \\ \nabla \wedge \mathbf{j}_{t_0+s} \end{pmatrix} = 0$$

by Lemma 2.11p.10. Terms of this kind will simply be added to the homogeneous solution while here we denote this sum by the same symbol F_t^{hom}. This way we arrive at two solution formulas. One being suitable for our forwards initial value problem, i.e. $t - t_0 > 0$,

$$F_t = F_t^{hom} + 4\pi \int_0^{t-t_0} ds\, K^-_{t-t_0-s} * \begin{pmatrix} -\nabla\rho_{t_0+s} - \partial_s \mathbf{j}_{t_0+s} \\ \nabla \wedge \mathbf{j}_{t_0+s} \end{pmatrix},$$

and the other suitable for the backwards initial value problem, i.e. $t - t_0 < 0$,

$$F_t = F_t^{hom} + 4\pi \int_{t-t_0}^{0} ds\, K^+_{t-t_0-s} * \begin{pmatrix} -\nabla\rho_{t_0+s} - \partial_s \mathbf{j}_{t_0+s} \\ \nabla \wedge \mathbf{j}_{t_0+s} \end{pmatrix}.$$

As a last step one needs to identify the homogeneous solutions which satisfy the given initial conditions (18p.11). With Corollary 2.13p.11 we have given the explicit representation formula:

$$F_t^{hom} := \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t-t_0} * F^0.$$

Therefore, using the definition of $K_t = \sum_\pm \mp K_t^\pm$ and a substitution in the integration variable, we finally arrive at the expression for $t \in \mathbb{R}$:

$$F_t = \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t-t_0} * F^0 + K_{t-t_0} * \begin{pmatrix} -4\pi \mathbf{j}_{t_0} \\ 0 \end{pmatrix} + 4\pi \int_{t_0}^{t} ds\, K_{t-s} * \begin{pmatrix} -\nabla\rho_s - \partial_s \mathbf{j}_s \\ \nabla \wedge \mathbf{j}_s \end{pmatrix}.$$

Theorem 2.14 (Maxwell Solutions)**.** *Let $(\rho, \mathbf{j}) \in \mathcal{D}$ be a given charge-current density.*

(i) Given $(\mathbf{E}^0, \mathbf{B}^0) \in \mathcal{F}^1$ fulfilling the Maxwell constraints $\nabla \cdot \mathbf{E}^0 = 4\pi\rho_{t_0}$ and $\nabla \cdot \mathbf{B}_{t_0} = 0$, then for any $t_0 \in \mathbb{R}$ the mapping $t \mapsto F_t = (\mathbf{E}_t, \mathbf{B}_t)$ with

$$\begin{pmatrix} \mathbf{E}_t \\ \mathbf{B}_t \end{pmatrix} := \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t-t_0} * \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{B}^0 \end{pmatrix} + K_{t-t_0} * \begin{pmatrix} -4\pi \mathbf{j}_{t_0} \\ 0 \end{pmatrix} + 4\pi \int_{t_0}^{t} ds\, K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_s \\ \mathbf{j}_s \end{pmatrix}$$

for all $t \in \mathbb{R}$ is $\mathcal{F}^1$ valued, infinitely often differentiable and a solution to the Maxwell equations (12p.9) with t_0 initial value F^0.

(ii) Furthermore, if for fixed $t_0, t^* \in \mathbb{R}$ and $\mathbf{x}^* \in \mathbb{R}^3$ it holds that

(20) $$K_{t-t_0} * \varrho_{t_0} = 0 \quad \text{and} \quad K_{t-t_0} * \mathbf{j}_{t_0} = 0$$

for all $t \in B_1(t^*)$ and $\mathbf{x} \in B_1(\mathbf{x}^*)$, then statement (i) restricted to such $(t, \mathbf{x})$ is also true for initial fields $(\mathbf{E}^0, \mathbf{B}^0) = 0$.

Proof. The regularity for the first two terms is given by Lemma 2.11p.10. The third term is well-defined by Definition 2.6p.9. Lemma 2.11p.10 states that its integrand is infinitely often differentiable in t and $\mathbf{x}$. As the integral goes over a compact set it inherits the regularity from the integrand. In the following we treat both cases (i) and (ii) together. We shall frequently commute spatial differential operators with integrals which is justified because the integrals go over compact sets and the integrand is continuously differentiable. It is convenient to make partial integrations in the third term first to yield:

$$K_{t-t_0} * \begin{pmatrix} -4\pi \mathbf{j}_{t_0} \\ 0 \end{pmatrix} + 4\pi \int_{t_0}^{t} ds\, K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_s \\ \mathbf{j}_s \end{pmatrix} = 4\pi \int_{t_0}^{t} ds\, \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho_s \\ \mathbf{j}_s \end{pmatrix}.$$

The spatial partial integrations hold by Definition 2.10p.9. The partial integration in s holds as, according to Lemma 2.11p.10, the boundary terms give $4\pi[K_{t-s} * \mathbf{j}_s]_{s=t_0}^{s=t} = -4\pi K_{t-t_0} * \mathbf{j}_{t_0}$. Next we verify the Maxwell constraints. At first for the electric field:

$$\nabla \cdot \mathbf{E}_t = \partial_t K_{t-t_0} * \nabla \cdot \mathbf{E}^0 + 4\pi \int_{t_0}^{t} ds\, \left[-\triangle K_{t-s} * \rho_s - \partial_t K_{t-s} * \nabla \cdot \mathbf{j}_s \right].$$

Applying the continuity equation, cf. 2.6p.9, in the last term we get

$$\ldots = \partial_t K_{t-t_0} * \nabla \cdot \mathbf{E}^0 + 4\pi \int_{t_0}^{t} ds \ \left[-\triangle K_{t-s} * \rho_s + \partial_t K_{t-s} * \partial_s \rho_s \right]$$

After a partial integration in the last term we find

$$\ldots = \partial_t K_{t-t_0} * \nabla \cdot \mathbf{E}^0 + 4\pi \left[\partial_s K_{t-s} * \rho_s \right]_{s=t_0}^{s=t} + 4\pi \int_{t_0}^{t} ds \, \square K_{t-s} * \rho_s.$$

Lemma 2.11p.10 identifies the middle term $4\pi \left[\partial_s K_{t-s} * \rho_s \right]_{s=t_0}^{s=t} = 4\pi\rho_t - 4\pi\partial_t K_{t-t_0} * \rho_{t_0}$ and states that the last term is zero. Therefore,

$$\ldots = \partial_t K_{t-t_0} * \nabla \cdot \mathbf{E}^0 - 4\pi\partial_t K_{t-t_0} * \rho_{t_0} + 4\pi\rho_t.$$

In the case (i) we have $\nabla \cdot \mathbf{E}^0 = 4\pi\varrho_{t_0}$ and the first two terms cancel each other. In the case (ii) these two terms are identically zero because of (20p.12). Hence, we get for both cases $\nabla \cdot \mathbf{E}_t = 4\pi\rho_t$. Second, for the magnetic field we immediately get $\nabla \cdot \mathbf{B}_t = \partial_t K_{t-t_0} * \nabla \cdot \mathbf{B}_0 = 0$ because in the case (i) $\nabla \cdot \mathbf{B}_0 = 0$ and in the case (ii) $\mathbf{B}_0 = 0$. Therefore, the Maxwell constraints are fulfilled in both cases. Next we verify the rest of the Maxwell equations:

$$\begin{aligned}
\boxed{1} := \left(\partial_t - \begin{pmatrix} 0 & \nabla\wedge \\ -\nabla\wedge & 0 \end{pmatrix} \right) \begin{pmatrix} \mathbf{E}_t \\ \mathbf{B}_t \end{pmatrix} &= \begin{pmatrix} \triangle + \nabla \wedge (\nabla \wedge \cdot) & 0 \\ 0 & \triangle + \nabla \wedge (\nabla \wedge \cdot) \end{pmatrix} K_{t-t_0} * \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{B}^0 \end{pmatrix} \\
&+ 4\pi\partial_t \int_{t_0}^{t} ds \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho_s \\ \mathbf{j}_s \end{pmatrix} - 4\pi \int_{t_0}^{t} ds \begin{pmatrix} 0 & \nabla \wedge (\nabla \wedge \cdot) \\ 0 & \partial_t \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho_s \\ \mathbf{j}_s \end{pmatrix} \\
&=: \boxed{2} + \boxed{3} + \boxed{4}
\end{aligned}$$

where we have used Equation (16p.10) from Lemma 2.11p.10 in the first term, which together with $\nabla \cdot \mathbf{B}^0 = 0$ further reduces to

$$\boxed{2} = \nabla K_{t-t_0} * \begin{pmatrix} \nabla \cdot \mathbf{E}^0 \\ 0 \end{pmatrix}.$$

The time derivative in the second term gives

$$\begin{aligned}
\boxed{3} = 4\pi\partial_t \int_{t_0}^{t} ds \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho_s \\ \mathbf{j}_s \end{pmatrix} &= 4\pi \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho_s \\ \mathbf{j}_s \end{pmatrix} \Bigg|_{s\to t} \\
&+ 4\pi \int_{t_0}^{t} ds \begin{pmatrix} -\partial_t \nabla & -\partial_t^2 \\ 0 & \partial_t \nabla\wedge \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho_s \\ \mathbf{j}_s \end{pmatrix}
\end{aligned}$$

where Lemma 2.11p.10 states that the first term on the right-hand side equals $-4\pi \begin{pmatrix} \mathbf{j}_t \\ 0 \end{pmatrix}$. Therefore, with $\nabla \wedge (\nabla \wedge \cdot) = \nabla(\nabla \cdot (\cdot)) - \triangle$ we yield

$$\boxed{1} = \nabla K_{t-t_0} * \begin{pmatrix} \nabla \cdot \mathbf{E}^0 \\ 0 \end{pmatrix} + \begin{pmatrix} -4\pi\mathbf{j}_t \\ 0 \end{pmatrix} + 4\pi \int_{t_0}^{t} ds \begin{pmatrix} -\partial_t \nabla & -\square - \nabla(\nabla\cdot) \\ 0 & 0 \end{pmatrix} K_{t-s} * \begin{pmatrix} \rho_s \\ \mathbf{j}_s \end{pmatrix}.$$

According to Lemma 2.11p.10, the term involving the $\square$ is zero. Inserting the continuity equation for the current, i.e. $\nabla \cdot \mathbf{j}_t = -\partial_t \rho_t$, together with another partial integration in the last term, the electric (first) component of this vector equals

$$\ldots = -4\pi\mathbf{j}_t + \left(\nabla K_{t-t_0} * \nabla \cdot \mathbf{E}^0 + 4\pi \left[K_{t-s} * \nabla\rho_s \right]_{s=t_0}^{s=t} \right).$$

Again, by Lemma 2.11p.10 the braket yields

$$K_{t-t_0} * \nabla \cdot \mathbf{E}^0 - 4\pi K_{t-t_0} * \nabla \rho_{t_0}$$

In the case (i) $\nabla \cdot \mathbf{E}^0 = 4\pi\rho_{t_0}$ so that both terms cancel while in case (ii) both terms are identically zero by $\mathbf{E}^0 = 0$ and (20p.12). Hence,

$$\boxed{1} = \begin{pmatrix} -4\pi \mathbf{j}_t \\ 0 \end{pmatrix},$$

and, thus, $t \to (\mathbf{E}_t, \mathbf{B}_t)$ solves the Maxwell equations (12p.9). The initial values can be computed with Lemma 2.11p.10

$$\begin{pmatrix} \mathbf{E}_t \\ \mathbf{B}_t \end{pmatrix}\Bigg|_{t=t_0} = \lim_{t\to t_0} \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t-t_0} * \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{B}^0 \end{pmatrix} = \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{B}^0 \end{pmatrix}.$$

□

REMARK 2.15. *Clearly one needs less regularity of the initial values in order to get a strong solution. However, we will only need initial values in $\mathcal{F}^1$. The explicit formula of the solutions (after the additional partial integration as noted in the beginning of the proof) was already found in* [KS00]*[(A.24),(A.25)]*[1] *where it was derived with the help of the Fourier transform.*

Theorem 2.14p.12 gives rise to the following definition:

Definition 2.16 (Maxwell Time Evolution)**.** *Let $(\rho, \mathbf{j})$ be the $\varrho \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ induced charge-current density of a given a charge trajectory $(\mathbf{q}, \mathbf{p}) \in \mathcal{T}^1_\vee$ with mass $m \neq 0$, cf. Definition 2.7p.9. Then denote the solution $t \mapsto F_t$ of the Maxwell equations given by Theorem 2.14p.12 corresponding to $(\rho, \mathbf{j})$ and t_0 initial values $F^0 = (\mathbf{E}_0, \mathbf{B}_0) \in \mathcal{F}^1$ by*

$$t \mapsto M_{\varrho,m}[F^0, (\mathbf{q}, \mathbf{p})](t, t_0) := F_t.$$

The second result of this section puts the well-known Liénard-Wiechert field formulas of time-like charge trajectories on mathematical rigorous grounds.

Definition 2.17 (Liénard-Wiechert Fields)**.** *Let $(\mathbf{q}, \mathbf{p}) \in \mathcal{T}^1_\triangledown$ be a strictly time-like charge trajectory and $(\rho, \mathbf{j})$ the $\varrho \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ induced charge-current density for some mass $m \neq 0$, cf. Definitions 2.5p.8 and 2.7p.9. Then we define*

$$t \mapsto M_{\varrho,m}[(\mathbf{q}, \mathbf{p})](t, \pm\infty) := 4\pi \int_{\pm\infty}^t ds\; K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_s \\ \mathbf{j}_s \end{pmatrix}$$

which we call the advanced and retarded Liénard-Wiechert fields of the charge trajectory $(\mathbf{q}, \mathbf{p})$.

That this definition makes sense for charge trajectories in $\mathcal{T}^1_\triangledown$ as the Maxwell time evolution forgets its asymptotic $\mathcal{F}^1$ initial data is part of the content of the next theorem:

Theorem 2.18 (Liénard-Wiechert Fields)**.** *Let $(\mathbf{q}, \mathbf{p}) \in \mathcal{T}^1_\triangledown$ be a strictly time-like charge trajectory and $(\rho, \mathbf{j})$ the $\varrho \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ induced charge-current density for some mass $m \neq 0$, cf. Definitions 2.5p.8 and 2.7p.9. Furthermore, let $F^0 = (\mathbf{E}^0, \mathbf{B}^0) \in \mathcal{F}^1$ be fields which fulfill the Maxwell constraints $\nabla \cdot \mathbf{E}^0 = 4\pi\rho_{t_0}$ and $\nabla \cdot \mathbf{B}_{t_0} = 0$ as well as*

$$\|\mathbf{E}^0(\mathbf{x})\| + \|\mathbf{B}^0(\mathbf{x})\| + \|\mathbf{x}\| \sum_{i=1}^3 \left(\|\partial_{\mathbf{x}_i} \mathbf{E}^0(\mathbf{x})\| + \|\partial_{\mathbf{x}_i} \mathbf{B}^0(\mathbf{x})\| \right) = \underset{\|\mathbf{x}\|\to\infty}{\mathrm{O}} \left(\|\mathbf{x}\|^{-\epsilon}\right) \tag{21}$$

[1] There seems to be a misprint in equation [KS00][(A.24)]. However, (A.20) from which it is derived is correct.

for some $\epsilon > 0$. *Then for all* $t \in \mathbb{R}$

$$\begin{aligned} M_{\varrho,m}[(\mathbf{q},\mathbf{p})](t,\pm\infty) &= \text{pw-lim}_{t_0\to\pm\infty} M_{\varrho,m}[F^0,(\mathbf{q},\mathbf{p})](t,t_0) \\ &= 4\pi\int_{\pm\infty}^t ds\,\left[K_{t-s} * \begin{pmatrix}-\nabla & -\partial_s \\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\rho_s \\ \mathbf{j}_s\end{pmatrix}\right] = \int d^3z\,\varrho(\mathbf{z})\begin{pmatrix}\mathbf{E}_t^{LW\pm}(\cdot-\mathbf{z}) \\ \mathbf{B}_t^{LW\pm}(\cdot-\mathbf{z})\end{pmatrix} \end{aligned} \tag{22}$$

is in $\mathcal{F}^1$ *for*

$$\mathbf{E}_t^{LW\pm}(\mathbf{x}-\mathbf{z}) := \left[\frac{(\mathbf{n}\pm\mathbf{v})(1-\mathbf{v}^2)}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}\|^2(1\pm\mathbf{n}\cdot\mathbf{v})^3} + \frac{\mathbf{n}\wedge[(\mathbf{n}\pm\mathbf{v})\wedge\mathbf{a}]}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}\|(1\pm\mathbf{n}\cdot\mathbf{v})^3}\right]^{\pm} \tag{23}$$

$$\mathbf{B}_t^{LW\pm}(\mathbf{x}-\mathbf{z}) := \mp[\mathbf{n}\wedge\mathbf{E}_t(\mathbf{x}-\mathbf{z})]^{\pm} \tag{24}$$

and

$$\begin{array}{llllll} \mathbf{q}^{\pm} & := \mathbf{q}_{t^{\pm}} & \mathbf{v}^{\pm} & := \mathbf{v}(\mathbf{p}_{t^{\pm}}) & \mathbf{a}^{\pm} := \dot{\mathbf{v}}^{\pm} \\ \mathbf{n}^{\pm} & := \frac{\mathbf{x}-\mathbf{z}-\mathbf{q}^{\pm}}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}^{\pm}\|} & t^{\pm} & = t\pm\|\mathbf{x}-\mathbf{z}-\mathbf{q}^{\pm}\|. & \end{array} \tag{25}$$

In this context pw-lim *denotes the point-wise limit in* $\mathbb{R}^3$.

For the proof we need the following lemma:

Lemma 2.19. *Given a strictly time-like charge trajectory* $(\mathbf{q},\mathbf{p}) \in \mathcal{T}_\triangledown^1$ *and a function* f *on* $\mathbb{R}^3$ *with* $\operatorname{supp} f \subseteq B_R(0)$ *for some* $R > 0$ *and* $\mathbf{x}^* \in \mathbb{R}^3$ *there exists a* $T_{max} > 1$ *so that*

$$K_r * f(\cdot - \mathbf{q}_{t\pm r}) = 0$$

for all $\mathbf{x} \in B_1(\mathbf{x}^*)$ *and* $|r| > T_{max}$.

Proof. Since

$$K_r * f(\cdot - \mathbf{q}_{t\pm r}) = r \fint_{\partial B_{|r|}(\mathbf{x})} d\sigma(y)\, f(\mathbf{y} - \mathbf{q}_{t\pm r})$$

this expression is zero if $\partial B_{|r|}(\mathbf{x}) \cap B_R(\mathbf{q}_{t\pm r}) = \emptyset$. On the one hand, for $\mathbf{x} \in B_1(\mathbf{x}^*)$, $\mathbf{y} \in \partial B_{|r|}(\mathbf{x})$ gives

$$\|\mathbf{x}-\mathbf{y}\| \geq \|\mathbf{x}^*-\mathbf{y}\| - \|\mathbf{x}-\mathbf{x}^*\| < |r| - 1. \tag{26}$$

In the following we consider $|r| > 1$ such that the right-hand side above is positive. On the other hand, if $\mathbf{y} \in B_R(\mathbf{q}_{t\pm r})$, we have

$$\|\mathbf{x}-\mathbf{y}\| \leq \|\mathbf{x}^*-\mathbf{q}_{t\pm r}\| + 1 + \|\mathbf{q}_{t\pm r}-\mathbf{y}\| \leq \|\mathbf{x}^*-\mathbf{q}_{t\pm r}\| + 1 + R \leq \|\mathbf{x}^*-\mathbf{q}_t\| + 1 + v_{max}|r| + R.$$

The last estimate is due to the strictly time-like nature of the charge trajectory; cf. Definition 2.5p.8. Combining this estimate with (26) we get $\partial B_{|r|}(\mathbf{x}) \cap B_R(\mathbf{q}_{t\pm r}) = \emptyset$ whenever

$$|r| > \max\left\{1, \frac{\|\mathbf{x}^*-\mathbf{q}_t\| + 2 + R}{1 - v_{max}}\right\} =: T_{max}.$$

□

*Proof of Theorem 2.18**p.14*. Fix $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^3$. By Theorem 2.14p.12 for every $t_0, t \in \mathbb{R}$

$$\begin{aligned} M_{\varrho,m}[F^0,(\mathbf{q},\mathbf{p})](t,t_0) &:= \begin{pmatrix}\partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t\end{pmatrix} K_{t-t_0} * \begin{pmatrix}\mathbf{E}^0 \\ \mathbf{B}^0\end{pmatrix} + K_{t-t_0} * \begin{pmatrix}-4\pi\mathbf{j}_{t_0} \\ 0\end{pmatrix} \\ &\quad + 4\pi\int_{t_0}^t ds\, K_{t-s} * \begin{pmatrix}-\nabla & -\partial_s \\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\rho_s \\ \mathbf{j}_s\end{pmatrix} =: \boxed{5} + \boxed{6} + \boxed{7} \end{aligned}$$

is in $\mathcal{F}^1$. At first we show that for $t_0 \to \pm\infty$ the terms $\boxed{5}$ and $\boxed{6}$ vanish with the help of ($21_{p.14}$), which ensures that there is a constant $1 \le C_1 < \infty$ such that for $\|\mathbf{x}\|$ large enough

$$\left(\|\mathbf{E}^0(\mathbf{x})\| + \|\mathbf{B}^0(\mathbf{x})\| + \|\mathbf{x}\| \sum_{i=1}^{3} \left(\|\partial_{\mathbf{x}_i} \mathbf{E}^0(\mathbf{x})\| + \|\partial_{\mathbf{x}_i} \mathbf{B}^0(\mathbf{x})\| \right) \right) \|\mathbf{x}\|^{\epsilon} \le C_1.$$

By Definition $2.10_{p.9}$ and for large enough t_0 we get:

$$\begin{aligned} \left\| [\nabla \wedge K_{t-t_0} * \mathbf{E}^0](\mathbf{x}) \right\| &\le |t - t_0| \fint_{\partial B_{|t-t_0|}(\mathbf{x})} d\sigma(y) \, \frac{\left\|\nabla \wedge \mathbf{E}^0(\mathbf{y})\right\| \left\|\mathbf{y}\right\|^{1+\epsilon}}{\left\|\mathbf{y}\right\|^{1+\epsilon}} \\ &\le |t - t_0| \fint_{\partial B_1(0)} d\sigma(y) \, \frac{C_1}{\left\|\mathbf{x} - |t - t_0| y\right\|^{-(1+\epsilon)}} \le \frac{C_1 |t - t_0|}{(|t - t_0| - \|\mathbf{x}\|)^{1+\epsilon}} \xrightarrow[t_0 \to \pm\infty]{\mathbb{R}} 0 \end{aligned}$$

where the constant $C_1 < \infty$ is given by ($21_{p.14}$). By Equation ($15_{p.10}$) we have

$$\left\| [\partial_t K_{t-t_0} * \mathbf{E}^0](\mathbf{x}) \right\| \le \fint_{\partial B_{|t-t_0|}(\mathbf{x})} d\sigma(y) \, \|\mathbf{E}^0(\mathbf{y})\| + |t - t_0| \fint_{\partial B_1(0)} d\sigma(y) \, \|\mathbf{y} \cdot \nabla \mathbf{E}^0(\mathbf{x} - |t - t_0| \mathbf{y})\|.$$

Let again t_0 be sufficiently large. The first term on the right-hand side equals

$$\fint_{\partial B_{|t-t_0|}(\mathbf{x})} d\sigma(y) \, \frac{\|\mathbf{E}^0(\mathbf{y})\| \, \|\mathbf{y}\|^{\epsilon}}{\|\mathbf{y}\|^{\epsilon}} \le \frac{C_1}{(|t - t_0| - \|\mathbf{x}\|)^{\epsilon}} \xrightarrow[t_0 \to \pm\infty]{\mathbb{R}} 0,$$

while the second term is smaller or equals

$$\begin{aligned} &|t - t_0| \fint_{\partial B_1(0)} d\sigma(y) \, \frac{\sum_{i=1}^{3} \|\partial_{\mathbf{x}_i} \mathbf{E}^0(\mathbf{x} - |t - t_0| \mathbf{y})\| \, \|\mathbf{x} - |t - t_0| \mathbf{y}\|^{1+\epsilon}}{\|\mathbf{x} - |t - t_0| \mathbf{y}\|^{1+\epsilon}} \\ &\le \frac{C_1 |t - t_0|}{(|t - t_0| - \|\mathbf{x}\|)^{1+\epsilon}} \xrightarrow[t_0 \to \pm\infty]{\mathbb{R}} 0. \end{aligned}$$

Next we show that in the limit $t_0 \to \pm\infty$ the term $\boxed{6}$ also vanishes. As $(\mathbf{q}, \mathbf{p})$ is a time-like charge trajectory we can apply Lemma $2.19_{p.15}$ for $r = t - t_0$ which yields

$$\|[K_{t-t_0} * \mathbf{j}_{t_0}](\mathbf{x})\| = 0$$

for large enough $|t_0|$. Therefore, we can conclude that term $\boxed{6}$ is zero for t_0 large enough. The same holds with $\mathbf{E}^0$ replaced by $\mathbf{B}^0$, and therefore we find

$$\begin{aligned} &\lim_{t_0 \to \pm\infty} \boxed{7} = 4\pi \int_{\pm\infty}^{t} ds \left[K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_s \\ \mathbf{j}_s \end{pmatrix} \right](\mathbf{x}) = \left(M_{\varrho,m}[(\mathbf{q}, \mathbf{p})](t, \pm\infty) \right)(\mathbf{x}) \\ &= 4\pi \int_0^{\infty} dr \left[K_r * \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \rho_{t\pm r} \\ \mathbf{j}_{t\pm r} \end{pmatrix} \right](\mathbf{x}) = \int d^3y \begin{pmatrix} -\nabla & -\partial_t \\ 0 & \nabla\wedge \end{pmatrix} \frac{\rho(\mathbf{x} - \mathbf{y} - \mathbf{q}_{t\pm\|\mathbf{y}\|})}{\|\mathbf{y}\|} \begin{pmatrix} 1 \\ \mathbf{v}_{t\pm r} \end{pmatrix} \\ &=: \begin{pmatrix} \mathbf{E}_t^{\pm}(\mathbf{x}) \\ \mathbf{B}_t^{\pm}(\mathbf{x}) \end{pmatrix}. \end{aligned} \tag{27}$$

Let us first compute the electric fields

$$\mathbf{E}_t^\pm(\mathbf{x}) = \int d^3y \left[\frac{-\nabla\rho(\mathbf{x}-\mathbf{y}-\mathbf{q}_{t\pm\|\mathbf{y}\|})}{\|\mathbf{y}\|} + \frac{\mathbf{v}_{t\pm\|\mathbf{y}\|}\cdot\nabla\rho(\mathbf{x}-\mathbf{y}-\mathbf{q}_{t\pm\|\mathbf{y}\|})\,\mathbf{v}_{t\pm\|\mathbf{y}\|}}{\|\mathbf{y}\|} - \frac{\rho(\mathbf{x}-\mathbf{y}-\mathbf{q}_{t\pm\|\mathbf{y}\|})\,\mathbf{a}_{t\pm\|\mathbf{y}\|}}{\|\mathbf{y}\|}\right].$$

In order to simplify this expression we make a transformation of the integration variable:

$$\mathbf{y} \to \mathbf{z}(\mathbf{y}) := \mathbf{x}-\mathbf{y}-\mathbf{q}_{t\pm\|\mathbf{y}\|} \tag{28}$$

Here, we use that $(\mathbf{q},\mathbf{p}) \in \mathcal{T}^1_\triangledown$ is a strictly time-like charge trajectory. We observe that $\mathbf{z}(\cdot)$ is a diffeomorphism because, first, it is bijective since for $\sup_{t\in\mathbb{R}}\|\mathbf{v}_t\| \le v_{max} < 1$ the equation $\mathbf{y}(\mathbf{z}) = \mathbf{x}-\mathbf{z}-\mathbf{q}_{t\pm\|\mathbf{y}(\mathbf{z})\|}$ has a unique solution $\mathbf{y}(\mathbf{z})$ for all $\mathbf{z}\in\mathbb{R}^3$ which is given by $\{\mathbf{q}^\pm\} = \bigcup_{r\ge 0}(\partial B_r(\mathbf{x}-\mathbf{z})\cap\{\mathbf{q}_{t\pm r}\})$, i.e. the intersection of the charge trajectory and the forward, respectively backward, light cone of $\mathbf{x}-\mathbf{z}$. And second, $\mathbf{z}(\cdot)$ is continuously differentiable with $(\partial_{y_i}\mathbf{z}_j(\mathbf{y}))_{1\le i,j\le 3} = -\delta_{ij} \pm \mathbf{v}_{j,t\pm\|\mathbf{y}\|}\frac{\mathbf{y}_i}{\|\mathbf{y}\|}$ such that it has a non-zero determinant which equals $(-1 \pm \mathbf{v}_{t\pm\|\mathbf{y}\|}\cdot\frac{\mathbf{y}}{\|\mathbf{y}\|})$, again because $\sup_{t\in\mathbb{R}}\|\mathbf{v}_t\| \le v_{max} < 1$, and therefore the inverse of $\mathbf{z}(\cdot)$ is also continuously differentiable. In order to make the notation more readable we shall use the abbreviations ($25_{p.15}$). We then get

$$\begin{aligned}\mathbf{E}_t^\pm(\mathbf{x}) &= \int d^3z\,\frac{-\nabla\rho(\mathbf{z}) + \mathbf{v}^\pm\cdot\nabla\rho(\mathbf{z})\,\mathbf{v}^\pm - \rho(\mathbf{z})\,\mathbf{a}^\pm}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}^\pm\|(1\pm\mathbf{n}^\pm\cdot\mathbf{v}^\pm)}\\ &= \int d^3z\,\rho(\mathbf{z})\left[\nabla_\mathbf{z}\frac{1}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}^\pm\|(1\pm\mathbf{n}^\pm\cdot\mathbf{v}^\pm)} - \sum_{k=1}^{3}\partial_{z_k}\frac{v_k^\pm\,\mathbf{v}^\pm}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}^\pm\|(1\pm\mathbf{n}^\pm\cdot\mathbf{v}^\pm)} - \frac{\mathbf{a}^\pm}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}^\pm\|(1\pm\mathbf{n}^\pm\cdot\mathbf{v}^\pm)}\right].\end{aligned} \tag{29}$$

after a partial integration. Note that for this we only need almost everywhere differentiability. Doing the same for the magnetic field yields

$$\mathbf{B}_t^\pm(\mathbf{x}) = \int d^3z\,\rho(\mathbf{z})\left[-\nabla\wedge\frac{\mathbf{v}^\pm}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}^\pm\|(1\pm\mathbf{n}^\pm\cdot\mathbf{v}^\pm)}\right] \tag{30}$$

After a tedious but not really interesting computation(see Computation A.$1_{p.39}$) one finds that Equation ($22_{p.15}$) holds. Since we can represent the Maxwell solution by a convolution with a $\varrho \in C_c^\infty(\mathbb{R}^3,\mathbb{R})$ function it is immediate that $F_t^\pm \in \mathcal{F}^1$. This concludes the proof. □

REMARK 2.20. *Condition ($21_{p.14}$) guarantees that in the limit $t_0 \to \pm\infty$ the initial value F^0 are forgotten by the time evolution of the Maxwell equations. Note that in order to compute the Liénard-Wiechert fields the strictly time-like nature of the charge trajectory is sufficient for the limit to exists $t_0 \to \pm\infty$. This condition could be softened into an integrability condition for more general ρ and $\mathbf{j}$, e.g. one must only demand that the right-hand side of ($27_{p.16}$) is finite. However, the Liénard-Wiechert fields for time-like charge trajectories would then in general not be given by ($22_{p.15}$) since (28) does not have to be bijective anymore. This fact is indicated by the blow up of the factors $(1\pm\mathbf{n}\cdot\mathbf{v})^{-3}$ in Equation ($23_{p.15}$) for $\mathbf{v}\to 1$.*

Theorem 2.21 (Liénard-Wiechert Fields Solve the Maxwell Equations)**.** *Let $(\mathbf{q},\mathbf{p}) \in \mathcal{T}^1_\triangledown$ be a strictly time-like charge trajectory and $(\rho,\mathbf{j})$ the $\varrho \in C_c^\infty(\mathbb{R}^3,\mathbb{R})$ induced charge-current density for some mass $m \neq 0$, cf. Definitions 2.$5_{p.8}$ and 2.$7_{p.9}$. Then the Liénard-Wiechert fields $M_{\varrho,m}[(\mathbf{q},\mathbf{p})](t,\pm\infty)$ are a solution to the Maxwell equations ($12_{p.9}$) including the Maxwell constraints for all $t\in\mathbb{R}$.*

Proof. $(\rho, \mathbf{j})$ is the $\varrho \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ induced charge-current density of the strictly time-like charge trajectory $(\mathbf{q}, \mathbf{p}) \in \mathcal{T}^1_\triangledown$. Hence, for any $t \in \mathbb{R}$

$$\rho_t = \varrho(\cdot - \mathbf{q}_t) \qquad \text{and} \qquad \mathbf{j}_t = \mathbf{v}(\mathbf{p}_t)\varrho(\cdot - \mathbf{q}_t).$$

Therefore, Lemma 2.19p.15, for the choice $r = t - s$, states that for all $t^* \in \mathbb{R}$ and $\mathbf{x}^* \in \mathbb{R}^3$ there exists a constant $1 < T_{max} < \infty$ such that: For all $t \in B_1(t^*)$ and $\mathbf{x} \in B_1(\mathbf{x}^*)$

$$\left[K_{t-s} * \begin{pmatrix}\rho_s \\ \mathbf{j}_s\end{pmatrix}\right](\mathbf{x}) = 0 \text{ if } |s| > T := T_{max} + |t^*| + 1.$$

This allows for any $t \in B_1(t^*)$ and $\mathbf{x} \in B_1(\mathbf{x}^*)$ to rewrite Equation (27p.16) into

$$\left(M_{\varrho,m}[(\mathbf{q},\mathbf{p})](t, \pm\infty)\right)(\mathbf{x}) = \begin{pmatrix}\mathbf{E}_t^\pm(\mathbf{x}) \\ \mathbf{B}_t^\pm(\mathbf{x})\end{pmatrix} = 4\pi \int_{\pm\infty}^t ds \left[K_{t-s} * \begin{pmatrix}-\nabla & -\partial_s \\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\rho_s \\ \mathbf{j}_s\end{pmatrix}\right](\mathbf{x}) \tag{31}$$

$$= 4\pi \int_{\pm T}^t ds \left[K_{t-s} * \begin{pmatrix}-\nabla & -\partial_s \\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\rho_s \\ \mathbf{j}_s\end{pmatrix}\right](\mathbf{x}). \tag{32}$$

So, for $t_0 = \pm T$, the right-hand side of (31) equals

$$K_{t-t_0} * \begin{pmatrix}-4\pi\mathbf{j}_{t_0} \\ 0\end{pmatrix} + 4\pi \int_{t_0}^t ds \left[K_{t-s} * \begin{pmatrix}-\nabla & -\partial_s \\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\rho_s \\ \mathbf{j}_s\end{pmatrix}\right](\mathbf{x})$$

which by Theorem 2.14p.12(ii) solves the Maxwell Equation including the Maxwell constraints 12p.9 for all $t \in B_1(t^*)$ and $\mathbf{x} \in B_1(\mathbf{x}^*)$. Since $t^* \in \mathbb{R}$ and $\mathbf{x}^* \in \mathbb{R}^3$ are arbitrary, the Maxwell Equation including the Maxwell constraints are fulfilled for all $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^3$ which concludes the proof. □

From their explicit expressions we immediately get a simple bound on the Liénard-Wiechert fields:

Corollary 2.22 (Liénard-Wiechert estimate)**.** *Let* $(\mathbf{q}, \mathbf{p}) \in \mathcal{T}^1_\triangledown$ *be a strictly time-like charge trajectory and* $(\rho, \mathbf{j})$ *the* $\varrho \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ *induced charge-current density for some mass* $m \neq 0$*, cf. Definitions 2.5p.8 and 2.7p.9. Furthermore, assume there exists an* $a_{max} < \infty$ *such that* $\sup_{t\in\mathbb{R}} \|\partial_t \mathbf{v}(\mathbf{p}_t)\| \leq a_{max}$*, we then get a simple estimate for the Liénard-Wiechert fields for all* $\mathbf{x} \in \mathbb{R}^3$*,* $t \in \mathbb{R}$ *and multi-index* $\alpha \in \mathbb{N}^3$*:*

$$\|D^\alpha \mathbf{E}_t^\pm(\mathbf{x})\| + \|D^\alpha \mathbf{B}_t^\pm(\mathbf{x})\| \leq \frac{C_2^{(\alpha)}}{(1 - v_{max})^3}\left(\frac{1}{1 + \|\mathbf{x} - \mathbf{q}_t\|^2} + \frac{a_{max}}{1 + \|\mathbf{x} - \mathbf{q}_t\|}\right)$$

for

$$\begin{pmatrix}\mathbf{E}_i^\pm \\ \mathbf{B}_i^\pm\end{pmatrix} := M_{\varrho,m}[(\mathbf{q},\mathbf{p})](t, \pm\infty),$$

a family of finite constants $(C_2^{(\alpha)})_{\alpha\in\mathbb{N}^3}$ *and* v_{max} *as defined in Definition 2.5p.8.*

Proof. From Theorem 2.18p.14 we know that for this sub-light charge trajectory the Liénard-Wiechert fields take the form

$$\begin{pmatrix}\mathbf{E}_i^\pm(\mathbf{x}) \\ \mathbf{B}_i^\pm(\mathbf{x})\end{pmatrix} = \int d^3z\, \varrho(\mathbf{x} - \mathbf{z}) \begin{pmatrix}\mathbf{E}_i^{LW\pm}(\mathbf{z}) \\ \mathbf{B}_i^{LW\pm}(\mathbf{z})\end{pmatrix}. \tag{33}$$

As the integrand is infinitely often differentiable in $\mathbf{x}$ and has compact support, the derivatives for any multi-index $\alpha \in \mathbb{N}^3$ are given by

$$D^\alpha F_t^\pm(\mathbf{x}) = \int d^3z\, D^\alpha \varrho_i(\mathbf{x} - \mathbf{z}) \begin{pmatrix}\mathbf{E}_i^\pm(\mathbf{z}) \\ \mathbf{B}_i^\pm(\mathbf{z})\end{pmatrix}.$$

First, we take a look at (23p.15) and (24p.15) for given $\mathbf{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$. As we have a strictly time-like charge trajectory $(\mathbf{q}, \mathbf{p}) \in \mathcal{T}^1_\triangledown$, $\|\mathbf{x} - \mathbf{z} - \mathbf{q}^\pm\|$ is the smallest if we assume the worst case, i.e. that from time t on the rigid charge moves into the future (respectively into the past) with the speed of light towards the point $\mathbf{x}-\mathbf{z}$. Therefore, $\|\mathbf{x}-\mathbf{z}-\mathbf{q}^{\pm}\| \leq \frac{1}{2}\|\mathbf{x}-\mathbf{z}-\mathbf{q}_t\|$ and, hence,

$$\|\mathbf{B}_t^{LW\pm}(\mathbf{x}-\mathbf{z})\| + \|\mathbf{E}_t^{LW\pm}(\mathbf{x}-\mathbf{z})\| \leq \frac{2}{(1-v_{max})^3}\left[\frac{1}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}_t\|^2} + \frac{a_{max}}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}_t\|}\right]^{\pm} \tag{34}$$

because $\sup_{t\in\mathbb{R}} \|\mathbf{v}_t\| \leq v_{max} < 1$. The rest is straightforward computation (see Computation A.2p.40). □

2.2. **Unique Characterization by ML-SI Cauchy Data.** The goal of this section is to prove Theorem 2.1p.5 which states that a class of solutions to the WF equations can be uniquely characterized by ML-SI Cauchy data, i.e. positions, momenta and fields at one time instant. Using the results of Section 2.1p.8 we can give a sensible definition of what we mean by solutions to the WF equations (4p.3) and (5p.3). We restrict the class of possible WF solutions to:

Definition 2.23 (Class of WF solutions)**.** *Let $\mathcal{T}_{\mathrm{WF}}$ denote the set of strictly time-like charge trajectories $(\mathbf{q}_i,\mathbf{p}_i)_{1\leq i\leq N} \in \mathcal{T}_{\triangledown}$ with masses $m_i \neq 0$, $1 \leq i \leq N$ and with the properties:*

- *(i) There exists an $a_{max} < \infty$ such that $\sup_{t\in\mathbb{R}} \|\partial_t \mathbf{v}(\mathbf{p}_{i,t})\| \leq a_{max}$, i.e. the accelerations of the charges are bounded.*
- *(ii) for all times $t \in \mathbb{R}$ solve the WF equations (4p.3) and (5p.3).*

REMARK 2.24. *(i) Note that this definition is sensible because with $(\mathbf{q}_i,\mathbf{p}_i)_{1\leq i\leq N} \in \mathcal{T}_{\triangledown}$, equations (5p.3) for $1 \leq i \leq N$ can by Definition 2.17p.14 be rewritten as:*

$$(\mathbf{E}_{i,t}^{\mathrm{WF}}, \mathbf{B}_{i,t}^{\mathrm{WF}}) = \frac{1}{2}\sum_{\pm} M_{\varrho_i,m_i}[(\mathbf{q}_i,\mathbf{p}_i)](t,\pm\infty).$$

Theorem 2.18p.14 guarantees that the right-hand side is well-defined. Furthermore, charge trajectories in $\mathcal{T}_{\triangledown}^1$ are once continuously differentiable so that the left-hand side of (4p.3) is also well-defined. The bound on the acceleration will give us a bound on the WF fields in a suitable norm; see Lemma 2.26p.20.

(ii) Furthermore, it is highly expected that $\mathcal{T}_{\mathrm{WF}}$ is non-empty for two reasons: 1. In the point particle case there are explicit solutions to the WF equations known, i.e. the Schild solutions [Sch63] *and the solutions of Bauer's existence theorem* [Bau97]*, which yield strictly time-like charge trajectories with bounded accelerations. 2. Physically, one would expect that in general scattering solutions have accelerations that decay at $t \to \pm\infty$.*

For the proof of Theorem 2.1p.5 we need the following two lemmas. First, we give an example of a suitable weight w in $\mathcal{W}^\infty$.

Lemma 2.25 (Explicit Expression for the Weight w)**.** *For $\mathbf{x} \mapsto w(\mathbf{x}) := (1-\|\mathbf{x}\|^2)^{-1}$ it holds $w \in \mathcal{W}^\infty$; cf. Equation (62p.43).*

Proof. As computed in Part I [BDD10] w is in $\mathcal{W}$. Thus, it is left to show that this w is also in $\mathcal{W}^k$ for any $k \in \mathbb{N}$. To see this let us consider

$$\begin{aligned}
0 &= D^\alpha\left(w(\mathbf{x})(1+\|\mathbf{x}\|^2)\right)\\
&= \sum_{k_1,k_2,k_3=0}^{\alpha_1,\alpha_2,\alpha_3}\binom{\alpha_1}{k_1}\binom{\alpha_2}{k_2}\binom{\alpha_3}{k_3}\partial_1^{\alpha_1-k_1}\partial_2^{\alpha_2-k_2}\partial_3^{\alpha_3-k_3}w(\mathbf{x})\partial_1^{k_1}\partial_2^{k_2}\partial_3^{k_3}(1+\|\mathbf{x}\|^2)\\
&= \left(D^\alpha w(\mathbf{x})\right)(1+\|\mathbf{x}\|^2) + \sum_{i=1}^{3}\alpha_i\left(\partial_i^{\alpha_i-1}w(\mathbf{x})\right)2x_i + \sum_{i=1}^{3}\frac{1}{2}\alpha_i(\alpha_i-1)\left(\partial_i^{\alpha_i-2}w(\mathbf{x})\right)2
\end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ is a multi-index. This leads to the recursive estimate

$$|D^\alpha w(\mathbf{x})| \leq w(\mathbf{x}) \left(\sum_{i=1}^{3} 2\alpha_i \left| \partial_i^{\alpha_i - 1} w(\mathbf{x}) \right| |x_i| + \sum_{i=1}^{3} \alpha_i(\alpha_i - 1) \left| \partial_i^{\alpha_i - 2} w(\mathbf{x}) \right| \right)$$

in the sense that terms involving ∂^l for negative l equal zero. Hence, the left-hand side can be bounded by lower derivatives, and therefore, by induction over the multi-index α, we get constants $C^\alpha < \infty$ such that $|D^\alpha w(\mathbf{x})| \leq C^\alpha w(\mathbf{x})$. Furthermore, from the computation

$$D^\alpha w(\mathbf{x}) = D^\alpha \left(\sqrt{w(\mathbf{x})} \sqrt{w(\mathbf{x})} \right) = \sum_{k_1, k_2, k_3 = 0}^{\alpha_1, \alpha_2, \alpha_3} \binom{\alpha_1}{k_1} \binom{\alpha_2}{k_2} \binom{\alpha_3}{k_3} \partial_1^{\alpha_1 - k_1} \partial_2^{\alpha_2 - k_2} \partial_3^{\alpha_3 - k_3} \sqrt{w(\mathbf{x})} \partial_1^{k_1} \partial_2^{k_2} \partial_3^{k_3} \sqrt{w(\mathbf{x})}$$

and with $I_\alpha := \{k \in \mathbb{N}^3 \mid 0 \leq k_i \leq \alpha_i, i = 1, 2, 3\} \setminus \{(0,0,0), \alpha\}$ we get the recursive formula

$$\left| D^\alpha \sqrt{w(\mathbf{x})} \right| \leq \frac{1}{2} \Bigg[C^\alpha \sqrt{w(\mathbf{x})} + \frac{1}{\sqrt{w(\mathbf{x})}} \sum_{(k_1, k_2, k_3) \in I_\alpha} \binom{\alpha_1}{k_1} \binom{\alpha_2}{k_2} \binom{\alpha_3}{k_3} \times \times \left| \partial_1^{\alpha_1 - k_1} \partial_2^{\alpha_2 - k_2} \partial_3^{\alpha_3 - k_3} \sqrt{w(\mathbf{x})} \right| \left| \partial_1^{k_1} \partial_2^{k_2} \partial_3^{k_3} \sqrt{w(\mathbf{x})} \right| \Bigg].$$

where we have used the above established estimate $|D^\alpha w(\mathbf{x})| \leq C^\alpha w(\mathbf{x})$. Again, the left-hand side can be bounded by lower derivatives, and therefore, by induction over the multi-index α, we yield finite constants C_α such that also $|D^\alpha \sqrt{w}| \leq C_\alpha w$. Therefore, $w \in \mathcal{W}^k$ for any $k \in \mathbb{N}$ and, thus, $w \in \mathcal{W}^\infty$. □

Second, we show that this weight w decays quickly enough such that all Liénard-Wiechert fields of strictly time-like charge trajectories in $\mathcal{T}_\triangledown^1$ with bounded accelerations lie in $D_w(A^\infty)$.

Lemma 2.26 (Regularity of the Liénard-Wiechert Fields)**.** *Let* $(\mathbf{q}_i, \mathbf{p}_i)_{1 \leq i \leq N} \in \mathcal{T}_\triangledown^1$ *with masses* $m_i \neq 0$, $1 \leq i \leq N$, *and assume there exists an* $a_{max} < \infty$ *such that* $\sup_{t \in \mathbb{R}} \|\partial_t \mathbf{v}(\mathbf{p}_{i,t})\| \leq a_{max}$. *Define* $t \mapsto (\mathbf{E}_{i,t}, \mathbf{B}_{i,t}) := M_{\varrho_i, m_i}[(\mathbf{q}_i, \mathbf{p}_i)](t, \pm\infty)$. *Then there exists a* $w \in \mathcal{W}^\infty$ *such that for any* $\mathbf{q}_i, \mathbf{p}_i \in \mathbb{R}^3$, $1 \leq i \leq N$, *it is true that*

$$(\mathbf{q}_i, \mathbf{p}_i, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1 \leq i \leq N} \in D_w(A^\infty), \ \textit{for all } t \in \mathbb{R}.$$

Proof. The charge trajectories are in $\mathcal{T}_\triangledown^1$ and therefore strictly time-like. Furthermore, they have bounded accelerations. Therefore, by Corollary 2.22p.18, for $1 \leq i \leq N$ and each multi-index $\alpha \in \mathbb{N}^3$ there exists a constant $C_2^{(\alpha)} < \infty$ such that

$$\|D^\alpha \mathbf{E}_{i,t}^\pm(\mathbf{x})\| + \|D^\alpha \mathbf{B}_{i,t}^\pm(\mathbf{x})\| \leq \frac{C_2}{(1 - v_{max})^3} \left(\frac{1}{1 + \|\mathbf{x} - \mathbf{q}_t\|^2} + \frac{a_{max}}{1 + \|\mathbf{x} - \mathbf{q}_t\|} \right).$$

Hence, for $w(\mathbf{x}) = \frac{1}{1 + \|\mathbf{x}\|^2}$ we get

$$\left\| A^n(\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}^\pm, \mathbf{B}_{i,t}^\pm) \right\|_{\mathcal{H}_w} \leq \sum_{i=1}^{N} \sum_{|\alpha| \leq n} \left(\|\mathbf{q}_{i,t}\| + \|\mathbf{p}_{i,t}\| + \int d^3x \, w(\mathbf{x}) \left(\|D^\alpha \mathbf{E}_{i,t}^\pm(\mathbf{x})\|^2 + \|D^\alpha \mathbf{B}_{i,t}^\pm(\mathbf{x})\|^2 \right) \right)$$

which is finite, so that for any $t \in \mathbb{R}$ we have $\varphi_t \in D_w(A^\infty)$. □

We finally come to the proof of Theorem 2.1p.5.

*Proof of Theorem 2.1**p.5* *(Characterization of WF Solutions).* (i) First, as the charge trajectories fulfill the WF equations (4p.3), they also fulfill the Lorentz force law (7p.4). Second, by Theorem 2.21p.17 the fields $(\mathbf{E}_{i,t}^\pm, \mathbf{B}_{i,t}^\pm)$ solve the Maxwell equations including the Maxwell constraints, both given in the set of equations (6p.4). Therefore, $t \mapsto \varphi_t$ is a solution to the ML-SI equations, i.e. the coupled set of equations (7p.4) plus

($6_{p.4}$). By Lemma 2.26 for any t_0 we yield $\varphi_{t_0} \in D_w(A^\infty)$ so that the existence assertion of Theorem B.$6_{p.44}$ states that there is a solution $t \mapsto \widetilde{\varphi}_t$ of the ML-SI equations with $\varphi_{t_0} = \widetilde{\varphi}_{t_0}$ while the uniqueness assertion of that theorem states that if $\varphi_{t_0} = \widetilde{\varphi}_{t_0}$ for any $t_0 \in \mathbb{R}$, we have $\varphi_t = \widetilde{\varphi}_t$ for all $t \in \mathbb{R}$. Therefore, we conclude $\varphi_t = \widetilde{\varphi}_t = M_L[\varphi_{t_0}](t, t_0)$ for all $t \in \mathbb{R}$.

(ii) Let $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N}, (\widetilde{\mathbf{q}}_i, \widetilde{\mathbf{p}}_i)_{1\le i\le N} \in \mathcal{T}_{\text{WF}}$ and $t_0 \in \mathbb{R}$. Define $\varphi_t := (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}^{\text{WF}}_{i,t}, \mathbf{B}^{\text{WF}}_{i,t})_{1\le i\le N}$ and $\widetilde{\varphi}_t := (\widetilde{\mathbf{q}}_{i,t}, \widetilde{\mathbf{p}}_{i,t}, \widetilde{\mathbf{E}}^{\text{WF}}_{i,t}, \widetilde{\mathbf{B}}^{\text{WF}}_{i,t})_{1\le i\le N}$ for all $t \in \mathbb{R}$ as in (i). By Lemma $2.26_{p.20}$ there is a $w \in \mathcal{W}^\infty$ such that $\varphi_{t_0}, \widetilde{\varphi}_{t_0} \in D_w(A^\infty)$ and therefore the range of i_{t_0} is a subset of $D_w(A^\infty)$. From (i) we know in addition that for all $t \in \mathbb{R}$, $\varphi_t = M_L[\varphi_{t_0}](t, t_0)$ and $\widetilde{\varphi}_t = M_L[\widetilde{\varphi}_{t_0}](t, t_0)$. Assume $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N} \neq (\widetilde{\mathbf{q}}_i, \widetilde{\mathbf{p}}_i)_{1\le i\le N}$, i.e. there exist $t \in \mathbb{R}$ such that we have $(\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1\le i\le N} \neq (\widetilde{\mathbf{q}}_{i,t}, \widetilde{\mathbf{p}}_{i,t})_{1\le i\le N}$. For such t we have then $M_L[\varphi_{t_0}](t, t_0) = \varphi_t \neq \widetilde{\varphi}_t = M_L[\widetilde{\varphi}_{t_0}](t, t_0)$. The uniqueness assertion of Theorem B.$6_{p.44}$ then states $\varphi_{t_0} \neq \varphi_{t_0}$. By construction $\varphi_{t_0} = i_{t_0}((\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N}$ and $\widetilde{\varphi}_{t_0} = i_{t_0}(\widetilde{\mathbf{q}}_i, \widetilde{\mathbf{p}}_i)_{1\le i\le N}$. Hence $i_{t_0} : \mathcal{T}_{\text{WF}} \to D_w(A^\infty)$ is injective. □

REMARK 2.27. *Note that the weight function w could be chosen to decay faster than the choice in Lemma $2.25_{p.19}$. This freedom allows to generalize Theorem $2.1_{p.5}$ also for possible WF solutions whose acceleration is not bounded but may grow with $t \to \pm\infty$. This is due to the fact that growth of the acceleration* $\mathbf{a}$ *in equations ($22_{p.15}$) can be suppressed by the weight w. However, by the conditions of Theorem B.$6_{p.44}$ the weight w must be at least in $\mathcal{W}^1$ one can only allow the acceleration* $\mathbf{a}$ *to grow slower than exponentially.*

As byproduct we can use the same technique to provide global existence and uniqueness to the Synge equations:

Proof of Theorem $2.2_{p.6}$ (Existence and Uniqueness of Synge Solutions). (i) is proven in exact the same way as $2.1_{p.5}$(i) as the proof holds for any linear combination of Liénard-Wiechert fields.

(ii) Define the fields $(\mathbf{E}^{\text{SY}}_{i,t_0}, \mathbf{B}^{\text{SY}}_{i,t_0})_{1\le i\le N} := M_{\varrho_i,m_i}[(\mathbf{q}^-_i, \mathbf{p}^-_i)](t_0, -\infty)$ by Theorem $2.18_{p.14}$. By (i) these fields fulfill

$$\varphi^0 := (\mathbf{q}^-_{i,t_0}, \mathbf{p}^-_{i,t_0}, \mathbf{E}^{\text{SY}}_{i,t_0}, \mathbf{B}^{\text{SY}}_{i,t_0})_{1\le i\le N} \in D_w(A^\infty).$$

Define $(\mathbf{q}^+_{i,t}, \mathbf{p}^+_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N} := M_L[\varphi^0](t, t_0)$ for $t \ge t_0$; see Definition B.$10_{p.45}$. Concatenate the past and future pieces of the charge trajectories according to ($9_{p.6}$). For (i) one needs to check $(\mathbf{q}, \mathbf{p}) \in C^1$ which is guaranteed by Theorem B.$6_{p.44}$ and the fact that the Synge equations hold at time t_0. For (ii) let us consider the difference between $(\mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N}$ and $(\mathbf{E}^{\text{SY}}_{i,t}, \mathbf{B}^{\text{SY}}_{i,t})_{1\le i\le N} := M_{\varrho_i,m_i}[(\mathbf{q}^-_i, \mathbf{p}^-_i)](t_0, -\infty)$ for $t \ge t_0$. By Theorem B.$6_{p.44}$ and Theorem $2.21_{p.17}$ this difference solves the homogeneous Maxwell equations for $t \ge t_0$ for initial value 0. Hence, this difference is zero for all time $t \ge t_0$ from what we infer that

$$\begin{aligned}
\partial_t \mathbf{p}^+_{i,t} &= \sum_{j\neq i} \int d^3x\, \varrho_i(\mathbf{x} - \mathbf{q}^+_{i,t}) \left(\mathbf{E}_{j,t}(\mathbf{x}) + \mathbf{v}(\mathbf{q}^+_{i,t}) \wedge \mathbf{B}_{j,t}(\mathbf{x})\right) \\
&= \sum_{j\neq i} \int d^3x\, \varrho_i(\mathbf{x} - \mathbf{q}^+_{i,t}) \left(\mathbf{E}^{\text{SY}}_{j,t}(\mathbf{x}) + \mathbf{v}(\mathbf{q}^+_{i,t}) \wedge \mathbf{B}^{\text{SY}}_{j,t}(\mathbf{x})\right)
\end{aligned}$$

for $t \ge t_0$ which concludes the proof as the uniqueness follows by the uniqueness of the ML-SI equations; see Theorem B.$6_{p.44}$. □

2.3. **Existence of WF Solutions on Finite Time Intervals.** We shall now come to the question of existence of WF solutions. We shall formalize the map $S^{p,X^\pm}_T$ and prove the existence of a fixed point. The proof will rely on the explicit expressions of the Maxwell fields of ML-SI solutions given in Theorem B.$6_{p.44}$ in terms of the Kirchoff's formulas given in Section $2.1_{p.8}$. Therefore, we inserted a small intermediate paragraph before the main proof which will provide all necessary formulas.

Once and for all we fix the parameters:

Definition 2.28 (Global Definition of w, ϱ_i, m_i and e_{ij})**.** *To the very end of this chapter we fix the charge distributions $\varrho_i \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ such that $\operatorname{supp} \varrho_i \subset B_R(0) \subset \mathbb{R}^3$ for one finite $R > 0$ and the masses $m_i \neq 0$, $1 \leq i \leq N$. Furthermore, we choose a weight $w \in \mathcal{W}^\infty$ for which Theorem 2.1p.5 holds.*

The Maxwell Fields of the Maxwell-Lorentz Dynamics. This intermediate section is supposed to bring quickly together the solution theories of the ML-SI equations, cf. Theorem B.6p.44, on $D_w(A)$ and the Maxwell equations (Section 2.1p.8) on $\mathcal{F}^1$. In particular, it will provide explicit formulas for the Maxwell solutions expressed by $(W_t)_{t\in\mathbb{R}}$ and J on a suitable domain. We recall the Newtonian phase space $\mathcal{P} = \mathbb{R}^{6N}$, the space of weighted square integrable fields $\mathcal{F}_w$, the phase space $\mathcal{H}_w = \mathcal{P} \oplus \mathcal{F}_w$ of the Maxwell-Lorentz equations, cf. Definition B.3p.43, the definition of the operator A on $D_w(A) \subset \mathcal{H}_w$, cf. Definition B.4p.43, as well as the one of the operator J on $\mathcal{H}_w$, cf. Definition B.5p.44. In order not to blow up the notation we use the following:

Notation 2.29 (Projectors $\mathtt{P}, \mathtt{Q}, \mathtt{F}$)**.** *For any $\varphi = (\mathbf{q}_i, \mathbf{p}_i, \mathbf{E}_i, \mathbf{B}_i)_{1\leq i\leq N} \in \mathcal{H}_w$ we define the projectors $\mathtt{Q}, \mathtt{P}, \mathtt{F}$ by*

$$\mathtt{Q}\varphi = (\mathbf{q}_i, 0, 0, 0)_{1\leq i\leq N}, \qquad \mathtt{P}\varphi = (0, \mathbf{p}_i, 0, 0)_{1\leq i\leq N}, \qquad \mathtt{F}\varphi = (0, 0, \mathbf{E}_i, \mathbf{B}_i)_{1\leq i\leq N}.$$

Wherever formal type errors do not lead to ambiguities we sometimes forget about or add the zero components and write, e.g.,

$$(\mathbf{q}_i, \mathbf{p}_i)_{1\leq i\leq N} = (\mathtt{Q} + \mathtt{P})\varphi \qquad \text{or} \qquad (\mathbf{q}_i, \mathbf{p}_i, 0, 0)_{1\leq i\leq N} = (\mathtt{Q} + \mathtt{P})(\mathbf{q}_i, \mathbf{p}_i)_{1\leq i\leq N}.$$

As we now treat N fields simultaneously, we need to extend $\mathcal{F}^1$, cf. Definition 2.8p.9, according to:

Definition 2.30 (Space of N Smooth Fields)**.** $\mathcal{F} := \bigoplus_{i=1}^N C^\infty(\mathbb{R}^3, \mathbb{R}^3) \oplus C^\infty(\mathbb{R}^3, \mathbb{R}^3)$.

Furthermore, we recall that A is the generator of a γ contractive group $(W_t)_{t\in\mathbb{R}}$ on $D_w(A)$ which was the content of Definition B.8p.45 and its preceding lemma. Since we shall mainly work in field spaces, we need the projections of the operators A, W_t and J onto field space $\mathcal{F}_w$:

Definition 2.31 (Projection of A, W_t, J to Field Space $\mathcal{F}_w$)**.** *For all $t \in \mathbb{R}$ and $\varphi \in \mathcal{H}_w$ we define*

$$\mathsf{A} := \mathtt{F}A\mathtt{F}, \qquad \mathsf{W}_t := \mathtt{F}W_t\mathtt{F} \qquad \text{and} \qquad \mathsf{J} := \mathtt{F}J(\varphi).$$

The natural domain of A, W_t is given by $D_w(\mathsf{A}) := \mathtt{F}D_w(A) \subset \mathcal{F}_w$. We shall also need $D_w(\mathsf{A}^n) := \mathtt{F}D_w(A^n) \subset \mathcal{F}_w$ for every $n \in \mathbb{N} \cup \{\infty\}$. Clearly, the operator A on $D_w(\mathsf{A})$ is also closed and inherits also the resolvent properties from A on $D_w(A)$. Furthermore, this implies $(\mathtt{Q} + \mathtt{P})W_t = \mathrm{id}_{\mathcal{P}}$ and $\mathtt{F}W_t = \mathsf{W}_t$ so that $(\mathsf{W}_t)_{t\in\mathbb{R}}$ is also a γ contractive group on the smaller space $D_w(\mathsf{A})$. Finally, note also that by the definition of J we have $\mathsf{J}(\varphi) = \mathsf{J}((\mathtt{Q} + \mathtt{P})\varphi)$ for all $\varphi \in \mathcal{H}_w$, i.e. J does not depend on the field components $\mathtt{F}\varphi$.

The following corollary translates the explicit Kirchoff formulas for free Maxwell solutions computed in Section 2.1p.8 into the language of the group $(\mathsf{W}_t)_{t\in\mathbb{R}}$. We have used Kirchoff's formulas for initial fields in $\mathcal{F}$ while the group $(\mathsf{W}_t)_{t\in\mathbb{R}}$ operates on $D_w(\mathsf{A})$. Therefore, by uniqueness, we expect to be able to express free Maxwell solution generated by the group by Kirchoff's formulas as long as the initial conditions come from $\mathcal{F} \cap D_w(\mathsf{A})$.

Corollary 2.32 (Kirchoff's formulas in terms of $(W_t)_{t\in\mathbb{R}}$)**.** *Let $w \in \mathcal{W}^1$, $F \in D_w(\mathsf{A}^n) \cap \mathcal{F}$ for some $n \in \mathbb{N}$, and*

$$(\mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\leq i\leq N} := \mathsf{W}_t F, \text{ for all } t \in \mathbb{R}.$$

Then

$$\begin{pmatrix} \widetilde{\mathbf{E}}_{i,t} \\ \widetilde{\mathbf{B}}_{i,t} \end{pmatrix} = \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_t * \begin{pmatrix} \mathbf{E}_{i,0} \\ \mathbf{B}_{i,0} \end{pmatrix} - \int_0^t ds\, K_{t-s} * \begin{pmatrix} \nabla\nabla\cdot\mathbf{E}_{i,0} \\ \nabla\nabla\cdot\mathbf{B}_{i,0} \end{pmatrix}.$$

fulfill $\mathbf{E}_{i,t} = \widetilde{\mathbf{E}}_{i,t}$ *and* $\mathbf{B}_{i,t} = \widetilde{\mathbf{B}}_{i,t}$ *for all* $t \in \mathbb{R}$ *and* $1 \le i \le N$ *in the* L^2_w *sense. Furthermore, for all* $t \in \mathbb{R}$ *it holds also that* $(\mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N} \in D_w(\mathsf{A}^n) \cap \mathcal{F}$.

Proof. By the group properties $\mathsf{W}_t F \in D_w(\mathsf{A}^n)$ and by Definition 2.31p.22 and B.4p.43

$$\partial_t \mathsf{W}_t F = \mathsf{A}\mathsf{W}_t F = (0, 0, \nabla \wedge \mathbf{B}_{i,t}, -\nabla \wedge \mathbf{E}_{i,t})_{1\le i\le N}.$$

Since $(\mathbf{E}_{i,0}, \mathbf{B}_{i,0}) \in \mathcal{F}$, a straight-forward computation together with the properties of K_t from Lemma 2.11p.10 yields

$$\begin{aligned} \boxed{8} &= \left(\partial_t - \begin{pmatrix} 0 & \nabla\wedge \\ -\nabla\wedge & 0 \end{pmatrix}\right)\begin{pmatrix} \mathbf{E}_{i,t} \\ \mathbf{B}_{i,t} \end{pmatrix} \\ &= -\partial_t \int_{t_0}^t ds\, K_{t-s} * \begin{pmatrix} \nabla\nabla \cdot \mathbf{E}_{i,0} \\ \nabla\nabla \cdot \mathbf{B}_{i,0} \end{pmatrix} + \begin{pmatrix} \partial_t^2 + \nabla \wedge (\nabla \wedge \cdot) & 0 \\ 0 & \partial_t^2 + \nabla \wedge (\nabla \wedge \cdot) \end{pmatrix} K_t * \begin{pmatrix} \mathbf{E}_{i,0} \\ \mathbf{B}_{i,0} \end{pmatrix} \end{aligned}$$

Applying $\nabla \wedge (\nabla \wedge \cdot) = \nabla(\nabla\cdot) - \triangle$ and Lemma 2.11p.10 again gives

$$(\partial_t^2 - \triangle) K_t * \begin{pmatrix} \mathbf{E}_{i,0} \\ \mathbf{B}_{i,0} \end{pmatrix} = 0$$

and

$$\begin{aligned} \partial_t \int_0^t ds\, K_{t-s} * \begin{pmatrix} \nabla\nabla \cdot \mathbf{E}_{i,0} \\ \nabla\nabla \cdot \mathbf{B}_{i,0} \end{pmatrix} &= K_{t-s} * \begin{pmatrix} \nabla\nabla \cdot \mathbf{E}_{i,0} \\ \nabla\nabla \cdot \mathbf{B}_{i,0} \end{pmatrix}\Bigg|_{s\to t} - \left[K_{t-s} * \begin{pmatrix} \nabla\nabla \cdot \mathbf{E}_{i,0} \\ \nabla\nabla \cdot \mathbf{B}_{i,0} \end{pmatrix}\right]_{s\to 0}^{s\to t} \\ &= K_t * \begin{pmatrix} \nabla\nabla \cdot \mathbf{E}_{i,0} \\ \nabla\nabla \cdot \mathbf{B}_{i,0} \end{pmatrix}. \end{aligned}$$

Hence, we get $\boxed{8} = 0$ and, therefore, for $\widetilde{F}_t := (\widetilde{\mathbf{E}}_{i,t}, \widetilde{\mathbf{B}}_{i,t})_{1\le i\le N}$ it is true that $\partial_t \widetilde{F}_t = \mathsf{A}\widetilde{F}_t$ in the strong sense. By the group properties W_t and A commute on $D_w(\mathsf{A})$ which implies

$$\partial_t \left(\mathsf{W}_{-t}\widetilde{F}_t\right) = -\mathsf{A}\mathsf{W}_{-t}\widetilde{F}_t + \mathsf{W}_{-t}\mathsf{A}\widetilde{F}_t = 0.$$

Therefore, $\widetilde{F}_t = \mathsf{W}_t \widetilde{F}_0 = \mathsf{W}_t F_0 = \chi_t$ as by definition $F_0 = F = \widetilde{F}_0$. This means in particular that $\mathbf{E}_{i,t} = \widetilde{\mathbf{E}}_{i,t}$ and $\mathbf{B}_{i,t} = \widetilde{\mathbf{B}}_{i,t}$ for all $t \in \mathbb{R}$ and $1 \le i \le N$ in the L^2_w sense. Furthermore, as $F \in D_w(\mathsf{A}^n) \cap \mathcal{F}$, Lemma 2.11p.10 states that $\widetilde{F}_t \in \mathcal{F}$, and by the group properties of $(\mathsf{W}_t)_{t\in\mathbb{R}}$ we also have $F_t \in D_w(\mathsf{A}^n)$ for all $t \in \mathbb{R}$. Hence, $F_t = \widetilde{F}_t \in D_w(\mathsf{A}^n) \cap \mathcal{F}$ for all $t \in \mathbb{R}$ which concludes the proof. □

A ready application of this corollary is the following lemma which allows to express the smooth inhomogeneous Maxwell solutions of Section 2.1p.8 in terms of $(\mathsf{W}_t)_{t\in\mathbb{R}}$.

Lemma 2.33 (The Maxwell Solutions in Terms of $(\mathsf{W}_t)_{t\in\mathbb{R}}$ and J)**.** *Let* $t, t_0 \in \mathbb{R}$ *be given times,* $F = (F_i)_{1\le i\le N} \in D_w(\mathsf{A}^n) \cap \mathcal{F}$ *for some* $n \in \mathbb{N}$ *be given initial fields and* $(\mathbf{q}_i, \mathbf{p}_i) \in \mathcal{T}^1_\vee$ *time-like charge trajectories for* $1 \le i \le N$*. If in addition the initial fields* $F_i = (\mathbf{E}_i, \mathbf{B}_i)$ *fulfill the Maxwell constraints*

$$\nabla \cdot \mathbf{E}_i = 4\pi\varrho_i(\cdot - \mathbf{q}_{i,t_0}) \qquad \text{and} \qquad \nabla \cdot \mathbf{B}_i = 0$$

for $1 \le i \le N$*, then for all* $t \in \mathbb{R}$

$$F_t := \mathsf{W}_{t-t_0}\chi + \int_{t_0}^t ds\, \mathsf{W}_{t-s}\mathsf{J}(\varphi_s) \in D_w(\mathsf{A}^n) = (M_{\varrho_i,m_i}[F_i, (\mathbf{q}_i, \mathbf{p}_i)](t, t_0))_{1\le i\le N}.$$

in the L^2_w *sense where* $\varphi_s := (\mathsf{Q} + \mathsf{P})(\mathbf{q}_{i,s}, \mathbf{p}_{i,s})_{1\le i\le N}$ *for* $s \in \mathbb{R}$*. Furthermore,* $F_t \in D_w(\mathsf{A}^n) \cap \mathcal{F}$ *for all* $t \in \mathbb{R}$*.*

Proof. This can be computed by applying Corollary 2.32p.22 twice and using one partial integration. □

WF Solutions for Prescribed Newtonian Cauchy Data. The strategy will be to use Banach's and Schauder's fixed point theorem to prove the existence of a fixed point of S_T. The following normed spaces will prove to be suitable for this problem:

Definition 2.34 (Hilbert Spaces for the Fixed Point Theorem)**.** *For $n \in \mathbb{N}$ let $\mathcal{F}_w^n$ be the linear space of functions $F \in D_w(\mathsf{A}^n)$ with*

$$\|F\|_{\mathcal{F}_w^n(B)} := \left(\sum_{k=0}^{n} \|\mathsf{A}^k F\|_{\mathcal{F}_w} \right)^{\frac{1}{2}}.$$

with $B = \mathbb{R}^3$ in which case we simply write $\|\cdot\|_{\mathcal{F}_w^n}$ instead of $\|\cdot\|_{\mathcal{F}_w^n(\mathbb{R})}$. For other $B \subset \mathbb{R}^3$ we shall use this notation to split up integration domains. We shall use this notation also for $\mathcal{F}_w = \mathcal{F}_w^0$.

Lemma 2.35. *For $n \in \mathbb{N}$, $\mathcal{F}_w^n$ is a Hilbert space.*

Proof. This is an immediate consequence of Theorem 2.11 of Part I [BDD10] which relies on the fact that A is closed on $D_w(\mathsf{A})$ and $\sqrt{w}d^3x$ is absolute continuous with respect to the Lebesgue measure. □

Next we specify the class of boundary fields $(X_{i,\pm T}^{\pm})_{1\le i\le N}$ which we want to allow.

Definition 2.36 (The Class of Boundary Fields $\mathcal{A}_w^n$, $\widetilde{\mathcal{A}}_w^n$ and $\mathcal{A}_w^{\mathrm{Lip}}$)**.** *For weight $w \in \mathcal{W}$ and $n \in \mathbb{N}$ let $\mathcal{A}_w^n$ be the set of maps*

$$X : \mathbb{R} \times D_w(A) \to D_w(\mathsf{A}^\infty) \cap \mathcal{F}, \qquad (T, \varphi) \mapsto X_T[\varphi]$$

which have the following properties for all $p \in \mathcal{P}$ and $T \in \mathbb{R}$:

(i) There is a function $C_3 \in \mathtt{Bounds}$ such that for all $\varphi \in D_w(A)$ with $(\mathtt{Q}+\mathtt{P})\varphi = p$ it is true that $\|X_T[\varphi]\|_{\mathcal{F}_w^n} \le C_3^{(n)}(|T|, \|p\|)$.

(ii) The map $F \mapsto X_T[p, F]$ as $\mathcal{F}_w^1 \to \mathcal{F}_w^1$ is continuous.

(iii) For $(\mathbf{E}_{i,T}, \mathbf{B}_{i,T})_{1\le i\le N} := X_T[\varphi]$ and $(\mathbf{q}_{i,T}, \mathbf{p}_{i,T})_{1\le i\le N} := (\mathtt{Q}+\mathtt{P})M_L[\varphi](T,0)$ one has $\nabla\cdot\mathbf{E}_{i,T} = 4\pi\varrho_i(\cdot - \mathbf{q}_{i,T})$ and $\nabla\cdot\mathbf{B}_{i,T} = 0$.

Let the subset $\widetilde{\mathcal{A}}_w^n \subset \mathcal{A}_w^n$ comprise such maps X that fulfill:

(iv) For balls $B_\tau := B_\tau(0) \subset \mathbb{R}^3$ with radius $\tau > 0$ around the origin and any bounded set $M \subset D_w(\mathsf{A})$ it holds that $\lim_{\tau\to\infty} \sup_{F\in M} \|X_T[p, F]\|_{\mathcal{F}_w^n(B_\tau^c)} = 0$.

Furthermore, let the subset $\mathcal{A}_w^{\mathrm{Lip}} \subset \mathcal{A}_w^1$ comprise such maps X that fulfill:

(v) There is a function $C_4 \in \mathtt{Bounds}$ such that for all $\varphi, \widetilde{\varphi} \in D_w(A)$ with $(\mathtt{Q}+\mathtt{P})\varphi = p = (\mathtt{Q}+\mathtt{P})\widetilde{\varphi}$ it is true that $\|X_T[\varphi] - X_T[\widetilde{\varphi}]\|_{\mathcal{F}_w^1} \le |T| C_4(|T|, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w})\, \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}$.

REMARK 2.37. *The boundary fields needed are now encoded via $(X_{i,\pm T}^{\pm})_{1\le i\le N} := X_{\pm T}^{\pm}[\varphi]$ for two elements $X^\pm \in \mathcal{A}_w^n$ and some $\varphi \in D_w(A)$. The dependence of $X_{\pm T}^{\pm}$ on a $\varphi \in D_w(A)$ instead of the charge trajectories $(\mathbf{q}_i, \mathbf{p}_i)$, $1 \le i \le N$, in $\mathcal{T}_\triangledown^1$ makes sense as φ carries the whole information about the charge trajectories by $t \mapsto (\mathtt{Q}+\mathtt{P})M_L[\varphi](t,0)$ which are the charge trajectories of the ML-SI solutions. As we shall discuss after showing that these classes are not empty, one can imagine their elements to be the Liénard-Wiechert fields of any charge trajectories in $\mathcal{T}_\triangledown^1$ which continue the ML-SI charge trajectories on either the time interval $(-\infty, -T]$ or $[T, \infty)$ for the given $T \in \mathbb{R}$. Finally, the reason why we define three classes $\mathcal{A}_w^n$, $\widetilde{\mathcal{A}}_w^n$ and $\mathcal{A}^{\mathrm{Lip}}$ is to distinguish clearly the properties needed, first, to define what we mean by a bWF solution, second, to show existence of bWF solutions, and third, to show uniqueness of the bWF solution for small enough T. Note also that $\mathcal{A}_w^{n+1} \subset \mathcal{A}_w^n$ as well as $\widetilde{\mathcal{A}}_w^{n+1} \subset \widetilde{\mathcal{A}}_w^n$ for $n \in \mathbb{N}$.*

Having this we can formalize what we mean by a WF solution for given Newtonian Cauchy data and boundary fields.

Definition 2.38. *[bWF Solutions for Newtonian Cauchy Data and Boundary Fields] Let $T > 0$, $p \in \mathcal{P}$ and two boundary fields $X^\pm \in \mathcal{A}^1_w$ be given. We define $\mathcal{T}^{p,X^\pm}_T$ to be the set of time-like charge trajectories in $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N} \in \mathcal{T}_\vee$ which solve the WF equations in the form (4p.3)-(11p.7) for Newtonian Cauchy data $p = (\mathbf{q}_{i,t}, \mathbf{q}_{i,t})_{1\le i\le N}|_{t=0}$. We shall call every element of $\mathcal{T}^{p,X^\pm}_T$ a bWF solution for initial value p and boundary fields $X^\pm$ at time T.*

REMARK 2.39. *By Definition 2.36p.24(iv) the boundary fields fulfill the Maxwell constraints at time $\pm T$. This is important as our formulas for the Maxwell solutions of Section 2.1p.8 are only valid if the Maxwell constraints are fulfilled. Though this requirement could be loosened by refining the formulas for the Maxwell fields it is natural to stick with it because the fields of true WF solution fulfills the Maxwell equations including the constraints, and the final goal is to find solutions for $T \to \infty$.*

Now we can define a convenient fixed point map whose fixed points are the bWF solutions.

Definition 2.40 (The Fixed Point Map S_T)**.** *For finite time $T > 0$, Newtonian Cauchy data $p \in \mathcal{P}$ and boundary fields $X^\pm \in \mathcal{A}^1_w$, we define*

$$S^{p,X^\pm}_T : D_w(\mathsf{A}) \to D_w(\mathsf{A}^\infty), \qquad F \mapsto S^{p,X^\pm}_T[F]$$

for

$$S^{p,X^\pm}_T[F] := \frac{1}{2}\sum_\pm \left[\mathsf{W}_{\mp T} X^\pm_{\pm T}[p,F] + \int_{\pm T}^t ds\, \mathsf{W}_{-s}\mathsf{J}(\varphi_s[p,F])\right]$$

where $\varphi_s[p,F] := M_L[p,F](s,0)$ for $s \in \mathbb{R}$ is the ML-SI solution, cf. Definition B.10p.45, for initial value $(p,F) \in D_w(A)$.

We got to make sure that the fixed point map is well-defined and that its possible fixed points have the desired properties, i.e. their corresponding charge trajectories are in $\mathcal{T}^{p,X^\pm}_T$.

Theorem 2.41 (The Map S_T and its Fixed Points)**.** *For finite time $T > 0$, Newtonian Cauchy data $p \in \mathcal{P}$ and boundary fields $X^\pm \in \mathcal{A}^1_w$ the following is true:*
(i) The map $S^{p,X^\pm}_T$ introduced in Definition 2.40 is well-defined.
(ii) For $F \in D_w(\mathsf{A})$, setting $(X^\pm_{i,\pm T})_{1\le i\le N} := X^\pm_{\pm T}[p,F]$ and denoting the ML-SI charge trajectories

$$t \mapsto (\mathbf{q}_{i,t}, \mathbf{q}_{i,t})_{1\le i\le N} := (\mathsf{Q}+\mathsf{P})M_L[p,F](t,0)$$

by $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N}$ it holds that

$$S^{p,X^\pm}_T[F] = \frac{1}{2}\sum_\pm \Big(M_{\varrho_i,m_i}[X^\pm_{i,\pm T}, (\mathbf{q}_i,\mathbf{p}_i)](0,\pm T)\Big)_{1\le i\le N} \in D_w(\mathsf{A}^\infty)\cap\mathcal{F}.$$

(iii) For any $F = S^{p,X^\pm}_T[F]$ it is true that and that the corresponding charge trajectories $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N}$ as defined in (ii) are in $\mathcal{T}^{p,X^\pm}_T$.

Proof. (i) Let $F \in D_w(\mathsf{A})$, then $(p,F) \in D_w(A)$ and, hence, by the ML±SI existence and uniqueness Theorem B.6p.44 $t \mapsto \varphi_t := M_L[\varphi](t,0)$ is a once continuously differentiable map $\mathbb{R} \to D_w(A) \subset \mathcal{H}_w$. By properties of J stated in Lemma 2.23 of Part I [BDD10] we know that $A^k J : \mathcal{H}_w \to D_w(A^\infty) \subset \mathcal{H}_w$ is locally Lipschitz continuous for any $k \in \mathbb{N}$. By projecting onto field space $\mathcal{F}_w$, cf. Definition2.31p.22, we yield that also $\mathsf{A}^k\mathsf{J} : \mathcal{H}_w \to D_w(\mathsf{A}^\infty) \subset \mathcal{F}_w$ is locally Lipschitz continuous. Hence, by the group properties of $(\mathsf{W}_t)_{t\in\mathbb{R}}$ we know that $s \mapsto \mathsf{W}_{-s}\mathsf{A}^k\mathsf{J}(\varphi_s)$ for any $k \in \mathbb{N}$ is continuous. Therefore, we may apply Corollary A.3p.41 which states that

$$\mathsf{A}^k \int_{\pm T}^0 ds\ \mathsf{W}_{-s}\mathsf{J}(\varphi_s) = \int_{\pm T}^0 ds\ \mathsf{W}_{-s}\mathsf{A}^k\mathsf{J}(\varphi_s)$$

while the integral on the right-hand side exists because the integrand is continuous and the integral goes over a compact set. As this holds for any $k \in \mathbb{N}$, $\int_{\pm T}^0 ds\ \mathsf{W}_{-s}\mathsf{J}(\varphi_s) \in D_w(\mathsf{A}^\infty)$. Furthermore, by Definition 2.36p.24 the term $X^\pm_{\pm T}[p,F]$ is in $D_w(\mathsf{A}^\infty)$ and therefore $\mathsf{W}_{\mp T}X^\pm_{\pm T}[p,F] \in D_w(\mathsf{A}^\infty)$ by the group properties. Hence, the map $S_T^{\phi,X^\pm}$ is well-defined as a map $D_w(A) \to D_w(\mathsf{A}^\infty)$.

(ii) For $F \in D_w(\mathsf{A})$ let $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N}$ denote the charge trajectories $t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1\le i\le N} = (\mathsf{Q}+\mathsf{P})\varphi_t$ of $t \mapsto \varphi_t := M_L[p,F](t,0)$. Since $(p,F) \in D_w(A)$, we know again by Theorem B.6p.44 that these charge trajectories are once continuously differentiable as $\mathbb{R} \to D_w(A) \subset \mathcal{H}_w$. As the absolute value of the velocity is given by $\|\mathbf{v}(\mathbf{p}_{i,t})\| = \frac{\|\mathbf{p}_{i,t}\|}{\sqrt{m_i^2+\mathbf{p}_{i,t}^2}} < 1$, we conclude that $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N}$ are once continuously differentiable and time-like and therefore in $\mathcal{T}_\vee$, cf. Definition 2.5p.8. Furthermore, the boundary fields $X^\pm_{\pm T}[p,F]$ are in $D_w(\mathsf{A}^\infty) \cap \mathcal{F}$ and obey the Maxwell constraints by the definition of $\mathcal{A}^n_w$. So we can apply Lemma 2.33p.23 which states for $(X^\pm_{i,\pm T})_{1\le i\le N} := X^\pm_{\pm T}[p,F]$ that

$$(M_{\varrho_i,m_i}[X_i, (\mathbf{q}_i,\mathbf{p}_i](t,\pm T))_{1\le i\le N} = \mathsf{W}_{t\mp T}X^\pm_\pm[p,F] + \int_{\pm T}^t ds\ \mathsf{W}_{t-s}\mathsf{J}(\varphi_s) \in D_w(\mathsf{A}) \cap \mathcal{F}. \tag{35}$$

For $t = 0$ this proves claim (ii).

(iii) Finally, assume there is an $F \in \mathcal{F}_w$ such that $F = S_T^{p,X^\pm}[F]$. By (ii) this implies $F \in D_w(\mathsf{A}^\infty) \cap \mathcal{F}$. Let $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N}$ and $t \mapsto \varphi_t$ be as defined in the proof of (ii) which now is infinitely often differentiable as $\mathbb{R} \to \mathcal{H}_w$ since $(p,F) \in D_w(A^\infty)$. We shall show that the following integral equality holds

$$\varphi_t = (p,0) + \int_0^t ds\ (\mathsf{Q}+\mathsf{P})J(\varphi_s) + \frac{1}{2}\sum_\pm \left[W_{t\mp T}(0, X^\pm_{\pm T}[p,F]) + \int_{\pm T}^t ds\ W_{t-s}\mathsf{F}J(\varphi_s) \right] \tag{36}$$

for all $t \in \mathbb{R}$; keep in mind that $t \mapsto \varphi_t$ depends also on (p,F). Then differentiation with respect to time t of the phase space components of $(\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N} := \varphi_t$ yields $\partial_t(\mathsf{Q}+\mathsf{P})\varphi_t = (\mathsf{Q}+\mathsf{P})J(\varphi_t)$ which by definition of J yields

$$\begin{aligned} \partial_t\mathbf{q}_{i,t} &= \mathbf{v}(\mathbf{p}_{i,t}) := \frac{\mathbf{p}_{i,t}}{\sqrt{m_i^2+\mathbf{p}_{i,t}^2}} \\ \partial_t\mathbf{p}_{i,t} &= \sum_{j\ne i}\int d^3x\ \varrho_i(\mathbf{x}-\mathbf{q}_{i,t})\left(\mathbf{E}_{j,t}(\mathbf{x}) + \mathbf{v}(\mathbf{q}_{i,t}) \wedge \mathbf{B}_{j,t}(\mathbf{x})\right). \end{aligned} \tag{37}$$

Furthermore, the field components fulfill

$$\begin{aligned} \mathsf{F}\varphi_t &= \mathsf{F}\frac{1}{2}\sum_\pm \left[W_{t\mp T}(0, X^\pm_{\pm T}[\varphi]) + \int_{\pm T}^t ds\ W_{t-s}\mathsf{F}J(\varphi_s) \right] \\ &= \frac{1}{2}\sum_\pm \left[\mathsf{W}_{t\mp T}X^\pm_{\pm T}[p,F] + \int_{\pm T}^t ds\ \mathsf{W}_{t-s}\mathsf{J}(\varphi_s) \right] \end{aligned}$$

where we only used the definition of the projectors, cf. Definition 2.31p.22. Hence, by (35) we know

$$(\mathbf{E}_{i,t}, \mathbf{B}_{i,t}) = \frac{1}{2}\sum_\pm M_{\varrho_i,m_i}[F_i, (\mathbf{q}_i,\mathbf{p}_i](t,\pm T). \tag{38}$$

Finally, we have

$$(\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1\le i\le N}\Big|_{t=0} = p = (\mathbf{q}_i^0, \mathbf{p}_i^0)_{1\le i\le N}. \tag{39}$$

Equations (37p.26), (38p.26) and (39) are exactly the WF equations (4p.3) and (11p.7) for Newtonian Cauchy data p and boundary fields $X^\pm$. Hence, since in (ii) we proved that $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N}$ are in $\mathcal{T}_\vee$, we conclude that they are also in $\mathcal{T}_T^{p,X^\pm}$, cf. Definition 2.38p.25.

Now it is only left to prove that the integral equation (36p.26) holds. By Definition B.10p.45, φ_t fulfills

$$\varphi_t = W_t(p,F) + \int_0^t ds\ W_{t-s}J(\varphi_s)$$

for all $t \in \mathbb{R}$. Inserting the fixed point equation $F = S_T^{p,X^\pm}[F]$, i.e.

$$F = \mathsf{W}_{t\mp T}X_\pm^\pm[p,F] + \int_{\pm T}^t ds\ \mathsf{W}_{t-s}\mathsf{J}(\varphi_s),$$

we find

$$\varphi_t = (p,0) + \frac{1}{2}\sum_\pm W_{t\mp T}(0, X_{\pm T}^\pm[p,F]) + \frac{1}{2}\sum_\pm W_t\int_{\pm T}^0 ds\ W_{-s}(0,\mathsf{J}(\varphi_s)) + \int_0^t ds\ W_{t-s}J(\varphi_s).$$

Using the continuity of the integrands we may apply Lemma A.4p.41 to commute W_t with the integral. This together with $J = (\mathtt{Q}+\mathtt{P})J + \mathtt{F}J$ and that $(\mathtt{Q}+\mathtt{P})W_t = \mathrm{id}_{\mathcal{P}}$ yields the desired result (36p.26) for all $t\in\mathbb{R}$ which concludes the proof. □

In the next Lemma we discuss a simple element $C \in \mathcal{A}_w^n$ and thereby show that the classes of boundary fields $\widetilde{\mathcal{A}}_w^n$ and $\mathcal{A}_w^{\mathrm{Lip}}$ are not empty.

Definition 2.42 (Coulomb Boundary Field)**.** *For $n\in\mathbb{N}$ define $C : \mathbb{R}\times\mathcal{H}_w^n \to D_w(\mathsf{A}^n)$, $(T,\varphi)\mapsto C_T[\varphi]$ to be*

$$C_T[\varphi] := \left(\mathbf{E}_i^C(\cdot - \mathbf{q}_{i,T}), 0\right)_{1\le i\le N}$$

where $(\mathbf{q}_{i,T})_{1\le i\le N} := \mathtt{Q}M_L[\varphi](T,0)$ and the Coulomb field

$$(\mathbf{E}_i^C, 0) := M_{\varrho_i,m_i}[t\mapsto(0,0)](0,-\infty) = \left(\int d^3z\ \varrho_i(\cdot - \mathbf{z})\frac{\mathbf{z}}{\|\mathbf{z}\|^3}, 0\right).$$

Note that the last equality holds by Theorem 2.18p.14.

Lemma 2.43 (The Class of Boundary Fields is Non-Empty)**.** *For any $n\in\mathbb{N}$ and any $w\in\mathcal{W}$ the set $C\in\mathcal{A}_w^n\cap\mathcal{A}_w^{\mathrm{Lip}}$.*

Proof. We need to show the properties given in Definition 2.36p.24. Fix $T>0$ and $p\in\mathcal{P}$. Recall the definition of C_T as introduced in Definition 2.42. Let $\varphi\in D_w(A)$ and $F = \mathtt{F}\varphi$ for $(\mathtt{Q}+\mathtt{P})\varphi = p$. Furthermore, we define $(\mathbf{q}_{i,T})_{1\le i\le N} := QM_L[\varphi](T,0)$. Since $\mathbf{E}^C$ is a Liénard-Wiechert field of the charge trajectory $t\mapsto(\mathbf{q}_{i,T},0)$ in $\mathcal{T}_\triangledown^1$, we can apply Corollary 2.22p.18 to yield the following estimate for any multi-index $\alpha\in\mathbb{N}^3$ and $\mathbf{x}\in\mathbb{R}^3$

$$\left\|D^\alpha\mathbf{E}^C(\mathbf{x})\right\|_{\mathbb{R}^3} \le \frac{C_5^{(\alpha)}}{1+\|\mathbf{x}\|^2}. \tag{40}$$

which allows to define the finite constants $C_6^{(\alpha)} := \|D^\alpha \mathbf{E}_C\|_{L^2_w}$. Using the properties of the weight $w \in \mathcal{W}$ we find

$$\|C_T[\varphi]\|^2_{\mathcal{F}^n_w} \le \sum_{k=0}^{n} \|\mathsf{A}^k C_T[\varphi]\|_{\mathcal{F}_w} \le \sum_{k=0}^{n}\sum_{i=1}^{N} \left\|(\nabla\wedge)^k \mathbf{E}^C_i(\cdot - \mathbf{q}_{i,T})\right\|_{L^2_w} \le \sum_{k=0}^{n}\sum_{|\alpha|\le k}\sum_{i=1}^{N} \left\|D^\alpha \mathbf{E}^C_i\right\|_{L^2_w}$$
$$\le \sum_{k=0}^{n}\sum_{|\alpha|\le k}\sum_{i=1}^{N} \left(1 + C_w \left\|\mathbf{q}_{i,T}\right\|\right)^{\frac{P_w}{2}} \left\|D^\alpha \mathbf{E}^C_i\right\|_{L^2_w} \le \sum_{k=0}^{n}\sum_{|\alpha|\le k}\sum_{i=1}^{N} \left(1 + C_w \left\|\mathbf{q}_{i,T}\right\|\right)^{\frac{P_w}{2}} C_6^{(\alpha)} < \infty.$$

This implies $C_T \in D_w(A^\infty) \cap \mathcal{F}$ and that $C : \mathbb{R} \times D_w(A) \to D_w(\mathsf{A}^\infty) \cap \mathcal{F}$ is well-defined.

Note that the right-hand side depends only on $\left\|\mathbf{q}_{i,T}\right\|$ which is bounded by $\|p\| + |T|$ since the maximal velocity is below one. Hence, property (i) holds for

$$C_3^{(n)}(|T|, \|p\|) := \sum_{k=0}^{n}\sum_{|\alpha|\le k}\sum_{i=1}^{N} (1 + C_w(\|p\| + |T|))^{\frac{P_w}{2}} C_6^{(\alpha)}.$$

Instead of showing property (ii), we prove the stronger property (v). For this let $\widetilde{\varphi} \in D_w(A)$ such that $(\mathsf{Q} + \mathsf{P})\varphi = (\mathsf{Q} + \mathsf{P})\widetilde{\varphi}$, $(\widetilde{\mathbf{q}}_{i,T})_{1\le i\le N} := \mathsf{Q}M_L[\widetilde{\varphi}](T, 0)$. Starting with

$$\|C_T[\varphi] - C_T[\widetilde{\varphi}]\|_{\mathcal{F}^1_w} \le \sum_{i=1}^{N}\sum_{|\alpha|\le 1} \left\|D^\alpha \left(\mathbf{E}^C(\cdot - \mathbf{q}_{i,T}) - \mathbf{E}^C(\cdot - \widetilde{\mathbf{q}}_{i,T})\right)\right\|_{L^2_w}$$

we compute

$$\left\|D^\alpha \left(\mathbf{E}^C(\cdot - \mathbf{q}_{i,T}) - \mathbf{E}^C(\cdot - \widetilde{\mathbf{q}}_{i,T})\right)\right\|_{L^2_w} = \left\|\int_0^1 d\lambda\, (\widetilde{\mathbf{q}}_{i,T} - \mathbf{q}_{i,t}) \cdot \nabla D^\alpha \mathbf{E}^C_{i,T}(\cdot - \widetilde{\mathbf{q}}_{i,T} + \lambda(\widetilde{\mathbf{q}}_{i,T} - \mathbf{q}_{i,t}))\right\|_{L^2_w}$$
$$\le \int_0^1 d\lambda\, \left\|(\mathbf{q}_{i,t} - \widetilde{\mathbf{q}}_{i,T}) \cdot \nabla D^\alpha \mathbf{E}^C(\cdot - \widetilde{\mathbf{q}}_{i,T} + \lambda(\widetilde{\mathbf{q}}_{i,T} - \mathbf{q}_{i,t}))\right\|_{L^2_w}$$

where in the last step we have used Minkowski's inequality. Therefore, for all $|\alpha| \le 1$ we get

$$\sum_{|\alpha|\le 1} \left\|D^\alpha \left(\mathbf{E}^C(\cdot - \mathbf{q}_{i,T}) - \mathbf{E}^C(\cdot - \widetilde{\mathbf{q}}_{i,T})\right)\right\|_{L^2_w}$$
$$\le \|\mathbf{q}_{i,T} - \widetilde{\mathbf{q}}_{i,T}\|_{\mathbb{R}^3} \sup_{0\le\lambda\le 1} \sum_{|\beta|\le 2} \left\|D^\beta \mathbf{E}^C(\cdot + \lambda(\mathbf{q}_{i,T} - \widetilde{\mathbf{q}}_{i,T}))\right\|_{L^2_w}.$$

The estimate ($40_{\text{p.27}}$) and the properties of $w \in \mathcal{W}$ yield

$$\left\|D^\beta \mathbf{E}^C(\cdot - \widetilde{\mathbf{q}}_{i,T} + \lambda(\widetilde{\mathbf{q}}_{i,T} - \mathbf{q}_{i,t}))\right\|_{L^2_w} \le \left(1 + C_w \lambda \left\|\widetilde{\mathbf{q}}_{i,T} + \lambda(\widetilde{\mathbf{q}}_{i,T} - \mathbf{q}_{i,t})\right\|_{\mathbb{R}^3}\right)^{\frac{P_w}{2}} \left\|D^\beta \mathbf{E}^C\right\|_{L^2_w}$$
$$\le (1 + C_w(\|\mathbf{q}_i\|_{\mathbb{R}^3} + \|\widetilde{\mathbf{q}}_i\|_{\mathbb{R}^3} + 2|T|)^{\frac{P_w}{2}} C_6^{(\beta)}$$

Furthermore, since the maximal velocity is smaller than one, property (v) holds for

$$C_4(|T|, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) := N \sum_{|\beta|\le 2} (1 + C_w(\|\mathsf{Q}\varphi\|_{\mathbb{R}^3} + \|\mathsf{Q}\widetilde{\varphi}\|_{\mathbb{R}^3} + 2|T|)^{\frac{P_w}{2}} C_6^{(\beta)}.$$

(iii) holds by Theorem $2.14_{\text{p.12}}$.

(iv) Let $B_\tau(0) \subset \mathbb{R}^3$ be a ball of radius $\tau > 0$ around the origin. For any $F \in D_w(\mathsf{A})$ we define $(\mathbf{q}_{i,T})_{1\le i\le N} := QM_L[\varphi](T,0)$ and yield

$$\|C^T[p,F]\|_{\mathcal{F}_w^n(B_\tau^c(0))} \le \sum_{i=1}^N \sum_{|\alpha|\le n} \left\|D^\alpha \mathbf{E}^C(\cdot - \mathbf{q}_{i,T})\right\|_{L^2_w(B^c_\tau(0))}$$
$$\le \sum_{i=1}^N \sum_{|\alpha|\le n} (1 + C_w\|\mathbf{q}_{i,T}\|)^{\frac{P_w}{2}} \left\|D^\alpha \mathbf{E}^C\right\|_{L^2_w(B^c_\tau(\mathbf{q}_{i,T}))}.$$

Note again that the maximal velocity is smaller than one so that $\|\mathbf{q}_{i,T} \le \|\mathbf{q}_i^0\| + T$. Hence, for $\tau > \|\mathbf{q}_i^0\| + T$ define $r(\tau) := \tau - \|\mathbf{q}_i^0\| + T$ such that it holds

$$\sup_{F\in D_w(\mathsf{A})} \|C^T[p,F]\|_{\mathcal{F}_w^n(B_\tau^c(0))} \le \sum_{i=1}^N \sum_{|\alpha|\le n} (1 + C_w\|\mathbf{q}_{i,T}\|)^{\frac{P_w}{2}} \left\|D^\alpha \mathbf{E}^C\right\|_{L^2_w(B^c_{r(\tau)}(0))} \xrightarrow[\tau\to\infty]{} 0$$

To summarize we have shown that for all $n \in \mathbb{N}$ the map C as introduced in Definition 2.42p.27 is an element of $\widetilde{\mathcal{A}}_w^n \cap \mathcal{A}^{\mathrm{Lip}}$ which is a subset of $\mathcal{A}_w^n$. □

REMARK 2.44. *In view of (11**p.7**) the boundary fields are a guess of how the charge trajectories* $(\mathbf{q}_i^0, \mathbf{p}_i)_{1\le i\le N}$ *continue on the intervals* $(-\infty, -T]$ *and* $[T, \infty)$*. Instead of the Coulomb fields of a charge at rest we could have also taken the Liénard-Wiechert fields of a charge trajectory which starts at* $\mathbf{q}_{i,T}$ *and has constant momentum* $\mathbf{p}_{i,T}$ *for* $(\mathbf{q}_{i,T}, \mathbf{p}_{i,T})_{1\le i\le N} := (\mathsf{Q}+\mathsf{P})M_L[\varphi](T,0)$ *with only minor modification (the result would be a Lorentz boosted Coulomb field). Such boundary fields would also be in* $\mathcal{A}_w^{\mathrm{Lip}}$ *as for* $(\mathbf{p}_{i,T})_{1\le i\le N} := \mathsf{P}M_L[\widetilde{\varphi}](T,0)$ *we have*

$$\|\mathbf{p}_{i,T} - \widetilde{\mathbf{p}}_{i,T}\| \le \int_0^T ds\, \|\dot{\mathbf{p}}_{i,s} - \dot{\widetilde{\mathbf{p}}}_{i,s}\| \le T \sup_{s\in[0,T]} \|\dot{\mathbf{p}}_{i,s} - \dot{\widetilde{\mathbf{p}}}_{i,s}\|$$

while the supremum exists because the charge trajectories are smooth thanks to $\varphi, \widetilde{\varphi} \in D_w(A)$*. Only if one wanted to continue the charge trajectories* $(\mathbf{q}_i^0, \mathbf{p}_i)_{1\le i\le N}$ *in (11**p.7**) more smoothly, for example also continuously in the acceleration, the resulting boundary fields would not lie in* $\mathcal{A}_w^{\mathrm{Lip}}$ *anymore but rather in* $\widetilde{\mathcal{A}}_w^1$ *since in general different initial value for the ML-SI equations yield different accelerations at time zero.*

We come to the proof of the existence theorem of WF solutions for finite times, i.e. Theorem 2.3p.7. The strategy for the proof is to use Banach's and Schauder's fixed point theorem. Before we give a proof of Theorem 2.3p.7 we collect the needed estimates and properties of $S_T^{p,X^\pm}$ in the following three lemmas.

Lemma 2.45 (Estimates on $\mathcal{F}_w^n$)**.** *For* $n \in \mathbb{N}_0$ *the following is true:*

(i) For all $t \in \mathbb{R}$ *and* $F \in D_w(\mathsf{A}^n)$ *it holds that* $\|\mathsf{W}_t F\|_{\mathcal{F}_w^n} \le e^{\gamma|t|}\|F\|_{\mathcal{F}_w^n}$.

(ii) For all $\varphi \in \mathcal{H}_w$ *there is a constant* $C_7^{(n)} \in \mathtt{Bounds}$ *such that*

$$\|\mathsf{J}(\varphi)\|_{\mathcal{F}_w^n} \le C_7^{(n)}(\|\mathsf{Q}\varphi\|_{\mathcal{H}_w}).$$

(iii) For all $\varphi, \widetilde{\varphi} \in \mathcal{H}_w$ *there is a* $C_8^{(n)} \in \mathtt{Bounds}$ *such that*

$$\|\mathsf{J}(\varphi) - \mathsf{J}(\widetilde{\varphi})\|_{\mathcal{F}_w^n} \le C_8^{(n)}(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w})\|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}.$$

Proof. (i) As shown in Part I [BDD10], A on $D_w(A)$ generates a γ-contractive group $(W_t)_{t\in\mathbb{R}}$; cf. Definition B.8p.45. Hence, A and W_t commute for any $t \in \mathbb{R}$ which implies for all $F \in D_w(\mathsf{A}^n)$ that

$$\|\mathsf{W}_t F\|^2_{\mathcal{F}_w^n} = \sum_{k=0}^n \|\mathsf{A}^k \mathsf{W}_t F\|^2_{\mathcal{F}_w} = \sum_{k=0}^n \|\mathsf{W}_t \mathsf{A}^k F\|^2_{\mathcal{F}_w} \le e^{\gamma|t|} \sum_{k=0}^n \|\mathsf{A}^k F\|^2_{\mathcal{F}_w} = e^{\gamma|t|}\|F\|_{\mathcal{F}_w^n}.$$

For (ii) let $(\mathbf{q}_i, \mathbf{p}_i, \mathbf{E}_i, \mathbf{B}_i)_{1\leq i\leq N} = \varphi \in \mathcal{H}_w$. Using then the definitions of J, cf. Definition 2.31p.22 and B.5p.44, we find

$$\|\mathsf{J}(\varphi)\|_{\mathcal{F}_w^n} \leq \sum_{k=0}^{n} \|(\nabla\wedge)^k \mathbf{v}(\mathbf{p}_i)\varrho_i(\cdot - \mathbf{q}_i)\|_{L^2_w}.$$

By applying the triangular inequality one finds a constant C_9, e.g. $C_9 = 4\sqrt{6}$, for which

$$\|(\nabla\wedge)^k \mathbf{v}(\mathbf{p}_i)\varrho_i(\cdot - \mathbf{q}_i)\|_{L^2_w} \leq (C_9)^n \sum_{|\alpha|\leq n} \|\mathbf{v}(\mathbf{p}_i) D^\alpha \varrho_i(\cdot - \mathbf{q}_i)\|_{L^2_w} \leq (C_9)^n \sum_{|\alpha|\leq n} \|D^\alpha \varrho_i(\cdot - \mathbf{q}_i)\|_{L^2_w}$$

whereas in the last step we used the fact that the maximal velocity is smaller than one. Using the properties of the weight function $w \in \mathcal{W}$, cf. Definition B.1p.43, we conclude

$$\|D^\alpha \varrho_i(\cdot - \mathbf{q}_i)\|_{L^2_w} \leq (1 + C_w\|\mathbf{q}_i\|)^{\frac{P_w}{2}} \|D^\alpha \varrho_i\|_{L^2_w}.$$

Collecting these estimates we yield that claim (ii) holds for

$$C_7^{(n)}(\|\mathsf{Q}\varphi\|_{\mathcal{H}_w}) := (C_9)^n \sum_{i=1}^{N} (1 + C_w\|\mathbf{q}_i\|)^{\frac{P_w}{2}} \sum_{|\alpha|\leq n} \|D^\alpha \varrho_i\|.$$

Claim (iii) is shown by repetitively applying estimate of Lemma 2.23(i) of Part I [BDD10] on the right-hand side of

$$\|\mathsf{J}(\varphi) - \mathsf{J}(\widetilde{\varphi})\|_{\mathcal{F}_w^n} \leq \sum_{k=0}^{n} \|A^k[J(\varphi) - J(\widetilde{\varphi})]\|_{\mathcal{H}_w}$$

which yields a constant $C_8^{(n)} := \sum_{k=0}^n C_{10}^{(k)}(\|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w})$ where $C_{10} \in \mathtt{Bounds}$ can be taken from Lemma 2.23(i) of Part I [BDD10]. This concludes the proof. □

Lemma 2.46 (Properties of $S_T^{p,X^\pm}$)**.** *Let $T > 0$, $p \in \mathcal{P}$ and $X^\pm \in \mathcal{A}_w^n$ for $n \in \mathbb{N}$. Then it holds:*

(i) There is a function $C_{11} \in \mathtt{Bounds}$ such that for all $F \in \mathcal{F}_w^1$ we have

$$\|S_T^{p,X^\pm}[p, F]\|_{\mathcal{F}_w^n} \leq C_{11}^{(n)}(T, \|p\|).$$

(ii) $F \mapsto S_T^{p,X^\pm}[F]$ as $\mathcal{F}_w^1 \to \mathcal{F}_w^1$ is continuous.

If $X^\pm \in \mathcal{A}_w^{\mathrm{Lip}}$, it is also true that:

(iii) There is a function $C_{12} \in \mathtt{Bounds}$ such that for all $F, \widetilde{F} \in \mathcal{F}_w^1$ we have

$$\|S_T^{p,X^\pm}[F] - S_T^{p,X^\pm}[\widetilde{F}]\|_{\mathcal{F}_w^1} \leq T C_{12}(T, \|p\|, \|F\|_{\mathcal{F}_w}, \|\widetilde{F}\|_{\mathcal{F}_w})\|F - \widetilde{F}\|_{\mathcal{F}_w}.$$

Proof. Fix $T > 0$, $p \in \mathcal{P}$, $X^\pm \in \mathcal{A}_w^n$ for $n \in \mathbb{N}$. Before we prove the claims we preliminarily recall the relevant estimates of the ML-SI dynamics. Throughout the proof and for for any $F, \widetilde{F} \in \mathcal{F}_w^n$ we define $D_w(A^n) \ni \varphi = (p, F)$ and $D_w(A^n) \ni \widetilde{\varphi} = (p, \widetilde{F})$ and furthermore the ML-SI solutions $\varphi_t := M_L[\varphi](t, 0)$ and $\widetilde{\varphi}_t := M_L[\widetilde{\varphi}](t, 0)$ for any $t \in \mathbb{R}$. Recall the estimate (65p.44) from the ML±SI existence and uniqueness Theorem B.6p.44 which gives the following T dependent upper bounds on these ML-SI solutions:

(41) $$\sup_{t\in[-T,T]} \|\varphi_t - \widetilde{\varphi}_t\|_{\mathcal{H}_w} \leq C_{20}(T, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w})\|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w},$$

(42) $$\sup_{t\in[-T,T]} \|\varphi_t\|_{\mathcal{H}_w} \leq C_{20}(T, \|\varphi\|_{\mathcal{H}_w}, 0)\|\varphi\|_{\mathcal{H}_w} \quad \text{and} \quad \sup_{t\in[-T,T]} \|\widetilde{\varphi}_t\|_{\mathcal{H}_w} \leq C_{20}(T, \|\widetilde{\varphi}\|_{\mathcal{H}_w}, 0)\|\widetilde{\varphi}\|_{\mathcal{H}_w}.$$

To prove claim (i) we estimate

$$\|S_T^{p,X^\pm}[F]\|_{\mathcal{F}_w^n} \leq \left\|\frac{1}{2}\sum_\pm \mathsf{W}_{\mp T} X^\pm_{\pm T}[p,F]\right\|_{\mathcal{F}_w^n} + \left\|\frac{1}{2}\sum_\pm \int_{\pm T}^0 ds\, \mathsf{W}_{-s}\mathsf{J}(\varphi_s)\right\|_{\mathcal{F}_w^n} =: \boxed{9} + \boxed{10},$$

cf. Definition 2.40p.25 where $S_T^{p,X^\pm}$ was defined. By the estimate given in Lemma 2.45p.29(i) and the property given in Definition 2.36p.24(i) of the boundary fields we find

$$\boxed{9} \leq \frac{1}{2}\sum_\pm \|\mathsf{W}_{\mp T} X^\pm_{\pm T}[p,F]\|_{\mathcal{F}_w^n} \leq e^{\gamma T}\|X^\pm_{\pm T}[p,F]\|_{\mathcal{F}_w^n} \leq e^{\gamma T} C_3{}^{(n)}(T, \|\phi\|_{\mathcal{H}_w}).$$

Furthermore, using in addition the estimates given in Lemma 2.45p.29(i-ii) we get a bound for the next term by

$$\boxed{10} \leq Te^{\gamma T} \sup_{s\in[-T,T]} \|\mathsf{J}(\varphi_s)\|_{\mathcal{F}_w^n} \leq Te^{\gamma T} \sup_{s\in[-T,T]} C_7(\|\mathsf{Q}\varphi_s\|_{\mathcal{H}_w}) \leq Te^{\gamma T} C_7(\|p\| + T)$$

whereas the last step is implied by the fact that the maximal velocity is below one. These estimates prove claim (i) for

$$C_{11}{}^{(n)}(T, \|\phi\|_{\mathcal{H}_w^n}) := e^{\gamma T}\left(C_3{}^{(n)}(T, \|p\|) + TC_7(\|p\| + T)\right).$$

Next we prove claim (ii). Therefore, we regard

$$\begin{aligned}\|S_T^{p,X^\pm}[F] - S_T^{p,X^\pm}[\widetilde{F}]\|_{\mathcal{F}_w^n} &\leq e^{\gamma T}\|X^\pm_{\pm T}[\varphi] - X^\pm_{\pm T}[\widetilde{\varphi}]\|_{\mathcal{F}_w^n} + Te^{\gamma T}\sup_{s\in[-T,T]}\|\mathsf{J}(\varphi_s) - \mathsf{J}(\widetilde{\varphi}_s)\|_{\mathcal{F}_w^n} \\ &=: \boxed{11} + \boxed{12}\end{aligned}$$

where we have already applied Lemma 2.45p.29(i). Next we use Lemma 2.45p.29(iii) on $\boxed{12}$ and yield

$$\boxed{12} \leq Te^{\gamma T} \sup_{s\in[-T,T]} C_8{}^{(n)}(\|\varphi_s\|_{\mathcal{H}_w}, \|\widetilde{\varphi}_s\|_{\mathcal{H}_w})\|\varphi_s - \widetilde{\varphi}_s\|_{\mathcal{H}_w}$$

Finally, by the ML-SI estimates (41p.30) and (42p.30) we yield

$$\boxed{12} \leq TC_{13}(T, \|p\|, \|F\|_{\mathcal{F}_w^n}, \|\widetilde{F}\|_{\mathcal{F}_w^n})\|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w} \tag{43}$$

for

$$\begin{aligned}C_{13}(T, \|p\|, \|F\|_{\mathcal{F}_w^n}, \|\widetilde{F}\|_{\mathcal{F}_w^n}) :=& e^{\gamma T} C_8{}^{(n)}\Big(C_{20}(T, \|\varphi\|_{\mathcal{H}_w}, 0)\|\varphi\|_{\mathcal{H}_w}, C_{20}(T, 0, \|\widetilde{\varphi}\|_{\mathcal{H}_w})\|\varphi\|_{\mathcal{H}_w}\Big)\times \\ &\times C_{20}(T, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}).\end{aligned}$$

For $\widetilde{F} \to F$ in $\mathcal{F}_w^1$ these estimates imply $S_T^{p,X^\pm}[\widetilde{F}] \to S_T^{p,X^\pm}[F]$ in $\mathcal{F}_w^1$ since here $\|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w} = \|F - \widetilde{F}\|_{\mathcal{F}_w}$ which proves claim (ii).

(iii) Let now $X^\pm \in \mathcal{A}_w^{\mathrm{Lip}}$. Term $\boxed{11}$ then behaves by Definition 2.36p.24 as

$$\boxed{11} \leq TC_4{}^{(n)}(|T|, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w})\ \|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w}$$

Together with the estimate (43) this proves claim (ii) for

$$C_{12}{}^{(n)}(T, \|p\|, \|F\|_{\mathcal{F}_w}, \|\widetilde{F}\|_{\mathcal{F}_w}) := C_4{}^{(n)}(|T|, \|\varphi\|_{\mathcal{H}_w}, \|\widetilde{\varphi}\|_{\mathcal{H}_w}) + C_{13}(T, \|p\|, \|F\|_{\mathcal{F}_w^n}, \|\widetilde{F}\|_{\mathcal{F}_w^n})$$

since in our case $\|\varphi - \widetilde{\varphi}\|_{\mathcal{H}_w} = \|F - \widetilde{F}\|_{\mathcal{F}_w}$. □

Before we proof the main theorem of this section we need a last lemma which gives a criterion for precompactness of sequences in L_w^2.

Lemma 2.47 (Criterion for Precompactness)**.** *Let* $(\mathbf{F}_n)_{n\in\mathbb{N}}$ *be a sequence in* $L_w^2(\mathbb{R}^3, \mathbb{R}^3)$ *such that*

(i) *The sequence* $(\mathbf{F}_n)_{n\in\mathbb{N}}$ *is uniformly bounded in* $\mathcal{H}_w^{\triangle}$.
(ii) $\lim_{\tau\to\infty}\sup_{n\in\mathbb{N}}\|\mathbf{F}n\|_{L^2_w(B^c_\tau(0))} = 0$.

Then the sequence $(F_n)_{n\in\mathbb{N}}$ *is precompact, i.e. it contains a convergent subsequence.*

Proof. (see Appendix Ap.42) The idea for the proof is based on [Lie01, Chapter 8, Proof of Theorem 8.6, p.208]. □

REMARK 2.48. *Of course one only needs to control solely the gradient, however, the Laplace turns out to be more convenient for the later application of the lemma.*

Now we can prove the first main theorem of this section.

*Proof of Theorem 2.3*p.7 *(Existence of WF Solution for Finite Times).* Fix $p\in\mathcal{P}$.

(i) Let $X^\pm\in\mathcal{A}^{\mathrm{Lip}}_w\subset\mathcal{A}^1_w$, then Lemma 2.46p.30(i) states

$$\|S_T^{p,X^\pm}[p,F]\|_{\mathcal{F}^1_w}\le C_{11}{}^{(1)}(T,\|p\|) =: r.$$

Hence, the map $S_T^{p,X^\pm}$ restricted to the ball $B_r(0)\subset\mathcal{F}^1_w$ with radius r around the origin is a nonlinear self-mapping. Lemma 2.46p.30(iii) states for all $T>0$ and $F,\widetilde{F}\in B_r(0)\subset D_w(\mathsf{A})$ that

$$\begin{aligned}\|S_T^{p,X^\pm}[F]-S_T^{p,X^\pm}[\widetilde{F}]\|_{\mathcal{F}^1_w} &\le TC_{12}(T,\|p\|,\|F\|_{\mathcal{F}_w},\|\widetilde{F}\|_{\mathcal{F}_w})\|F-\widetilde{F}\|_{\mathcal{F}_w}\\ &\le TC_{12}(T,\|p\|,r,r)\|F-\widetilde{F}\|_{\mathcal{F}_w}.\end{aligned}$$

where we have also used that $C_{12}\in\texttt{Bounds}$ is a continuous and strictly increasing function of its arguments. Hence, for T sufficiently small we have $TC_{12}(T,\|p\|,r,r)<1$ such that $S_T^{p,X^\pm}$ is a contraction on $B_r(0)\subset\mathcal{F}^1_w$. By Banach's fixed point theorem $S_T^{p,X^\pm}$ has a unique fixed point in $B_r(0)\subset\mathcal{F}^1_w$.

(ii) Given a finite $T>0$, $p\in\mathcal{P}$ and $X^\pm\in\widetilde{\mathcal{A}}^3_w$ Lemma 2.41p.25(i) states for all $F\in\mathcal{F}^1_w$

$$\|S_T^{p,X^\pm}[p,F]\|_{\mathcal{F}^1_w}\le\|S_T^{p,X^\pm}[p,F]\|_{\mathcal{F}^3_w}\le C_{11}{}^{(3)}(T,\|p\|) =: r. \tag{44}$$

Let K be the closed convex hull of $M := \{S_T^{p,X^\pm}[F]\mid F\in\mathcal{F}^1_w\}\subset B_r(0)\subset\mathcal{F}^1_w$. By Lemma 2.41p.25(ii) we know that the map $S_T^{p,X^\pm}: K\to K$ is continuous as a map $\mathcal{F}^1_w\to\mathcal{F}^1_w$. If M is compact, it implies that K is compact, and hence, Schauder's fixed point Theorem ensures the existence of a fixed point.

It is left to show that M is compact. Therefore, let $(G_m)_{m\in\mathbb{N}}$ be a sequence in M. We need to show that it contains an $\mathcal{F}^1_w$ convergent subsequence. To show this we intend to use Lemma 2.47p.31. By definition there is a sequence $(F_m)_{m\in\mathbb{N}}$ in $B_r(0)\subset\mathcal{F}^1_w$ such that $G_m := S_T^{p,X^\pm}[F_m]$, $m\in\mathbb{N}$. We define for $m\in\mathbb{M}$

$$(\mathbf{E}_i^{(m)},\mathbf{B}_i^{(m)})_{1\le i\le N} := S_T^{p,X^\pm}[F_m].$$

Recall the definition of the norm of $\mathcal{F}^n_w$, cf. Definition 2.34p.24, for some $(\mathbf{E}_i,\mathbf{B}_i)_{1\le i\le N}=F\in\mathcal{F}^n_w$ and $n\in\mathbb{N}$

$$\|F\|^2_{\mathcal{F}^n_w}=\sum_{k=0}^n\|\mathsf{A}^kF\|^2_{\mathcal{F}_w}=\sum_{k=0}^n\sum_{i=1}^N\left(\|(\nabla\wedge)^k\mathbf{E}_i\|^2_{L^2_w}+\|(\nabla\wedge)^k\mathbf{B}_i\|^2_{L^2_w}\right). \tag{45}$$

Therefore, since A on $D_w(\mathsf{A})$ is closed, $(G_m)_{m\in\mathbb{N}}$ has an $\mathcal{F}^1_w$ convergent subsequence if and only if all the sequences $((\nabla\wedge)^k\mathbf{E}_i^{(m)})_{m\in\mathbb{N}}$, $((\nabla\wedge)^k\mathbf{B}_i^{(m)})_{m\in\mathbb{N}}$ for $k=0,1$ and $1\le i\le n$ have a common convergent subsequence in L^2_w.

To show this we first provide the bounds needed for Lemma 2.47p.31(i). Estimate (44) implies that

$$\sum_{k=0}^3\sum_{i=1}^N\left(\|(\nabla\wedge)^k\mathbf{E}_i^{(m)}\|^2_{L^2_w}+\|(\nabla\wedge)^k\mathbf{B}_i^{(m)}\|^2_{L^2_w}\right)=\|G_m\|^2_{\mathcal{F}^3_w}\le r^2 \tag{46}$$

for all $m \in \mathbb{N}$. Furthermore, by Lemma 2.41p.25(ii) the fields $(\mathbf{E}_i^{(m)}, \mathbf{B}_i^{(m)})_{1\le i\le N}$ are a solution to the Maxwell equations at time zero and hence, by Theorem 2.14p.12 fulfill the Maxwell constraints for $(\mathbf{q}_i^0, \mathbf{p}_i^0)_{1\le i\le N} := p$ which read

$$\nabla \cdot \mathbf{E}^{(m)} = 4\pi\varrho_i(\cdot - \mathbf{q}_i^0) \qquad \text{and} \qquad \nabla \cdot \mathbf{B}_i^{(m)} = 0.$$

Also by Theorem 2.14p.12, G_m is in $\mathcal{F}$ so that for every $k \in \mathbb{N}_0$

$$(\nabla\wedge)^{k+2}\mathbf{E}_i^{(m)} = 4\pi\delta_{k0}\nabla\varrho_i(\cdot - \mathbf{q}_i^0) - \triangle(\nabla\wedge)^k\mathbf{E}_i^{(m)} \qquad \text{and} \qquad (\nabla\wedge)^{k+2}\mathbf{B}_i^{(m)} = -\triangle(\nabla\wedge)^k\mathbf{B}_i^{(m)}$$

where δ_{k0} is the Kronecker delta which is zero except for $k = 0$. Estimate (46p.32) implies for all $m \in \mathbb{N}$ that

$$\begin{aligned}
&\sum_{k=0}^{1}\sum_{i=1}^{N}\left(\|\triangle(\nabla\wedge)^k\mathbf{E}_i^{(m)}\|_{L^2_w}^2 + \|\triangle(\nabla\wedge)^k\mathbf{B}_i^{(m)}\|_{L^2_w}^2\right)\\
&\le 2\sum_{k=0}^{1}\sum_{i=1}^{N}\left(\|(\nabla\wedge)^{k+2}\mathbf{E}_i^{(m)}\|_{L^2_w}^2 + \|(\nabla\wedge)^{k+2}\mathbf{B}_i^{(m)}\|_{L^2_w}^2\right) + 2\sum_{i=1}^{N}\|4\pi\nabla\varrho_i(\cdot - \mathbf{q}_i^0)\|_{L^2_w}\\
&\le 2r^2 + 8\pi\sum_{i=1}^{N}\left(1 + C_w\left\|\mathbf{q}_i^0\right\|\right)^{P_w}\|\nabla\varrho_i\|_{L^2_w}^2
\end{aligned}$$

where we made use of the properties of the weight $w \in \mathcal{W}$. Note that the right-hand does not depend on m. Therefore, all the sequences $((\nabla\wedge)^k\mathbf{E}_i^{(m)})_{m\in\mathbb{N}}$, $(\triangle(\nabla\wedge)^k\mathbf{E}_i^{(m)})_{m\in\mathbb{N}}$, $((\nabla\wedge)^k\mathbf{B}_i^{(m)})_{m\in\mathbb{N}}$, $(\triangle(\nabla\wedge)^k\mathbf{B}_i^{(m)})_{m\in\mathbb{N}}$ for $k = 0, 1$ and $1 \le i \le N$ are uniformly bounded in L^2_w.

Second, we need to show that all the sequences $((\nabla\wedge)^k\mathbf{E}_i^{(m)})_{m\in\mathbb{N}}$, $((\nabla\wedge)^k\mathbf{B}_i^{(m)})_{m\in\mathbb{N}}$ for $k = 0, 1$ and $1 \le i \le N$ decay uniformly at infinity to meet condition (ii) of Lemma 2.47p.31. Define $(\mathbf{E}_{i,\pm T}^{(m),\pm}, \mathbf{B}_{i,\pm T}^{(m),\pm})_{1\le i\le N} := X_{\pm T}^{\pm}[p, F_m]$ for $m \in \mathbb{N}$ and denote the ith charge trajectory $t \mapsto (\mathbf{q}_{i,t}^{(m)}, \mathbf{p}_{i,t}^{(m)}) := (\mathsf{Q} + \mathsf{P})M_L[p, F_m](t, 0)$ by $(\mathbf{q}_i^{(m)}, \mathbf{p}_i^{(m)})$, $1 \le i \le N$. Using Lemma 2.41p.25(ii) and afterwards Lemma 2.14p.12 we can write the fields as

$$\begin{aligned}
\begin{pmatrix}\mathbf{E}_i^{(m)}\\ \mathbf{B}_i^{(m)}\end{pmatrix} &= \frac{1}{2}\sum_{\pm} M_{\varrho,m_i}[(\mathbf{E}_{i,\pm T}^{\pm}, \mathbf{E}_{i,\pm T}^{\pm}), (\mathbf{q}_i^{(m)}, \mathbf{p}_i^{(m)})](0, \pm T)\\
&= \frac{1}{2}\sum_{\pm}\Bigg[\begin{pmatrix}\partial_t & \nabla\wedge\\ -\nabla\wedge & \partial_t\end{pmatrix}K_{t\mp T} * \begin{pmatrix}\mathbf{E}_{i,\pm T}^{(m),\pm}\\ \mathbf{B}_{i,\pm T}^{(m),\pm}\end{pmatrix} + K_{t\mp T} * \begin{pmatrix}-4\pi\mathbf{j}_{i,\pm T}^{(m)}\\ 0\end{pmatrix}\\
&\quad + 4\pi\int_{\pm T}^{t} ds\; K_{t-s} * \begin{pmatrix}-\nabla & -\partial_s\\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\rho_{i,s}^{(m)}\\ \mathbf{j}_{i,s}^{(m)}\end{pmatrix}\Bigg]_{t=0} =: \boxed{13} + \boxed{14} + \boxed{15}
\end{aligned}$$

where $\rho_{i,t}^{(m)} := \varrho_i(\cdot - \mathbf{q}_{i,t}^{(m)})$ and $\mathbf{j}_{i,t}^{(m)} := \mathbf{v}(\mathbf{p}_{i,t}^{(m)})\rho_{i,t}$ for all $t \in \mathbb{R}$.

We shall show that there is a $\tau^* > 0$ such that for all $m \in \mathbb{N}$ the terms $\boxed{14}$ and $\boxed{15}$ are point-wise zero on $B^c_{\tau^*}(0) \subset \mathbb{R}^3$. Recalling the computation rules for K_t from Lemma 2.11p.10 we calculate

$$\|4\pi[K_{\mp T} * \mathbf{j}_{\pm T}^{(m)}](\mathbf{x})\|_{\mathbb{R}^3} \le 4\pi T \fint_{B_T(\mathbf{x})} d\sigma(y)\; \varrho_i(\mathbf{y} - \mathbf{q}_{\pm T}^{(m)}).$$

The right-hand side is zero for all $\mathbf{x} \in \mathbb{R}^3$ such that $\partial B_T(\mathbf{x}) \cap \operatorname{supp}\varrho_i(\cdot - \mathbf{q}_{\pm T}) = \emptyset$. Because the charge distributions have compact support there is a $R > 0$ such that $\operatorname{supp}\varrho_i \subseteq B_R(0)$ for all $1 \le i \le N$. Now for any $1 \le i \le N$ and $m \in \mathbb{N}$ we have

$$\operatorname{supp}\varrho_i(\cdot - \mathbf{q}_{i,\pm T}^{(m)}) \subseteq B_R(\mathbf{q}_{i,\pm T}^{(m)}) \subseteq B_{R+T}(\mathbf{q}_i^0)$$

since the supremum of the velocities of the charge $\sup_{t\in[-T,T],m\in\mathbb{N}}\|\mathbf{v}(\mathbf{p}^{(m)}_{i,t})\|$ is smaller or equal one. Hence, $\partial B_T(\mathbf{x})\cap B_{R+T}(\mathbf{q}^0_i)=\emptyset$ for all $\mathbf{x}\in B^c_\tau(0)$ with $\tau>\|p\|+R+2T$.

Considering $\boxed{15}$ we have

$$\left\|4\pi\int_{\pm T}^0 ds\,\left[K_{-s}*\begin{pmatrix}-\nabla & -\partial_s\\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\rho^{(m)}_{i,s}\\ \mathbf{j}^{(m)}_{i,s}\end{pmatrix}\right](\mathbf{x})\right\|_{\mathbb{R}^3\oplus\mathbb{R}^2}\leq 4\pi\int_{\pm T}^0 ds\; s\fint_{\partial B_{|s|}(\mathbf{x})} d\sigma(y)\,\|\mathbf{G}(\mathbf{y}-\mathbf{q}^{(m)}_s)\|_{\mathbb{R}^3\oplus\mathbb{R}^3} \tag{47}$$

where we used the abbreviation

$$\mathbf{G}:=\begin{pmatrix}-\nabla & -\partial_s\\ 0 & \nabla\wedge\end{pmatrix}\begin{pmatrix}\rho^{(m)}_{i,s}\\ \mathbf{j}^{(m)}_{i,s}\end{pmatrix}$$

and the computation rules for K_t given in Lemma 2.11p.10. As $\operatorname{supp}\mathbf{G}\subseteq\operatorname{supp}\varrho_i\subseteq B_R(0)$, the right-hand side of (47) is zero for all $\mathbf{x}\in\mathbb{R}$ such that

$$\bigcup_{s\in[-T,T]}\left[\partial B_{|s|}(\mathbf{x})\cap B_R(\mathbf{q}^{(m)}_{i,s})\right]=\emptyset.$$

Now the left-hand side is subset equal

$$\cup_{s\in[-T,T]}\partial B_{|s|}(\mathbf{x})\bigcap\cup_{s\in[-T,T]}B_R(\mathbf{q}^{(m)}_{i,s})\qquad\subseteq\qquad B_T(\mathbf{x})\cap B_{R+T}(\mathbf{q}^0_i)$$

which is equal the empty set for all $\mathbf{x}\in B^c_\tau(0)$ with $\tau>\|p\|+R+2T$.

Hence, setting $\tau^*:=\|p\|+R+2T$ we conclude that for all $\tau>\tau^*$ the terms $\boxed{14}$ and $\boxed{15}$ and all their derivatives are zero on $B^c_\tau(0)\subset\mathbb{R}^3$. That means in order to show that all the sequences $((\nabla\wedge)^k\mathbf{E}^{(m)}_i)_{m\in\mathbb{N}}$, $((\nabla\wedge)^k\mathbf{B}^{(m)}_i)_{m\in\mathbb{N}}$ for $k=0,1$ and $1\leq i\leq N$ decay uniformly at spatial infinity, it suffices to show

$$\lim_{\tau\to\infty}\sup_{m\in\mathbb{N}}\sum_{k=0}^n\sum_{i=1}^N\left(\|(\nabla\wedge)^k\mathbf{e}^{(m)}_i\|_{L^2_w(B^c_\tau(0))}+\|(\nabla\wedge)^k\mathbf{b}^{(m)}_i\|_{L^2_w(B^c_\tau(0))}\right)=0. \tag{48}$$

for

$$\begin{pmatrix}\mathbf{e}^{(m)}_i\\ \mathbf{b}^{(m)}_i\end{pmatrix}:=\boxed{13}=\begin{pmatrix}\partial_t & \nabla\wedge\\ -\nabla\wedge & \partial_t\end{pmatrix}K_{t\mp T}*\begin{pmatrix}\mathbf{E}^{(m),\pm}_{i,\pm T}\\ \mathbf{B}^{(m),\pm}_{i,\pm T}\end{pmatrix}\Bigg|_{t=0}$$

for $1\leq i\leq N$. Let $\mathbf{F}\in C^\infty(\mathbb{R}^3,\mathbb{R}^3)$ and $\tau>0$. By computation rules for K_t given in Lemma 2.11p.10 we then yield

$$\begin{aligned}
\|\nabla\wedge K_{\mp T}*\mathbf{F}\|_{L^2_w(B^c_{\tau+T}(0))}&=\|K_{\mp T}*\nabla\wedge\mathbf{F}\|_{L^2_w(B^c_{\tau+T}(0))}\leq\left\|T\fint_{\partial B_T(0)}d\sigma(y)\,\nabla\wedge\mathbf{F}(\cdot-\mathbf{y})\right\|_{L^2_w(B^c_{\tau+T}(0))}\\
&\leq T\fint_{\partial B_T(0)}d\sigma(y)\,\|\nabla\wedge\mathbf{F}(\cdot-\mathbf{y})\|_{L^2_w(B^c_{\tau+T}(0))}\leq T\sup_{\mathbf{y}\in\partial B_T(0)}\|\nabla\wedge\mathbf{F}(\cdot-\mathbf{y})\|_{L^2_w(B^c_{\tau+T}(0))}\\
&\leq T\sup_{\mathbf{y}\in\partial B_T(0)}(1+C_w\|\mathbf{y}\|)^{\frac{P_w}{2}}\|\nabla\wedge\mathbf{F}(\cdot-\mathbf{y})\|_{L^2_w(B^c_{\tau+T}(0))}\leq T(1+C_wT)^{\frac{P_w}{2}}\|\nabla\wedge\mathbf{F}\|_{L^2_w(B^c_\tau(0))}.
\end{aligned}$$

We also estimate using the computation rules for K_t given in Lemma 2.11p.10 the term

$$\|\partial_t K_{t\mp T}|_{t=0} * \mathbf{F}\|_{L^2_w(B^c_{\tau+T}(0))} = \left\| \fint_{\partial B_T(0)} d\sigma(y)\ \mathbf{F}(\cdot - \mathbf{y}) + \frac{T^2}{3} \fint_{B_T(0)} d^3y\ \triangle \mathbf{F}(\cdot - \mathbf{y}) \right\|_{L^2_w(B^c_{\tau+T}(0))}$$

$$\leq \fint_{\partial B_T(0)} d\sigma(y)\ \|\mathbf{F}(\cdot - \mathbf{y})\|_{L^2_w(B^c_{\tau+T}(0))} + \frac{T^2}{3} \fint_{B_T(0)} d^3y\ \|\triangle \mathbf{F}(\cdot - \mathbf{y})\|_{L^2_w(B^c_{\tau+T}(0))}$$

$$\leq (1 + C_w T)^{\frac{P_w}{2}} \|\mathbf{F}\|_{L^2_w(B^c_\tau(0))} + \frac{T^2}{3}(1 + C_w T)^{\frac{P_w}{2}} \|\triangle \mathbf{F}\|_{L^2_w(B^c_\tau(0))}.$$

Substituting $\mathbf{F}$ with $(\nabla\wedge)^k \mathbf{E}^{(m),\pm}_{i,\pm T}$ and $(\nabla\wedge)^k \mathbf{B}^{(m),\pm}_{i,\pm T}$ for $k = 0, 1$ and $1 \leq i \leq N$ in the two estimates above yields

$$\begin{aligned} &\sum_{k=0}^{n} \sum_{i=1}^{N} \left(\|(\nabla\wedge)^k \mathbf{e}^{(m)}_i\|_{L^2_w(B^c_{\tau+T}(0))} + \|(\nabla\wedge)^k \mathbf{b}^{(m)}_i\|_{L^2_w(B^c_{\tau+T}(0))} \right) \\ &\leq (1 + C_w T)^{\frac{P_w}{2}} \Bigg(\|(\nabla\wedge)^k \mathbf{E}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))} + \|(\nabla\wedge)^k \mathbf{B}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))} + \\ &\qquad + \frac{T^2}{3} \left(\|(\nabla\wedge)^k \triangle \mathbf{E}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))} + \|(\nabla\wedge)^k \triangle \mathbf{B}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))} \right) + \\ &\qquad + T \left(\|(\nabla\wedge)^{k+1} \mathbf{E}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))} + \|(\nabla\wedge)^{k+1} \mathbf{B}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))} \right) \Bigg). \end{aligned} \tag{49}$$

Now $X^\pm$ lie in $\widetilde{\mathcal{A}}^3_w \subset \mathcal{A}^3_w$ which means that the fields $\mathbf{E}^{(m),\pm}_{i,\pm T}$ and $\mathbf{B}^{(m),\pm}_{i,\pm T}$ for $1 \leq i \leq N$ fulfill the Maxwell constraints so that

$$\|(\nabla\wedge)^k \triangle \mathbf{E}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))} = \|(\nabla\wedge)^{k+2} \mathbf{E}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))} + 4\pi \|(\nabla\wedge)^k \nabla \varrho_i(\cdot - \mathbf{q}^{(m)}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))}$$

and

$$\|(\nabla\wedge)^k \triangle \mathbf{B}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))} = \|(\nabla\wedge)^{k+2} \mathbf{B}^{(m),\pm}_{i,\pm T}\|_{L^2_w(B^c_\tau(0))}.$$

Applying Definition 2.36p.24(iv) yields

$$\lim_{\tau\to\infty} \sup_{m\in\mathbb{N}} \sum_{j=0}^{3} \sum_{i=1}^{N} \left\| (\nabla\wedge)^j \mathbf{E}^{(m),\pm}_{i,\pm T}\|^2_{L^2_w(B^c_\tau(0))} + \|(\nabla\wedge)^j \mathbf{B}^{(m),\pm}_{i,\pm T}\|^2_{L^2_w(B^c_\tau(0))} \right) \leq \lim_{\tau\to\infty} \sup_{m\in\mathbb{N}} \|\chi^\pm_{\pm T}[p, F_m]\|^2_{\mathcal{H}^n_w} = 0$$

because $F_m \in B_r(0) \subset \mathcal{F}^1_w$ for all $m \in \mathbb{N}$, which implies (48p.34) by the above estimates. By the above estimate (49) we conclude that equation (48p.34) holds which we proved to be sufficient to show the uniform decay at spatial infinity of all the sequences $((\nabla\wedge)^k \mathbf{E}^{(m)}_i)_{m\in\mathbb{N}}$, $((\nabla\wedge)^k \mathbf{B}^{(m)}_i)_{m\in\mathbb{N}}$ for $k = 0, 1$ and $1 \leq i \leq N$.

Let us summarize using the abbreviations $\mathbf{E}^{(m,k)}_i := (\nabla\wedge)^k \mathbf{E}^{(m)}_i$ and $\mathbf{B}^{(m,k)}_i := (\nabla\wedge)^k \mathbf{B}^{(m)}_i$ for $k = 0, 1$, $1 \leq i \leq N$ and $m \in \mathbb{N}$: First, we have shown that the sequences $(\mathbf{E}^{(m,k)}_i)_{m\in\mathbb{N}}$, $(\mathbf{B}^{(m,k)}_i)_{m\in\mathbb{N}}$, $(\triangle \mathbf{E}^{(m,k)}_i)_{m\in\mathbb{N}}$ and $(\triangle \mathbf{B}^{(m,k)}_i)_{m\in\mathbb{N}}$ are all uniformly bounded in L^2_w. Second, we have shown that the sequences $(\mathbf{E}^{(m,k)}_i)_{m\in\mathbb{N}}$, $(\mathbf{B}^{(m,k)}_i)_{m\in\mathbb{N}}$, $(\triangle \mathbf{E}^{(m,k)}_i)_{m\in\mathbb{N}}$ decay uniformly at spatial infinity.

Having this we can now successively apply Lemma 2.47p.31 to yield the common $\mathcal{F}^1_w$ convergent subsequence: Fix $1 \leq i \leq N$. Let $(\mathbf{E}^{(m^0_l,0)}_i)_{l\in\mathbb{N}}$ be the L^2_w convergent subsequence of $(\mathbf{E}^{(m,0)}_i)_{m\in\mathbb{N}}$ and $(\mathbf{E}^{(m^1_l,1)}_i)_{l\in\mathbb{N}}$ the L^2_w convergent subsequence of $(\mathbf{E}^{(m^0_l,1)}_i)_{l\in\mathbb{N}}$. In the same way we proceed with the other indices $1 \leq i \leq N$ and the magnetic fields, every time choosing a further subsequence of the previous one. Let us denote the final subsequence by $(m_l)_{l\in\mathbb{N}} \subset \mathbb{N}$. Then we have constructed sequences $(G_{m_l})_{l\in\mathbb{N}}$ as well as $(\mathsf{A}G_{m_l})_{l\in\mathbb{N}}$ which are convergent in $\mathcal{F}_w$. However, A on $D_w(\mathsf{A})$ is closed so that this implies convergence of $(G_{m_l})_{l\in\mathbb{N}}$ in $\mathcal{F}^1_w$.

As $(G_m)_{m\in\mathbb{N}}$ was arbitrary, we conclude that every sequence in M has an $\mathcal{F}^1_w$ convergent subsequence and therefore M is compact which had to be shown. □

Having established the existence of a fixed point F for all times $T > 0$, Newtonian Cauchy data $p \in \mathcal{P}$ and boundary fields $(X^\pm_{i,\pm T})_{1\le i\le N} = X^\pm \in \widetilde{\mathcal{A}}^3_w$, Theorem 2.41p.25(iii) states that the charge trajectories $t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1\le i\le N} := (\mathtt{Q} + \mathtt{P})M_L[p, F](t, 0)$ are in $\mathcal{T}^{p,X^\pm}_T$, i.e. they are time-like charge trajectories that solve the WF equations (4p.3)-(11p.7) for all times $t \in \mathbb{R}$. It remains to show Theorem 2.4p.8 which ensures that we see true advanced and delayed interactions between the charges.

Definition 2.49 (Partial WF solutions)**.** *For Newtonian Cauchy data $p \in \mathcal{P}$ we define $\mathcal{T}^L_{\mathrm{WF}}$ to be the set of time-like charge trajectories in $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N} \in \mathcal{T}_\vee$ which solve the WF equations in the form (4p.3)-(11p.7) for time $t \in [-L, L]$ and initial conditions $(\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1\le i\le N}|_{t=0} = p$. We shall call every element of $\mathcal{T}^L_{\mathrm{WF}}$ a partial WF solution for initial value p.*

In order to see that a bWF solution $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N} \in \mathcal{T}^{p,X^\pm}_T$ is also a partial WF solution we have to regard the difference

$$
\begin{aligned}
&M_{\varrho_i,m_i}[X^\pm_{i,\pm T}, (\mathbf{q}_i, \mathbf{p}_i)](t, \pm T) - M_{\varrho_i,m_i}[(\mathbf{q}_i, \mathbf{p}_i)](t, \pm\infty)\\
&= \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t\mp T} * X^\pm_{i,\pm T} + K_{t\mp T} * \begin{pmatrix} -4\pi \mathbf{v}(\mathbf{p}_{i,\pm T})\varrho_i(\cdot - \mathbf{q}_{i,\pm T}) \\ 0 \end{pmatrix}\\
&\quad - 4\pi \int_{\pm\infty}^{\pm T} ds\, K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \varrho_i(\cdot - \mathbf{q}_{i,s}) \\ \mathbf{v}(\mathbf{p}_{i,s})\varrho_i(\cdot - \mathbf{q}_{i,s}) \end{pmatrix}.
\end{aligned}
\tag{50}
$$

where we used Definition 2.16p.14 with Theorem 2.14p.12 as well as Definition 2.17p.14. Whenever the difference is zero everywhere within the tubes around the positions of the $j \ne i$ charge trajectories for $t \in [-L, L]$, the charge trajectories $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N}$ are in $\mathcal{T}^L_{\mathrm{WF}}$. This is certainly not true for all boundary fields $X^\pm \in \widetilde{\mathcal{A}}^3_w$. However, it is the case for the advanced, respectively retarded, Liénard-Wiechert fields of any charge trajectories which continue $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N}$ on the time interval $[T, \infty)$, respectively $(-\infty, -T]$, and we shall show this in the particular case of the Coulomb boundary fields C, cf. Definition 2.42p.27.

In fact, for the Coulomb boundary fields $C \in \widetilde{\mathcal{A}}^3_w \cap \mathcal{A}^{\mathrm{Lip}}_w$ we find that the difference discussed above is for "+" zero everywhere on the backward light-cone of the spacetime point $(T, \mathbf{q}_{i,T})$ as well as for "−" everywhere on the forward light-cone of $(-T, \mathbf{q}_{i,-T})$.

Lemma 2.50 (Shadows of the Boundary Fields and WF fields)**.** *Let $\mathbf{q}, \mathbf{v} \in \mathbb{R}^3$, $\varrho \in C^\infty_c(\mathbb{R}^3, \mathbb{R})$ such that* $\operatorname{supp}\varrho \subseteq B_R(0)$ *for some finite $R > 0$. Furthermore, let $\mathbf{E}^C$ be the Coulomb field of a charge at rest at the origin*

$$
\mathbf{E}^C := \int d^3z\, \varrho(\cdot - \mathbf{z}) \frac{\mathbf{z}}{\|\mathbf{z}\|^3}
$$

Then for $T > R$

$$
\left[\begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t\mp T} * \begin{pmatrix} \mathbf{E}^C(\cdot - \mathbf{q}) \\ 0 \end{pmatrix} + K_{t\mp T} * \begin{pmatrix} -4\pi\mathbf{v}\varrho(\cdot - \mathbf{q}) \\ 0 \end{pmatrix}\right](\mathbf{x}) = 0
\tag{51}
$$

and

$$
\int_{\pm\infty}^{\pm T} ds\, K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \varrho_i(\cdot - \mathbf{q}_{i,s}) \\ \mathbf{v}(\mathbf{p}_{i,s})\varrho_i(\cdot - \mathbf{q}_{i,s}) \end{pmatrix} (\mathbf{x}) = 0
\tag{52}
$$

for $t \in (-T + R, T - R)$ and $\mathbf{x} \in B_{|t\mp T|-R}(\mathbf{q})$.

Proof. Let $t \in [-T+R, T-R]$. With regard to the second term we compute

$$\left\|-4\pi\mathbf{v}\left[K_{t\mp T} * \varrho(\cdot - \mathbf{q})\right](\mathbf{x})\right\| = 4\pi\|\mathbf{v}\|\left|(t\mp T)\fint_{\partial B_{|t\mp t|}(0)} d\sigma(y)\,\varrho(\mathbf{x}-\mathbf{y}-\mathbf{q})\right|$$
$$\leq 4\pi\|\mathbf{v}\||t\mp T|\sup|\varrho|\fint_{\partial B_{|t\mp t|}(\mathbf{q})} d\sigma(y)\,\mathbb{1}_{B_R(\mathbf{x})}(\mathbf{y})$$

where we used Definition 2.10p.9 for $K_{t\mp T}$. Now $\mathbf{x} \in B_{|t\mp T|-R}(\mathbf{q})$ implies $\partial B_{|t\mp T|}(\mathbf{q}) \cap B_R(\mathbf{x}) = \emptyset$ and hence that the term above is zero.

With regard to the first term we note that the only non-zero contribution is $\partial_t K_{t\mp T} * \mathbf{E}_i^C$ since $\nabla\wedge\mathbf{E}^C = 0$. We shall need the computation rules for K_t as given in Lemma 2.11p.10 and in particular equation (15p.10) which in our case reads

$$\left[\partial_t K_{t\mp T} * \mathbf{E}^C(\cdot - \mathbf{q})\right](\mathbf{x}) = \fint_{\partial B_{|t\mp T|}(0)} d\sigma(y)\,\mathbf{E}^C(\mathbf{x}-\mathbf{y}-\mathbf{q}) + (t\mp T)\partial_t \fint_{\partial B_{|t\mp T|}(0)} d\sigma(y)\,\mathbf{E}^C(\mathbf{x}-\mathbf{y}-\mathbf{q}) \tag{53}$$

$$= \fint_{\partial B_{|t\mp T|}(0)} d\sigma(y)\,\mathbf{E}^C(\mathbf{x}-\mathbf{y}-\mathbf{q}) + \frac{(t\mp T)^2}{3}\fint_{B_{|t\mp T|}(0)} d^3y\,\triangle\mathbf{E}^C(\mathbf{x}-\mathbf{y}) =: \boxed{16} + \boxed{17}. \tag{54}$$

Using Lebesgue's theorem we start with

$$\boxed{16} = \mathbf{E}^C(\mathbf{x}-\mathbf{q}) + \int_0^{|t\mp T|} ds\,\partial_s \fint_{\partial B_s(0)} d\sigma(y)\,\mathbf{E}^C(\mathbf{x}-\mathbf{y}-\mathbf{q})$$
$$= \mathbf{E}^C(\mathbf{x}-\mathbf{q}) + \int_0^{|t\mp T|} dr\,\frac{r}{3}\fint_{B_r(0)} d^3y\,\triangle\mathbf{E}^C(\mathbf{x}-\mathbf{y}-\mathbf{q}).$$

Furthermore, we know that $0 = (\nabla\wedge)^2\mathbf{E}^C = \nabla(\nabla\cdot\mathbf{E}^C) - \triangle\mathbf{E}^C$ and $\nabla\cdot\mathbf{E}^C = 4\pi\varrho$. So we continue the computation with

$$\boxed{16} = \mathbf{E}^C(\mathbf{x}-\mathbf{q}) + \int_0^{|t\mp T|} dr\,\frac{r}{3}\fint_{B_r(0)} d^3y\,4\pi\nabla\varrho(\mathbf{x}-\mathbf{y}-\mathbf{q})$$
$$= \mathbf{E}^C(\mathbf{x}-\mathbf{q}) - \int_0^{|t\mp T|} dr\,\frac{1}{r^2}\int_{\partial B_r(0)} d\sigma(y)\,\frac{\mathbf{y}}{r}\varrho(\mathbf{x}-\mathbf{y}-\mathbf{q})$$

where we have used (53) to evaluate the derivative and in addition used Stoke's Theorem. Note that the minus sign in the last line is due to the fact that ∇ acts on $\mathbf{x}$ and not $\mathbf{y}$. Inserting the definition of the Coulomb field $\mathbf{E}^C$ we finally get

$$\boxed{16} = \int_{B^c_{|t\mp T|}(0)} d^3y\,\varrho(\mathbf{x}-\mathbf{y}-\mathbf{q})\frac{\mathbf{y}}{\|\mathbf{y}\|^3}.$$

This integral is zero if, for example, $B^c_{|t\mp T|}(\mathbf{q}) \cap B_R(\mathbf{x}) = \emptyset$ and this is the case for $\mathbf{x} \in B_{|t\mp T|-R}(\mathbf{q})$. So it remains to show that $\boxed{17}$ also vanishes. Therefore, using $\triangle \mathbf{E}^C = 4\pi\nabla\varrho$ as before, we get

$$\boxed{17} = -\int_{\partial B_{|t\mp T|}(0)} d\sigma(y)\ \frac{\mathbf{y}}{(t\mp T)^2}\varrho(\mathbf{x}-\mathbf{y}-\mathbf{q}).$$

This expression is zero, for example, when $\partial B_{|t\mp T|}(\mathbf{q}) \cap B_R(\mathbf{x}) = \emptyset$ which is true for $\mathbf{x} \in B_{|t\mp T|-R}(\mathbf{q})$. Hence, we have shown that for $t \in (-T+R, T-R)$ and $\mathbf{x} \in B_{|t\mp T|}(\mathbf{q})$ the term (51p.36) is zero.

Looking at the support of the integrand and the integration domain in term (52p.36) we find that for all $t \in (-T+R, T-R)$ it is zero for all $\mathbf{x} \in \mathbb{R}^3$ such that

$$\bigcup_{|s|>T} (\partial B_{|t-s|}(\mathbf{x}) \cap B_R(\mathbf{q}_s)) = \emptyset. \tag{55}$$

Hence, for $t \in (-T+R, T-R)$ and $\mathbf{x} \in B_{|t\mp T|}(\mathbf{q})$ the term (52p.36) is also zero which concludes the proof. □

REMARK 2.51. *This lemma directly applies to the difference (50p.36) we were discussing before. By looking at the explicit formulas for the Maxwell solutions given in Theorem 2.14p.12 we recognize that this difference term is in some sense the free time evolution of the initial fields. This time evolution makes sure that the initial fields coming from a charge at rest have to make way for the new fields generated by the current of the charge during the time interval* $[-T, T]$*. This will certainly hold for all boundary fields which are Liénard-Wiechert fields of given charge trajectories on the intervals* $(-\infty, T]$ *and* $[T, \infty)$ *not only for the case of a charge at rest.*

Now that we know a big region where the difference (50p.36) is zero we have to make sure that the charge trajectories spend the time interval $[-L, L]$ there. For this we need a uniform momentum estimate:

Lemma 2.52 (Uniform Velocity Bound)**.** *For finite* $T > 0$ *and* $r > 0$ *there is a continuous and strictly increasing map* $v^{a,b} : \mathbb{R}^+ \to [0,1)$, $T \mapsto v^{a,b}_T$ *such that*

$$\sup\left\{\|\mathbf{v}(\mathbf{p}_{i,t})\|_{\mathbb{R}^3}\ \middle|\ t \in [-T,T], \|p\| \le a, F \in \operatorname{Range} S^{p,C}_T, \|\varrho_i\|_{L^2_w} + \|w^{-1/2}\varrho_i\|_{L^2} \le b, 1 \le i \le N\right\} \le v^{a,b}_T.$$

for $(\mathbf{p}_{i,t})_{1\le i\le N} := \mathtt{P} M_L[p,F](t,0)$ *for all* $t \in \mathbb{R}$.

Proof. Recall the estimate (64p.44) from the ML±SI existence and uniqueness Theorem B.6p.44 which gives the following T dependent upper bounds on these ML-SI solutions for all $\varphi \in D_w(A)$:

$$\sup_{t\in[-T,T]} \|M_L[\varphi](t,0)\|_{\mathcal{H}_w} \le C_{19}\left(T, \|\varrho_i\|_{L^2_w}, \|w^{-1/2}\varrho_i\|_{L^2}, 1 \le i \le N\right)\ \|\varphi\|_{\mathcal{H}_w}. \tag{56}$$

Note further that by Lemma 2.46p.30 since $C \in \mathcal{A}^1_w$, there is a $C_{11}{}^{(1)} \in \mathtt{Bounds}$ such that fields $F \in \operatorname{Range} S^{p,C}_T \in D_w(\mathsf{A}^\infty)$ fulfill

$$\|F\|_{\mathcal{F}_w} \le C_{11}{}^{(1)}(T, \|p\|) \le C_{11}{}^{(1)}(T, a).$$

Therefore, setting $c := a + C_{11}{}^{(1)}(T,a)$ we estimate the maximal momentum of the charges by

$$\begin{aligned}
&\sup\left\{\|\mathbf{v}(\mathbf{p}_{i,t})\|_{\mathbb{R}^3}\ \middle|\ t \in [-T,T], \|p\| \le a, F \in \operatorname{Range} S^{p,C}_T, \|\varrho_i\|_{L^2_w} + \|w^{-1/2}\varrho_i\|_{L^2} \le b, 1 \le i \le N\right\}\\
&\le \sup\left\{\|\mathbf{v}(\mathbf{p}_{i,t})\|_{\mathbb{R}^3}\ \middle|\ t \in [-T,T], \varphi \in D_w(A), \|\varphi\|_{\mathcal{H}_w} \le c, \|\varrho_i\|_{L^2_w} + \|w^{-1/2}\varrho_i\|_{L^2} \le b, 1 \le i \le N\right\}\\
&\le C_{19}\,(T, b, b,)\, c =: p^{a,b}_T < \infty.
\end{aligned}$$

Now, since C_{20} as well as $C_{11}{}^{(1)}$ are in Bounds the map $T \mapsto p_T^{a,b}$ as $\mathbb{R}^+ \to \mathbb{R}^+$ is continuous and strictly increasing. We conclude that claim is fulfilled for the choice

$$v_T^{a,b} := \frac{p_T^{a,b}}{\sqrt{m^2 + (p_T^{a,b})^2}}$$

and $m := \min_{1\le i\le N} |m_i|$. □

With this we can formulate our last result, i.e. Theorem 2.4p.8.

Proof of Theorem 2.4p.8. Let F be a fixed point $F = S_T^{p,C}[F]$ which exists by Theorem 2.3p.7. Define the charge trajectories $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N}$ by $t \mapsto (\mathbf{q}_{i,t}, \mathbf{p}_{i,t})_{1\le i\le N} := (\mathtt{Q}+\mathtt{P})M_L[p, F](t, 0)$. By the fixed point properties of F we know that these trajectories are in $\mathcal{T}_T^{p,C}$ and therefore solve the WF equations (4p.3)-(11p.7) for Newtonian Cauchy data p and boundary fields C. In order to show that the charge trajectories $(\mathbf{q}_i, \mathbf{p}_i)_{1\le i\le N}$ are also in $\mathcal{T}_{\mathrm{WF}}^L$ for the given L we need to show that the difference (50p.36)

$$\begin{aligned}
&M_{\varrho_i,m_i}[X^\pm_{i,\pm T}, (\mathbf{q}_i, \mathbf{p}_i)](t, \pm T) - M_{\varrho_i,m_i}[(\mathbf{q}_i, \mathbf{p}_i)](t, \pm\infty)\\
&= \begin{pmatrix} \partial_t & \nabla\wedge \\ -\nabla\wedge & \partial_t \end{pmatrix} K_{t\mp T} * X^\pm_{i,\pm T} + K_{t\mp T} * \begin{pmatrix} -4\pi \mathbf{v}(\mathbf{p}_{i,\pm T})\varrho_i(\cdot - \mathbf{q}_{i,\pm T}) \\ 0 \end{pmatrix}\\
&\quad - 4\pi \int_{\pm\infty}^{\pm T} ds\, K_{t-s} * \begin{pmatrix} -\nabla & -\partial_s \\ 0 & \nabla\wedge \end{pmatrix} \begin{pmatrix} \varrho_i(\cdot - \mathbf{q}_{i,s}) \\ \mathbf{v}(\mathbf{p}_{i,s})\varrho_i(\cdot - \mathbf{q}_{i,s}) \end{pmatrix}.
\end{aligned}$$

is zero for times $t \in [-L, L]$ at least for all points $\mathbf{x}$ in a tube around the position of the $j \neq i$ charge trajectories. Lemma 2.50p.36 states that this expression is zero for all $t \in [-T + R, T - R]$ and $\mathbf{x} \in B_{|t\mp T|-R}(\mathbf{q}_{i,\pm T})$. So it is sufficient to show that the charge trajectories spend the time interval $[-L, L]$ in this particular spacetime region. Clearly, the position $\mathbf{q}_i^0$ at time zero is in $B_{T-R}(\mathbf{q}_{i,\pm T})$. Hence, we need to compute the earliest exit time L of this spacetime region of a charge trajectory j in the worst case. The exit time L is the time when the jth charge trajectory leaves the region $B_{|L\mp T|-R}(\mathbf{q}_{i,\pm T})$. By Lemma 2.52p.38 the charges can in the worst case move apart from each other with velocity $v_T^{a,b}$ during the time interval $[-T, T]$. Putting the origin at $\mathbf{q}_i^0$ we can compute the exit time L by

$$-v_T^{a,b} T = \|\mathbf{q}_j^0 - \mathbf{q}_i^0\| + 2R + v_T^{a,b} L - (T - L)$$

This gives $L := \frac{(1-v_T^{a,b})T - \triangle q_{max} - 2R}{1+v_T^{a,b}} > 0$ as long as $\triangle q_{max} < (1 - v_T^{a,b})T$ wich is the case. □

Appendix A. Missing Proofs and Computations

Computation A.1. *Here we compute the differentiation which was not performed in Theorem 2.18p.14, Equation (29p.17). At first we compute the derivative of $t^\pm$ defined in (25p.15). Recall that all entities with a superscript $\pm$ depend on $t^\pm$. For any $k = 1, 2, 3$*

$$\partial_{z_k} t^\pm = \pm\partial_{z_k} \|\mathbf{x} - \mathbf{z} - \mathbf{q}^\pm\|.$$

Now

$$\partial_{z_k} \|\mathbf{x} - \mathbf{z} - \mathbf{q}^\pm\| = \frac{x_j - z_j - q_j^\pm}{\|\mathbf{x} - \mathbf{z} - \mathbf{q}^\pm\|}(-\delta_{kj} - \partial_{z_k} q_j^\pm) = -n_k^\pm - n_j^\pm\, \partial_{z_k} q_j^\pm$$

where $(\delta_{ij})_{1\le i,j\le 3}$ is the Kronecker delta, i.e. the identity on the space of $\mathbb{R}^{3x3}$ matrices, and we have used Einstein's summation convention (we sum over double indices). On the other hand $\partial_{z_k} q^\pm = v^\pm\, \partial_{z_k} t^\pm$, such *that we can plug all of these equations together and find*

$$\partial_{z_k} t^{\pm} = \frac{\mp n_k^{\pm}}{1 \pm \mathbf{n}^{\pm} \cdot \mathbf{v}^{\pm}} \qquad \text{and in return} \qquad \partial_{z_k} \frac{1}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}^{\pm}\|} = \frac{n_k^{\pm}}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}^{\pm}\|^2 (1 \pm \mathbf{n}^{\pm} \cdot \mathbf{v}^{\pm})}.$$

With these formulas at hand it is straightforward to compute the rest. Let us drop the superscript $\pm$ *in order to make the following formulas more readable. We find*

$$\partial_{z_k} \frac{1}{1 \pm \mathbf{n} \cdot \mathbf{v}} = \frac{\pm v_k + \mathbf{n} \cdot \mathbf{v}\, v_k - \mathbf{v}^2\, n_k \mp \mathbf{n} \cdot \mathbf{v}\, n_k}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}\|(1 \pm \mathbf{n} \cdot \mathbf{v})^3} + \frac{\mathbf{n} \cdot \mathbf{a}\, n_k}{(1 \pm \mathbf{n} \cdot \mathbf{v})^3}.$$

Let us denote the three integrands on the right-hand side of Equation (29p.17) by $\boxed{18}$, $\boxed{19}$ *and* $\boxed{20}$. *Plugging in the above equations we find*

$$\boxed{18} = \frac{\mathbf{n}}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}\|^2 (1 \pm \mathbf{n} \cdot \mathbf{v})^2} + \frac{\pm \mathbf{v} + \mathbf{n} \cdot \mathbf{v}\, \mathbf{v} - \mathbf{v}^2\, \mathbf{n} \mp \mathbf{n} \cdot \mathbf{v}\, \mathbf{n}}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}\|^2 (1 \pm \mathbf{n} \cdot \mathbf{v})^3} + \frac{\mathbf{n} \cdot \mathbf{a}\, \mathbf{n}}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}\|(1 \pm \mathbf{n} \cdot \mathbf{v})^3},$$

$$\begin{aligned} \boxed{19} = & \frac{-\mathbf{n} \cdot \mathbf{v}\, \mathbf{v}}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}\|^2 (1 \pm \mathbf{n} \cdot \mathbf{v})^2} + \frac{\mp \mathbf{v}^2\, \mathbf{v} \pm (\mathbf{n} \cdot \mathbf{v})^2\, \mathbf{v}}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}\|^2 (1 \pm \mathbf{n} \cdot \mathbf{v})^3} \\ & + \frac{-\mathbf{n} \cdot \mathbf{a}\, \mathbf{n} \cdot \mathbf{v}\, \mathbf{v}}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}\|(1 \pm \mathbf{n} \cdot \mathbf{v})^3} + \frac{\pm \mathbf{n} \cdot \mathbf{a}\, \mathbf{v} \pm \mathbf{n} \cdot \mathbf{v}\, \mathbf{a}}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}\|(1 \pm \mathbf{n} \cdot \mathbf{v})^2} \end{aligned}$$

and

$$\boxed{20} = \frac{-\mathbf{a}}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}\|(1 \pm \mathbf{n} \cdot \mathbf{v})}.$$

These three terms add up to the right-hand side of (23p.15). Furthermore, let us denote the integrand of the right-hand side of Equation (30p.17) by $\boxed{21}$, *then*

$$\begin{aligned} \boxed{21} = & \frac{-\mathbf{n} \wedge \mathbf{v}}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}\|^2 (1 \pm \mathbf{n} \cdot \mathbf{v})^2} + \frac{\mathbf{v}^2\, \mathbf{n} \wedge \mathbf{v} \pm \mathbf{n} \cdot \mathbf{v}\, \mathbf{n} \wedge \mathbf{v}}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}\|^2 (1 \pm \mathbf{n} \cdot \mathbf{v})^3} \\ & + \frac{-\mathbf{n} \cdot \mathbf{a}\, \mathbf{n} \wedge \mathbf{v}}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}\|(1 \pm \mathbf{n} \cdot \mathbf{v})^3} + \frac{\pm \mathbf{n} \wedge \mathbf{a}}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}\|(1 \pm \mathbf{n} \cdot \mathbf{v})^2} \end{aligned}$$

which after appropriate insertion of factors of the form $\mathbf{n} \wedge \mathbf{n} = 0$ *gives the right-hand side of (24p.15).*

Computation A.2. *We only consider the case for* $\varrho \in C_c^\infty$. *Substitution of* ϱ *by* $D^\alpha \varrho \in C_c^\infty$ *for any multi-index* $\alpha \in \mathbb{N}^3$ *yields the desired estimates for the general case for which only the constants* C_5 *change according to Equation (58). It suffices to show that for* $n \leq 2$ *there exist positive constants* ${C_{14}}^{(n)} < \infty$ *such that*

$$\left| \int d^3z\, \frac{\varrho(\mathbf{z})}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}_t\|^n} \right| \leq \frac{{C_{14}}^{(n)}}{1 + \|\mathbf{x}-\mathbf{q}_t\|^n}. \tag{57}$$

Since $\varrho \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ *there exists a* $R < \infty$ *such that* $\operatorname{supp} \varrho \subseteq B_R(0)$. *So for some* $\epsilon > 0$ *we have*

$$\left| \int d^3z\, \frac{\varrho(\mathbf{z})}{\|\mathbf{x}-\mathbf{z}-\mathbf{q}_t\|^n} \right| \leq \sup_{\mathbf{y} \in \mathbb{R}^3} |\varrho(\mathbf{y})| \left[\int_{B_\epsilon^c(0) \cap B_R(\mathbf{x}-\mathbf{q}_t)} d^3y\, \frac{1}{\|\mathbf{y}\|^n} + \int_{B_\epsilon(0) \cap B_R(\mathbf{x}-\mathbf{q}_t)} d^3y\, \frac{1}{\|\mathbf{y}\|^n} \right] =: \boxed{22} + \boxed{23}$$

which involved a substitution in the integration variable, and we have used the notation $B_\epsilon^c(0) := \mathbb{R}^3 \setminus B_\epsilon(0)$. *For* $\mathbf{x} \in B_{R+\epsilon}^c(\mathbf{q}_t)$ *the term* $\boxed{23}$ *is zero and*

$$\boxed{22} \leq \frac{\sup_{\mathbf{y} \in \mathbb{R}^3} |\varrho(\mathbf{y})|\, \frac{4}{3}\pi R^3}{(\|\mathbf{x}-\mathbf{q}_t\| - R)^n} =: \frac{C_{15}}{(\|\mathbf{x}-\mathbf{q}_t\| - R)^n}. \tag{58}$$

On the other hand for $\mathbf{x} \in B_{R+\epsilon}(\mathbf{q}_t)$ *and* $\epsilon < R$ *we find*

$$\boxed{22} \leq \frac{C_{15}}{\epsilon^n} \text{and} \qquad\qquad \boxed{23} \leq 4\pi \int_0^\epsilon dr\, r^{2-n} =: C_{16}^{(n)}.$$

Plugging these estimates in the left-hand side of (57p.40) we find

$$\left| \int d^3z \, \frac{\varrho(\mathbf{z})}{\|\mathbf{x} - \mathbf{z} - \mathbf{q}_t\|^n} \right| \leq \begin{cases} \frac{C_{15}}{\epsilon^n} + C_{17}^{(n)} & \text{for } \mathbf{x} \in B_{R+\epsilon}(\mathbf{q}_t) \\ \frac{C_{15}}{(\|\mathbf{x}-\mathbf{q}_t\|-R)^n} & \text{otherwise.} \end{cases}$$

Clearly one finds an appropriate constant $C_{18}^{(n)} < \infty$ *such that*

$$\left| \int d^3z \, \frac{\varrho(\mathbf{z})}{\|\mathbf{x} - \mathbf{z} - \mathbf{q}_t\|^n} \right| \leq \frac{C_{18}^{(n)}}{1 + \|\mathbf{x} - \mathbf{q}_t\|^n}.$$

This together with (34p.19) gives $C_5 := 2(C_{18}^{(n=2)} + C_{18}^{(n=1)})$.

Corollary A.3. *Let* A *and* J *be the operators defined in Definition 2.31p.22, i.e. the projection of* A *and* J *to field space. Furthermore, for some* $n \geq 1$ *let* $t \mapsto \mathsf{A}^k F_t$ *be a continuous map* $\mathbb{R} \to D_w(\mathsf{A}^{n-k}) \subset \mathcal{F}_w$ *for* $0 \leq k \leq n$. *Then it is also true that:*

$$\mathsf{A}^k \int_0^t ds\, F_s = \int_0^t ds\, \mathsf{A}^k F_s \qquad \text{and} \qquad \mathsf{W}_r \int_0^t ds\, F_s = \int_0^t ds\, \mathsf{W}_r F_s$$

for all $t, r \in \mathbb{R}$.

Proof. By Definition B.4p.43 we have $A = (0, \mathsf{A})$ on $D_w(A)$ so that $W_t = (\mathrm{id}_{\mathcal{P}}, \mathsf{W}_t)$ on $D_w(A)$ for all $t \in \mathbb{R}$. Apply Lemma A.4 and $t \mapsto (0, F_t)$ and project to field space $\mathcal{F}_w$ to yield the claim. □

Lemma A.4. *Let* A *be the operator defined in Definitions B.4p.43. Furthermore, for some* $n \geq 1$ *let* $t \mapsto A^k \varphi_t$ *be a continuous map* $\mathbb{R} \to D_w(A^{n-k}) \subset \mathcal{H}_w$ *for* $0 \leq k \leq n$. *Then it is true that:*

$$A^k \int_0^t ds\, \varphi_s = \int_0^t ds\, A^k \varphi_s \qquad \text{and} \qquad W_r \int_0^t ds\, \varphi_s = \int_0^t ds\, W_r \varphi_s$$

for all $t, r \in \mathbb{R}$.

Proof. First, we show the equality on the left-hand side of the claim. Since the integrands are continuous, we can define the integrals as $\mathcal{H}_w$ limits $N \to \infty$ of the Riemann sums for all $t \in \mathbb{R}$

$$\sigma_N^k = \frac{t}{N} \sum_{j=1}^N A^k \varphi_{\frac{t}{N} j}$$

for $k \leq n$. By Lemma 2.20 of Part I [BDD10] the operator A is closed on $D_w(A)$ so that A^k is closed on $D_w(A^k)$. Since $(\sigma_N^k)_{N \in \mathbb{N}}$ converge to, say, σ^k in $\mathcal{H}_w$, we get $\sigma^0 \in D_w(A^k)$ and $A^k \sigma^0 = \sigma^k$ which is exactly the equality on the left-hand side of the claim.

Second, we show the equality on the right-hand side. Therefore, for any $r, t \in \mathbb{R}$ we get

$$\frac{d}{dr} W_{-r} \int_0^t ds\, W_r \varphi_s = -A W_{-r} \int_0^t ds\, W_r \varphi_s + W_{-r} \int_0^t ds\, A W_r \varphi_s = 0$$

by the equality on the left-hand side of the claim. Hence,

$$W_{-r} \int_0^t ds\, W_r \varphi_s = \int_0^t ds\, \varphi_s \qquad \text{or} \qquad W_r \int_0^t ds\, \varphi_s = \int_0^t ds\, W_r \varphi_s.$$

This proves the right-hand side of the claim and concludes the proof. □

Proof of Lemma 2.47$_{p.31}$. Since by (i) the sequence $(\mathbf{F}_n)_{n\in\mathbb{N}}$ is uniformly bounded in the Hilbert space $\mathcal{H}_w^\triangle$ the Banach-Alaoglu Theorem states that it has a weakly convergent subsequence in $H_w^\triangle$ which we denote by $(\mathbf{G}_n)_{n\in\mathbb{N}}$. Let the convergence point be denoted by $\mathbf{F} \in H_w^\triangle$. We have to show that under the assumptions this subsequence is also strongly convergent in L_w^2. The idea is the following: Far away from the origin (ii) makes sure that the formation of spikes is suppressed while oscillations can be controlled by the Laplace which behave nicely by (i). So let $\epsilon > 0$ and divide the integration domain for $\tau > 0$

$$\|\mathbf{F} - \mathbf{G}_n\|_{L_w^2} \le \|\mathbf{F} - \mathbf{G}_n\|_{L_w^2(B_\tau(0))} + \|\mathbf{F} - \mathbf{G}_n\|_{L_w^2(B_\tau^c(0))}.$$

Now by assumption (ii) we know for τ large enough it holds for all $n \in \mathbb{N}$ that

$$\|\mathbf{F} - \mathbf{G}_n\|_{L_w^2(B_\tau^c(0))} < \epsilon.$$

By Lemma 2.12 of Part I [BDD10] the norm on $L_w^2(B_\tau(0))$ is equivalent to the one on $L^2(B_\tau(0))$ so that it suffices to show that there is an $N \in \mathbb{N}$ such that

$$\|\mathbf{F} - \mathbf{G}_n\|_{L^2(B_\tau(0))} < \epsilon \tag{59}$$

for all $n > N$. Before we do this let us introduce a tool to control possible oscillations. We define for any $\mathbf{H} \in L_{loc}^1$ the heat kernel

$$(e^{\triangle t}\mathbf{H})(\mathbf{x}) = h_t * \mathbf{G} := \frac{1}{(4\pi t)^{\frac{3}{2}}} \int d^3y \ \exp\left(-\frac{\|\mathbf{x} - \mathbf{y}\|^2}{4t}\right)\mathbf{H}(\mathbf{y}).$$

Denoting the Fourier transformation $\widehat{\cdot}$ and using Plancherel's Theorem we find

$$\begin{aligned} \|(1 - e^{\triangle t})\mathbf{H}\|_{L^2}^2 &= \|(1 - \widehat{h_t})\widehat{\mathbf{H}}\|_{L_w^2}^2 = \int d^3k \ \|\widehat{\mathbf{H}}(\mathbf{k})\|^2 \left(1 - \exp(-\mathbf{k}^2 t)\right)^2 \\ &\le |t| \ \|k^2\widehat{\mathbf{H}}\|_{L_w^2}^2 = |t| \ \|\triangle\widehat{\mathbf{H}}\|_{L_w^2}^2. \end{aligned} \tag{60}$$

Hence, we expand by triangle inequality

$$\begin{aligned} \|\mathbf{F} - \mathbf{G}_n\|_{L^2(B_\tau(0))} &\le \|(1 - e^{\triangle t})\mathbf{G}_n\|_{L^2(B_\tau(0))} + \|(1 - e^{\triangle t})\mathbf{F}\|_{L^2(B_\tau(0))} + \|(1 - e^{\triangle t})(\mathbf{F} - \mathbf{G}_n)\|_{L^2(B_\tau(0))} \\ &=: \boxed{24} + \boxed{25} + \boxed{26}. \end{aligned}$$

We start with the first term. Using the estimate (60) for small enough $t > 0$ yields

$$\boxed{24} \le \sqrt{t} \ \|\triangle G_n\|_{L^2(B_\tau(0))} < \frac{\epsilon}{3}$$

because $(\triangle G_n)_{n\in\mathbb{N}}$ is uniformly bounded in L_w^2 by (i). The same procedure for the second term yield

$$\boxed{24} \le \sqrt{t} \ \|\triangle G_n\|_{L^2(B_\tau(0))} \le \sqrt{t} \ \liminf_{n\to\infty} \|\triangle G_n\|_{L^2(B_\tau(0))} < \frac{\epsilon}{3}$$

where we use the lower semi-continuity of the norm and again (i). By weak convergence in L_w^2 we get the pointwise convergence for all $\mathbf{x} \in \mathbb{R}^3$ that

$$\left\|\mathbb{1}_{B_\tau(0)}(\mathbf{x})\left[e^{\triangle t}(\mathbf{F} - \mathbf{G}_n)\right](\mathbf{x})\right\|_{\mathbb{R}^3} \xrightarrow[n\to\infty]{} 0.$$

Furthermore, by Schwarz's inequality we get the estimate

$$\left\|\mathbb{1}_{B_\tau(0)}(\mathbf{x})\left[e^{\triangle t}\mathbf{G}_n\right](\mathbf{x})\right\|_{\mathbb{R}^3} \le \mathbb{1}_{B_\tau(0)}\|h_t\|_{L^2(B_\tau(0))}\|\mathbf{G}_n\|_{L^2(B_\tau(0))}.$$

Again the right-hand side is uniformly bounded by (i). Hence, by dominated convergence $(e^{\triangle t}\mathbf{G}_n)_{n\in\mathbb{N}}$ converges in $L^2(B_\tau(0))$ to $e^{\triangle t}\mathbf{F}$. Therefore, for an $N \in \mathbb{N}$ large enough we have

$$\boxed{27} = \|(1 - e^{\triangle t})(\mathbf{F} - \mathbf{G}_n)\|_{L^2(B_\tau(0))} \le \frac{\epsilon}{3}.$$

The estimate for the three terms prove claim (59). Thus, we conclude that $(G_n)_{n\in\mathbb{N}}$ is a strongly convergent subsequence of $(F_n)_{n\in\mathbb{N}}$ in L^2_w. □

Appendix B. Summary of Part I

We briefly summarize the results from Part I [BDD10] on the ML-SI equations ($6_{p.4}$)-($7_{p.4}$):

Definition B.1. *Let*

$$\mathcal{W} := \left\{ w \in C^\infty(\mathbb{R}^3, \mathbb{R}^+ \setminus \{0\}) \,\middle|\, \exists\, C_w \in \mathbb{R}^+, P_w \in \mathbb{N} : w(\mathbf{x}+\mathbf{y}) \leq (1 + C_w\|\mathbf{x}\|)^{P_w} w(\mathbf{y}) \right\} \tag{61}$$

be the class of weight functions. For any $w \in \mathcal{W}$ and $\Omega \subseteq \mathbb{R}^3$ we define the space of weighted square integrable functions $\Omega \to \mathbb{R}^3$ by

$$L^2_w(\Omega, \mathbb{R}) := \left\{ \mathbf{F} : \Omega \to \mathbb{R}^3 \,\middle|\, \int d^3x\, w(\mathbf{x})\|\mathbf{F}(\mathbf{x})\|^2 < \infty \right\}.$$

For global regularity arguments we need more conditions on the weight functions which for $k \in \mathbb{N}$ gives rise to the definitions:

$$\mathcal{W}^k := \left\{ w \in \mathcal{W} \,\middle|\, \exists\, C_\alpha \in \mathbb{R}^+ : |D^\alpha \sqrt{w}| \leq C_\alpha \sqrt{w}, |\alpha| \leq k \right\} \tag{62}$$

and

$$\mathcal{W}^\infty := \{ w \in \mathcal{W} \mid w \in \mathcal{W}^k \text{ for any } k \in \mathbb{N} \}.$$

REMARK B.2. *As computed in Part I* [BDD10], *$\mathcal{W} \ni w(\mathbf{x}) := (1 + \|\mathbf{x}\|^2)^{-1}$.*

The space of initial values is then given by:

Definition B.3 (Phase Space for the ML Equations of Motion)**.** *We define the Newtonian phase space $\mathcal{P} := \mathbb{R}^{6N}$, the field space*

$$\mathcal{F}_w := L^2_w(\mathbb{R}^3, \mathbb{R}^3) \oplus L^2_w(\mathbb{R}^3, \mathbb{R}^3)$$

and the phase space for the ML equation of motion

$$\mathcal{H}_w := \mathcal{P} \oplus \mathcal{F}_w.$$

Any element $\varphi \in \mathcal{H}_w$ consists of the components $\varphi = (\mathbf{q}_i, \mathbf{p}_i, \mathbf{E}_i, \mathbf{B}_i)_{1\leq i\leq N}$, i.e. positions $\mathbf{q}_i$, momenta $\mathbf{p}_i$ and electric and magnetic fields $\mathbf{E}_i, \mathbf{B}_i$ for each of the $1 \leq i \leq N$ charges.

Wherever not noted otherwise, any spatial derivative will for the rest of this section be understood in the distribution sense, and the Latin indices $i, j, \ldots$ shall run over the charge labels $1 \ldots N$. We shall also need the weighted Sobolev spaces $H^{curl}_w(\mathbb{R}^3, \mathbb{R}^3) := \{\mathbf{F} \in L^2_w(\mathbb{R}^3, \mathbb{R}^3) \mid \nabla \wedge \mathbf{F} \in L^2_w(\mathbb{R}^3, \mathbb{R}^3)\}$ and $H^k_w(\mathbb{R}^3, \mathbb{R}^3) := \{\mathbf{F} \in L^2_w(\mathbb{R}^3, \mathbb{R}^3) \mid D^\alpha \mathbf{F} \in L^2_w(\mathbb{R}^3, \mathbb{R}^3)\ \forall\, |\alpha| \leq k\}$ for any $k \in \mathbb{N}$. Furthermore, we define the following operators:

Definition B.4 (Operator A)**.** *Let $\mathtt{A}$ and A be given by the expressions*

$$A\varphi = \left(0, 0, \mathtt{A}(\mathbf{E}_i, \mathbf{B}_i)\right)_{1\leq i\leq N} := \left(0, 0, -\nabla \wedge \mathbf{E}_i, \nabla \wedge \mathbf{B}_i\right)_{1\leq i\leq N}.$$

for a $\varphi = (\mathbf{q}_i, \mathbf{p}_i, \mathbf{E}_i, \mathbf{B}_i)_{1\leq i\leq N}$. The natural domain is given by

$$D_w(A) := \bigoplus_{i=1}^N \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus H^{curl}_w(\mathbb{R}^3, \mathbb{R}^3) \oplus H^{curl}_w(\mathbb{R}^3, \mathbb{R}^3) \subset \mathcal{H}_w.$$

Furthermore, for any $n \in \mathbb{N} \cup \{\infty\}$ we define

$$D_w(A^n) := \{\varphi \in D_w(A) \,|\, A^k\varphi \in D_w(A) \text{ for } k = 0, \ldots, n-1\}.$$

Definition B.5 (Operator J)**.** *Together with* $\mathbf{v}(\mathbf{p}_i) := \frac{\mathbf{p}_i}{\sqrt{\mathbf{p}_i^2+m^2}}$ *we define* $J : \mathcal{H}_w \to D_w(A^\infty)$ *by the expression*

$$J(\varphi) = \left(\mathbf{v}(\mathbf{p}_i), \sum_{j=1}^{N} e_{ij} \int d^3x\, \varrho_i(\mathbf{x} - \mathbf{q}_i) \left(\mathbf{E}_j(\mathbf{x}) + \mathbf{v}(\mathbf{p}_i) \wedge \mathbf{B}_j(x) \right), -4\pi \mathbf{v}(\mathbf{p}_i)\varrho_i(\cdot - \mathbf{q}_i), 0 \right)_{1\le i\le N}$$

for a $\varphi = (\mathbf{q}_i, \mathbf{p}_i, \mathbf{E}_i, \mathbf{B}_i)_{1\le i\le N} \in \mathcal{H}_w$*.*

Note that J is well-defined because $\varrho_i \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$. With these definitions the Lorentz force law ($7_{p.4}$), the Maxwell equations ($6_{p.4}$), neglecting the Maxwell constraints, can be collected in the form

$$\dot{\varphi}_t = A\varphi_t + J(\varphi_t) \tag{63}$$

The two main theorems are:

Theorem B.6 (Global Existence and Uniqueness for the ML Equations)**.** *Let the space* $\mathcal{H}_w$ *and the operators* $A : D_w(A) \to \mathcal{H}_w$*,* $J : \mathcal{H}_w \to D_w(A^\infty)$ *be the ones introduced in Definitions B.3*$_{p.43}$*, B.4*$_{p.43}$ *and B.5. Let the weight function* $w \in \mathcal{W}^1$ *and let* $n \in \mathbb{N}$ *and* $\varphi^0 = (\mathbf{q}_i^0, \mathbf{p}_i^0, \mathbf{E}_i^0, \mathbf{B}_i^0)_{1\le i\le N} \in D_w(A^n)$ *be given. Then the following holds:*

(i) (global existence) *There exists an n times continuously differentiable mapping*

$$\varphi_{(\cdot)} : \mathbb{R} \to \mathcal{H}_w, \qquad t \mapsto \varphi_t = (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N}$$

such that $\frac{d^j}{dt^j}\varphi_t \in D_w(A^{n-j})$ *for all* $t \in \mathbb{R}$ *and* $0 \le j \le n$*, which solves the ML equations (63) for initial value* $\varphi_t|_{t=0} = \varphi^0$*.*

(ii) (uniqueness) *The solution* φ *is unique in the sense that if for any interval* $\Lambda \subset \mathbb{R}$ *there is any once continuously differentiable function* $\widetilde{\varphi} : \Lambda \to D_w(A)$ *which solves the equation (63) on* Λ *and there is some* $t^* \in \Lambda$ *such that* $\widetilde{\varphi}_{t^*} = \varphi_{t^*}$ *then* $\varphi_t = \widetilde{\varphi}_t$ *holds for all* $t \in \Lambda$*. In particular, for any* $T \ge 0$ *such that* $[-T, T] \subseteq \Lambda$ *there exist* $C_{19}, C_{20} \in \texttt{Bounds}$ *such that*

$$\sup_{t\in[-T,T]} \|\varphi_t\|_{\mathcal{H}_w} \le C_{19}\left(T, \|\varrho_i\|_{L^2_w}, \|w^{-1/2}\varrho_i\|_{L^2}, 1 \le i \le N\right) \|\varphi^0\|_{\mathcal{H}_w}. \tag{64}$$

and

$$\sup_{t\in[-T,T]} \|\varphi_t - \widetilde{\varphi}_t\|_{\mathcal{H}_w} \le C_{20}(T, \|\varphi_{t_0}\|_{\mathcal{H}_w}, \|\widetilde{\varphi}_{t_0}\|_{\mathcal{H}_w})\|\varphi_{t_0} - \widetilde{\varphi}_{t_0}\|_{\mathcal{H}_w}. \tag{65}$$

(iii) (constraints) *If the solution* $t \mapsto \varphi_t = (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N}$ *obeys the Maxwell constraints*

$$\nabla \cdot \mathbf{E}_{i,t} = 4\pi\varrho(\cdot - \mathbf{q}_{i,t}), \qquad \nabla \cdot \mathbf{B}_{i,t} = 0 \tag{66}$$

for one $t = t^* \in \mathbb{R}$*, then they are obeyed for all times* $t \in \mathbb{R}$*.*

Theorem B.7 (Regularity of the ML Solutions)**.** *Assume the same conditions as in Theorem B.6 hold. In addition, let* $w \in \mathcal{W}^2$*. Let* $t \mapsto \varphi_t = (\mathbf{q}_{i,t}, \mathbf{p}_{i,t}, \mathbf{E}_{i,t}, \mathbf{B}_{i,t})_{1\le i\le N}$ *be the solution to the Maxwell equations 63 for initial value* $\varphi_t|_{t=0} = \varphi^0 \in D_w(A^n)$*. Now let* $n = 2m$ *for some* $m \in \mathbb{N}$*, then for all* $1 \le i \le N$*:*

(i) It holds for any $t \in \mathbb{R}$ *that* $\mathbf{E}_{i,t}, \mathbf{B}_{i,t} \in \mathcal{H}_w^{\triangle^m}$*.*

(ii) The electromagnetic fields viewed as maps $\mathbf{E}_i : (t, \mathbf{x}) \mapsto \mathbf{E}_{i,t}(\mathbf{x})$ *and* $\mathbf{B}_i : (t, \mathbf{x}) \mapsto \mathbf{B}_{i,t}(\mathbf{x})$ *are in* $L^2_{loc}(\mathbb{R}^4, \mathbb{R}^3)$ *and have a representative in* $C^{n-2}(\mathbb{R}^4, \mathbb{R}^3)$ *in their equivalence class, respectively.*

(iii) For $w \in \mathcal{W}^k$ *for* $k \ge 2$ *and every* $t \in \mathbb{R}$ *we have also* $\mathbf{E}_{i,t}, \mathbf{B}_{i,t} \in H^n_w$ *and* $C < \infty$ *such that:*

$$\sup_{\mathbf{x}\in\mathbb{R}^3} \sum_{|\alpha|\le k} \|D^\alpha \mathbf{E}_{i,t}(\mathbf{x})\| \le C\|\mathbf{E}_{i,t}\|_{H^k_w} \qquad \text{and} \qquad \sup_{\mathbf{x}\in\mathbb{R}^3} \sum_{|\alpha|\le k} \|D^\alpha \mathbf{B}_{i,t}(\mathbf{x})\| \le C\|\mathbf{B}_{i,t}\|_{H^k_w}. \tag{67}$$

As shown in Part I [BDD10], A on $D_w(A)$ generates a γ-contractive group $(W_t)_{t\in\mathbb{R}}$:

Definition B.8 (Free Maxwell Time Evolution)**.** *We denote by* $(W_t)_{t\in\mathbb{R}}$ *the* γ*-contractive group on* $\mathcal{H}_w$ *generated by* A *on* $D_w(A)$.

REMARK B.9. *The* γ*-contractive group* $(W_t)_{t\in\mathbb{R}}$ *comes with a standard bound* $\|W_t\|_{\mathfrak{L}(L^2_w)} \le e^{\gamma|t|}$ *which we shall use often.*

The above existence and uniqueness result induces:

Definition B.10 (ML Time Evolution)**.** *We define the non-linear operator*

$$M_L : \mathbb{R}^2 \times D_w(A) \to D_w(A), \qquad (t, t_0, \varphi^0) \to M_L(t, t_0)[\varphi^0] = \varphi_t = W_{t-t_0}\varphi^0 + \int_{t_0}^{t} W_{t-s} J(\varphi_s)$$

which encodes the time evolution of the charges as well as their electromagnetic fields from time t_0 *to time* t.

REMARK B.11. *For times* $t_0, t_1, t \in \mathbb{R}$ *and* $\varphi^0 \in D_w(A)$ *it holds*

$$M_L(t, t_0)[\varphi^0] = M_L(t, t_1)\left[M_L(t_1, t_0)[\varphi^0]\right]$$

by uniqueness.

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