Existence of translating solutions to the flow by powers of mean curvature on unbounded domains *

Huai-Yu Jian and Hong-Jie Ju
Department of Mathematics
Tsinghua University, Beijing 100084, P.R.China

Abstract. In this paper, we prove the existence of classical solutions of the Dirichlet problem for a class of quasi-linear elliptic equations on unbounded domains like a cone or a U-type domain in $R^n (n \geq 2)$. This problem comes from the study of mean curvature flow and its generalization, the flow by powers of mean curvature. Our approach is a modified version of the classical Perron method, where the solutions to the minimal surface equation are used as sub-solutions and a family auxiliary functions are constructed as super-solutions.

Key Words. Dirichlet problem, mean curvature flow, elliptic equation, unbounded domain.

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Correspondence author: Huai-Yu Jian
Email: hjian@math.tsinghua.edu.cn
Tel: 86-10-62772864 Fax: 86-10-62785847

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1. Introduction

Given a constant $\alpha > 0$ and a function $\varphi \in C^0(\partial\Omega)$. Consider the Dirichlet problem:

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = -\left(\frac{1}{\sqrt{1+|Du|^2}}\right)^{\alpha} \quad \text{in } \Omega, \tag{1.1}$$

$$u = \varphi \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is an unbounded domain in $R^n (n \geq 2)$ with $C^{2,\gamma}(0 < \gamma < 1)$ boundary.

The motivation to study this problem comes from the well-known mean curvature flow and its generalization, *H^k-flow*, i.e., the flow of hypersurfaces by powers of mean curvature. Locally, a *H^k-flow* of hypersurfaces in R^{n+1} can be described by the nonlinear parabolic equation,

$$\frac{\partial V}{\partial t} = \sqrt{1+|DV|^2}\left[\operatorname{div}\left(\frac{DV}{\sqrt{1+|DV|^2}}\right)\right]^k. \tag{1.3}$$

When $k = 1$, it is the well-known mean curvature flow, which has been studied strongly since the Huisken's work in 1984. See [5,6,9,18,21] and the references therein.

A function $u = u(x)$ is called a *translating solution* to the H^k-flow if the function $V(x,t) = u(x) + t$ solves (1.3). Equivalently, $-u$ is a solution to equation (1.1) with $\alpha = \frac{1}{k}$. When $k = 1$, the translating solutions play a key role in studying the singularity of mean curvature flows [5,6,8,18,21,22]. Scaling the space and time variables in a proper way near type II-singularity points on the surfaces evolved by mean curvature vector with a mean convex initial surface, Huisken-Sinestrari [5,6] and White [22] proved that the limit flow can be represented as $M_t = \{(x, u(x) + t) \in R^{n+1} : x \in R^n, t \in R\}$, where $-u$ is a solution to equation (1.1) with $\alpha = 1$. Therefore, the study of type II-singularity of mean curvature flow is reduced to studying the behavior of the solutions of equation (1.1) with $\alpha = 1$. Xu-Jia Wang [21] proved that when $\alpha = 1$, any complete strictly convex solution of (1.1) in R^n is radially symmetric for $n = 2$ and constructed a non-radially symmetric solution on a strip region for $n \geq 2$. Sheng and Wang [18] used a direct argument to study the Singularity profile in mean curvature flow, and the stability was studied in [1] for the radially symmetric solution for mean curvature flow.

For general $k > 0$, H^k-flow (1.3) was studied in [15,16]. It was found to have important applications in minimal surfaces [2] and isoperimetric inequalities [16]. It was proved in [19] that when the initial surfaces are mean convex compact without boundary, the flow (1.3) must blow up in finite time, and similarly as in [5,6], the type II-singularity is reduced to the understanding solutions of equation (1.1) for general $\alpha > 0$.

When Ω is a bounded domain, Marquardt [14] proved that when $\alpha \geq 1$, there exists a solution in $C^0(\bar{\Omega}) \cap C^2(\Omega)$ to problem (1.1)-(1.2) if $\partial\Omega \in C^{2,\gamma}$, $\mathrm{H}_{\partial\Omega} > 0$ and $|\Omega| \leq n^n \alpha_n$.

Here and below, *$H_{\partial\Omega}$ always denotes the mean curvature on $\partial\Omega$ with respect to the inner normal, and α_n denotes the volume of unit ball in R^n.*

In [4], Gui and the authors obtained an interior gradient estimate, a Liouville type theorem and the asymptotic behavior at infinity of the radially symmetric solutions to (1.1).

In this article, we prove the existence of classical solutions of problem (1.1)-(1.2) for unbounded domains Ω like U-type or a cone in R^n. To be precise, we assume that Ω satisfy the following $(\Omega 1) - (\Omega 4)$.

Assumption for Ω :

(Ω1) *there exists a sequence of bounded domains $\{\Omega_j\}$ in R^n such that $\Omega_j \subset \Omega_{j+1} \subset \Omega$ for any $j \geq 1$ and $\Omega = \cup_{j=1}^{\infty} \Omega_j$;*

(Ω2) *there exists a $\gamma \in (0,1)$ such that each $\partial\Omega_j \in C^{2,\gamma}$ and $\mathrm{H}_{\partial\Omega_j} > 0$;*

(Ω3) *$dist(0, \Omega \setminus \Omega_j) \to \infty$ as $j \to \infty$;*

(Ω4) *$H_{\partial\Omega} > 0$.*

The main results of this paper are the following two theorems.

Theorem 1.1 *Suppose that (Ω1)-(Ω4) are satisfied and there are a constant N and a positive constant M such that*

$$\Omega \subset C_N(M) := \{x = (x_1, x_2, \cdots, x_n) \in R^n |\ x_1 > N, x_2^2 + \cdots + x_n^2 < M^2\}$$

and $\partial\Omega \cap \partial C_N(M) = \emptyset$. If $\alpha > 0$ and $\varphi \in C^0(\partial\Omega)$, then there exists a solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ to problem (1.1)-(1.2).

Theorem 1.2 *Assume that* (Ω1) -(Ω4) *are satisfied and there is a constant* $\theta \in (0, \frac{\pi}{2})$ *such that*

$$\Omega \subset C(\theta) := \{x = (x_1, x_2, \cdots, x_n) \in R^n |\ x_1 > 0, x_2^2 + \cdots + x_n^2 < (x_1 \tan\theta)^2\}$$

and $\partial\Omega \cap \partial C(\theta) = \emptyset$. *If* $\alpha > 0$ *and* $\varphi \in C^0(\partial\Omega)$, *then there exists a solution* $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *to problem (1.1)-(1.2) .*

The paper is organized as follows. In section 2, we prove the existence of Dirichlet problem (1.1)-(1.2) with $\alpha > 0$ on bounded domain, extending the main result in [14] for the case of $\alpha \geq 1$. Note that when $0 < \alpha < 1$, the hypothesis (sc) of the corresponding theorem in [14] can not be satisfied and the techniques in [14] can not be applied directly. In section 3, we construct a family of auxiliary functions which will be used as super-solutions. In section 4, we define the lifting function so as to construct the class of subfunctions and prove the properties of the subfunctions which is necessary for the proofs of Theorems 1.1 and 1.2. Theorems 1.1 and 1.2 will be proved in section 5 by a modified version of the classical Perron method. The interior gradient estimate for (1.1) derived recently by Gui and the authors in [4] plays an important role.

2. Existence for the solutions on bounded domains

In this section, we prove the existence of the Dirichlet problem (1.1)-(1.2) with $\alpha > 0$ on bounded domains, which is necessary in the proofs of theorems 1.1 and 1.2. For this purpose, we need the interior gradient estimates for equation (1.1), which was obtained in [4] recently by Gui and the authors using the idea of Xu-Jia Wang [20].

Lemma 2.1 [4] *Suppose* $u \in C^3(B_r(0))$ *is a nonnegative solution of equation (1.1), then*

$$|\nabla u(0)| \leq \exp\{C_1 + C_2 \frac{m^2}{r^2}\},$$

where $m = \sup_{x \in B_r(0)} u(x)$, C_1 *and* C_2 *are constants depending only on* n *and* α.

Lemma 2.2 *Let $\Omega_0 \subset R^n$ be a bounded domain with $C^{2,\gamma}$ boundary for some $\gamma \in (0,1)$ and $|\Omega_0| < n^n \alpha_n$. Suppose that $H_{\partial\Omega_0} > 0$ and $\varphi \in C^0(\partial\Omega_0)$. Then the Dirichlet problem (1.1)-(1.2) with Ω_0 instead of Ω has a unique solution $u \in C^0(\bar{\Omega}_0) \cap C^2(\Omega_0)$.*

Proof. Firstly, we suppose $\varphi \in C^{2,\gamma}(\bar{\Omega}_0)$ and prove the Dirichlet problem (1.1)-(1.2) has a solution $u \in C^{2,\gamma}(\bar{\Omega}_0)$. This was proved in [14] for the case of $\alpha \geq 1$, so we assume $\alpha \in (0,1)$ below.

Write (1.1)-(1.2) as

$$\begin{aligned} Qu : &= a^{ij}(Du)D_{ij}u + b(Du) = 0 \quad in\ \Omega_0 && (2.1) \\ u &= \varphi \quad on\ \partial\Omega_0 && (2.2) \end{aligned}$$

where

$$\begin{aligned} a^{ij}(p) : &= (1+|p|^2)\delta_{ij} - p_i p_j, \\ b(p) : &= (1+|p|^2)^{\frac{3-\alpha}{2}}. \end{aligned}$$

By virtue of Theorem 13.8 in [3], it suffices to prove the C^1-estimate for the solutions $u \in C^{2,\gamma}(\bar{\Omega}_0)$ of (2.1)-(2.2).

It follows from the assumption $|\Omega_0| < n^n \alpha_n$ and Theorem 10.5 in [3] that

$$\begin{aligned} \sup_{\Omega_0} |u| &\leq \sup_{\partial\Omega_0} |u| + C diam \Omega_0 \\ &= \sup_{\partial\Omega_0} |\varphi| + C diam \Omega_0, && (2.3) \end{aligned}$$

where constant C depends only on n and Ω_0.

Applying Theorem 15.1 in [3], a maximum principle for the gradient, we can obtain

$$\sup_{\Omega_0} |Du| = \sup_{\partial\Omega_0} |Du|. \tag{2.4}$$

Therefore, we need only to estimate $\sup_{\partial\Omega_0} |Du|$, which will be proved by constructing global upper and lower barriers for u as follows.

Let

$$\Gamma := \{x \in \bar{\Omega}_0 \mid d(x) := dist(x, \partial\Omega_0) < d_1\}$$

with $0 < d_1 < 1$ which will be determined later. Denote $m := \sup_{\bar{\Omega}_0} |u|$ and $a := \sup_{\bar{\Gamma}} |\varphi|$. We want to find a function ψ, such that $w^{\pm} := \varphi \pm \psi \circ d$ are global upper and lower barriers for u and operator Q in domain Γ, i.e.,

$$w^{\pm} = u \quad \text{on } \partial\Omega_0, \tag{2.5}$$

$$w^{-} \leq u \leq w^{+} \quad \text{on } \partial\Gamma \setminus \partial\Omega_0, \tag{2.6}$$

$$\pm Q w^{\pm} < 0 \quad \text{in } \Gamma \setminus \partial\Omega_0. \tag{2.7}$$

Assuming $\psi''(d) \leq 0$ and $\psi'(d) \geq \nu$ for some constant $\nu > 0$ which will be determined by Ω_0, α and $\|\varphi\|_{C^1(\bar{\Omega}_0)}$. For $x \in \Gamma$, there is a $y \in \partial\Omega_0$ such that $d(x) = |x-y|$. Hence,

$$\begin{aligned}
\pm & \quad a^{ij}(Dw^{\pm}) D_{ij} w^{\pm} \\
= & \quad \pm[(1+|Dw^{\pm}|^2)\delta_{ij} - D_i w^{\pm} D_j w^{\pm}][D_{ij}\varphi \pm \psi'' D_i d D_j d \pm \psi' D_{ij} d] \\
= & \quad \pm(1+|Dw^{\pm}|^2)\sum_{i=1}^{n} D_{ii}\varphi \mp D_i w^{\pm} D_j w^{\pm} D_{ij}\varphi \\
& \quad +\psi'' + \psi''[|Dw^{\pm}|^2 - D_i w^{\pm} D_j w^{\pm} D_i d D_j d] \\
& \quad +\psi'(1+|Dw^{\pm}|^2)\sum_{i=1}^{n} D_{ii} d - \psi' D_i w^{\pm} D_j w^{\pm} D_{ij} d, \quad \forall x \in \Gamma,
\end{aligned} \tag{2.8}$$

where we have used the fact $|Dd| = 1$. Noting that $\psi' \geq \nu$ we have

$$\begin{aligned}
\pm & \quad (1+|Dw^{\pm}|^2)\sum_{i=1}^{n} D_{ii}\varphi \mp D_i w^{\pm} D_j w^{\pm} D_{ij}\varphi \\
\leq & \quad 2n^2(1+|D\varphi \pm \psi' Dd|^2)\sup_{\bar{\Gamma}} |D^2\varphi| \\
\leq & \quad 2n^2(1+2|D\varphi|^2 + 2\psi'^2)\sup_{\bar{\Gamma}} |D^2\varphi| \\
\leq & \quad [2n^2(\frac{1+2\sup_{\bar{\Gamma}} |D\varphi|^2}{\nu} + 2)\sup_{\bar{\Gamma}} |D^2\varphi|]\psi'^2 \\
:= & \quad c_1 \psi'^2, \quad \forall x \in \Gamma.
\end{aligned} \tag{2.9}$$

By Schwarz's inequality,

$$D_i w^{\pm} D_j w^{\pm} D_i d D_j d \leq |Dw^{\pm}|^2. \tag{2.10}$$

Since $D_i d D_j d D_{ij} d = 0$, then

$$- \quad \psi' D_i w^{\pm} D_j w^{\pm} D_{ij} d$$

$$\begin{aligned}
&= -\psi'(D_i\varphi D_j\varphi + 2\psi' D_i d D_j\varphi) D_{ij} d \\
&\leq [\sup_{\bar\Gamma} |D^2 d| (\frac{n^2 \sup_{\bar\Gamma} |D\varphi|^2}{\nu} + 2n \sup_{\bar\Gamma} |D\varphi|)]\psi'^2 \\
&:= c_2\psi'^2, \quad \forall x \in \Gamma. \qquad (2.11)
\end{aligned}$$

From Lemma 14.17 in [3],

$$[D^2 d(x)] = diag\left[\frac{-k_1}{1-k_1 d}, \cdots, \frac{-k_{n-1}}{1-k_{n-1}d}, 0\right]$$

where $k_1, \cdots, k_{n-1}$ are the principal curvatures of $\partial\Omega_0$ at y, then we have

$$\sum_{i=1}^{n} D_{ii} d(x) \leq -(n-1) H_{\partial\Omega_0}(y)$$

if d_1 is small enough. Since Ω_0 is a bounded set with $C^{2,\gamma}$ boundary and $H_{\partial\Omega_0} > 0$, $H_0 := \min_{y\in\partial\Omega_0} H_{\partial\Omega_0}(y) = H_{\partial\Omega_0}(y_0) > 0$ for some point y_0. Therefore,

$$\sum_{i=1}^{n} D_{ii} d(x) \leq -(n-1)H_0, \quad \forall\ x \in \Gamma. \qquad (2.12)$$

Now, inserting (2.9)-(2.12) into (2.8), we obtain

$$\pm a^{ij}(Dw^{\pm}) D_{ij} w^{\pm} \leq \psi'' + (c_1 + c_2)\psi'^2 - (n-1)H_0\psi'(1 + |Dw^{\pm}|^2). \qquad (2.13)$$

On the other hand, by the assumption $\alpha \in (0,1)$ we have

$$\begin{aligned}
|b(Dw^{\pm})| &= (1 + |Dw^{\pm}|^2)^{\frac{3-\alpha}{2}} \\
&\leq (1 + |Dw^{\pm}|^2)[(\frac{1 + 2\sup_{\bar\Gamma} |D\varphi|^2}{\nu^2} + 2)\psi'^2]^{\frac{1-\alpha}{2}} \\
&= (1 + |Dw^{\pm}|^2)(\frac{1 + 2\sup_{\bar\Gamma} |D\varphi|^2}{\nu^2} + 2)^{\frac{1-\alpha}{2}}\psi'^{1-\alpha}. \qquad (2.14)
\end{aligned}$$

Combining (2.13) and (2.14), we obtain

$$\begin{aligned}
\pm Q w^{\pm} &\leq \psi'' + (c_1 + c_2)\psi'^2 - (n-1)H_0\psi'(1 + |Dw^{\pm}|^2) \\
&+ (1 + |Dw^{\pm}|^2)(\frac{1 + 2\sup_{\bar\Gamma} |D\varphi|^2}{\nu^2} + 2)^{\frac{1-\alpha}{2}}\psi'^{1-\alpha} \\
&= \psi'' + (c_1 + c_2)\psi'^2 - \psi'(1 + |Dw^{\pm}|^2) \cdot \\
&\quad [(n-1)H_0 - (\frac{1 + 2\sup_{\bar\Gamma} |D\varphi|^2}{\nu^2} + 2)^{\frac{1-\alpha}{2}}\psi'^{-\alpha}].
\end{aligned}$$

Note that $\psi' \geq \nu$, $H_0 > 0$ and $\alpha \in (0,1)$. Choose some large number $\nu > 0$ such that

$$\begin{aligned}
& (n-1)H_0 - (\frac{1+2\sup_{\bar{\Gamma}}|D\varphi|^2}{\nu^2}+2)^{\frac{1-\alpha}{2}}\psi'^{-\alpha} \\
\geq \;& (n-1)H_0 - (\frac{1+2\sup_{\bar{\Gamma}}|D\varphi|^2}{\nu^2}+2)^{\frac{1-\alpha}{2}}\nu^{-\alpha} \\
> \;& 0.
\end{aligned}$$

Consequently,

$$\pm Qw^{\pm} < \psi'' + (c_1+c_2)\psi'^2 =: \psi'' + c_3\psi'^2. \tag{2.15}$$

Thus, (2.5)-(2.7) is reduced to finding a function ψ such that $\psi'' + c_3\psi'^2 = 0$, $\psi'(d) \geq \nu$, $\psi(d) \geq 0$ for $d \in (0, d_1)$, and $\psi(d_1) \geq m + a$.

Now choose the function

$$\psi(d) = \frac{1}{c_3}\ln(1+kd), \quad k > 0.$$

Then

$$\psi'' + c_3\psi'^2 = 0, \quad \psi(0) = 0, \quad \psi(d) > 0, \quad \forall\, d \in (0, d_1].$$

Fix a small $d_1 \in (0, \frac{1}{\nu c_3})$ and set

$$k = \frac{e^{c_3(a+m)} - 1}{d_1} + \frac{\nu c_3}{1 - \nu c_3 d_1},$$

then

$$1 + kd_1 \geq e^{c_3(a+m)}, \quad k \geq \nu c_3(1+kd_1).$$

Thus

$$\psi(d_1) = \frac{1}{c_3}\ln(1+kd) \geq a + m$$

and

$$\psi'(d) = \frac{k}{c_3(1+kd)} \geq \frac{k}{c_3(1+kd_1)} \geq \nu, \quad \text{for } 0 < d \leq d_1.$$

In this way, we have constructed barriers $w^{\pm}$ such that (2.5)-(2.7) are satisfied.

Applying a maximum principle to (2.5)-(2.7) we see that

$$w^- \leq u \leq w^+ \quad \text{on } \partial\Gamma.$$

This, together with (2.5) again, implies

$$\sup_{\partial\Omega_0}|Du| \leq \sup_{\partial\Omega_0}|D\varphi| + \psi'(0) = \sup_{\partial\Omega_0}|D\varphi| + \frac{k}{c_3}. \tag{2.16}$$

Combining (2.3), (2.4) and (2.16), we have

$$\|u\|_{C^1(\bar{\Omega}_0)} = \sup_{\Omega_0}|u| + \sup_{\Omega_0}|Du| \leq C, \tag{2.17}$$

where constant $C = C(n, \alpha, \Omega_0, \| \ d \ \|_{C^2(\bar{\Gamma})}, \| \ \varphi \ \|_{C^2(\bar{\Omega}_0)})$. Hence, by Theorem 13.8 in [3], the Dirichlet problem (1.1)-(1.2) has a solution $u \in C^{2,\gamma}(\bar{\Omega}_0)$ with boundary value $\varphi \in C^{2,\gamma}(\bar{\Omega}_0)$.

If $\varphi \in C^0(\partial\Omega_0)$, we choose a sequence of functions $\varphi_m \in C^{2,\gamma}(\bar{\Omega}_0)$ which is bounded in $C^0(\bar{\Omega}_0)$ and approximates φ in $C^0(\partial\Omega_0)$. As above, the Dirichlet problem (1.1)-(1.2) has solution $u_m \in C^{2,\gamma}(\bar{\Omega}_0)$ with boundary value φ_m. Applying a comparison principle, $\{u_m\}$ converges uniformly to some function $u \in C^0(\bar{\Omega}_0)$ with $u = \varphi$ on $\partial\Omega_0$. The interior gradient estimates (Lemma 2.1), interior Hölder estimate (Theorem 13.1 in [3]) and standard Schauder estimate imply that there is a subsequence of $\{u_m\}$ such that it converges to u in $C^{2,\gamma}(\bar{\Omega}_1)$ for any $\Omega_1 \subset\subset \Omega_0$ by Arzelà-Ascoli theorem. Thus, $u \in C^0(\bar{\Omega}_0) \cap C^2(\Omega_0)$ solves (1.1)-(1.2). The uniqueness follows directly from a comparison principle (Theorem 10.2 in [3]). In this way, Lemma 2.2 has been proved. □

3. A family of auxiliary functions

In this section, we will construct a family of auxiliary functions which will be used as supersolutions for problem (1.1)-(1.2).

Recall the definition of Qu in (2.1), namely,

$$Qu := ((1 + |Du|^2)\delta_{ij} - D_iuD_ju)D_{ij}u + (1 + |Du|^2)^{\frac{3-\alpha}{2}}.$$

We want to construct a family of functions $\{w_k\}$ and a family of sets $\{A_k\}$ which covers the domains in Theorems 1.1 and 1.2, such that $Qw_k \leq 0$ in A_k for each $k \geq 1$. The construction method was introduced in [17] and was used again in [10,11] for the existence of the prescribed mean curvature equations in unbounded domains. Also see [13] for the existence of the constant mean curvature equations in unbounded convex domains.

Set

$$\Phi(\rho)=\begin{cases}\rho^{-2}, & \text{if } 0<\rho<1\\ n-1, & \text{if } \rho\geq 1,\end{cases}$$

and define a function ξ by

$$\xi(t)=\int_t^\infty \frac{d\rho}{\rho^3\Phi(\rho)} \quad \text{for}\, t>0.$$

Let η be the inverse of ξ. It is easy to check that

$$\eta(\beta)=\begin{cases}\frac{1}{\sqrt{2(n-1)\beta}}, & \text{if } 0<\beta<\frac{1}{2(n-1)}\\ e^{-\beta+\frac{1}{2(n-1)}}, & \text{if } \frac{1}{2(n-1)}\leq\beta<+\infty,\end{cases}$$

and

$$\int_0^\infty \eta(\beta)d\beta<\infty.$$

For positive constants L,μ,τ with $\tau>L$ (which will be determined), we define

$$h(r)=h_{\mu,\tau}(r)=\int_r^\tau \eta\left(\mu\ln\frac{t}{L}\right)dt, \quad \text{for } r\in[L,\tau]. \tag{3.1}$$

Then h is a positive, monotonically decreasing function, satisfying

$$h(\tau)=0, \quad h'(L)=-\infty, \quad h(L)=\int_L^\tau \eta\left(\mu\ln\frac{t}{L}\right)dt<\infty$$

and

$$\frac{h''}{(h')^3}=-\frac{\mu}{r}\Phi(-h') \quad \text{for } r\in(L,\tau). \tag{3.2}$$

Since $\eta(\beta)\to\infty$ as $\beta\to 0^+$, for any $H^*>1$ there is a constant $c(H^*,\eta)$ such that $\eta(\beta)\geq H^*$ for all $0<\beta<c(H^*,\eta)$. Note that we may assume $c(H^*,\eta)$ is decreasing in H^*. Letting $d=\frac{c(H^*,\eta)}{\mu}$, we have

$$|h'(r)|=\eta\left(\mu\ln\frac{r}{L}\right)\geq H^*, \quad \forall\, r\in(L,Le^d). \tag{3.3}$$

Now set $\vec{x}_0=(x_1^0,0,\cdots,0)$, $r(\vec{x})=|\vec{x}-\vec{x}_0|$, and

$$w(\vec{x})=w_{\vec{x}_0}(\vec{x})=h(r(\vec{x})). \tag{3.4}$$

Then for any $\vec{x} \in \{\vec{x} \in R^n | \ r(\vec{x}) \in (L, Le^d)\}$, we have

$$Dw(\vec{x}) = h'(r(\vec{x}))\frac{\vec{x} - \vec{x}_0}{r(\vec{x})}, \quad |Dw(\vec{x})| = |h'(r(\vec{x}))| \geq H^*$$

and

$$\begin{aligned} Qw &= ((1 + |Dw|^2)\delta_{ij} - D_i w D_j w)D_{ij}w + (1 + |Dw|^2)^{\frac{3-\alpha}{2}} \\ &= h'' + (n-1)(1 + h'^2)\frac{h'}{r} + (1 + h'^2)^{\frac{3-\alpha}{2}} \\ &= -\frac{\mu}{r}h'^3\Phi(-h') + (n-1)(1 + h'^2)\frac{h'}{r} + (1 + h'^2)^{\frac{3-\alpha}{2}} \\ &= |h'|^3\{\frac{(n-1)\mu}{r} - \frac{n-1}{rh'^2} - \frac{n-1}{r} + \frac{1}{|h'|^3}(1 + h'^2)^{\frac{3-\alpha}{2}}\}, \end{aligned} \tag{3.5}$$

where we have used (3.2) and (3.3).

In order to construct the local super-solutions to equation (1.1), we distinguish two cases which correspond to the domains in theorems 1.1 and 1.2 respectively.

Case 1: *Ω is inside the cylinder $C_N(M)$ as in Theorem 1.1.*

Fix $0 < \mu < 1$. Let $L = M$ and $\tau = Me^d$, where $d = \frac{c(H^*,\eta)}{\mu}$ (which will be determined by H^*). Note that for any fixed $\alpha > 0$,

$$\frac{1}{t^3}(1 + t^2)^{\frac{3-\alpha}{2}} \to 0 \quad \text{as } t \to \infty. \tag{3.6}$$

By (3.3) we can choose some large $H^* > 1$ such that for all $\vec{x} \in \{\vec{x} \in R^n | \ r(\vec{x}) \in (M, Me^d)\}$,

$$\begin{aligned} \frac{1}{|h'|^3}(1 + h'^2)^{\frac{3-\alpha}{2}} &\leq \frac{(n-1)(1-\mu)}{Me^d} \\ &\leq \frac{(n-1)(1-\mu)}{r}. \end{aligned} \tag{3.7}$$

Replacing this inequality in (3.5) we have proved

Claim 1 *For any $\mu \in (0,1)$, there is a $H^* > 1$ such that $Qw(\vec{x}) \leq 0$ for all $\vec{x} \in \{\vec{x} \in R^n | \ r(\vec{x}) \in (M, Me^d)\}$, where w is defined by (3.1) and (3.4) with $d = c(H^*, \eta)/\mu$, $L = M$ and $\tau = Me^d$.*

For a sequence $\{a_k\}$, define $\vec{x}_k = (a_k, 0, \cdots, 0)$ and

$$A(\vec{x}_k) = \{\vec{x} = (x_1, x_2, \cdots, x_n) \in C_N(M) |\ M < |\vec{x} - \vec{x}_k| < Me^d, x_1 < a_k\}. \quad (3.8)$$

By Lemma A.1 in Appendix, we can find a small number $\varepsilon > 0$ and a sequence $\{a_k\}$ satisfying

$$a_1 = N, \quad 0 < a_{k+1} - a_k \leq \varepsilon M(e^d - 1), \quad k = 1, 2, \cdots$$

such that

$$\bigcup_{k=1}^{\infty} A(\vec{x}_k) = C_N(M)$$

and

$$\partial A(\vec{x}_{k+1}) \bigcap \{\vec{x} \in C_N(M) |\ |\vec{x} - \vec{x}_{k+1}| = Me^d, x_1 < a_{k+1}\} \subset A(\vec{x}_k).$$

On each domain $A(\vec{x}_k)$, we define a function w_k as follows. Let $h_k(r(\vec{x})) = h(|\vec{x} - \vec{x}_k|)$, where $h(r)$ is the function defined by (3.1) with $L = M, \tau = Me^d$. Set

$$w_k(\vec{x}) = h_k(r(\vec{x})) + (k-1)h(M) + \sup\{|\varphi(\vec{x})| \mid \vec{x} \in \partial\Omega, x_1 \leq a_k\}. \quad (3.9)$$

It follows from Claim 1 that each w_k is well defined in $A(\vec{x}_k)$ and satisfies

$$Qw_k \leq 0 \quad in \quad A(\vec{x}_k). \quad (3.10)$$

Furthermore, by the obvious properties of h, we see that

$$\begin{aligned} w_k(\vec{x}) &\leq h(M) + (k-1)h(M) + \sup\{|\varphi(\vec{x})| \mid \vec{x} \in \partial\Omega, x_1 \leq a_k\} \\ &\leq h_{k+1}(r(\vec{x})) + kh(M) + \sup\{|\varphi(\vec{x})| \mid \vec{x} \in \partial\Omega, x_1 \leq a_{k+1}\} \\ &= w_{k+1}(\vec{x}), \quad \forall\ \vec{x} \in A(\vec{x}_k) \cap A(\vec{x}_{k+1}), \end{aligned} \quad (3.11)$$

where the r in $h_{k+1}(r)$ is $|\vec{x} - \vec{x}_{k+1}|$.

Case 2: *Ω is inside the cone $C(\theta)$ as in Theorem 1.2.*

Recall that for any $L > 0$, $0 < \mu < 1$ and $H^* > 1$ there is a constant $c^*(H^*, \eta)$ such that (3.3) holds for $d = \frac{c(H^*,\eta)}{\mu}$, which means that for any $0 < d \leq \frac{c(H^*,\eta)}{\mu}$,

$$|h'(r)| = \eta\left(\mu ln\frac{r}{L}\right) \geq H^* \quad \text{for } L < r \leq Le^d. \quad (3.12)$$

For a number $b > 0$, setting $L = b\sin\theta$, $\tau = be^d \sin\theta$ in (3.1) where $0 < d \leq \frac{c(H^*,\eta)}{\mu}$, we have obtained the function h. Then let $\vec{x}_0 = (b, 0, \cdots, 0)$, $r(\vec{x}) = |\vec{x} - \vec{x}_0|$, and $w(\vec{x}) = h(r(\vec{x}))$. It follows from (3.12) that for any $d \in (0, \frac{c(H^*,\eta)}{\mu})$,

$$|h'(r(\vec{x}))| \geq H^*, \quad \forall \vec{x} \in \{\vec{x} \in R^n |\ L < r(\vec{x}) < Le^d\}.$$

Then as (3.6)-(3.7), we have

$$\begin{aligned} \frac{1}{|h'|^3}(1+h'^2)^{\frac{3-\alpha}{2}} &\leq \frac{(n-1)(1-\mu)}{Le^d} \\ &\leq \frac{(n-1)(1-\mu)}{r}, \quad \forall \vec{x} \in \{\vec{x} \in R^n |\ L < r(\vec{x}) < Le^d\}. \end{aligned}$$

Hence, we have proved

Claim 2 *For any $b > 0$, $0 < \mu < 1$ and $\theta \in (0, \frac{\pi}{2})$, there exists $H^* > 1$ such that for any $0 < d \leq c(H^*, \eta)/\mu$, $Qw \leq 0$ for all $\vec{x} \in \{\vec{x} \in R^n \mid L < r(\vec{x}) < Le^d\}$, where w is defined by (3.1) and (3.4) with $L = b\sin\theta$ and $\tau = b\sin\theta e^d$.*

Since $\partial\Omega \cap \partial C(\theta) = \emptyset$, the vertex of $C(\theta)$, $0 \notin \partial\Omega$. Hence, we can find a small $b_1 > 0$ such that the ball centered at $\vec{x}_1 = (b_1, 0, \cdots, 0)$ with radius b_1 does not intersect with Ω. Choose a $d \in (0, \frac{c(H^*,\eta)}{\mu})$ such that $1 - e^d \sin\theta > 0$, and then take a δ_0 such that

$$1 < \delta_0 < \frac{1 - \sin\theta}{1 - e^d \sin\theta}. \tag{3.13}$$

For $k \geq 1$, let $b_k = \delta_0^{k-1} b_1$, $L_k = b_k \sin\theta$, $\vec{x}_k = \delta_0^{k-1}\vec{x}_1$ and

$$\tilde{A}(\vec{x}_k) = \{\vec{x} = (x_1, x_2, \cdots, x_n) \in C(\theta) \mid L_k < |\vec{x} - \vec{x}_k| < L_k e^d,\ x_1 < b_k, \}. \tag{3.14}$$

By Lemma A.2 in Appendix, we have

$$\Omega \subset \bigcup_{k=1}^{\infty} \tilde{A}(\vec{x}_k), \tag{3.15}$$

and the part of $\partial\tilde{A}(\vec{x}_k)$, $S_k := \{\vec{x} \in C(\theta) \mid |\vec{x} - \vec{x}_k| = L_k,\ x_1 < b_k\}$, is completely covered by $\tilde{A}(\vec{x}_{k+1})$.

On each domain $\tilde{A}(\vec{x}_k)$, we define a function $\tilde{w}_k$ as follows. Let $h_k(r)$ be the function defined by (3.1) with $L = L_k = b_k \sin\theta$ and $\tau = L_k e^d = b_k e^d \sin\theta$. Namely,

$$h_k(r) = \int_r^{b_k e^d \sin\theta} \eta\left(\mu ln \frac{t}{b_k \sin\theta}\right) dt, \quad r \in [L_k, L_k e^d].$$

Denote

$$\tilde{B}_k = h_k(L_k) = \int_{b_k \sin\theta}^{b_k e^d \sin\theta} \eta\left(\mu ln \frac{t}{b_k \sin\theta}\right) dt$$

and define

$$\tilde{w}_k(\vec{x}) = h_k(r(\vec{x})) + \sum_{j=1}^{k-1} \tilde{B}_j + \sup\{|\varphi(\vec{x})| \mid \vec{x} \in \partial\Omega,\ x_1 \le b_k\}, \tag{3.16}$$

where $r(\vec{x}) = |\vec{x} - \vec{x}_k|$. Then by Claim 2 we see that $\tilde{w}_k$ is well defined in $\tilde{A}(\vec{x}_k)$ and satisfies

$$Q\tilde{w}_k \le 0 \quad \text{in } \tilde{A}(\vec{x}_k). \tag{3.17}$$

Moreover,

$$\begin{aligned}
\tilde{w}_k(\vec{x}) &\le h_k(L_k) + \sum_{j=1}^{k-1} \tilde{B}_j + \sup\{|\varphi(\vec{x})| \mid \vec{x} \in \partial\Omega,\ x_1 \le b_k\} \\
&\le h_{k+1}(r(\vec{x})) + \sum_{j=1}^{k} \tilde{B}_j + \sup\{|\varphi(\vec{x})| \mid \vec{x} \in \partial\Omega,\ x_1 \le b_{k+1}\} \\
&= \tilde{w}_{k+1}(\vec{x}) \quad \text{in } \tilde{A}(\vec{x}_k) \cap \tilde{A}(\vec{x}_{k+1}),
\end{aligned} \tag{3.18}$$

where the r in $h_{k+1}(r)$ is $|\vec{x} - \vec{x}_{k+1}|$.

4. The lifting and subfunction

In this section, we define the lifting of a function and the class of subfunctions which contains the solutions of minimal surface equations. We show a few properties which will be used to prove the supreme function for all the subfunctions is a solution to (1.1)-(1.2) in the next section.

Let Π be the family of all bounded open sets $O \subset \Omega$ satisfying $\partial O \in C^{2,\gamma}$, $H_{\partial O} > 0$ and $|O| < n^n \alpha_n$. φ, $C_N(M)$ and $C(\theta)$ are the same as in Theorems 1.1 and 1.2.

Definition 4.1 *Let $v \in C^0(\bar{\Omega})$. For each $O \in \Pi$, define a new function $M_O(v)$, called the lifting of v over O, as follows:*

$$M_O(v)(\vec{x}) = \begin{cases} v(\vec{x}), & \text{if } \vec{x} \in \Omega \setminus O \\ z(\vec{x}), & \text{if } \vec{x} \in O \end{cases}$$

where $z(\vec{x})$ is the solution of the boundary-value problem

$$\begin{cases} Qz = 0, & \text{in } O, \\ z = v, & \text{on } \partial O. \end{cases}$$

Note that the definition is well-defined by Lemma 2.2.

Definition 4.2 *The subfunction class F is defined as follows: a function v is in F if and only if*

(1) $v \in C^0(\bar{\Omega})$ and $v \leq \varphi$ on $\partial\Omega$;

(2) for any $O \in \Pi$, $v \leq M_O(v)$;

(3) if $\Omega \subseteq C_N(M)$, then $v \leq w_k$ in $\Omega \cap A(\vec{x}_k)$ for $k \geq 1$;

(4) if $\Omega \subseteq C(\theta)$, then $v \leq \tilde{w}_k$ in $\Omega \cap \tilde{A}(\vec{x}_k)$ for $k \geq 1$.

Let Ω_1 be a domain in R^n, $\{c_k\}_{k=1}^{\infty}$ be a non-negative, non-decreasing sequence. If Ω_1 is inside the cylinder $C_N(M)$, we set

$$w_k^1(\vec{x}) = h_k(r(\vec{x})) + (k-1)h(M) + c_k \quad \text{in } A(\vec{x}_k), \tag{4.1}$$

where h_k and $A(\vec{x}_k)$ are the same as those defined in (3.8) and (3.9). Thus, w_k^1 satisfies (3.10) and (3.11) in $A(\vec{x}_k)$.

If Ω_1 is inside the cone $C(\theta)$, we set

$$\tilde{w}_k^1(\vec{x}) = h_k(r(\vec{x})) + \sum_{j=1}^{k-1} \tilde{B}_j + c_k \quad \text{in } \tilde{A}(\vec{x}_k), \tag{4.2}$$

where h_k, $\tilde{B}_j$ and $\tilde{A}(\vec{x}_k)$ are the same as in (3.14) and (3.16). Thus, $\tilde{w}_k^1$ satisfies (3.17) and (3.18) in $\tilde{A}(\vec{x}_k)$.

Lemma 4.1 *Suppose $u \in C^2(\Omega_1) \cap C^0(\bar{\Omega}_1)$ and $Qu \geq 0$ in Ω_1.*

(i) When $\Omega_1 \subset C_N(M)$ and $\partial\Omega_1 \cap \partial C_N(M) = \emptyset$, if

$$u \leq w_k^1 \quad \text{on } A(\vec{x}_k) \cap \partial\Omega_1 \quad \text{for } k \geq 1, \tag{4.3}$$

then

$$u \le w_k^1 \quad \text{in } A(\vec{x}_k) \cap \Omega_1 \quad \text{for } k \ge 1.$$

(ii) When $\Omega_1 \subset C(\theta)$ *and* $\partial\Omega_1 \cap \partial C(\theta) = \emptyset$, *if*

$$u \le \tilde{w}_k^1 \quad \text{on } \tilde{A}(\vec{x}_k) \cap \partial\Omega_1 \quad \text{for } k \ge 1, \tag{4.4}$$

then

$$u \le \tilde{w}_k^1 \quad \text{in } \tilde{A}(\vec{x}_k) \cap \Omega_1 \quad \text{for } k \ge 1.$$

Proof At first, let us prove (i).

Among the family of domains $A(\vec{x}_k)$, let $A(\vec{x}_{k_0})$ be the first one (i.e. smallest k) which intersects with Ω_1. We conclude that

$$u \le w_{k_0}^1 \quad \text{in } A(\vec{x}_{k_0}) \cap \Omega_1. \tag{4.5}$$

In fact, by (4.3),

$$u \le w_{k_0}^1 \qquad \text{on } A(\vec{x}_{k_0}) \cap \partial\Omega_1.$$

Note that $\partial A(\vec{x}_{k_0}) \cap \Omega_1 \cap \{|\vec{x} - \vec{x}_{k_0}| = Me^d\}$ is empty. Otherwise, from the fact that $\partial A(\vec{x}_{k_0}) \cap \{|\vec{x} - \vec{x}_{k_0}| = Me^d\}$ is covered by $A(\vec{x}_{k_0-1})$ (Lemma A.1), we see that $A(\vec{x}_{k_0})$ will not be the first to intersect with Ω_1, a contradiction.

Also, $\partial A(\vec{x}_{k_0}) \cap \Omega_1 \cap \{M < |\vec{x} - \vec{x}_{k_0}| < Me^d\}$ is empty, which follows from the fact that $\partial A(\vec{x}_{k_0}) \cap \{M < |\vec{x} - \vec{x}_{k_0}| < Me^d\}$ is a part of $\partial C_N(M)$ and $\partial\Omega_1 \cap \partial C_N(M) = \emptyset$ by the assumption.

On $\partial A(\vec{x}_{k_0}) \cap \Omega_1 \cap \{|\vec{x} - \vec{x}_{k_0}| = M\}$, it follows from the fact $h'(M) = -\infty$ that the outer normal derivative of $w_{k_0}^1$ is $+\infty$. Thus, $u - w_{k_0}^1$ cannot achieve a maximum on this part of the boundary.

Therefore,

$$u \le w_{k_0}^1 \ \text{ on } \partial(A(\vec{x}_{k_0}) \cap \Omega_1).$$

Furthermore, (3.10) and the assumption imply

$$Qw_{k_0}^1 \le Qu \quad \text{in } A(\vec{x}_{k_0}) \cap \Omega_1.$$

Hence (4.5) follows from the standard maximum principle [3].

We now compare u with $w_{k_0+1}^1$ on $A(\vec{x}_{k_0+1}) \cap \Omega_1$. By (4.3),

$$u \le w_{k_0+1}^1 \quad \text{on } A(\vec{x}_{k_0+1}) \cap \partial\Omega_1.$$

Since $\partial A(\vec{x}_{k_0+1}) \cap \Omega_1 \cap \{|\vec{x} - \vec{x}_{k_0+1}| = Me^d\}$ is covered by $A(\vec{x}_{k_0})$(Lemma A.1), then $u \le w^1_{k_0} \le w^1_{k_0+1}$ on this part, by (4.5) and (3.11).

As above, $\partial A(\vec{x}_{k_0+1}) \cap \Omega_1 \cap \{M < |\vec{x} - \vec{x}_{k_0+1}| < Me^d\}$ is also empty.

On $\partial A(\vec{x}_{k_0+1}) \cap \Omega_1 \cap \{|\vec{x} - \vec{x}_{k_0+1}| = M\}$, the outer normal derivative of $w^1_{k_0+1}$ is $+\infty$. Thus, $u - w^1_{k_0+1}$ cannot achieve a maximum on this part of the boundary.

Since

$$Qw^1_{k_0+1} \le Qu \quad \text{in } A(\vec{x}_{k_0+1}) \cap \Omega_1,$$

by the standard maximum principle [3] we obtain

$$u - w^1_{k_0+1} \le 0 \quad \text{in } A(\vec{x}_{k_0+1}) \cap \Omega_1. \tag{4.6}$$

Repeating the above procedure, we can obtain

$$u \le w^1_k \quad \text{in } A(\vec{x}_k) \cap \Omega_1, \quad \forall k \ge 1.$$

The proof of (ii) is almost the same, and we write as follows just for the completeness. In the family of domains $\tilde{A}(\vec{x}_k)$, let $\tilde{A}(\vec{x}_{k_0})$ be the first one (i.e.smallest k) to intersect with Ω_1. We first conclude that

$$u \le \tilde{w}^1_{k_0} \quad \text{in } \tilde{A}(\vec{x}_{k_0}) \cap \Omega_1. \tag{4.7}$$

In fact, by (4.4) we have

$$u \le \tilde{w}^1_{k_0} \qquad \text{on } \tilde{A}(\vec{x}_{k_0}) \cap \partial\Omega_1.$$

Note that $\partial\tilde{A}(\vec{x}_{k_0}) \cap \Omega_1 \cap \{|\vec{x} - \vec{x}_{k_0}| = L_{k_0}e^d\}$ is empty. Otherwise, by the fact that $\partial\tilde{A}(\vec{x}_{k_0}) \cap \{|\vec{x} - \vec{x}_{k_0}| = L_{k_0}e^d\}$ is covered by $\tilde{A}(\vec{x}_{k_0-1})$ (Lemma A.2), we see that $\tilde{A}(\vec{x}_{k_0})$ will not be the first to intersect with Ω_1, a contradiction.

$\partial\tilde{A}(\vec{x}_{k_0}) \cap \Omega_1 \cap \{L_{k_0} < |\vec{x} - \vec{x}_{k_0}| < L_{k_0}e^d\}$ is also empty, since $\partial\tilde{A}(\vec{x}_{k_0}) \cap \{L_{k_0} < |\vec{x} - \vec{x}_{k_0}| < L_{k_0}e^d\}$ is a part of $\partial C(\theta)$ and $\partial\Omega_1 \cap \partial C(\theta) = \emptyset$ by the assumption.

On $\partial\tilde{A}(\vec{x}_{k_0}) \cap \Omega_1 \cap \{|\vec{x} - \vec{x}_{k_0}| = L_{k_0}\}$, the outer normal derivative of $\tilde{w}^1_{k_0}$ is $+\infty$. Thus, $u - \tilde{w}^1_{k_0}$ cannot achieve a maximum on this part of the boundary.

Since

$$Q\tilde{w}^1_{k_0} \le Qu \quad \text{in } \tilde{A}(\vec{x}_{k_0}) \cap \Omega_1,$$

by a maximum principle we obtain

$$u - \tilde{w}^1_{k_0} \le 0 \quad \text{in } \tilde{A}(\vec{x}_{k_0}) \cap \Omega_1. \tag{4.8}$$

We now compare u with $\tilde{w}^1_{k_0+1}$ in $\tilde{A}(\vec{x}_{k_0+1}) \cap \Omega$. By (4.4) again,

$$u \le \tilde{w}^1_{k_0+1} \quad \text{on } \tilde{A}(\vec{x}_{k_0+1}) \cap \partial\Omega_1.$$

Since $\partial\tilde{A}(\vec{x}_{k_0+1}) \cap \Omega_1 \cap \{|\vec{x} - \vec{x}_{k_0+1}| = L_{k_0+1}e^d\}$ is covered by $\tilde{A}(\vec{x}_{k_0})$ (Lemma A.2), then $u \le \tilde{w}^1_{k_0} \le \tilde{w}^1_{k_0+1}$ on this part, by (4.8) and (3.18).

As above, $\partial\tilde{A}(\vec{x}_{k_0+1}) \cap \Omega_1 \cap \{L_{k_0+1} < |\vec{x} - \vec{x}_{k_0+1}| < L_{k_0+1}e^d\}$ is also empty.

On $\partial\tilde{A}(\vec{x}_{k_0+1}) \cap \Omega_1 \cap \{|\vec{x} - \vec{x}_{k_0+1}| = L_{k_0+1}\}$, the outer normal derivative of $\tilde{w}^1_{k_0+1}$ is $+\infty$. Thus, $u - \tilde{w}^1_{k_0+1}$ cannot achieve a maximum on this part of the boundary.

Since

$$Q\tilde{w}^1_{k_0+1} \le Qu \quad \text{in } \tilde{A}(\vec{x}_{k_0+1}) \cap \Omega_1,$$

by a maximum principle we obtain

$$u - \tilde{w}^1_{k_0+1} \le 0 \quad \text{in } \tilde{A}(\vec{x}_{k_0+1}) \cap \Omega_1. \tag{4.9}$$

Repeating the above procedure as necessary, we arrive at

$$u \le \tilde{w}^1_k \quad \text{in } \tilde{A}(\vec{x}_k) \cap \Omega_1, \quad \forall k \ge 1.$$

□

Corollary 4.1 *Let Ω be the same domain as in Theorem 1.1. If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be a solution of the problem*

$$((1 + |Du|^2)\delta_{ij} - D_i u D_j u) D_{ij} u = 0 \quad \textit{in } \Omega \tag{4.10}$$

$$u = \varphi \quad \textit{on } \partial\Omega, \tag{4.11}$$

then

$$|u(\vec{x})| \le w_k(\vec{x}) \quad \textit{in } A(\vec{x}_k) \cap \Omega, \ \textit{for } k \ge 1.$$

Proof Note that

$$Qu \ge 0 \quad \text{in } A(\vec{x}_k) \cap \Omega,$$

$$Qw_k \le 0 \quad \text{in } A(\vec{x}_k) \cap \Omega$$

and $u = \varphi \le \sup\{|\varphi(\vec{x})| \mid \vec{x} \in \partial\Omega, \ x_1 < a_k\} \le w_k(\vec{x})$ on $A(\vec{x}_k) \cap \partial\Omega$. By the conclusion (i) of Lemma 4.1, we can obtain

$$u \le w_k \quad \text{in } A(\vec{x}_k) \cap \Omega, \ k \ge 1.$$

On the other hand, $v = -u$ is also a solution of (4.10)-(4.11) with φ replaced by $-\varphi$, we can get

$$v = -u \le w_k \quad \text{in } A(\vec{x}_k) \cap \Omega, \ k \ge 1.$$

Therefore,

$$|u| \le w_k \quad \text{in } A(\vec{x}_k) \cap \Omega, \ k \ge 1.$$

□

Similarly, we have

Corollary 4.2 *Let Ω be the same domain as in Theorem 1.2. If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be a solution of the problem*

$$\begin{aligned} ((1 + |Du|^2)\delta_{ij} - D_i u D_j u) D_{ij} u &= 0 \quad \text{in } \Omega \\ u &= \varphi \quad \text{on } \partial\Omega, \end{aligned}$$

then

$$|u(\vec{x})| \le \tilde{w}_k \quad \text{in } \tilde{A}(\vec{x}_k) \cap \Omega, \ \text{for } k \ge 1.$$

Corollary 4.3 *Let Ω be the same domains as in Theorems 1.1 or 1.2. Then F is not empty.*

Proof It follows from Lemma 4.5 in [9] that under the assumption $(\Omega 1) - (\Omega 3)$, the boundary-value problem (4.10)-(4.11) has a solution $v_0 \in C^2(\Omega) \cap C^0(\bar{\Omega})$. By Corollaries 4.1 or 4.2, we can see that $v_0 \in F$.

□

Next, we show a few properties of subfunctions which will be necessary in the proofs of theorems 1.1 and 1.2. For this purpose, we assume that Ω is one of the following cases:

Case (i) *$\Omega \subset C_N(M)$ and $\partial\Omega \cap \partial C_N(M) = \emptyset$;*

Case (ii) *$\Omega \subset C(\theta)$ and $\partial\Omega \cap \partial C(\theta) = \emptyset$.*

First, we assume case (i) and prove the following three lemmas, which also hold for case (ii).

Lemma 4.2 *If $v_1, v_2 \in C^0(\bar{\Omega})$ and $v_1 \le v_2$ in Ω, then $M_O(v_1) \le M_O(v_2)$ for any $O \in \Pi$.*

Proof By the definition of $M_O(v)$, we have

$$M_O(v_1) = v_1 \le v_2 = M_O(v_2) \quad \text{on } \Omega \setminus O,$$

thus, we need only to prove $M_O(v_1) \le M_O(v_2)$ on O.
Since $z_i := M_O(v_i)(i = 1, 2)$ satisfies the boundary-value equation

$$\begin{aligned} Qz_i &= 0 \quad \text{in } O \\ z_i &= v_i \quad \text{on } \partial O \end{aligned}$$

and $z_1 = v_1 \le v_2 = z_2$ on ∂O, then by a comparison principle we obtain $z_1 \le z_2$ in O. Therefore, $M_O(v_1) \le M_O(v_2)$ on Ω. □

Lemma 4.3 *If* $v_i \in F \quad (i = 1, 2)$*, then* $\max\{v_1, v_2\} \in F$*.*

Proof By the definition of F, $\max\{v_1, v_2\} \in C^0(\bar{\Omega})$, $\max\{v_1, v_2\} \le \varphi$ on $\partial\Omega$ and $\max\{v_1, v_2\} \le w_k$ in $A(x_k) \cap \Omega$ for $k \ge 1$. So we need only to check that for any $O \in \Pi$, $\max\{v_1, v_2\} \le M_O(\max\{v_1, v_2\})$.
Since $v_i \le \max\{v_1, v_2\} \quad (i = 1, 2)$, by Lemma 4.2 we have that for any $O \in \Pi$,

$$M_O(v_i) \le M_O(\max\{v_1, v_2\}) \quad (i = 1, 2).$$

Since $v_i \in F$ imply that $v_i \le M_O(v_i)$, we obtain $v_i \le M_O(\max\{v_1, v_2\}) \quad (i = 1, 2)$. Namely, $\max\{v_1, v_2\} \le M_O(\max\{v_1, v_2\})$. □

Lemma 4.4 *If* $v \in F$*, then* $M_O(v) \in F$ *for any* $O \in \Pi$*.*

Proof By the definition, $M_O(v) \in C^0(\bar{\Omega})$ and $M_O(v) = v \le \varphi$ on $\partial\Omega$.

First we prove that, for any $O_1 \in \Pi$,

$$M_O(v) \le M_{O_1}(M_O(v)). \tag{4.12}$$

Observe that

$$M_{O_1}(M_O(v)) = M_O(v) \quad \text{in } \Omega \setminus O_1. \tag{4.13}$$

It is enough to prove that (4.12) holds on O_1.

Since $v \le M_O(v)$ on Ω, then we have $M_{O_1}(v) \le M_{O_1}(M_O(v))$ by Lemma 4.2. Moreover, we have

$$M_O(v) = v \le M_{O_1}(M_O(v)) \quad \text{in } O_1 \setminus O. \tag{4.14}$$

Denote $z_1 = M_O(v)$ and $z_2 = M_{O_1}(M_O(v))$. We see that

$$Qz_i = 0 \quad \text{in } O_1 \cap O, \quad i = 1, 2.$$

It follows from (4.13), (4.14) and the continuity of z_i that

$$z_1 = M_O(v) \le M_{O_1}(M_O(v)) = z_2 \quad \text{on } \partial(O_1 \cap O). \tag{4.15}$$

Then a comparison principle implies that $z_1 \le z_2$ in $O_1 \cap O$. Thus, (4.12) is true in $O_1 \cap O$ and hence in O_1 by (4.14).

It remains to prove that $M_O(v) \le w_k$ in $A(\vec{x}_k) \cap \Omega$ for all $k \ge 1$. Since $v \in F$, we find that

$$M_O(v) = v \le w_k \quad \text{in } A(\vec{x}_k) \cap \partial O, \quad \forall k \ge 1.$$

Thus, the assumption (4.3) in Lemma 4.1 is satisfied for $\Omega_1 = O$, $w_k^1 = w_k$ and $c_k = \sup\{|\varphi(\vec{x})| \mid \vec{x} \in \partial\Omega, x_1 \le a_k\}$. Apply this lemma to $u = M_O(v)$ we conclude that $M_O(v) \le w_k$ in $A(\vec{x}_k) \cap O$.

□

If case (ii) happens, replacing w_k and $A(\vec{x}_k)$ by $\tilde{w}_k$ and $\tilde{A}(\vec{x}_k)$, respectively, without changing the rest of the proof, we see that Lemmas 4.2, 4.3 and 4.4 also hold.

5. Proofs of Theorems 1.1 and 1.2

We are in the position to use Perron's method to prove the theorems.

Proof of Theorem 1.1: Set $u(\vec{x}) = \sup\{v(\vec{x}) \mid v \in F\}$ for $\vec{x} \in \bar{\Omega}$. We will show that u is in $C^0(\bar{\Omega}) \cap C^2(\Omega)$ and satisfies (1.1)-(1.2).

For any $\vec{x}_0 \in \Omega$, by the definition of $u(\vec{x}_0)$, there is a sequence of functions $\{v_i\}_{i=1}^{\infty} \subset F$ such that

$$u(\vec{x}_0) = \lim_{i\to\infty} v_i(\vec{x}_0).$$

Let v_0 be a solution of (4.10)-(4.11). Then by the proof of Corollary 4.3, we have

$$v_0 \in F \quad \textit{and} \quad u \ge v_0 \ \ \textit{in} \ \ \Omega. \tag{5.1}$$

Replacing v_i by $\max\{v_i,\ v_0\}$, we may assume that $v_i \geq v_0$ on Ω by Lemma 4.3. For any $O \in \Pi$ such that $\vec{x}_0 \in O$, replacing v_i by $M_O(v_i)$, we then obtain a sequence of functions $z_i = M_O(v_i)$ such that

$$u(\vec{x}_0) = \lim_{i\to\infty} z_i(\vec{x}_0),$$

$$\begin{aligned} Qz_i &= 0 \quad \text{in } O, \\ z_i &= v_i \quad \text{on } \partial O. \end{aligned}$$

Since, for all k and i,

$$v_0 \leq v_i \leq z_i \leq w_k \quad \text{in } O \cap A(\vec{x}_k) \tag{5.2}$$

and O can be covered by the finitely many domains $A(\vec{x}_k)$, there is a constant K_1 such that

$$v_0 \leq z_i \leq K_1 \quad \text{in } O, \quad \forall i \geq 1.$$

Using Lemma 2.1 first, then the standard interior Hölder estimate of the gradients [3, Theorem 13.1] and finally standard Schauder estimates [3], by Arzelà-Ascoli theorem we can choose a subsequence of z_i (denoted still by z_i) converging to a function $z \in C^2(O)$ and so $z(\vec{x})$ satisfies

$$Qz = 0 \quad \text{in } O. \tag{5.3}$$

Obviously, $u(\vec{x}_0) = z(\vec{x}_0)$ and $u(\vec{x}) \geq z(\vec{x})$ in O.

Next, we prove that $u \equiv z$ on O. Indeed, if there is another point $\vec{x}_1 \in O$ such that $u(\vec{x}_1) > z(\vec{x}_1)$, then there is a function $u_0 \in F$ such that

$$z(\vec{x}_1) < u_0(\vec{x}_1) \leq u(\vec{x}_1).$$

Setting $\bar{z}_i = M_O(\max\{u_0, M_O(v_i)\})$, we have, for all k and i, that

$$v_0 \leq v_i \leq \bar{z}_i \leq w_k \quad \text{in } O \cap A(\vec{x}_k)$$

and $Q\bar{z}_i = 0$ in O. Repeating the arguments from (5.2) to (5.3), we obtain a subsequence of $\{\bar{z}_i\}$ (denoted still by $\bar{z}_i$) which converges to a function $\bar{z}$ in $C^2(O)$ and $Q\bar{z} = 0$ on O. Obviously

$$z_i = M_O(v_i) \leq M_O(\max\{u_0, M_O(v_i)\}) = \bar{z}_i.$$

Hence,

$$z \le \bar{z} \quad \text{in } O,$$

$$z(\vec{x}_1) < u_0(\vec{x}_1) \le \bar{z}(\vec{x}_1)$$

and

$$z(\vec{x}_0) = u_0(\vec{x}_0) = \bar{z}(\vec{x}_0).$$

That is, $\bar{z}(\vec{x}) - z(\vec{x})$ is non-negative, not identically zero in O and attains its minimum value zero inside O. However, it follows from the equations satisfied by z and $\bar{z}$, we find that

$$\begin{aligned} & ((1+|D\bar{z}|^2)\delta_{pq} - D_p\bar{z}D_q\bar{z})D_{pq}(\bar{z}-z) \\ = \; & E(x, z, \bar{z}, Dz, D\bar{z}, D^2z, D^2\bar{z})D(\bar{z}-z) \quad \text{in } O \end{aligned}$$

for some continuous function E. Then, by the standard maximum principle, we have got a contradiction. Thus, $u \equiv z$ in O. Since O can be arbitrary, $u \in C^2(\Omega)$ and $Qu = 0$ in Ω.

Finally, it remains to prove that

$$u \in C^0(\bar{\Omega}) \;\; \textit{and} \;\; u = \varphi \;\; \textit{on} \;\; \partial\Omega.$$

For any point $\vec{x}_2 \in \partial\Omega$, we can find a bounded $C^{2,\gamma}$ domain $\Omega_1 \subset \Omega$ such that $\partial\Omega_1 \cap \partial\Omega$ is an open neighborhood of $\vec{x}_2$ in $\partial\Omega$, $|\Omega_1| < n^n\alpha_n$ and $H_{\partial\Omega_1} > 0$.

Since Ω_1 is covered by finitely many $A(\vec{x}_k)$, there is a constant $K_3 > 0$ such that

$$v \le K_3 \quad \text{in } \bar{\Omega}_1, \quad \forall v \in F. \tag{5.4}$$

Now on $\partial\Omega_1$, we choose a continuous function φ^* as follows: $\varphi^* = K_3$ on $\partial\Omega_1 \cap \Omega$; $\varphi^* = \varphi$ in a neighbourhood of $\vec{x}_2$ in $\partial\Omega_1 \cap \partial\Omega$; and $\varphi^* \ge \varphi$ on the rest of $\partial\Omega_1$. Consider the boundary value problem

$$Qu = 0 \quad \text{in } \Omega_1, \tag{5.5}$$

$$u = \varphi^* \quad \text{on } \partial\Omega_1, \tag{5.6}$$

which has a solution $u_1 \in C^2(\Omega_1) \cap C^0(\bar{\Omega}_1)$ by Lemma 2.2. Therefore, for any $v \in F$ we have $M_O(v) \le u_1$ in Ω_1. Hence, $u \le u_1$ in Ω_1, which together with (5.1), implies

$$v_0 \le u \le u_1 \quad \text{on } \Omega_1.$$

The continuity of u at $\vec{x}_2$ then follows from the fact that $v_0 = u_1 = \varphi$ on a neighbourhood of $\vec{x}_2$ in $\partial\Omega$ and both v_0 and u_1 are continuous in a neighbourhood of $\vec{x}_2$ in $\bar{\Omega}$. Since $\vec{x}_2 \in \partial\Omega$ can be arbitrary, we have proved $u \in C^0(\bar{\Omega})$ and $u = \varphi$ on $\partial\Omega$.

□

Proof of Theorem 1.2: In this case, Ω is inside $C(\theta)$. Replacing w_k and $A(\vec{x}_k)$ by $\tilde{w}_k$ and $\tilde{A}(\vec{x}_k)$ respectively, without changing the rest of the proof of Theorem 1.1, we can obtain Theorem 1.2. □

Appendix A

Lemma A.1 *Let M, d be positive constant. For a sequence $\{a_k\}$, set $\vec{x}_k = (a_k, 0, \cdots, 0)$ and*

$$A(\vec{x}_k) = \{\vec{x} = (x_1, \cdots, x_n) \in C_N(M) |\ M < |\vec{x} - \vec{x}_k| < Me^d, x_1 < a_k\}.$$

Then there exists a $\varepsilon \in (0,1)$ such that if $\{a_k\}$ satisfies

$$a_1 = N, \quad 0 < a_{k+1} - a_k \leq \varepsilon M(e^d - 1), \quad k = 1, 2, \cdots \tag{6.1}$$

then the part of the boundary of $A(\vec{x}_{k+1})$

$$\{\vec{x} = (x_1, \cdots, x_n) \in C_N(M) |\ |\vec{x} - \vec{x}_{k+1}| = Me^d, x_1 < a_{k+1}\}$$

is inside $A(\vec{x}_k)$. Thus,

$$C_N(M) = \bigcup_k A(\vec{x}_k).$$

Proof For $\vec{x} \in \{\vec{x} \in C_N(M) |\ |\vec{x} - \vec{x}_{k+1}| = Me^d, x_1 < a_{k+1}\}$, we have

$$(x_1 - a_{k+1})^2 + \sum_{i=2}^{n} x_i^2 = M^2 e^{2d} \tag{6.2}$$

and

$$x_1 < a_{k+1}. \tag{6.3}$$

We need only to prove that

$$M^2 < (x_1 - a_k)^2 + \sum_{i=2}^{n} x_i^2 < M^2 e^{2d} \tag{6.4}$$

and

$$x_1 \le a_k. \tag{6.5}$$

We first verify (6.5). In fact, by (6.2) and the definition of $C_N(M)$, we have

$$(x_1 - a_{k+1})^2 > M^2 e^{2d} - M^2,$$

which, together with (6.3), implies

$$x_1 < a_{k+1} - M\sqrt{e^{2d} - 1}.$$

In order to prove (6.5), it is sufficient to show that

$$a_{k+1} - a_k < M\sqrt{e^{2d} - 1}, \tag{6.6}$$

which holds true by (6.1) if we choose a small $\varepsilon \in (0, 1)$ such that

$$\varepsilon M(e^d - 1) < M\sqrt{e^{2d} - 1}.$$

Next, we want to prove (6.4). Since $a_k < a_{k+1}$, we have

$$(x_1 - a_k)^2 + \sum_{i=2}^{n} x_i^2 = M^2 e^{2d} + (a_{k+1} - a_k)(2x_1 - a_k - a_{k+1}) < M^2 e^{2d},$$

which is the second inequality in (6.4). The first inequality in (6.4) is reduced to

$$(x_1 - a_k)^2 + \sum_{i=2}^{n} x_i^2 = M^2 e^{2d} + (a_{k+1} - a_k)(2x_1 - a_k - a_{k+1}) > M^2,$$

which is equivalent to

$$x_1 > \frac{1}{2}[a_k + a_{k+1} + \frac{M^2(1 - e^{2d})}{a_{k+1} - a_k}]. \tag{6.7}$$

By the definition of set $\{\vec{x} \in C_N(M) | \, |\vec{x} - \vec{x}_{k+1}| = Me^d, x_1 < a_{k+1}\}$, we have

$$x_1 \ge a_{k+1} - Me^d.$$

Therefore, in order to prove (6.7), it is enough to show

$$a_{k+1} - Me^d > \frac{1}{2}[a_k + a_{k+1} + \frac{M^2(1 - e^{2d})}{a_{k+1} - a_k}],$$

which is equivalent to

$$(a_{k+1} - a_k)^2 - 2Me^d(a_{k+1} - a_k) - M^2(1 - e^{2d}) > 0,$$

i.e.,

$$a_{k+1} - a_k > M(e^d + 1) \ \text{ or } \ a_{k+1} - a_k < M(e^d - 1).$$

The last inequality is obvious by (6.1). Thus, the lemma is completed.

□

Lemma A.2 *Suppose that* $b_1 > 0$, $\theta \in (0, \frac{\pi}{2})$, $d \in (0, \frac{c(H^*,\eta)}{\mu})$ *such that* $1 - e^d \sin\theta > 0$, *and* δ_0 *satisfies*

$$1 < \delta_0 < \frac{1 - \sin\theta}{1 - e^d \sin\theta}. \tag{6.8}$$

Let $b_k = \delta_0^{k-1} b_1$, $L_k = b_k \sin\theta$, $\vec{x}_k = (b_k, 0, \cdots, 0)$ *and*

$$\tilde{A}(\vec{x}_k) = \{\vec{x} = (x_1, x_2, \cdots, x_n) \ \in \ C(\theta) \mid L_k < |\vec{x} - \vec{x}_k| < L_k e^d, \ x_1 < b_k, \}$$

for $k = 1, 2, \cdots$. *Then the part of the boundary of* $\tilde{A}(\vec{x}_k)$,

$$S_k := \{\vec{x} = (x_1, x_2, \cdots, x_n) \in C(\theta) \mid |\vec{x} - \vec{x}_k| = L_k, \ x_1 < b_k, \ L_k = b_k \sin\theta\}$$

is completely covered by $\tilde{A}(\vec{x}_{k+1})$. *Thus,*

$$C(\theta) = \bigcup_k \tilde{A}(\vec{x}_k).$$

Proof Denote

$$T_k := \{\vec{x} \in C(\theta) \mid |\vec{x} - \vec{x}_k| = L_k e^d, \ x_1 < b_k, \ L_k = b_k \sin\theta\}.$$

Obviously, The distances from $(0, 0, \cdots, 0)$ to S_k, T_{k+1}, S_{k+1} are $b_k(1 - \sin\theta)$, $b_{k+1}(1 - e^d \sin\theta)$, $b_{k+1}(1 - \sin\theta)$, respectively. By (6.8), we have

$$b_{k+1}(1 - e^d \sin\theta) < b_k(1 - \sin\theta) < b_{k+1}(1 - \sin\theta).$$

We need only to prove that $S_k \cap T_{k+1} = \emptyset$ and $S_k \cap S_{k+1} = \emptyset$.

At first, we will show that T_{k+1} does not touch S_k for $x_1 \leq b_k$. Indeed, the expressions of S_k and T_{k+1} are

$$\begin{aligned} (x_1 - b_k)^2 + \sum_{i=2}^{n} x_i^2 &= b_k^2 \sin^2\theta, \\ (x_1 - b_k\delta_0)^2 + \sum_{i=2}^{n} x_i^2 &= \delta_0^2 b_k^2 e^{2d} \sin^2\theta, \end{aligned}$$

respectively. Suppose $S_k \cap T_{k+1} \neq \emptyset$, by calculating, we can see that the coordinate on x_1-axis of the intersection point is

$$x_1 = \frac{1}{2} b_k [1 + \delta_0 - \frac{\sin^2\theta(e^{2d} - 1)}{\delta_0 - 1}].$$

We claim that

$$\frac{1}{2} b_k [1 + \delta_0 - \frac{\sin^2\theta(e^{2d} - 1)}{\delta_0 - 1}] < b_k(1 - \sin^2\theta). \tag{6.9}$$

In order to prove the claim (6.9), we need to prove the following inequality,

$$\frac{1 - \sin\theta}{1 - \sin\theta e^d} < \frac{1}{1 - \tan\theta\sqrt{e^{2d} - 1}}. \tag{6.10}$$

In fact, since $0 < \theta < \frac{\pi}{2}$ and $1 < e^d < \frac{1}{\sin\theta}$, then

$$1 - \tan\theta\sqrt{e^{2d} - 1} > 1 - \tan\theta\sqrt{(\frac{1}{\sin\theta})^2 - 1} = 0.$$

Thus, (6.10) is equivalent to

$$(1 - \sin\theta)(1 - \tan\theta\sqrt{e^{2d} - 1}) < 1 - \sin\theta e^d,$$

i.e.,

$$e^d - 1 < \frac{1 - \sin\theta}{\cos\theta}\sqrt{e^{2d} - 1}. \tag{6.11}$$

In order to prove (6.11), it is enough to prove

$$\cos\theta\sqrt{e^d - 1} < (1 - \sin\theta)\sqrt{e^d + 1},$$

i.e.,

$$2\sin\theta(1-\sin\theta)e^d < 2(1-\sin\theta),$$

which is obvious since $e^d < \frac{1}{\sin\theta}$. Therefore, (6.10) holds.

It follows from (6.10) and $1 < \delta_0 < \frac{1-\sin\theta}{1-\sin\theta e^d}$ that $1 < \delta_0 < \frac{1}{1-\tan\theta\sqrt{e^{2d}-1}}$, which implies $(\frac{\delta_0-1}{\delta_0})^2 < \tan^2\theta(e^{2d}-1)$, i.e.,

$$\delta_0^2 - 1 + \sin^2\theta - \sin^2\theta e^{2d}\delta_0^2 < 2\delta_0(1-\sin^2\theta) - 2(1-\sin^2\theta). \qquad (6.12)$$

Since $b_k > 0$ and $\delta_0 > 1$, by (6.12), we obtain (6.9).

However, it is obvious that the coordinate on x_1-axis of any point in S_k is larger than $b_k(1-\sin^2\theta)$. Thus, (6.9) can imply a contradiction with $S_k \cap T_{k+1} \neq \emptyset$, therefore T_{k+1} does not intersect S_k for $x_1 \leq b_k$.

Next, we prove that $S_k \cap S_{k+1} = \emptyset$. Write the expressions of S_k and S_{k+1} as follows:

$$\begin{aligned}(x_1 - b_k)^2 + \sum_{i=2}^n x_i^2 &= b_k^2 \sin^2\theta, \\ (x_1 - b_k\delta_0)^2 + \sum_{i=2}^n x_i^2 &= \delta_0^2 b_k^2 \sin^2\theta,\end{aligned}$$

respectively. Suppose $S_k \cap S_{k+1} \neq \emptyset$. By calculating, we see that the coordinate on x_1-axis of the intersection point is

$$x_1 = \frac{\delta_0 + 1}{2} b_k(1-\sin^2\theta),$$

which is larger than $b_k(1-\sin^2\theta)$ by (6.8),while, $b_k(1-\sin^2\theta)$ is the coordinate on x_1-axis of the intersection of S_k and $\partial C(\theta)$, a contradiction! Therefore, S_k does not intersect S_{k+1} and hence $\tilde{A}(\vec{x}_{k+1})$ covers S_k completely. The lemma has been proven. □

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