

**ON THE CHARACTER DEGREES OF SYLOW p -SUBGROUPS
OF CHEVALLEY GROUP OF TYPE $E(p^f)$**

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ABSTRACT. Let \mathbb{F}_q be a field of characteristic p with q elements. It is known that the degrees of the irreducible characters of the Sylow p -subgroup of $GL_n(\mathbb{F}_q)$ are powers of q , see Isaacs [4]. On the other hand Sangroniz [6] showed that this is true for a Sylow p -subgroup of a classical group defined over \mathbb{F}_q if and only if p is odd. For the classical groups of Lie type B , C and D the only bad prime is 2. For the exceptional groups there are others. In this paper we construct irreducible characters for the Sylow p -subgroups of the Chevalley groups $D_4(q)$ with $q = 2^f$ of degree $q^3/2$. Then we use an analogous construction for $E_6(q)$ with $q = 3^f$ to obtain characters of degree $q^7/3$, and for $E_8(q)$ with $q = 5^f$ to obtain characters of degree $q^{16}/5$. This helps to explain why the primes 2, 3 and 5 are bad for the Chevalley groups of type E in terms of the representation theory of the Sylow p -subgroup.

1. INTRODUCTION

Let G be a Chevalley group defined over a field \mathbb{F}_q of order q and characteristic $p > 0$. By α_0 we denote the highest root of the root system Σ of G . It is well known that α_0 is a positive integral linear combination of the fundamental roots of Σ . So without loss $\alpha_0 = \sum_{i=1}^r a_i \alpha_i$ where the α_i are fundamental roots of Σ . Recall that p is a bad prime for G if p is a divisor of some a_i .

It is well known that if G classical then the only possible bad prime for G is 2. On the other hand if G is exceptional of type E , then the prime 3 is also bad. The “badness” of the prime evidences itself in the classification of the unipotent conjugacy classes of G . Here we aim to explain why the primes 3 and 5 are bad for groups of type E in terms to the representation theory of the Sylow p -subgroup of $G = E_6(q)$ with prime 3 and $G = E_8(q)$ with prime 5. Let $UE_k(q)$ denote the unipotent radical of the standard Borel subgroup of $E_k(q)$ for $k = 6$ and 8; i.e. the subgroup generated by all the positive root groups of G . By U_k we denote the quotient $UE_k(q)/K_{k-1}$, where K_{k-1} is the normal subgroup of $UE_k(q)$ generated by all root groups X_α such that α has height $k - 1$ or more. Clearly any character of U_k inflates to a character of $UE_k(q)$. Abusing terminology slightly we call the image under the natural projection of a root group of $UE_k(q)$, a root group of U_k . We observe that $Z(U_k)$ is generated by the root groups of height $k - 2$ and hence $|Z(U_k)| = q^{k-1}$. We define the family

$$\mathcal{F}_k := \{\chi \in \text{Irr}(U_k) : X_\alpha \not\subset \text{Ker}(\chi) \text{ for all } X_\alpha \subset Z(U_k)\}.$$

Theorem 1.1. *The following are true.*

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- (a) If $q = 3^f$, then for all $\chi \in \mathcal{F}_6$ we have $\chi(1) \in \{q^7, q^7/3\}$. Moreover \mathcal{F}_6 contains exactly $(q-1)^5(q^2 - (q-1)/2)$ characters of degree q^7 and exactly $3^2(q-1)^6/2$ characters of degree $q^7/3$.
- (b) If $q = 5^f$, then for all $\chi \in \mathcal{F}_8$ we have $\chi(1) \in \{q^{16}, q^{16}/5\}$. Moreover \mathcal{F}_8 contains exactly $(q-1)^8(q^3 + q^2 + q + 3/4)$ characters of degree q^{16} and exactly $5^2(q-1)^8/4$ characters of degree $q^{16}/5$.

We remark that $9(q-1)^6/2$, $(q-1)^5(q^2 - (q-1)/2)$, $(q-1)^8(q^3 + q^2 + q + 3/4)$ and $25(q-1)^8/4$ are not in $\mathbb{Z}[q]$. On the other hand we remark also that $|\mathcal{F}_6| = (q-1)^5q^2 \in \mathbb{Z}[q]$ and every character in \mathcal{F}_6 has degree q^7 whenever $p \neq 3$, and that $|\mathcal{F}_8| = (q-1)^7q^4 \in \mathbb{Z}[q]$ and every character in \mathcal{F}_8 has degree q^{16} whenever $p \neq 5$. Taken together these remarks provide evidence for a generalization of Higman's conjecture for groups of type $UE_i(q)$, $i = 6, 7, 8$, see for example [2], namely that $|\text{Irr}(UE_i(q))| \notin \mathbb{Z}[q]$ if and only if p is a bad prime for $E_i(q)$.

To prove our main theorem we begin by analyzing our construction of the irreducible characters of the Sylow 2-subgroup of $D_4(2^f)$ from [3]. Our starting point is the quotient of $UD_4(q)/K_4$ where $UD_4(q)$ is the unipotent radical of the standard Borel subgroup of the universal Chevalley group $D_4(q)$ and K_4 is the normal subgroup of $UD_4(q)$ generated by the root groups of roots of height 4 and 5. We showed that when $p = 2$ we have a family of characters of degree $q^3/2$ of size $4(q-1)^4$. As $UD_4(q)$ is a quotient of $UE_i(q)$ for $i = 6, 7, 8$ we also have families of irreducible characters of degree $q^3/2$ for groups of type $UE_i(q)$, $i = 6, 7, 8$ and q is even.

Our construction is fairly elementary. Starting with large elementary abelian normal subgroups we construct our characters via induction, using Clifford theory. To compute the necessary stabilizers we critically use Proposition 1.3 and Lemma 1.5. Throughout this paper we fix a nontrivial homomorphism $\phi : (\mathbb{F}_q, +) \rightarrow \mathbb{C}^\times$. For each $a \in \mathbb{F}_q$, we define $\phi_a(x) := \phi(ax)$ for all $x \in \mathbb{F}_q$, and denote $\mathbb{F}_q^\times := \mathbb{F}_q - \{1\}$. Hence, $\{\phi_a : a \in \mathbb{F}_q^\times\}$ are all non-principal irreducible characters of \mathbb{F}_q .

Definition 1.2. For $a \in \mathbb{F}_q$, we define $\mathbb{T}_a := \{t^p - a^{p-1}t : t \in \mathbb{F}_q\}$.

We note that $\mathbb{T}_0 = \mathbb{F}_q$.

Proposition 1.3. The following are true.

- (a) $t^p - a^{p-1}t = \prod_{c \in \mathbb{F}_p} (t - ca)$.
- (b) If $a \in \mathbb{F}_q^\times$, then \mathbb{T}_a is an additive subgroup of \mathbb{F}_q of index p .
- (c) For each $a \in \mathbb{F}_q^\times$, there exists $b \in \mathbb{F}_q^\times$ such that $b\mathbb{T}_a = \ker \phi$. Furthermore, $cb\mathbb{T}_a = \ker(\phi)$ iff $c \in \mathbb{F}_p^\times$.
- (d) $\{\mathbb{T}_a : a \in \mathbb{F}_q^\times\} = \{\ker \phi_a : a \in \mathbb{F}_q^\times\}$ are all subgroups of index p in \mathbb{F}_q .

Proof. Part (a) is clear since the degree of the polynomial $t^p - a^{p-1}t$ is p and the \mathbb{F}_p -multiples of a are clearly zeros. As \mathbb{F}_q is of characteristic p , the map $\psi_a : \mathbb{F}_q \rightarrow \mathbb{F}_p$ defined by $\psi_a(t) = t^p - a^{p-1}t$ is \mathbb{F}_p -linear. By Part (a) the kernel of the map is 1-dimensional and thus (b) follows. Evidently (d) follows from (c). We defer the proof (c) to Subsection 5.1. \square

Definition 1.4. For each $a \in \mathbb{F}_q^\times$, we pick a_ϕ such that $a_\phi\mathbb{T}_a = \ker \phi$.

By Proposition 1.3 (c), a_ϕ exists and but is only determined up to a scalar in the prime field. In the definition above we make some arbitrary choice which will not change throughout the paper.

Throughout we fix notation as follows. Let G be a group. $G^\times := G - \{1\}$, $Irr(G)$ the set of all complex irreducible characters of G , and $Irr(G)^\times := Irr(G) - \{1_G\}$. For $H, K \leq G$, and $\xi \in Irr(H)$, define $Irr(G/K) := \{\chi \in Irr(G) : K \subset \ker(\chi)\}$, $Irr(G, \xi) := \{\chi \in Irr(G) : (\chi, \xi^G) \neq 0\}$, and $Irr(G/K, \xi) := Irr(G/K) \cap Irr(G, \xi)$. Furthermore, for a character χ of G , we denote its restriction to H by $\chi|_H$.

Lemma 1.5. *Let $N \trianglelefteq G$ and $1 \in X$ be a transversal of N in G . Suppose $N = ZYM$ where $Y \trianglelefteq N$, $Z \subset Z(N)$, $M \leq N$ and $X \subset N_G(ZY)$. If there is $\lambda \in Irr(ZY)$ such that $Y \subset \ker(\lambda)$, and ${}^u\lambda \neq {}^v\lambda$ for all $u \neq v \in X$, then the following are true.*

- (a) *For all $\chi \in Irr(N/Y, \lambda)$, $\chi^G \in Irr(G)$. Moreover, if $\chi_1 \neq \chi_2 \in Irr(N/Y, \lambda)$, then $\chi_1^G \neq \chi_2^G$.*
- (b) *The induction map from $Irr(N/Y, \lambda)$ to $Irr(G, \lambda)$ is bijective.*

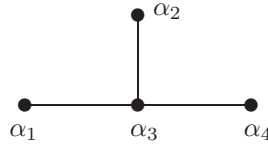
Proof. See Subsection 5.2. \square

We recall that a p -group P is monomial, i.e. for each $\chi \in Irr(P)$, there exist a subgroup H of P and a linear character λ of H such that $\chi = \lambda^P$. To construct irreducible characters whose degrees are not powers of $q = p^f$, $f > 1$ we construct subgroups $H \trianglelefteq P$ and $T \leq P$ such that T is a transversal of H . Then we find a linear character λ of H such that the order of the stabilizer $Stab_T(\lambda)$ of T is not a power of q . Moreover we insure that λ is extendable to the inertial group $I_P(\lambda) = HStab_T(\lambda)$. Let λ_I denote some extension of λ to $I_P(\lambda)$. By Clifford theory the induction of λ_I to P is irreducible, of degree not a power of q . The existence of a suitable pair (H, λ) is based on Proposition 1.3 because a polynomial of the form $x^p + a^{p-1}x$, $a \neq 0$, appears in the formulae of the action of elements of T on the characters of H .

We will now highlight the main steps of the constructions of our characters. We have deferred all of our proofs to Section 5.

2. SYLOW 2-SUBGROUPS OF THE CHEVALLEY GROUPS $D_4(2^f)$

Let \mathbb{F}_q be a field of order q and characteristic 2. Let $\Sigma := \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ be the root system of type D_4 , see Carter [1], Chapter 3. The Dynkin diagram of Σ is



The positive roots are those roots which can be written as linear combinations of the simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with nonnegative coefficients and we write Σ^+ for the set of positive roots. We use the notation $\begin{smallmatrix} 1 \\ 1 \ 2 \ 1 \end{smallmatrix}$ for the root $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$ and we use a similar notation for the remaining positive roots. The 12 positive roots of Σ are given in Table 1.

For $\alpha \in \Sigma$ we denote the corresponding root subgroup of the Chevalley group G by X_α whose elements we label by $x_\alpha(t)$ where $t \in \mathbb{F}_q$. We note that $X_\alpha \cong (\mathbb{F}_q, +)$.

We recall that the commutator formula $[x_\alpha(r), x_\beta(s)] = x_{\alpha+\beta}(-C_{\alpha,\beta}rs)$ if $\alpha+\beta \in \Sigma$, and $= 1$ otherwise, see Carter [1], Theorem 5.2.2. Since $p = 2$ we have $-1 = 1$ in \mathbb{F}_q , so all non-zero coefficients $C_{\alpha,\beta}$ are equal to 1. For positive roots, we use the abbreviation $x_i(t) := x_{\alpha_i}(t)$, $i = 1, 2, \dots, 12$. All nontrivial commutators are given in Table 2.

TABLE 1. Positive roots of the root system Σ of type D_4 .

Height	Roots
5	$\alpha_{12} := \begin{matrix} 1 \\ 1 & 2 & 1 \end{matrix}$
4	$\alpha_{11} := \begin{matrix} 1 \\ 1 & 1 & 1 \end{matrix}$
3	$\alpha_8 := \begin{matrix} 1 \\ 1 & 1 & 0 \end{matrix}$ $\alpha_9 := \begin{matrix} 0 \\ 1 & 1 & 1 \end{matrix}$ $\alpha_{10} := \begin{matrix} 1 \\ 0 & 1 & 1 \end{matrix}$
2	$\alpha_5 := \begin{matrix} 0 \\ 1 & 1 & 0 \end{matrix}$ $\alpha_6 := \begin{matrix} 1 \\ 0 & 1 & 0 \end{matrix}$ $\alpha_7 := \begin{matrix} 0 \\ 0 & 1 & 1 \end{matrix}$
1	α_1 α_2 α_3 α_4

TABLE 2. Commutator relations for type D_4 .

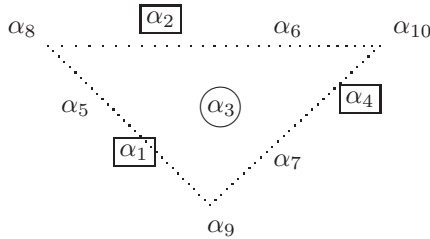
$$\begin{aligned}
[x_1(t), x_3(u)] &= x_5(tu), & [x_1(t), x_6(u)] &= x_8(tu), \\
[x_1(t), x_7(u)] &= x_9(tu), & [x_1(t), x_{10}(u)] &= x_{11}(tu), \\
[x_2(t), x_3(u)] &= x_6(tu), & [x_2(t), x_5(u)] &= x_8(tu), \\
[x_2(t), x_7(u)] &= x_{10}(tu), & [x_2(t), x_9(u)] &= x_{11}(tu), \\
[x_3(t), x_4(u)] &= x_7(tu), & [x_3(t), x_{11}(u)] &= x_{12}(tu), \\
[x_4(t), x_5(u)] &= x_9(tu), & [x_4(t), x_6(u)] &= x_{10}(tu), \\
[x_4(t), x_8(u)] &= x_{11}(tu), & [x_5(t), x_{10}(u)] &= x_{12}(tu), \\
[x_6(t), x_9(u)] &= x_{12}(tu), & [x_7(t), x_8(u)] &= x_{12}(tu)
\end{aligned}$$

The group UD_4 generated by all X_α for $\alpha \in \Sigma^+$ is a Sylow 2-subgroup of the Chevalley group $D_4(q)$. Each element $u \in UD_4$ can be written uniquely as

$$u = x_1(t_1)x_2(t_2)x_4(t_4)x_3(t_3)x_5(t_5) \cdots x_{12}(t_{12}) \text{ where } x_i(t_i) \in X_i.$$

So we write $\prod_{i=1}^{12} x_i(t_i)$ as this order. We note that our ordering of the roots is slightly non-standard as the positions of x_3 and x_4 are reversed.

We define $\mathcal{F}_4 := \{\chi \in \text{Irr}(UD_4(q)) : \chi|_{X_i} = \chi(1)\phi_{a_i} \text{ for each } a_8, a_9, a_{10} \in \mathbb{F}_q^\times\}$. If Ψ is a representation affording $\chi \in \mathcal{F}_4$, then we have $\Psi([x_8(t_8), x_4(t_4)]) = [\Psi(x_8(t_8)), \Psi(x_4(t_4))] = [\phi_{a_8}(t_8)\Psi(1), \Psi(x_4(t_4))] = \Psi(1)$ for all $t_4, t_8 \in \mathbb{F}_q$. Therefore, $X_{11} = [X_8, X_4] \subset \ker(\chi)$. Use the same argument for $X_{12} = [X_8, X_7] \subset \ker(\chi)$. Thus only the factor group $U = UD_4/X_{12}X_{11}$ acts on a module affording χ . Therefore, we may work with U which has order q^{10} , and $Z(U) = X_8X_9X_{10}$.

Figure $UD_4(q)$: Relations of Roots

Let $H := [U, U] = X_5X_6X_7X_8X_9X_{10}$, and $T = X_1X_2X_4$. It is clear that H, HX_3 and T are elementary abelian. The group U can be visualized in the above figure. The roots in boxes are in T , the others outside are in H , and α_3 , which is neither

in T nor in H , is in a circle. The disconnected lines demonstrate the relations of sum of roots equal to roots in center, e.g. $\alpha_2 + \alpha_5 = \alpha_6 + \alpha_1 = \alpha_8, \dots$. Each edge of the triangle contains four roots, two on the outside and two on the inside. Each of the three vertices of the triangle together with the four adjacent inside roots forms a hook of length 2, see [3]. The corresponding hook group is special of order q^{1+4} . The group generated by each hook group q^{1+4} and X_3 is isomorphic to a Sylow 2-subgroup of the general linear group $GL_4(\mathbb{F}_q)$.

To study those characters χ , we start with a linear character λ of H such that $\lambda|_{X_i} \neq 1_{X_i}$ for $i = 8, 9, 10$.

Definition 2.1. For $a_8, a_9, a_{10} \in \mathbb{F}_q^\times$ and $b_5, b_6, b_7 \in \mathbb{F}_q$, we define

- (a) $\lambda_{b_5, b_6, b_7}^{a_8, a_9, a_{10}} (\prod_{i=5}^{10} x_i(t_i)) := \phi(\sum_{i=5}^7 b_i t_i + \sum_{j=8}^{10} a_j t_j)$.
- (b) $S_{567} := \{x_{567}(t) := x_5(a_{10}t)x_6(a_9t)x_7(a_8t) : t \in \mathbb{F}_q\}$.
- (c) $S_{124} := \{x_{124}(t) := x_1(a_{10}t)x_2(a_9t)x_4(a_8t) : t \in \mathbb{F}_q\}$.
- (d) $A := a_8 a_9 a_{10}$ and $t_0 := \frac{1}{A}(b_5 a_{10} + b_6 a_9 + b_7 a_8)$.
- (e) $F_{124} := \{1, x_{124}(t_0)\}$.
- (f) $F_3 := \{1\}$ if $t_0 = 0$, and $F_3 := \{1, x_3(\frac{(t_0)_\phi}{A})\}$ otherwise.

It is easy to check that $S_{567}, S_{124}, F_{124}, F_3$ are subgroups of U . If $t_0 = 0$, then $F_{124} = F_3 = \{1\}$, otherwise $F_{124} \cong F_3 \cong (\mathbb{F}_2, +)$. Since $S_{124}, S_{567} \cong (\mathbb{F}_q, +)$, their linear characters are in the form $\phi_{b_i}(x_i(t)) = \phi(b_i t)$ where $i \in \{124, 567\}$ for all $b_i, t \in \mathbb{F}_q$. For each $\xi \in \text{Irr}(F_{124})$, $\xi = \phi_{b_{124}}|_{F_{124}}$ for some $\phi_{b_{124}} \in \text{Irr}(S_{124})$, $b_{124} \in \mathbb{F}_q$. If F_{124} is nontrivial, we choose $b_{124} \in \{0, a_{124}\} \cong (\mathbb{F}_2, +)$ where $\phi(a_{124}t_0) = -1$. The same for $F_3 \leq X_3$, for each $\xi \in \text{Irr}(F_3)$, $\xi = \phi_{b_3}|_{F_3}$ for some $\phi_{b_3} \in \text{Irr}(X_3)$ and $b_3 \in \{0, a_3\} \cong (\mathbb{F}_2, +)$ such that $\phi(a_3 \frac{(t_0)_\phi}{A}) = -1$ if $(t_0)_\phi$ exists.

For each $a_8, a_9, a_{10} \in \mathbb{F}_q^\times$, there are q^3 linear characters $\lambda_{-, -, -}^{a_8, a_9, a_{10}}$ of H . By the definition of t_0 , there are q^2 of them such that $t_0 = 0$ and the others $q^2(q-1)$ linears such that $t_0 \neq 0$. Therefore, there are q^2 cases where F_{124}, F_3 are trivial and $q^2(q-1)$ cases where F_{124}, F_3 are of order 2.

For all $x_1(t_1)x_2(t_2)x_4(t_4) \in T$, we have

$$x_1(t_1)x_2(t_2)x_4(t_4)(\lambda_{b_5, b_6, b_7}^{a_8, a_9, a_{10}}) = \lambda_{b_5 + a_8 t_2 + a_9 t_4, b_6 + a_8 t_1 + a_{10} t_4, b_7 + a_9 t_1 + a_{10} t_2}^{a_8, a_9, a_{10}}.$$

Hence, T acts on the set of linears $\{\lambda_{-, -, -}^{a_8, a_9, a_{10}}\}$. It is easy to check that t_0 is invariant under this action. All properties of $\lambda_{b_5, b_6, b_7}^{a_8, a_9, a_{10}}$ are known as follows.

Lemma 2.2. Set $\lambda := \lambda_{b_5, b_6, b_7}^{a_8, a_9, a_{10}}$. The following are true.

- (a) $S_{124} = \text{Stab}_T(\lambda)$ and $S_{567} = \{x \in X_5 X_6 X_7 : |\lambda^U(x)| = \lambda^U(1)\}$. Moreover, $\lambda^U|_{S_{567}} = \lambda^U(1)\phi_{A t_0}$.
- (b) λ extends to $H X_3 F_{124}$ and $H F_3 S_{124}$. Let λ_1, λ_2 be extensions of λ to $H X_3 F_{124}$. The inertia groups $I_U(\lambda_1) = H X_3 F_{124}$.
- (c) $\lambda_1^U = \lambda_2^U \in \text{Irr}(U)$ iff $\lambda_1|_{F_3} = \lambda_2|_{F_3}$ and $\lambda_1|_{F_{124}} = \lambda_2|_{F_{124}}$.

Proof. See Subsection 5.3.1.

Remark When q is odd, both $\{x \in X_5 X_6 X_7 : |\lambda^U(x)| = \lambda^U(1) = q^4\}$ and $\text{Stab}_T(\lambda)$ are trivial. Thus, λ extends to $H X_3$ and induces irreducibly to U of degree q^3 .

When $t_0 \neq 0$, the statement in Lemma 2.2 (c) makes sense since the dihedral subgroup $\langle F_{124}, F_3 \rangle \subset I_U(\lambda_1)$. By Lemma 2.2 (b), $X_3, S_{124} \subset I_U(\lambda)$ but λ is not able to extend to $H X_3 S_{124}$ since $[X_3, S_{124}] \not\subseteq \ker(\lambda)$.

By Lemma 2.2 (a), T acts on the set of q^3 linear $\lambda_{-,-,-}^{a_8,a_9,a_{10}}$ into q orbits, each has size q^2 . By Lemma 2.2 (b), all q^3 linear $\lambda_{-,-,-}^{a_8,a_9,a_{10}}$ extend to HX_3 and we obtain q^4 linear extensions, in which there are q^3 linear with $t_0 = 0$ and $q^3(q-1)$ linear with $t_0 \neq 0$.

If $t_0 = 0$, F_{124} is trivial. By Lemma 2.2 (b), λ extends to $I_U(\lambda) = HX_3 \trianglelefteq U$, as η . T is a transversal of HX_3 in U and acts regularly on these q^3 linear η with $t_0 = 0$. Therefore, $\eta^U \in \text{Irr}(U)$ of degree q^3 only depends on a_8, a_9, a_{10} , so we denote it by $\chi_{8,9,10,q^3}^{a_8,a_9,a_{10}} \in \text{Irr}(U)$. This character is the unique $\chi \in \mathcal{F}_4$ of degree q^3 such that $\chi|_{X_i} = \chi(1)\phi_{a_i}$ where $i = 8, 9, 10$. Furthermore, by Lemma 2.2 (a), this is the unique constituent χ of $(\lambda|_{X_8 X_9 X_{10}})^U$ such that $S_{567} \subset \ker(\chi)$.

If $t_0 \neq 0$, F_{124}, F_3 are isomorphic to \mathbb{F}_2 . By Lemma 2.2 (b), λ extends to $HX_3 F_{124}$ as λ_1 , and $\lambda_1^U \in \text{Irr}(U)$ of degree $\frac{q^3}{2}$. For each $t_0 \neq 0$, by Lemma 2.2 (c), all constituents λ_1^U of λ^U only depend on the restrictions of λ_1 to F_{124} and F_3 . Therefore, we denote these constituents of λ^U by $\chi_{8,9,10,\frac{q^3}{2}}^{b_{124},b_3,t_0,a_8,a_9,a_{10}}$ where $b_{124}, b_3 \in \mathbb{F}_2$, $t_0, a_8, a_9, a_{10} \in \mathbb{F}_q^\times$. For each $a_8, a_9, a_{10} \in \mathbb{F}_q^\times$, there are $4(q-1)$ characters $\chi \in \mathcal{F}_4$ of degree $\frac{q^3}{2}$ such that $\chi|_{X_i} = \chi(1)\phi_{a_i}$ where $i = 8, 9, 10$.

The next theorem lists generic character values of all $\chi \in \text{Irr}(U)$ such that $\chi|_{X_i} = \chi(1)\phi_{a_i}$ where $i = 8, 9, 10$.

Theorem 2.3. *For $a_8, a_9, a_{10} \in \mathbb{F}_q^\times$, suppose $\chi \in \text{Irr}(U)$ such that $\chi|_{X_i} = \chi(1)\phi_{a_i}$ where $i = 8, 9, 10$. Set $Z = F_{124}S_{567}X_8X_9X_{10}$ and the Kronecker $\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$. The following are true.*

(a) *If $\chi(1) = q^3$, then $\chi = \chi_{8,9,10,q^3}^{a_8,a_9,a_{10}}$ and*

$$\chi(\prod_{i=1}^{10} x_i(t_i)) = \delta_{0,t_1} \delta_{0,t_2} \delta_{0,t_4} \delta_{0,t_3} \delta_{a_8 t_5, a_{10} t_7} \delta_{a_8 t_6, a_9 t_7} q^3 \phi(\sum_{i=8}^{10} a_i t_i).$$

(b) *If $\chi(1) = \frac{q^3}{2}$, then $\chi = \chi_{8,9,10,\frac{q^3}{2}}^{b_{124},b_3,t_0,a_8,a_9,a_{10}}$ for some $b_{124}, b_3 \in \mathbb{F}_2$, $t_0 \in \mathbb{F}_q^\times$*

$$\text{and } \chi(\prod_{i=1}^{10} x_i(t_i)) =$$

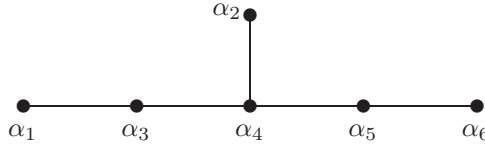
$$\begin{cases} \frac{q^3}{2} \phi(b_{124} \frac{t_1}{a_{10}} + At_0 \frac{t_7}{a_8} + \sum_{i=8}^{10} a_i t_i) & \text{if } \prod_{i=1}^{10} x_i(t_i) \in Z, \text{ and} \\ \delta_{a_8 t_1, a_{10} t_4} \delta_{a_8 t_2, a_9 t_4} \delta_{t_3, t_0} \frac{q^2}{2} \phi(b_{124} \frac{t_1}{a_{10}} + b_3 t_3 + At_0 \frac{t_7}{a_8} + (*) + \sum_{i=8}^{10} a_i t_i) & \text{otherwise,} \end{cases}$$

$$\text{where } t_0^\phi = \frac{(t_0)\phi}{A} \text{ and } (*) = \frac{A^2}{(t_0)\phi} (\frac{t_5}{a_{10}} + \frac{t_7}{a_8}) (\frac{t_6}{a_9} + \frac{t_7}{a_8}).$$

Proof. See Subsection 5.3.2.

3. SYLOW 3-SUBGROUPS OF THE CHEVALLEY GROUPS $E_6(3^f)$

Let \mathbb{F}_q be a field of order q and characteristic 3. We study $E_6(q)$ by its Lie root system. Let $\Sigma := \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \rangle$ be the root system of E_6 , see Carter [1], Chapter 3. The Dynkin diagram of Σ is



The positive roots are those roots which can be written as integral linear combinations of the simple roots $\alpha_1, \alpha_2, \dots, \alpha_6$ with nonnegative coefficients. We write

Σ^+ for the set of positive roots. Here, $|\Sigma^+| = 36$. We use the notation $\begin{smallmatrix} 2 \\ 1\ 2\ 3\ 2\ 1 \end{smallmatrix}$ for the root $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ and we use a similar notation for the remaining positive roots. Let $X_\alpha := \langle x_\alpha(t) \mid t \in \mathbb{F}_q \rangle$ be the root subgroup corresponding to $\alpha \in \Sigma$. The group generated by all X_α for $\alpha \in \Sigma^+$ is a Sylow 3-subgroup of the Chevally group $E_6(q)$, which we call UE_6 .

In this section, we are going to construct irreducible characters χ of degree $\frac{q^7}{3}$ by considering the following special family of irreducible characters of UE_6 .

$$\mathcal{F}_6 := \{ \chi \in \text{Irr}(UE_6) : \chi|_{X_\alpha} = \chi(1)\phi_a, \text{ ht}(\alpha) = 4, a \in \mathbb{F}_q^\times \}.$$

Let ψ be an affording representation of some $\chi \in \mathcal{F}_6$. Using the same argument as in Section 2 for all positive roots α with height greater than 4 to obtain $X_\alpha \subset \ker(\chi)$. Let K_5 be the normal subgroup of UE_6 generated by all root subgroups of height greater than 4. Thus only the factor group $U := UE_6/K_5$ acts on a module affording χ . By the nature of the canonical map from UE_6 to U , we can identify all root groups of root heights less than or equal 4 to their image groups. There are 21 roots $\alpha \in \Sigma^+$ with $\text{ht}(\alpha) \leq 4$. These 21 positive roots are given in Table 3. Therefore, the group U has order q^{21} and $Z(U) = X_{17}X_{18}X_{19}X_{20}X_{21} = \langle X_\beta : \text{ht}(\beta) = 4 \rangle$.

For positive roots, we use the abbreviation $x_i(t) = x_{\alpha_i}(t)$, $i = 1, 2, \dots, 21$. Each element $u \in U$ can be written uniquely as

$$u = x_2(t_2)x_1(t_1)x_3(t_3)x_4(t_4)x_5(t_5) \cdots x_{21}(t_{21}) \text{ where } x_i(t_i) \in X_i.$$

So we write $\prod_{i=1}^{21} x_i(t_i)$ as this order. It is noted that there is a permutation of x_2 .

 TABLE 3. Positive roots of the root system Σ of type E_6 .

Height	Roots
4	$\alpha_{20} := \begin{smallmatrix} 1 \\ 0\ 0\ 1\ 1\ 1 \end{smallmatrix}$ $\alpha_{21} := \begin{smallmatrix} 0 \\ 0\ 1\ 1\ 1\ 1 \end{smallmatrix}$
	$\alpha_{17} := \begin{smallmatrix} 1 \\ 1\ 1\ 1\ 0\ 0 \end{smallmatrix}$ $\alpha_{18} := \begin{smallmatrix} 0 \\ 1\ 1\ 1\ 1\ 0 \end{smallmatrix}$ $\alpha_{19} := \begin{smallmatrix} 1 \\ 0\ 1\ 1\ 1\ 0 \end{smallmatrix}$
	$\alpha_{15} := \begin{smallmatrix} 0 \\ 0\ 1\ 1\ 1\ 0 \end{smallmatrix}$ $\alpha_{16} := \begin{smallmatrix} 0 \\ 0\ 0\ 1\ 1\ 1 \end{smallmatrix}$
3	$\alpha_{12} := \begin{smallmatrix} 0 \\ 1\ 1\ 1\ 0\ 0 \end{smallmatrix}$ $\alpha_{13} := \begin{smallmatrix} 1 \\ 0\ 1\ 1\ 0\ 0 \end{smallmatrix}$ $\alpha_{14} := \begin{smallmatrix} 1 \\ 0\ 0\ 1\ 1\ 0 \end{smallmatrix}$
	$\alpha_{10} := \begin{smallmatrix} 0 \\ 0\ 0\ 1\ 1\ 0 \end{smallmatrix}$ $\alpha_{11} := \begin{smallmatrix} 0 \\ 0\ 0\ 0\ 1\ 1 \end{smallmatrix}$
2	$\alpha_7 := \begin{smallmatrix} 0 \\ 1\ 1\ 0\ 0\ 0 \end{smallmatrix}$ $\alpha_8 := \begin{smallmatrix} 1 \\ 0\ 0\ 1\ 0\ 0 \end{smallmatrix}$ $\alpha_9 := \begin{smallmatrix} 0 \\ 0\ 1\ 1\ 0\ 0 \end{smallmatrix}$
	α_2 α_1 α_3 α_4 α_5 α_6

For each $\alpha \in \Sigma$, since the lengths of α -chains of roots through a root are at most 1, the commutator formula $[x_\alpha(r), x_\beta(s)] = x_{\alpha+\beta}(-C_{\alpha,\beta}rs)$ if $\alpha + \beta \in \Sigma$, and $= 1$ otherwise, see Carter [1], Theorem 5.2.2. For each extraspecial pair (α, β) , we choose the coefficient $C_{\alpha,\beta} := -1$. By computing directly or using MAGMA [5] with the following codes, all nontrivial commutators are given in Table 4.

```
W:=RootDatum("E6");
R:=PositiveRoots(W); A:=R[1..21];
```

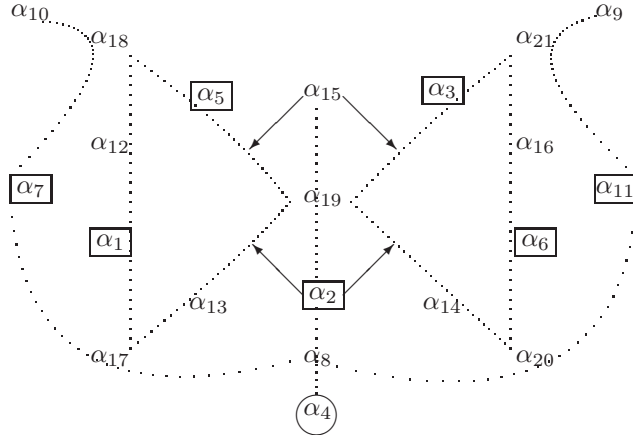
```

for i in [7..21] do
  for j in [1..(i-1)] do
    if (R[i]-R[j]) in A then
      k:=RootPosition(W,R[i]-R[j]);
      if k le j then print k,"+"",j,"="",i,"(",LieConstant.C(W,1,1,k,j),")"; end if;
    end if;
  end for;
end for;

```

TABLE 4. Commutator relations for type E_6 .

$[x_1(t), x_3(u)] = x_7(tu),$	$[x_2(t), x_4(u)] = x_8(tu),$	$[x_3(t), x_4(u)] = x_9(tu),$
$[x_4(t), x_5(u)] = x_{10}(tu),$	$[x_5(t), x_6(u)] = x_{11}(tu),$	$[x_1(t), x_9(u)] = x_{12}(tu),$
$[x_4(t), x_7(u)] = x_{12}(-tu),$	$[x_2(t), x_9(u)] = x_{13}(tu),$	$[x_3(t), x_8(u)] = x_{13}(tu),$
$[x_2(t), x_{10}(u)] = x_{14}(tu),$	$[x_5(t), x_8(u)] = x_{14}(-tu),$	$[x_3(t), x_{10}(u)] = x_{15}(tu),$
$[x_5(t), x_9(u)] = x_{15}(-tu),$	$[x_4(t), x_{11}(u)] = x_{16}(tu),$	$[x_6(t), x_{10}(u)] = x_{16}(-tu),$
$[x_1(t), x_{13}(u)] = x_{17}(tu),$	$[x_7(t), x_8(u)] = x_{17}(tu),$	$[x_2(t), x_{12}(u)] = x_{17}(tu),$
$[x_1(t), x_{15}(u)] = x_{18}(tu),$	$[x_7(t), x_{10}(u)] = x_{18}(tu),$	$[x_5(t), x_{12}(u)] = x_{18}(-tu),$
$[x_2(t), x_{15}(u)] = x_{19}(tu),$	$[x_3(t), x_{14}(u)] = x_{19}(tu),$	$[x_5(t), x_{13}(u)] = x_{19}(-tu),$
$[x_2(t), x_{16}(u)] = x_{20}(tu),$	$[x_8(t), x_{11}(u)] = x_{20}(tu),$	$[x_6(t), x_{14}(u)] = x_{20}(-tu),$
$[x_3(t), x_{16}(u)] = x_{21}(tu),$	$[x_9(t), x_{11}(u)] = x_{21}(tu),$	$[x_6(t), x_{15}(u)] = x_{21}(-tu).$

Figure $UE_6(q)$: Relations of Roots.

Let $H := \langle X_\alpha : \alpha \neq \alpha \in \Sigma^+, (\alpha, \alpha_4) > 0 \rangle = H_4 H_3 H_2$ where $H_4 := Z(U)$, $H_3 := \prod_{i=12}^{16} X_i$, $H_2 := \prod_{i=8}^{10} X_i$, and $T := \langle X_2, X_1, X_3, X_5, X_6 \rangle = X_2 X_1 X_3 X_7 X_5 X_6 X_{11}$. It is clear that $|H| = q^{13}$, $|T| = q^7$, H_k is generated by all root groups of root height k in H , and T is a transversal of HX_4 in U . Both H and HX_4 are elementary abelian and normal in U , and T is isomorphic to $UA_2(q) \times UA_2(q) \times UA_1(q)$, where $UA_k(q)$ is the unipotent subgroup of the standard Borel subgroup of the general linear group $GL_{k+1}(q)$. We can visualize the group U in the above figure. The roots in boxes are in T , the others outside are in H , and α_4 , which is neither in H nor in T , is in a circle. The disconnected lines demonstrate the relations between roots to give a sum root in center, e.g. $\alpha_7 + \alpha_{10} = \alpha_{18}$, $\alpha_7 + \alpha_8 = \alpha_{17}$... In addition, we have two triangles, as same as in Section 2 of $UD_4(q)$, namely $(\alpha_{17}, \alpha_{18}, \alpha_{19})$ and

$(\alpha_{19}, \alpha_{20}, \alpha_{21})$. These two triangles share a common pair of roots (α_2, α_{15}) where $\alpha_2 + \alpha_{15} = \alpha_{19}$.

We consider $\lambda \in \text{Irr}(H)$ such that $\lambda|_{X_i} = \phi_{a_i} \neq 1_{X_i}$ for $17 \leq i \leq 21$. Since the maximal split torus of $E_6(q)$ acts transitively on $\bigoplus_{i=17}^{21} \text{Irr}(X_i)^\times$, it is allowed to assume that $\lambda|_{X_i} = \phi$ for $17 \leq i \leq 21$. So we set $\lambda = \lambda_{b_8, b_9, b_{10}}^{b_{12}, b_{13}, b_{14}, b_{15}, b_{16}} \in \text{Irr}(H)$ such that $\lambda|_{X_i} = \phi_{b_i}$ where $b_i \in \mathbb{F}_q$ for all $8 \leq i \leq 16, i \neq 11$.

Definition 3.1. For $b_8, b_9, b_{10}, b_{12}, b_{13}, b_{14}, b_{15}, b_{16} \in \mathbb{F}_q$, we define

- (a) $S_1 := \{s_1(t, r, s) := x_2(t)x_1(t)x_3(-t)x_5(t)x_6(-t)x_7(r)x_{11}(s) : t, r, s \in \mathbb{F}_q\}$.
- (b) $S_2 := \{s_2(t) := s_1(t, 2t^2, 2t^2) : t \in \mathbb{F}_q\}$.
- (c) $R_3 := \{r_3(t) := x_{12}(t)x_{13}(-t)x_{14}(-t)x_{15}(t)x_{16}(t) : t \in \mathbb{F}_q\}$.
- (d) $R_2 := \{r_2(t) := x_8(-t)x_9(t)x_{10}(t) : t \in \mathbb{F}_q\}$.
- (e) $B_3 := b_{12} - b_{13} - b_{14} + b_{15} + b_{16}$.
- (f) $B_2 := b_{10} + b_9 - b_8$.
- (g) If $B_2 = c^2 \in \mathbb{F}_q^\times$, $F_2 := \{1, s_2(\pm c)\}$ and $F_4 := \{1, x_4(\pm c\phi)\}$.

We note that $R_k \leq H_k$ for $k = 2, 3$, $F_2 \leq S_2 \leq S_1 \leq T$, and $F_4 \leq X_4$. Since $R_k \cong \mathbb{F}_q$, for each $a \in \mathbb{F}_q$ we define $\phi_a(r_k(t)) = \phi_a(t)$ for all $r_k(t) \in R_k$. Hence, $\text{Irr}(R_k) = \{\phi_a : a \in \mathbb{F}_q\}$. Since $S_2 \cong \mathbb{F}_q$, we can define $\phi_a(s_2(t)) = \phi_a(t)$ for all $s_2(t) \in S_2$. When $B_2 = c^2 \in \mathbb{F}_q^\times$, for each linear $\xi \in \text{Irr}(F_2)$ there is $b_2 \in \{0, \pm a_2\} \cong (\mathbb{F}_3, +)$ such that $\xi = \phi_{b_2}|_{F_2}$ where $\phi_{b_2} \in \text{Irr}(S_2)$ and $\phi(a_2c) \neq 1$. Use the same argument for F_4 , for each $\xi \in \text{Irr}(F_4)$ there is $b_4 \in \{0, \pm a_4\} \cong (\mathbb{F}_3, +)$ such that $\xi = \phi_{b_4}|_{F_4}$, where $\phi_{b_4} \in \text{Irr}(X_4)$ and $\phi(a_4c\phi) \neq 1$.

Let $\overline{H_3}$ be the normal closure of H_3 in HX_4S_1 . Since HX_4 is abelian, $X_4 \subset \text{Stab}_U(\lambda)$. All properties of $\lambda = \lambda_{b_8, b_9, b_{10}}^{b_{12}, b_{13}, b_{14}, b_{15}, b_{16}}$ are known as follows.

Lemma 3.2. The following are true

- (a) $R_3 = \{x \in H_3 : |\lambda^U(x)| = \lambda^U(1)\}$ and $S_1 = \text{Stab}_T(\lambda|_{H_4H_3})$. Moreover, $\lambda^U|_{R_3} = \lambda^U(1)\phi_{B_3}$.
- (b) If $B_3 \neq 0$, then $\text{Stab}_T(\lambda) = \{1\}$. Hence, if η is an extension of λ to HX_4 , then $I_U(\eta) = HX_4$.
- (c) If $B_3 = 0$, then there exists $x \in T$ such that ${}^x\lambda = \lambda_{b'_8, b'_9, b'_{10}}^{0, 0, 0, 0, 0}$ for some $b'_8, b'_9, b'_{10} \in \mathbb{F}_q$. Furthermore, $\overline{H_3} \subset \ker({}^x\lambda)^{HX_4S_1}$ and the induction map from $\text{Irr}(HX_4S_1, {}^x\lambda)$ to $\text{Irr}(U, \lambda)$ is bijective.

Proof. See Subsection 5.4.1. \square

Remark If $\gcd(q, 3) = 1$, then $\{x \in H_3 : |\lambda^U(x)| = \lambda^U(1)\}$ and $\text{Stab}_T(\lambda)$ are trivial. Thus λ extends to HX_4 and hence induces up to U irreducibly.

By Lemma 3.2 (a), it is easy to see that T acts invariant on $B_3 = B_3(\lambda)$, i.e. $B_3(\lambda) = B_3({}^x\lambda)$ for all $x \in T$. As above we fix the actions of $\lambda|_{X_i} = \phi, 17 \leq i \leq 21$, H has q^8 linears, in which there are q^7 linears with $B_3 = 0$ and $q^7(q-1)$ linears with $B_3 \neq 0$.

By Lemma 3.2 (b), these $q^7(q-1)$ linears of H with $B_3 \neq 0$ extend to HX_4 to be $q^8(q-1)$ linears and induce irreducibly to U of degree $[U : HX_4] = q^7$. Therefore, there are $\frac{q^8(q-1)}{q^7} = q(q-1)$ irreducibles in this case and they are parametrized by (b_4, B_3) . So we denote them by $\chi_{q^7}^{b_4, B_3}$ where $b_4 \in \mathbb{F}_q$ and $B_3 \in \mathbb{F}_q^\times$.

Since $H \trianglelefteq U$, we have $\lambda, {}^x\lambda \in \text{Irr}(H)$ and $\text{Irr}(U, \lambda) = \text{Irr}(U, {}^x\lambda)$ for all $x \in T$, hence, by Lemma 3.2 (c), we suppose that $\lambda = \lambda_{b_8, b_9, b_{10}}^{0, 0, 0, 0, 0}$. Since $[U : HX_4S_1] = q^4$

and the induction map from HX_4S_1 to U is irreducible from all constituents of $Irr(HX_4S_1, \lambda)$, the above q^7 linears of H with $B_3 = 0$ are corresponding with these q^3 linears $\lambda_{b_8, b_9, b_{10}}^{0,0,0,0,0}$ when we observe them at level of HX_4S_1 .

Lemma 3.3. *The following are true.*

- (a) $R_2 = \{x \in H_2 : |\lambda^{HX_4S_1}(x)| = \lambda^{HX_4S_1}(1)\}$ and $S_2 = \text{Stab}_{S_1}(\lambda)$. Moreover, $\lambda^{HX_4S_1}|_{R_2} = \lambda^{HX_4S_1}(1)\phi_{B_2}$.
- (b) If $B_2 \notin \{c^2 : c \in \mathbb{F}_q^\times\}$ and let η be an extension of λ to HX_4 , then $I_{HX_4S_1}(\eta) = HX_4$. Therefore, S_2 acts transitively and faithfully on all extensions of λ to HX_4 .
- (c) If $B_2 = c^2 \in \mathbb{F}_q^\times$, then λ extends to HX_4F_2 and HF_4S_2 . Let λ_1, λ_2 be extensions of λ to HX_4F_2 . Then $I_{HX_4S_1}(\lambda_1) = HX_4F_2$. Moreover, $\lambda_1^{HX_4S_1} = \lambda_2^{HX_4S_1}$ iff $\lambda_1|_{F_2} = \lambda_2|_{F_2}$ and $\lambda_1|_{F_4} = \lambda_2|_{F_4}$.

Proof. See Subsection 5.4.2. \square

Remark When $B_2 = c^2 \neq 0$, we see that $HX_4F_3 \trianglelefteq U$ and $HF_4S_2 \not\trianglelefteq U$, and both have index $\frac{q^7}{3}$ in U . By Lemma 3.2(c) and Lemma 3.3 (c) all constituents of λ^U have degree $\frac{q^7}{3}$. Hence, if η is an extension of λ to HF_4S_2 , then $\eta^U \in Irr(U, \lambda)$. We have $X_4, S_2 \subset I_U(\lambda)$ and λ extends to HX_4F_3 and HF_4S_2 , but λ is not able to extend to HX_4S_2 .

The group HX_4 has q^4 linear characters λ such that $\lambda|_H = \lambda_{b_8, b_9, b_{10}}^{0,0,0,0,0}$. Since \mathbb{F}_q^\times is even and cyclic, there are $\frac{q^3(q+1)}{2}$ linears with $B_2 \notin \{c^2 : c \in \mathbb{F}_q^\times\}$, and $\frac{q^3(q-1)}{2}$ linears with $B_2 \in \{c^2 : c \in \mathbb{F}_q^\times\}$. Hence, by Lemma 3.3 (b), there are $\frac{q^3(q+1)}{2|S_1|} = \frac{q+1}{2}$ irreducibles of degree $|S_1| = q^3$ which are parametrized by $B_2 \notin \{c^2 : c \in \mathbb{F}_q^\times\}$. By Lemma 3.2 (c), we obtain $\frac{q+1}{2}$ irreducibles of degree $q^3[U : HX_4S_1] = q^7$ which are denoted by $\chi_{q^7}^{B_2}$ where $B_2 \in \mathbb{F}_q - \{c^2 : c \in \mathbb{F}_q^\times\}$. Therefore, together with characters $\chi_{q^7}^{b_4, B_3}$ as computed above, \mathcal{F}_6 has exactly $(q-1)q + \frac{q+1}{2}$ irreducible characters χ of degree q^7 such that $\chi|_{X_i} = \chi(1)\phi$ for all $X_i \subset Z(U)$.

By Lemma 3.3 (c), let λ_1 be an extension of λ to HX_4F_2 , then $\lambda_1^{HX_4S_1}$ is irreducible of degree $[HX_4S_1 : HX_4F_2] = \frac{q^3}{3}$. These $\lambda_1^{HX_4S_1}$ only depend on B_2 and their restrictions to F_2, F_4 . Hence, by Lemma 3.2 (c), $\lambda_1^U \in Irr(U)$ of degree $\frac{q^7}{3}$ is denoted by $\chi_{\frac{q^7}{3}}^{b_2, b_4, B_2}$ where $b_2, b_4 \in \mathbb{F}_3$ and $B_2 \in \{c^2 : c \in \mathbb{F}_q^\times\}$. Therefore, \mathcal{F}_6 has exactly $\frac{9(q-1)}{2}$ irreducibles of degree $\frac{q^7}{3}$ such that $\chi|_{X_i} = \chi(1)\phi$ for all $X_i \subset Z(U)$.

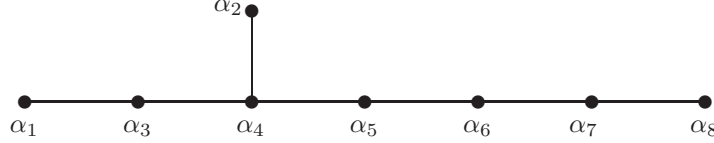
By the transitivity of the conjugate action of the maximal split torus T_0 of the Chevalley group $E_6(q)$ on $\bigoplus_{i=17}^{21} Irr(X_i)^\times$, there are $(q-1)^5(q^2 - q + \frac{q+1}{2})$ characters $\chi \in \mathcal{F}_6$ of degree q^7 , and $\frac{9(q-1)^6}{2}$ characters $\chi \in \mathcal{F}_6$ of degree $\frac{q^7}{3}$ such that $\chi|_{X_i} = \chi(1)\phi_{a_i}$, where $a_i \in \mathbb{F}_q^\times$, $17 \leq i \leq 21$. This gives the proof for the next theorem.

Theorem 3.4. *Let $\chi \in \mathcal{F}_6$. The following are true.*

- (a) If $\chi(1) = q^7$, then there exists $t \in T_0$ such that ${}^t\chi$ is either $\chi_{q^7}^{b_4, B_3}$ or $\chi_{q^7}^{B_2}$, for some $b_4 \in \mathbb{F}_q$, $B_3 \in \mathbb{F}_q^\times$, and $B_2 \in \mathbb{F}_q - \{c^2 : c \in \mathbb{F}_q^\times\}$.
- (b) If $\chi(1) = \frac{q^7}{3}$, then there exists $t \in T_0$ such that ${}^t\chi = \chi_{\frac{q^7}{3}}^{b_2, b_4, B_2}$, for some $b_3, b_4 \in \mathbb{F}_3$ and $B_2 \in \{c^2 : c \in \mathbb{F}_q^\times\}$.

4. SYLOW 5-SUBGROUPS OF THE CHEVALLEY GROUPS $E_8(5^f)$

Let \mathbb{F}_q be a field of order q and characteristic 5. We study $E_8(q)$ by its Lie root system. Let $\Sigma := \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \rangle$ be the root system of E_8 , see Carter [1], Chapter 3. The Dynkin diagram of Σ is



The positive roots are those roots which can be written as linear combinations of the simple roots $\alpha_1, \alpha_2, \dots, \alpha_8$ with nonnegative coefficients and we write Σ^+ for the set of positive roots. Here, $|\Sigma^+| = 120$. We use the notation $\begin{smallmatrix} 3 \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 \end{smallmatrix}$ for the root $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$ and we use a similar notation for the remaining positive roots. Let $X_\alpha := \langle x_\alpha(t) \mid t \in \mathbb{F}_q \rangle$ be the root subgroup corresponding to $\alpha \in \Sigma$. The group generated by all X_α for $\alpha \in \Sigma^+$ is a Sylow 5-subgroup of the Chevalley group $E_8(q)$, which we call UE_8 .

In this section, we are going to construct irreducible characters χ of degree $\frac{q^{16}}{5}$ by considering the following special family of irreducible characters of UE_8 .

$$\mathcal{F}_8 := \{ \chi \in \text{Irr}(UE_8) : \chi|_{X_\alpha} = \chi(1)\phi_\alpha, \text{ ht}(\alpha) = 6, a \in \mathbb{F}_q^\times \}.$$

Let ψ be an affording representation of some $\chi \in \mathcal{F}_8$. Using the same argument as in Section 2 for all positive roots α with height greater than 6 to obtain $X_\alpha \subset \ker(\chi)$. Let K_7 be the normal subgroup of UE_8 generated by all root subgroups of root heights greater than 6. Thus only the factor group $U := UE_8/K_7$ acts on a module affording χ . By the nature of the canonical map from UE_8 to U , we can identify all root groups of root heights less than or equal 6 to their image groups. There are 43 positive roots of height less than or equal 6. These 43 roots are given in Table 5.

For positive roots, we use the abbreviation $x_i(t) = x_{\alpha_i}(t)$, $i = 1, 2, \dots, 43$. Hence, $Z(U) = X_{37}X_{38}X_{39}X_{40}X_{41}X_{42}X_{43} = \langle X_\beta : \text{ht}(\beta) = 6 \rangle$. Each element $u \in U$ can be written uniquely as

$$u = x_2(t_2)x_1(t_1)x_3(t_3)x_4(t_4)x_5(t_5) \cdots x_{43}(t_{43}) \text{ where } x_i(t_i) \in X_i.$$

So we write $\prod_{i=1}^{43} x_i(t_i)$ as this order. It is noted that there is a permutation of x_2 .

For $\alpha, \beta \in \Sigma$, the commutator formula $[x_\alpha(r), x_\beta(s)] = x_{\alpha+\beta}(-C_{\alpha,\beta}rs)$ if $\alpha+\beta \in \Sigma$, $= 1$ otherwise, see Carter [1], Theorem 5.2.2. For each extraspecial pair (α, β) , we choose the coefficient $C_{\alpha,\beta} := -1$. By computing directly or using MAGMA [5], all nontrivial commutators are given in Table 6.

Let $H := \langle X_\alpha : \alpha_4 \neq \alpha \in \Sigma^+, (\alpha, \alpha_5) > 0 \rangle = H_6H_5H_4H_3H_2$ where $H_6 = Z(U)$, $H_5 = \prod_{i=30}^{36} X_i$, $H_4 = \prod_{i=24}^{29} X_i$, $H_3 = \prod_{i=18}^{21} X_i$ and $H_2 = X_{12}X_{13}$. Let $T := \langle X_1, X_3, X_4, X_2, X_6, X_7, X_8 \rangle = T_4T_3T_2T_1$ where $T_4 = X_{23}$, $T_3 = X_{16}X_{17}X_{22}$, $T_2 = X_9X_{10}X_{11}X_{14}X_{15}$ and $T_1 = X_1X_3X_4X_2X_6X_7X_8$. It is clear that $|H| = q^{26}$, $|T| = q^{16}$, H_k is generated by all root groups in H of root height k , as same as for T_k generated by all root subgroups in T of root height k , and T is a transversal of HX_5 in U . Both H and HX_5 are elementary abelian and normal in U . T is isomorphic to $UA_4(q) \times UA_3(q)$, where $UA_k(q)$ is the unipotent subgroup of the standard Borel subgroup of the general linear group $GL_{k+1}(q)$. It is noted that by letting $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ be a simple root set of type A_4 , the isomorphism from

TABLE 5. Positive roots of the root system Σ of type \widetilde{E}_8 .

Height	Roots
6	$\alpha_{43} := \begin{matrix} 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}$
	$\alpha_{40} := \begin{matrix} 1 \\ 0 & 1 & 2 & 1 & 1 & 0 & 0 \end{matrix}$ $\alpha_{41} := \begin{matrix} 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{matrix}$ $\alpha_{42} := \begin{matrix} 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}$
	$\alpha_{37} := \begin{matrix} 1 \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 \end{matrix}$ $\alpha_{38} := \begin{matrix} 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{matrix}$ $\alpha_{39} := \begin{matrix} 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{matrix}$
5	$\alpha_{36} := \begin{matrix} 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{matrix}$
	$\alpha_{33} := \begin{matrix} 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{matrix}$ $\alpha_{34} := \begin{matrix} 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{matrix}$ $\alpha_{35} := \begin{matrix} 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{matrix}$
	$\alpha_{30} := \begin{matrix} 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{matrix}$ $\alpha_{31} := \begin{matrix} 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{matrix}$ $\alpha_{32} := \begin{matrix} 1 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \end{matrix}$
4	$\alpha_{29} := \begin{matrix} 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{matrix}$
	$\alpha_{26} := \begin{matrix} 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{matrix}$ $\alpha_{27} := \begin{matrix} 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{matrix}$ $\alpha_{28} := \begin{matrix} 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{matrix}$
	$\alpha_{23} := \begin{matrix} 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{matrix}$ $\alpha_{24} := \begin{matrix} 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{matrix}$ $\alpha_{25} := \begin{matrix} 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{matrix}$
3	$\alpha_{22} := \begin{matrix} 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{matrix}$
	$\alpha_{19} := \begin{matrix} 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{matrix}$ $\alpha_{20} := \begin{matrix} 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{matrix}$ $\alpha_{21} := \begin{matrix} 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{matrix}$
	$\alpha_{16} := \begin{matrix} 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{matrix}$ $\alpha_{17} := \begin{matrix} 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{matrix}$ $\alpha_{18} := \begin{matrix} 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{matrix}$
2	$\alpha_{15} := \begin{matrix} 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{matrix}$
	$\alpha_{12} := \begin{matrix} 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{matrix}$ $\alpha_{13} := \begin{matrix} 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{matrix}$ $\alpha_{14} := \begin{matrix} 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{matrix}$
	$\alpha_9 := \begin{matrix} 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{matrix}$ $\alpha_{10} := \begin{matrix} 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{matrix}$ $\alpha_{11} := \begin{matrix} 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{matrix}$
1	α_2 α_1 α_3 α_4 α_5 α_6 α_7 α_8

$\langle X_1, X_3, X_4, X_2 \rangle$ to $UA_4(q)$ sends $x_1(t)$ to $x_{\beta_1}(t)$, $x_3(t)$ to $x_{\beta_2}(t)$, $x_4(t)$ to $x_{\beta_3}(t)$, and $x_2(t)$ to $x_{\beta_4}(-t)$ for all $t \in \mathbb{F}_q$.

We consider linear characters $\lambda \in \text{Irr}(H)$ such that $\lambda|_{X_i} = \phi_{a_i}$ for $37 \leq i \leq 43$ and $\lambda|_{X_j} = \phi_{b_j}$ for all appropriate $j \leq 36$ where $a_i \in \mathbb{F}_q^\times$ and $b_j \in \mathbb{F}_q$. Since the maximal split torus of the Chevalley group $E_8(q)$ acts transitively on $\otimes_{i=37}^{43} \text{Irr}(X_i)^\times$, it suffices to suppose that $\lambda|_{X_i} = \phi$ for all $37 \leq i \leq 43$.

Definition 4.1. For $b_i \in \mathbb{F}_q$ where $i \in [12..13, 18..21, 24..36]$ we define

- (a) $B_5 := b_{30} + b_{31} - b_{32} - b_{33} - 2b_{34} + 2b_{35} + 2b_{36}$.
- (b) $B_4 := 2b_{24} - 2b_{25} + b_{26} - b_{27} - b_{28} + b_{29}$.
- (c) $B_3 := b_{18} - b_{19} - b_{20} + b_{21}$.
- (d) $B_2 := b_{12} - b_{13}$.

TABLE 6. Commutator relations for type \widetilde{E}_8 .

$$\begin{array}{lll}
[x_1(t), x_3(u)] = x_9(tu), & [x_2(t), x_4(u)] = x_{10}(tu), & [x_3(t), x_4(u)] = x_{11}(tu), \\
[x_4(t), x_5(u)] = x_{12}(tu), & [x_5(t), x_6(u)] = x_{13}(tu), & [x_6(t), x_7(u)] = x_{14}(tu), \\
[x_7(t), x_8(u)] = x_{15}(tu), & [x_1(t), x_{11}(u)] = x_{16}(tu), & [x_4(t), x_9(u)] = x_{16}(-tu), \\
[x_2(t), x_{11}(u)] = x_{17}(tu), & [x_3(t), x_{10}(u)] = x_{17}(tu), & [x_2(t), x_{12}(u)] = x_{18}(tu), \\
[x_5(t), x_{10}(u)] = x_{18}(-tu), & [x_3(t), x_{12}(u)] = x_{19}(tu), & [x_5(t), x_{11}(u)] = x_{19}(-tu), \\
[x_4(t), x_{13}(u)] = x_{20}(tu), & [x_6(t), x_{12}(u)] = x_{20}(-tu), & [x_5(t), x_{14}(u)] = x_{21}(tu), \\
[x_7(t), x_{13}(u)] = x_{21}(-tu), & [x_6(t), x_{15}(u)] = x_{22}(tu), & [x_8(t), x_{14}(u)] = x_{22}(-tu), \\
[x_1(t), x_{17}(u)] = x_{23}(tu), & [x_2(t), x_{16}(u)] = x_{23}(tu), & [x_9(t), x_{10}(u)] = x_{23}(tu), \\
[x_1(t), x_{19}(u)] = x_{24}(tu), & [x_5(t), x_{16}(u)] = x_{24}(-tu), & [x_9(t), x_{12}(u)] = x_{24}(tu), \\
[x_2(t), x_{19}(u)] = x_{25}(tu), & [x_3(t), x_{18}(u)] = x_{25}(tu), & [x_5(t), x_{17}(u)] = x_{25}(-tu), \\
[x_2(t), x_{20}(u)] = x_{26}(tu), & [x_6(t), x_{18}(u)] = x_{26}(-tu), & [x_{10}(t), x_{13}(u)] = x_{26}(tu), \\
[x_3(t), x_{20}(u)] = x_{27}(tu), & [x_6(t), x_{19}(u)] = x_{27}(-tu), & [x_{11}(t), x_{13}(u)] = x_{27}(tu), \\
[x_4(t), x_{21}(u)] = x_{28}(tu), & [x_7(t), x_{20}(u)] = x_{28}(-tu), & [x_{12}(t), x_{14}(u)] = x_{28}(tu), \\
[x_5(t), x_{22}(u)] = x_{29}(tu), & [x_8(t), x_{21}(u)] = x_{29}(-tu), & [x_{13}(t), x_{15}(u)] = x_{29}(tu), \\
[x_1(t), x_{25}(u)] = x_{30}(tu), & [x_2(t), x_{24}(u)] = x_{30}(tu), & [x_5(t), x_{23}(u)] = x_{30}(-tu), \\
[x_9(t), x_{18}(u)] = x_{30}(tu), & [x_1(t), x_{27}(u)] = x_{31}(tu), & [x_6(t), x_{24}(u)] = x_{31}(-tu), \\
[x_9(t), x_{20}(u)] = x_{31}(tu), & [x_{13}(t), x_{16}(u)] = x_{31}(-tu), & [x_4(t), x_{25}(u)] = x_{32}(tu), \\
[x_{10}(t), x_{19}(u)] = x_{32}(-tu), & [x_{11}(t), x_{18}(u)] = x_{32}(-tu), & [x_{12}(t), x_{17}(u)] = x_{32}(-tu), \\
[x_2(t), x_{27}(u)] = x_{33}(tu), & [x_3(t), x_{26}(u)] = x_{33}(tu), & [x_6(t), x_{25}(u)] = x_{33}(-tu), \\
[x_{13}(t), x_{17}(u)] = x_{33}(-tu), & [x_2(t), x_{28}(u)] = x_{34}(tu), & [x_7(t), x_{26}(u)] = x_{34}(-tu), \\
[x_{10}(t), x_{21}(u)] = x_{34}(tu), & [x_{14}(t), x_{18}(u)] = x_{34}(-tu), & [x_3(t), x_{28}(u)] = x_{35}(tu), \\
[x_7(t), x_{27}(u)] = x_{35}(-tu), & [x_{11}(t), x_{21}(u)] = x_{35}(tu), & [x_{14}(t), x_{19}(u)] = x_{35}(-tu), \\
[x_4(t), x_{29}(u)] = x_{36}(tu), & [x_8(t), x_{28}(u)] = x_{36}(-tu), & [x_{12}(t), x_{22}(u)] = x_{36}(tu), \\
[x_{15}(t), x_{20}(u)] = x_{36}(-tu), & [x_1(t), x_{32}(u)] = x_{37}(tu), & [x_4(t), x_{30}(u)] = x_{37}(tu), \\
[x_{10}(t), x_{24}(u)] = x_{37}(-tu), & [x_{12}(t), x_{23}(u)] = x_{37}(-tu), & [x_{16}(t), x_{18}(u)] = x_{37}(-tu), \\
[x_1(t), x_{33}(u)] = x_{38}(tu), & [x_2(t), x_{31}(u)] = x_{38}(tu), & [x_6(t), x_{30}(u)] = x_{38}(-tu), \\
[x_9(t), x_{26}(u)] = x_{38}(tu), & [x_{13}(t), x_{23}(u)] = x_{38}(-tu), & [x_1(t), x_{35}(u)] = x_{39}(tu), \\
[x_7(t), x_{31}(u)] = x_{39}(-tu), & [x_9(t), x_{28}(u)] = x_{39}(tu), & [x_{14}(t), x_{24}(u)] = x_{39}(-tu), \\
[x_{16}(t), x_{21}(u)] = x_{39}(tu), & [x_4(t), x_{33}(u)] = x_{40}(tu), & [x_6(t), x_{32}(u)] = x_{40}(-tu), \\
[x_{10}(t), x_{27}(u)] = x_{40}(-tu), & [x_{11}(t), x_{26}(u)] = x_{40}(-tu), & [x_{17}(t), x_{20}(u)] = x_{40}(tu), \\
[x_2(t), x_{35}(u)] = x_{41}(tu), & [x_3(t), x_{34}(u)] = x_{41}(tu), & [x_7(t), x_{33}(u)] = x_{41}(-tu), \\
[x_{14}(t), x_{25}(u)] = x_{41}(-tu), & [x_{17}(t), x_{21}(u)] = x_{41}(tu), & [x_2(t), x_{36}(u)] = x_{42}(tu), \\
[x_8(t), x_{34}(u)] = x_{42}(-tu), & [x_{10}(t), x_{29}(u)] = x_{42}(tu), & [x_{15}(t), x_{26}(u)] = x_{42}(-tu), \\
[x_{18}(t), x_{22}(u)] = x_{42}(tu), & [x_3(t), x_{36}(u)] = x_{43}(tu), & [x_8(t), x_{35}(u)] = x_{43}(-tu), \\
[x_{11}(t), x_{29}(u)] = x_{43}(tu), & [x_{15}(t), x_{27}(u)] = x_{43}(-tu), & [x_{19}(t), x_{22}(u)] = x_{43}(tu),
\end{array}$$

(e) $R_5 := \{r_5(v) := x_{30}(v)x_{31}(v)x_{32}(-v)x_{33}(-v)x_{34}(-2v)x_{35}(2v)x_{36}(2v) : v \in \mathbb{F}_q\}$.

(f) $R_4 := \{r_4(v) := x_{24}(2v)x_{25}(-2v)x_{26}(v)x_{27}(-v)x_{28}(-v)x_{29}(v) : v \in \mathbb{F}_q\}$.

(g) $R_3 := \{r_3(v) := x_{18}(v)x_{19}(-v)x_{20}(-v)x_{21}(v) : v \in \mathbb{F}_q\}$.

(h) $R_2 := \{r_2(v) := x_{12}(v)x_{13}(-v) : v \in \mathbb{F}_q\}$.

(i) $L_1 := \{l_1(u) := x_2(2u)x_1(u)x_3(-2u)x_4(u)x_6(u)x_7(2u)x_8(-2u) : u \in \mathbb{F}_q\}$,
 $S_1 := L_1T_2T_3T_4$.

(j) $L_2 := \{l_2(u) := l_1(u)x_9(u^2)x_{10}(-u^2)x_{11}(u^2)x_{14}(-u^2)x_{15}(2u^2) : t \in \mathbb{F}_q\}$,
 $S_2 := L_2T_3T_4$.

(k) $L_3 := \{l_3(u) := l_2(u)x_{16}(4u^3)x_{17}(2u^3)x_{22}(3u^3) : u \in \mathbb{F}_q\}$, $S_3 := L_3T_4$

- (l) $S_4 := \{l_4(u) := l_3(u)x_{23}(3u^4) : u \in \mathbb{F}_q\}$.
(m) If $B_2 = c^4 \in \mathbb{F}_q^\times$, $F_4 := \{s_4(uc) : u \in \mathbb{F}_5\}$ and $F_5 := \{x_5(vc_\phi) : v \in \mathbb{F}_5\}$.

It is easy to check that for $k \in [2..5]$, $R_k \leq H_k$ of order q , $S_k \leq S_{k-1} \leq T$ with $S_5 = \{1\}$, and $F_4 \leq S_4$, $F_5 \leq X_5$ of order 5. It is noted that all B_i are defined for each λ as above, hence $B_i = B_i(\lambda)$. Since $R_k \cong \mathbb{F}_q$, for each $a \in \mathbb{F}_q$ we define $\phi_a(r_k(t)) = \phi_a(t)$ for all $r_k(t) \in R_k$. Hence, $\text{Irr}(R_k) = \{\phi_a : a \in \mathbb{F}_q\}$. Since $S_4 \cong \mathbb{F}_q$, we can define $\phi_a(s_4(t)) = \phi_a(t)$ for all $s_4(t) \in S_4$. When $B_2 = c^4 \in \mathbb{F}_q^\times$, for each linear $\xi \in \text{Irr}(F_4)$ there is $b_4 \in \{ta_4 : t \in \mathbb{F}_5\} \cong (\mathbb{F}_5, +)$ such that $\xi = \phi_{b_4}|_{F_4}$ where $\phi_{b_4} \in \text{Irr}(S_4)$ and $\phi(a_4c) \neq 1$. Use the same argument for $F_5 \leq X_5$, for each $\xi \in \text{Irr}(F_5)$ there is $b_5 \in \{ta_5 : t \in \mathbb{F}_5\} \cong (\mathbb{F}_5, +)$ such that $\xi = \phi_{b_5}|_{F_5}$, where $\phi_{b_5} \in \text{Irr}(X_5)$ and $\phi(a_5c_\phi) \neq 1$.

Let $\overline{H_5}$ be the normal closure of H_5 in HX_5S_1 . All properties of λ 's are known as follows. It is clear that $X_5 \subset \text{Stab}_U(\lambda)$.

Lemma 4.2. *The following are true*

- (a) $R_5 = \{x \in H_5 : |\lambda^U(x)| = \lambda^U(1)\}$ and $S_1 = \text{Stab}_T(\lambda|_{H_6H_5})$. Moreover, $\lambda^U|_{R_5} = \lambda^U(1)\phi_{B_5}$.
(b) If $B_5 \neq 0$, then $\text{Stab}_T(\lambda) = \{1\}$. Hence, if η is an extension of λ to HX_5 , then $I_U(\eta) = HX_5$. Furthermore, if η, η' are two extensions of $\lambda|_{H_6H_5H_4}$ to HX_5 , then $\eta^U = \eta'^U$ iff $B_i(\eta) = B_i(\eta')$ for $i = 2, 3$ and $\eta|_{X_5} = \eta'|_{X_5}$.
(c) If $B_5 = 0$, then there exists $x \in T$ such that ${}^x\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5$. Furthermore, $\overline{H_5} \subset \ker({}^x\lambda)^{HX_5S_1}$ and the induction map from $\text{Irr}(HX_5S_1, {}^x\lambda)$ to $\text{Irr}(U, \lambda)$ is bijective.

Proof. See Subsection 5.5.1. \square

Remark When $(q, 5) = 1$, both R_5 and $\text{Stab}_T(\lambda)$ are trivial. Hence, λ extends to HX_5 and induces irreducibly to U of degree $[U : HX_5] = q^{16}$.

Lemma 4.2 (a) can be observed by the following figure.

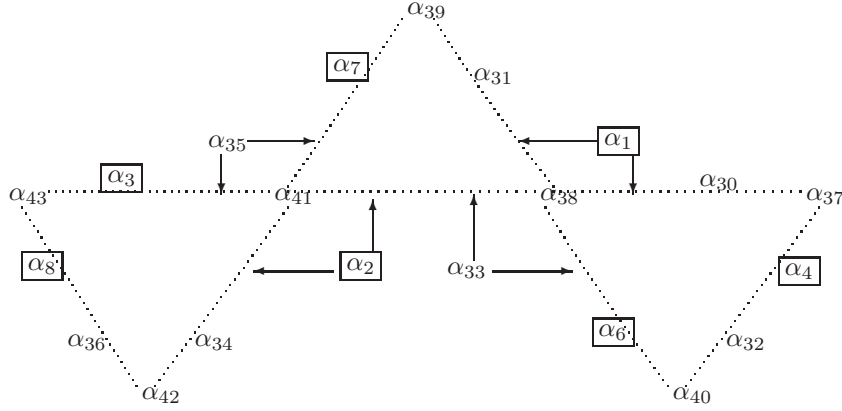


Figure $UE_8(q)$: Relations among roots of heights 5 in H and 1 in T .

We have q^{19} linear characters λ of H such that $\lambda|_{X_i} = \phi$ for all $X_i \subset Z(U)$. In these, there are q^{18} linears with $B_5 = 0$ and $q^{18}(q-1)$ linears with $B_5 \neq 0$. By Lemma 4.2 (a), it is clear that $B_5 = B_5(\lambda)$ is invariant under the action of T . Therefore, by Lemma 4.2 (b), these $q^{18}(q-1)$ linears with $B_5 \neq 0$ extend to HX_5 and induce irreducibly to U . Thus, we obtain $\frac{q^{19}(q-1)}{q^{16}} = q^3(q-1)$ irreducible

characters of U of degree q^{16} which are parametrized by (b_5, B_2, B_3, B_5) where $b_5, B_2, B_3 \in \mathbb{F}_q$ and $B_5 \in \mathbb{F}_q^\times$. Hence, we denote them by $\chi_{q^{16}}^{b_5, B_2, B_3, B_5}$.

Since $[U : HX_5S_1] = q^6$, by Lemma 4.2(c), the constituents of q^{18} linears of H with $B_5 = 0$ inducing to U correspond to the ones of q^{12} linears λ of H inducing to HX_5S_1 , where $\lambda|_{X_i} = \phi$ for all $X_i \subset H_6$, $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5$, and $\lambda|_{X_i} = \phi_{b_i}$ for all $X_i \subset H_4H_3H_2$ where $b_i \in \mathbb{F}_q$. Let $\lambda \in \text{Irr}(H)$ be one of these above q^{12} linears. Now we consider λ in the scenario of the subgroup HX_5S_1 . Let $\overline{H_5H_4}$ be the normal closure of H_5H_4 in HX_5S_2 .

Lemma 4.3. *The following are true*

- (a) $R_4 = \{x \in H_4 : |\lambda^{HX_5S_1}(x)| = \lambda^{HX_5S_1}(1)\}$ and $S_2 = \text{Stab}_{S_1}(\lambda|_{H_6H_5H_4})$. Moreover, $\lambda^{HX_5S_1}|_{R_4} = \lambda^{HX_5S_1}(1)\phi_{B_4}$.
- (b) If $B_4 \neq 0$, then $\text{Stab}_{S_1}(\lambda) = \{1\}$. Hence, if η is an extension of λ to HX_5 , then $I_{HX_5S_1}(\eta) = HX_5$. Furthermore, if η, η' are two extensions of $\lambda|_{H_6H_5H_4H_3}$ to HX_5 , then $\eta^{HX_5S_1} = \eta'^{HX_5S_1}$ iff $B_2(\eta) = B_2(\eta')$ and $\eta|_{X_5} = \eta'|_{X_5}$.
- (c) If $B_4 = 0$, then there exists $x \in S_1$ such that ${}^x\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5H_4$. Furthermore, $\overline{H_5H_4} \subset \ker({}^x\lambda)^{HX_5S_2}$ and the induction map from $\text{Irr}(HX_5S_2, {}^x\lambda)$ to $\text{Irr}(HX_5S_1, \lambda)$ is bijective.

Proof. See Subsection 5.5.2. \square

The main idea of Lemma 4.3 (a) can be visualized in the following figure.

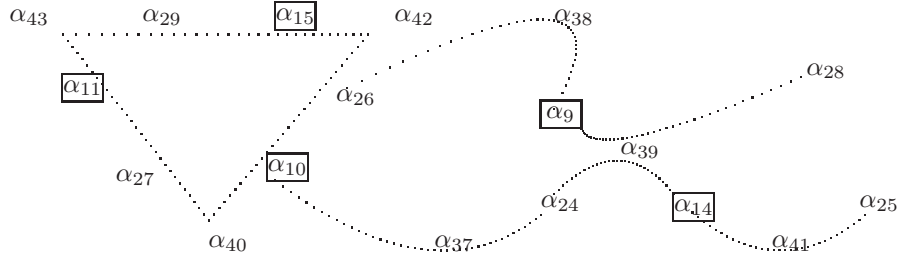


Figure $UE_8(q)$: Relations of between root heights 4 in H and 2 in T .

Recall that we have q^{12} linear characters λ of H such that $\lambda|_{X_i} = \phi$ for all $X_i \subset Z(U)$ and $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5$. In these, there are q^{11} linears with $B_4 = 0$ and $q^{11}(q-1)$ linears with $B_4 \neq 0$. By Lemma 4.3 (a), it is clear that $B_4 = B_4(\lambda)$ is invariant under the action of S_1 . Therefore, by Lemma 4.3 (b), these $q^{11}(q-1)$ linears with $B_4 \neq 0$ extend to HX_5 and induce irreducibly to HX_5S_1 . Thus, we obtain $\frac{q^{12}(q-1)}{|S_1|} = q^2(q-1)$ irreducible characters of HX_4S_1 of degree $|S_1| = q^{10}$ which are parametrized by (b_5, B_2, B_4) where $b_5, B_2 \in \mathbb{F}_q$ and $B_4 \in \mathbb{F}_q^\times$. By Lemma 4.2 (c), we obtain $q^2(q-1)$ characters of U of degree q^{16} which can be denoted by $\chi_{q^{16}}^{b_5, B_2, B_4}$.

Since $[HX_5S_1 : HX_5S_2] = 5$, by Lemma 4.3(c), the constituents of q^{11} linears of H with $B_4 = 0$ inducing to HX_5S_1 correspond to the ones of q^6 linears λ of H inducing to HX_5S_2 , where $\lambda|_{X_i} = \phi$ for all $X_i \subset H_6$, $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5H_4$, and $\lambda|_{X_i} = \phi_{b_i}$ for all $X_i \subset H_3H_2$ where $b_i \in \mathbb{F}_q$. Let $\lambda \in \text{Irr}(H)$ be one of above q^6 linears of H . Now we consider λ in the scenario of the subgroup HX_5S_2 . Let $\overline{H_5H_4H_3}$ be the normal closure of $H_5H_4H_3$ in HX_5S_3 .

Lemma 4.4. *The following are true*

- (a) $R_3 = \{x \in H_3 : |\lambda^{HX_5S_2}(x)| = \lambda^{HX_5S_2}(1)\}$ and $S_3 = \text{Stab}_{S_2}(\lambda|_{H_6H_5H_4H_3})$. Moreover, $\lambda^{HX_5S_2}|_{R_3} = \lambda^{HX_5S_2}(1)\phi_{B_3}$.
- (b) If $B_3 \neq 0$, then $\text{Stab}_{S_2}(\lambda) = \{1\}$. Hence, if η is an extension of λ to HX_5 , then $I_{HX_5S_2}(\eta) = HX_5$. Furthermore, if η, η' are two extensions of λ to HX_5 , then $\eta^{HX_5S_2} = \eta'^{HX_5S_2}$ iff $\eta|_{X_5} = \eta'|_{X_5}$.
- (c) If $B_3 = 0$, then there exists $x \in S_2$ such that ${}^x\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5H_4H_3$. Furthermore, $\overline{H_5H_4H_3} \subset \ker({}^x\lambda)^{HX_5S_3}$ and the induction map from $\text{Irr}(HX_5S_3, {}^x\lambda)$ to $\text{Irr}(HX_5S_2, \lambda)$ is bijective.

Proof. See Subsection 5.5.3. \square

The main idea of Lemma 4.4 (a) can be described as follows.

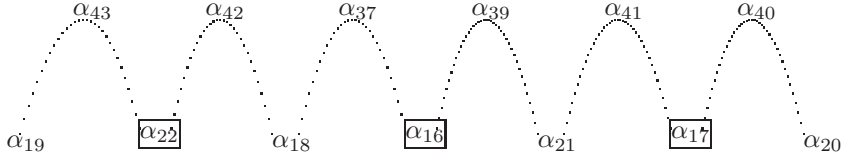


Figure $UE_8(q)$: Relations of between root heights 3 in H and 3 in T .

Recall that we have q^6 linear characters λ of H such that $\lambda|_{X_i} = \phi$ for all $X_i \subset Z(U)$ and $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5H_4$. In these, there are q^5 linears with $B_3 = 0$ and $q^5(q-1)$ linears with $B_3 \neq 0$. By Lemma 4.4 (a), it is clear that $B_3 = B_3(\lambda)$ is invariant under the action of S_2 . Therefore, by Lemma 4.4 (b), these $q^5(q-1)$ linears with $B_3 \neq 0$ extend to HX_5 and induce irreducibly to HX_5S_2 . Thus, we obtain $\frac{q^6(q-1)}{|S_2|} = q(q-1)$ irreducible characters of HX_4S_2 of degree $|S_2| = q^5$ which are parametrized by (b_5, B_3) where $b_5 \in \mathbb{F}_q$ and $B_3 \in \mathbb{F}_q^\times$. By Lemma 4.3 (c) and Lemma 4.2 (c), we obtain $q(q-1)$ characters of U of degree q^{16} which can be denoted by $\chi_{q^{16}}^{b_5, B_3}$.

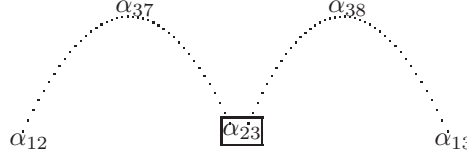
Since $[HX_5S_2 : HX_5S_3] = 3$, by Lemma 4.4(c), the constituents of q^5 linears of H with $B_3 = 0$ inducing to HX_5S_2 correspond to the ones of q^2 linears λ of H inducing to HX_5S_3 , where $\lambda|_{X_i} = \phi$ for all $X_i \subset H_6$, $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5H_4H_3$, and $\lambda|_{X_i} = \phi_{b_i}$ for all $X_i \subset H_2$ where $b_i \in \mathbb{F}_q$. Let $\lambda \in \text{Irr}(H)$ be one of above q^2 linears of H . Now we consider λ in the scenario of the subgroup HX_5S_3 .

Lemma 4.5. *The following are true.*

- (a) $R_2 = \{x \in H_2 : |\lambda^{HX_5S_3}(x)| = \lambda^{HX_5S_3}(1)\}$ and $S_4 = \text{Stab}_{S_3}(\lambda)$. Moreover, $\lambda^{HX_5S_3}|_{R_2} = \lambda^{HX_5S_3}(1)\phi_{B_2}$.
- (b) If $B_2 \notin \{c^4 : c \in \mathbb{F}_q^\times\}$ and let η be an extension of λ to HX_5 , then $I_{HX_5S_3}(\eta) = HX_5$. Therefore, S_4 acts transitively and faithfully on all extensions of λ to HX_5 .
- (c) If $B_2 = c^4 \in \mathbb{F}_q^\times$, then λ extends to HX_5F_4 and HF_5S_4 . Let λ_1, λ_2 be two extensions of λ to HX_5F_4 . Then $I_{HX_5S_3}(\lambda_1) = HX_5F_4$. Moreover, $\lambda_1^{HX_5S_3} = \lambda_2^{HX_5S_3}$ iff $\lambda_1|_{F_4} = \lambda_2|_{F_4}$ and $\lambda_1|_{F_5} = \lambda_2|_{F_5}$.

Proof. See Subsection 5.5.4. \square

It is noted that $\text{Stab}_{S_3}(\lambda) = \text{Stab}_T(\lambda)$. The main idea of Lemma 4.5 (a) can be visualized in the following figure.

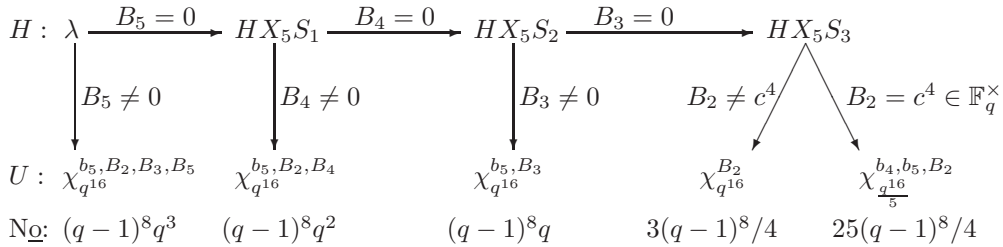

 Figure $UE_8(q)$: Relations of between root heights 2 in H and 4 in T .

Recall that we have q^2 linear characters λ of H such that $\lambda|_{X_i} = \phi$ for all $X_i \subset Z(U)$ and $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5H_4H_3$. By Lemma 4.5 (a), it is clear that $B_2 = B_2(\lambda)$ is invariant under the action of S_3 . Since F_q^\times is cyclic, we have $|\{c^4 : c \in \mathbb{F}_q^\times\}| = \frac{q-1}{4}$. Therefore, there are $\frac{q(q-1)}{4}$ linears with $B_2 = c^4 \in \mathbb{F}_q^\times$, and there are $\frac{3q(q-1)}{4}$ linears with $B_2 \notin \{c^4 : c \in \mathbb{F}_q^\times\}$. Thus, these linears with $B_2 \notin \{c^4 : c \in \mathbb{F}_q^\times\}$ extend to HX_5 and induce irreducibly to HX_5S_3 of degree $|S_3| = q^2$. By Lemma 4.4 (c), Lemma 4.3 (c) and Lemma 4.2 (c), we obtain $\frac{3(q-1)}{4}$ characters of U of degree q^{16} which can be denoted by $\chi_{q^{16}}^{B_2}$.

Now we sum up all irreducibles of degree q^{16} as counted above and denoted by $\chi_{q^{16}}^{b_5, B_2, B_3, B_5}$, $\chi_{q^{16}}^{b_5, B_2, B_4}$, $\chi_{q^{16}}^{b_5, B_3}$, and $\chi_{q^{16}}^{B_2}$. Therefore, \mathcal{F}_8 contains exactly $q^3(q-1) + q^2(q-1) + q(q-1) + \frac{3(q-1)}{5}$ characters χ of U of degree q^{16} such that $\chi|_{X_i} = \chi(1)\phi$ for all $X_i \subset Z(U)$.

By Lemma 4.5 (c), let λ_1 be an extension of λ to HX_5F_4 , then $\lambda_1^{HX_5S_3}$ is irreducible of degree $[HX_5S_3 : HX_5F_4] = \frac{q^2}{5}$. These $\lambda_1^{HX_5S_3}$ only depend on B_2 and their restrictions to F_4, F_5 . Hence, Lemma 4.4 (c), Lemma 4.3 (c) and Lemma 4.2 (c), $\lambda_1^U \in Irr(U)$ of degree $\frac{q^{16}}{5}$ is denoted by $\chi_{\frac{q^{16}}{5}}^{b_4, b_5, B_2}$ where $b_4, b_5 \in \mathbb{F}_5$ and $B_2 \in \{c^4 : c \in \mathbb{F}_q^\times\}$. Therefore, \mathcal{F}_8 has exactly $\frac{25(q-1)}{4}$ irreducibles of degree $\frac{q^{16}}{5}$ such that $\chi|_{X_i} = \chi(1)\phi$ for all $X_i \subset Z(U)$.

By the transitivity of the conjugate action of the maximal split torus T_0 of the Chevalley group $E_8(q)$ on $\bigoplus_{i=37}^{43} Irr(X_i)^\times$, there are $(q-1)^8(q^3 + q^2 + q + \frac{3}{4})$ characters $\chi \in \mathcal{F}_8$ of degree q^{16} , and $\frac{25(q-1)^8}{4}$ characters $\chi \in \mathcal{F}_8$ of degree $\frac{q^{16}}{5}$ such that $\chi|_{X_i} = \chi(1)\phi_{a_i}$, where $a_i \in \mathbb{F}_q^\times, 37 \leq i \leq 43$. The following diagram summarizes all the above arguments with their assumptions.


 Figure $UE_8(q)$: Summary on the branching rules of λ .

This gives the proof for the next theorem.

Theorem 4.6. *Let $\chi \in \mathcal{F}_8$. The following are true.*

- (a) *If $\chi(1) = q^{16}$, then there exists $t \in T_0$ such that ${}^t\chi$ is an element of $\{\chi_{q^{16}}^{b_5, B_2, B_3}, \chi_{q^{16}}^{b_5, B_2}, \chi_{q^{16}}^{b_5}, \chi_{q^{16}}^{B_2}\}$.*

(b) If $\chi(1) = q^{16}/5$, then there exists $t \in T_0$ such that ${}^t\chi = \chi_{\frac{q^{16}}{5}}^{b_4, b_5, B_2}$.

5. ALL PROOFS.

In all proofs, we use the following technique:

- (a) For all the decomposition of the commutator formula into product, we apply the formula $[a, bc] = [a, c][a, b]^c$.
- (b) For $H \leq G$ and $L \trianglelefteq G$, for each $\lambda \in Irr(L)$, $Stab_G(\lambda) := \{x \in G : {}^x\lambda = \lambda\}$, and $Stab_G(\lambda) \subset Stab_G(\lambda|_H) =: K$, hence, $Stab_G(\lambda) = Stab_K(\lambda)$.
- (c) For $K \leq G$ and $H \trianglelefteq G$, to extend a linear character λ of H to HK , we check if $[HK, HK] \subset ker(\lambda)$.

5.1. Proof of Proposition 1.3. Let $a \in \mathbb{F}_q^\times$. We are going to prove 1.3 (c).

Since $\gcd(q-1, p) = 1$, for each $b \in \mathbb{F}_q^\times$, there is $s \in \mathbb{F}_q^\times$ such that $b = s^p$. We have $b \prod_{c \in \mathbb{F}_p} (t - ca) = s^p \prod_{c \in \mathbb{F}_p} (t - ca) = \prod_{c \in \mathbb{F}_p} (st - csa) \in \mathbb{T}_{sa}$. Hence, \mathbb{F}_q^\times acts on $\{\mathbb{T}_a : a \in \mathbb{F}_q^\times\}$ such that $s^p \mathbb{T}_a = \mathbb{T}_{sa}$. We claim that $b\mathbb{T}_a = \mathbb{T}_a$ iff $b \in \mathbb{F}_p^\times$.

By (a), it is clear if $b \in \mathbb{F}_p^\times$. Suppose there exists $b \in \mathbb{F}_q - \mathbb{F}_p$ such that $b\mathbb{T}_a = \mathbb{T}_a$. Since $b \in \mathbb{F}_q - \mathbb{F}_p$, we have $b = s^p$ for some $s \in \mathbb{F}_q - \mathbb{F}_p$. Hence, $s^{p-1} \neq 1$. By induction, we have $\mathbb{T}_a = b\mathbb{T}_a = b^2\mathbb{T}_a = \dots = b^k\mathbb{T}_a$ for all $k \in \mathbb{N}$. Therefore, for each $t \in \mathbb{F}_q$, $t^p - (as^k)^{p-1}t \in \mathbb{T}_a$ for all $k \in \mathbb{N}$. For each $l \in \mathbb{N}^+$, we have

$$\begin{aligned} a^{p-1}t(s^{p-1} - 1)^l &= a^{p-1}t \sum_{k=0}^l \frac{l!}{k!(l-k)!} s^{(p-1)k} (-1)^{l-k} - t^p(1-1)^l \\ &= a^{p-1}t \sum_{k=0}^l \frac{l!}{k!(l-k)!} s^{(p-1)k} (-1)^{l-k} - t^p \sum_{k=0}^l \frac{l!}{k!(l-k)!} (-1)^{l-k} \\ &= - \sum_{k=0}^l (-1)^{l-k} \frac{l!}{k!(l-k)!} (t^p - (as^k)^{p-1}t) \in \mathbb{T}_a. \end{aligned}$$

Since $s^{p-1} - 1 \in \mathbb{F}_q^\times$, there exists $l \in \mathbb{N}^\times$ such that $(s^{p-1} - 1)^l = 1$. Therefore, $a^{p-1}t \in \mathbb{T}_a$ for all $t \in \mathbb{F}_q$. We have $(t^p - a^{p-1}t) + a^{p-1}t = t^p \in \mathbb{T}_a$. This makes a contradiction since $\{t^p : t \in \mathbb{F}_q\} = \mathbb{F}_q \not\subseteq \mathbb{T}_a$. So the claim holds. Thus $\mathbb{F}_q^\times/\mathbb{F}_p^\times$ acts faithfully and transitively on $\{\mathbb{T}_a : a \in \mathbb{F}_q^\times\}$, and $|\{\mathbb{T}_a : a \in \mathbb{F}_q\}| = \frac{q-1}{p-1}$.

It is easy to see that $ker\phi_a = ker\phi_{ca}$ for all $c \in \mathbb{F}_p^\times$ since $\mathbb{F}_p \cong \mathbb{Z}_p$ and $\phi_a(ku) = \phi_a(u)^k$ for all $k \in \mathbb{N}$. Therefore, $|\{ker\phi_a : a \in \mathbb{F}_q^\times\}| = \frac{q-1}{p-1}$.

Since $\{ker\phi_a : a \in \mathbb{F}_q^\times\}$ are all subgroups of index p in \mathbb{F}_q , $\{ker\phi_a : a \in \mathbb{F}_q^\times\} = \{\mathbb{T}_a : a \in \mathbb{F}_q^\times\}$. Therefore, for each $a \in \mathbb{F}_q^\times$, there exists $b \in \mathbb{F}_q^\times$ such that $b\mathbb{T}_a = ker\phi$, and $cb\mathbb{T}_a = ker\phi$ iff $c \in \mathbb{F}_p^\times$. \square

5.2. Proof of Proposition 1.5. (a) Suppose $\chi \in Irr(N/Y, \lambda)$, we are going to show that $\chi^G \in Irr(G)$ by showing that the inertia group $I_G(\chi) = N$.

Since $Y \subset ker(\chi)$ and $Z \subset Z(N)$, we have $\chi|_{ZY} = \chi(1)\lambda$. Since $X \subset N_G(ZY)$, for each $x \in X$, ${}^x\lambda \in Irr(ZY)$. Hence, for any $u \neq v \in X$ we have

$${}^u\chi|_{ZY} = \chi(1) {}^u\lambda \neq \chi(1) {}^v\lambda = {}^v\chi|_{ZY}, \text{ i.e. } {}^u\chi \neq {}^v\chi.$$

Therefore, $x \in X$ such that ${}^x\chi = \chi$ iff $x = 1$. Since X is a transversal of N in G , this shows that the inertia group $I_G(\chi) = N$.

The above argument also proves that for $\chi_1, \chi_2 \in Irr(N/Y, \lambda)$ and $u \neq v \in X$ we have ${}^u\chi_1 \neq {}^v\chi_2$. So by the Mackey formula for the double coset $N \backslash G/N = G/N$ represented by X , we have

$$(\chi_1^G, \chi_2^G) = (\chi_1^G|_N, \chi_2) = \sum_{x \in X} ({}^x\chi_1, \chi_2) = (\chi_1, \chi_2) = \begin{cases} 1 & \text{if } \chi_1 = \chi_2 \\ 0 & \text{otherwise} \end{cases}$$

(b) It is enough to show that the induction map is surjective, i.e. for each $\xi \in Irr(G, \lambda)$ there exists $\chi \in Irr(N/Y, \lambda)$ such that $\xi = \chi^G$.

Suppose $\xi|_N = \sum_{\chi_i \in S} a_i \chi_i$ where $a_i \in \mathbb{N}^\times$ and $S \subset Irr(H)$. By Frobenius reciprocity, $0 \neq (\xi, \lambda^G) = (\xi|_{ZY}, \lambda)$, there exists at least a constituent χ_0 of $\xi|_N$ such that $(\chi_0|_{ZY}, \lambda) \neq 0$, i.e. $\chi_0 \in Irr(N, \lambda)$.

Since $\lambda|_Y = \lambda(1)1_Y$ and $(\chi_0|_Y, \lambda|_Y) \geq (\chi_0|_{ZY}, \lambda) > 0$, we have χ_0 is a constituent of 1_Y^N . Since $Y \trianglelefteq N$, all constituents of 1_Y^N are $Irr(N/Y)$. Therefore, $\chi_0 \in Irr(N/Y, \lambda)$. By (a), $\chi_0^G \in Irr(G)$, hence it forces $\xi = \chi_0^G$. \square

5.3. Proofs of Sylow 2-subgroups of $D_4(2^f)$.

5.3.1. *Proof of Lemma 2.2.* Set $\lambda = \lambda_{b_3, b_5, b_6, b_7}^{a_8, a_9, a_{10}}$ for the whole proof.

(a) First we show that $Stab_T(\lambda) = S_{124}$. Since ${}^y\lambda(x) = \lambda(x)$ iff $\lambda(x^{-1}x^y) = \lambda([x, y]) = 1$ and $X_8X_9X_{10} \subset Z(U)$, it suffices to check for $[X_5X_6X_7, T]$. For all $t_i, s_j \in \mathbb{F}_q$, we have

$$[x_5(t_5)x_6(t_6)x_7(t_7), x_1(s_1)x_2(s_2)x_4(s_4)] = x_8(t_6t_1+t_5t_2)x_9(t_7t_1+t_5t_4)x_{10}(t_7t_2+t_6t_4).$$

Therefore, $x_1(s_1)x_2(s_2)x_4(s_4) \in Stab_T(\lambda)$ iff for all $t_5, t_6, t_7 \in \mathbb{F}_q$,

$$\begin{aligned} 1 &= \phi(a_8(t_6s_1 + t_5s_2) + a_9(t_7s_1 + t_5s_4) + a_{10}(t_7s_2 + t_6s_4)) \\ &= \phi(t_5(a_8s_2 + a_9s_4) + t_6(a_8s_1 + a_{10}s_4) + t_7(a_9s_1 + a_{10}s_2)) \end{aligned}$$

iff $a_8s_2 + a_9s_4 = a_8s_1 + a_{10}s_4 = a_9s_1 + a_{10}s_2 = 0$, i.e. $\frac{s_1}{a_{10}} = \frac{s_2}{a_9} = \frac{s_4}{a_8}$. So $Stab_T(\lambda) = S_{124}$.

To find all scalar points of $X_5X_6X_7$ on λ^U , since X_3T is a transversal of H in U and $[X_3, X_5X_6X_7] = \{1\}$, it is enough to find ones of $X_5X_6X_7$ on T , i.e. find $x_5x_6x_7 \in X_5X_6X_7$ such that $\lambda([x_5x_6x_7, x_1x_2x_4]) = 1$ for all $x_1x_2x_4 \in T$. Use above computation, for all $s_i \in \mathbb{F}_q$, we need

$$\begin{aligned} 1 &= \phi(a_8(t_6s_1 + t_5s_2) + a_9(t_7s_1 + t_5s_4) + a_{10}(t_7s_2 + t_6s_4)) \\ &= \phi(s_1(a_8t_6 + a_9t_7) + s_2(a_8t_5 + a_{10}t_7) + s_4(a_9t_5 + a_{10}t_6)) \end{aligned}$$

iff $a_8t_6 + a_9t_7 = a_8t_5 + a_{10}t_7 = a_9t_5 + a_{10}t_6 = 0$, i.e. $\frac{t_5}{a_{10}} = \frac{t_6}{a_9} = \frac{t_7}{a_8}$. Hence, $\prod_{i=5}^7 x_i(t_i) = x_{567}(\frac{t_7}{a_8}) \in S_{567}$. So $S_{567} = \{x \in X_5X_6X_7 : |\lambda^U(x)| = \lambda^U(1)\}$.

Now, to prove that $\lambda^U|_{S_{567}} = q^4\phi_{At_0}$, it suffices to check that $\lambda(x_{567}(t)) = \phi_{At_0}(t)$. For each $x_{567}(t) = x_5(a_{10}t)x_6(a_9t)x_7(a_8t) \in S_{567}$, we have

$$\lambda(x_{567}(t)) = \phi(t(b_5a_{10} + b_6a_9 + b_7a_8)) = \phi(tAt_0) = \phi_{At_0}(t).$$

(b) We study $Irr(U, \lambda)$ by two following ways. Let $K_1 := HX_3F_{124}$ and $K_2 := HS_{124}F_3$. Since $H = [U, U]$, it is clear that $H_1, K_1 \trianglelefteq U$.

$$\begin{array}{ccc} & H & \\ & \swarrow & \searrow \\ & HX_3 & HS_{124} \\ & \downarrow & \downarrow \\ K_1 = HX_3F_{124} & & K_2 = HS_{124}F_3 \\ & \searrow & \swarrow \\ & U & \end{array}$$

Since HX_3 is abelian, λ extends to HX_3 as η_1 . By (a), $S_{124} = Stab_T(\lambda)$, for all $x \in H, x_{124} \in S_{124}$, $\lambda([x, x_{124}]) = 1$, hence λ extends to HS_{124} as η_2 . To show that λ extends to K_1 and K_2 , we prove that $[K_1, K_1] \subset \ker(\eta_1)$, $[K_2, K_2] \subset \ker(\eta_2)$. We have

$[x_3(t_3), x_1(s_1)x_2(s_2)x_4(s_4)] = x_5(s_1t_3)x_6(s_2t_3)x_7(s_4t_3)x_8(s_1s_2t_3)x_9(s_1s_4t_3)x_{10}(s_2s_4t_3)$,
and $\lambda(x_5(s_1t_3)x_6(s_2t_3)x_7(s_4t_3)x_8(s_1s_2t_3)x_9(s_1s_4t_3)x_{10}(s_2s_4t_3))$
 $= \phi(t_3(b_5s_1 + b_6s_2 + b_7s_4 + a_8s_1s_2 + a_9s_1s_4 + a_{10}s_2s_4)) = (*)$.

Plug $s_1 = a_{10}t, s_2 = a_9t, s_4 = a_8t$ into $(*)$, we have

$$(*) = \phi(t_3(t(b_5a_{10} + b_6a_9 + b_7a_8) + t^2a_8a_9a_{10})) = \phi(t_3At(t_0 + t)).$$

Now we divide into two cases where $t_0 = 0$ and $t_0 \neq 0$. First, if $t_0 = 0$, $\phi(t_3At^2) = 1$ for all t_3 iff $t = 0$, hence, $Stab_T(\eta_1) = \{1\} = F_{124}$, i.e. $I_U(\eta_1) = HX_3$. And $\phi(t_3At^2) = 0$ for all t iff $t_3 = 0$, hence, $Stab_{X_3}(\eta_2) = \{1\} = F_3$, i.e. $I_U(\eta_2) = HS_{124}$.

If $t_0 \neq 0$, then $\phi(t_3At(t_0 + t)) = 1$ for all t_3 iff $t \in \{0, t_0\}$. Therefore, $[K_1, K_1] \subset \ker \lambda$. For each $\eta \in Irr(HX_3, \lambda)$, $Stab_T(\eta) = \{1, x_{124}(t_0)\} = F_{124}$.

We have $\phi(t_3At(t_0 + t)) = 1$ for all t iff $t_3 \in \{0, \frac{(t_0)\phi}{A}\}$, by Proposition 1.3. Hence, $[K_2, K_2] \subset \ker(\lambda)$. For each $\gamma \in Irr(HS_{124}, \lambda)$, $Stab_{X_3}(\gamma) = \{1, x_3(\frac{(t_0)\phi}{A})\} = F_3$.

So λ extends to K_1 and K_2 . For each $\lambda_i \in Irr(K_i, \lambda)$, $I_U(\lambda_i) = K_i$, $i = 1, 2$.

(c) Let λ_1, λ_2 be extensions of λ to K_1 . Let η be an extension of λ to K_2 . By (b), we have $\lambda_1^U, \lambda_2^U, \eta^U \in Irr(U, \lambda)$.

We choose $1 \in S \subset T$ as a representative set of the double coset $K_1 \backslash U / K_2$, by Mackey formula, since $K_1 \cap K_2 = HF_3F_{124}$ and $K_1 \trianglelefteq U$, we have

$$\begin{aligned} (\lambda_1^U, \eta^U) &= \sum_{s \in S} ({}^s\lambda_1|_{{}^sK_1 \cap K_2}, \eta|_{{}^sK_1 \cap K_2}) \\ &= \sum_{s \in S} ({}^s\lambda_1|_{HF_3F_{124}}, \eta|_{HF_3F_{124}}). \end{aligned}$$

For each $s \in S$, if ${}^s\lambda_1|_{HF_3F_{124}} = \eta|_{HF_3F_{124}}$, then ${}^s\lambda_1|_H = \eta|_H$. Since both are extensions of λ from H , we have ${}^s\lambda = \lambda$, i.e. $s \in Stab_T(\lambda) = S_{124}$. There is unique $s = 1 \in S \cap S_{124}$ since S is a representative set of $K_1 \backslash U / K_2$. Therefore, $(\lambda_1^U, \eta^U) = (\lambda_1|_{HF_3F_{124}}, \eta|_{HF_3F_{124}}) = 1$ iff $\lambda_1|_{F_i} = \eta|_{F_i}, i \in \{124, 3\}$.

So $\lambda_1^U = \eta^U = \lambda_2^U \in Irr(U, \lambda)$ iff $\lambda_1|_{F_i} = \lambda_2|_{F_i}, i \in \{124, 3\}$. \square

It is remarked that since $K_1, K_2 \trianglelefteq U$, the double coset $K_1 \backslash U / K_2$ equals $U / K_1 K_2 = U / HX_3 S_{124}$. Hence, we can pick above $S = X_1 X_2$ as a transversal of $U / K_1 K_2$.

5.3.2. *Proof of Theorem 2.3.* Fix $a_8, a_9, a_{10} \in \mathbb{F}_q^\times$ and set $\lambda = \lambda_{b_5, b_6, b_7}^{a_8, a_9, a_{10}}$ for some $b_5, b_6, b_7 \in \mathbb{F}_q$ in the whole proof. By Lemma 2.2 and using the same notations, we mainly find the generic character values: in (a) $\chi_{8,9,10,q^3}^{a_8, a_9, a_{10}} = \eta_1^U$ where $t_0 = 0$, and in (b) $\chi_{8,9,10,\frac{q^3}{2}}^{b_{124}, b_3, t_0, a_8, a_9, a_{10}} = \eta_1^U$ where $b_{124}, b_3 \in \mathbb{F}_2, t_0 \in \mathbb{F}_q^\times$.

(a) Suppose $t_0 = 0$ and $F_{124} = \{1\}$. Call η an extension of λ to HX_3 . By Lemma 2.2 (b), $I_U(\eta) = HX_3$. Therefore, $\eta^U \in Irr(U)$ and $\eta^U(1) = q^3$. By Lemma 2.2 (a), $S_{567}X_8X_9X_{10} \subset Z(\eta^U)$, hence $|\eta^U(x)| = q^3$ for all $x \in S_{567}X_8X_9X_{10}$. We have $|S_{567}X_8X_9X_{10}|q^3q^3 = q^{10} = |U|$. By the scalar product $(\eta^U, \eta^U) = 1$, it forces $\eta^U(x) = 0$ if $x \notin S_{567}X_8X_9X_{10}$. So we have the formula as stated.

(b) Suppose $t_0 \neq 0$, and $|F_3| = |F_{124}| = 2$. By Lemma 2.2 (b), let η_1, η_2 be extensions of λ to $K_1 := HX_3F_{124}$ and $K_2 := HS_{124}F_3$ respectively such that $\eta_1|_{F_i} = \eta_2|_{F_i} = \phi_{b_i}$, where $b_i \in \mathbb{F}_2, i \in \{124, 3\}$. By the proof of Lemma 2.2 (c), $\eta_1^U = \eta_2^U$.

We choose $V \subset T$ as a transversal of K_1 in U , and $1 \in S \subset X_3$ such that SX_1X_2 is a transversal of K_2 in U , so $|S| = q/2$. Since $K_1 \trianglelefteq U$, we have $\eta_1^U(\prod_{i=1}^{10} x_i) = \sum_{x \in V} x \eta_1(\prod_{i=1}^{10} x_i) = 0$ if $x_1x_2x_4 \notin K_1$. Since T is abelian, $[x, y] = 1$ for all $x \in V$ and $y \in F_{124}$. Therefore, $F_{124} \subset Z(\eta_1^U)$ and we have

$$\eta_1^U(\prod_{i=1}^{10} x_i(t_i)) = \delta_{a_8t_1, a_{10}t_4} \delta_{a_8t_2, a_9t_4} \phi(b_{124} \frac{t_1}{a_{10}}) \eta_1^U(x_3(t_3)) \prod_{i=5}^{10} x_i(t_i).$$

Since $K_2 \trianglelefteq U$, we have $\eta_2^U(x_3 \prod_{i=5}^{10} x_i) = \sum_{x \in SX_1X_2} {}^x\eta_2(x_3 \prod_{i=5}^{10} x_i) = 0$ if $x_3 \notin F_3$. Since $X_8X_9X_{10} \subset Z(U)$, we need to compute the two following cases: $\eta_2^U(\prod_{i=5}^7 x_i)$ and $\eta_2^U(x_3 \prod_{i=5}^7 x_i)$ with $x_3 \in F_3^\times$.

Since $[X_3, X_5X_6X_7] = \{1\}$, we have $\eta_2^U(x_5x_6x_7) = \sum_{x \in SX_1X_2} {}^x\eta_2(x_5x_6x_7) = \frac{q}{2} \sum_{x_1x_2 \in X_1X_2} {}^x\eta_2(x_5x_6x_7)$. Since $(x_5x_6x_7)^{x_1x_2} = x_5x_6x_7[x_5, x_2][x_6, x_1][x_7, x_1][x_7, x_2]$, plug in $x_5(t_5)x_6(t_6)x_7(t_7)$ and $x_1(s_1)x_2(s_2)$, we have

$$\begin{aligned} & \eta_2^U(\prod_{i=5}^7 x_i(t_i)) \\ &= \frac{q}{2} \sum_{s_1, s_2} \eta_2(x_5(t_5)x_6(t_6)x_7(t_7)x_8(t_5s_2 + t_6s_1)x_9(t_7s_1)x_{10}(t_7s_2)) \\ &= \frac{q}{2} \eta_2(x_5(t_5)x_6(t_6)x_7(t_7)) \sum_{s_1, s_2} \phi(a_8(t_5s_2 + t_6s_1) + a_9t_7s_1 + a_{10}t_7s_2) \\ &= \frac{q}{2} \eta_2(x_5(t_5)x_6(t_6)x_7(t_7)) \sum_{s_1, s_2} \phi(s_1(a_8t_6 + a_9t_7) + s_2(a_8t_5 + a_{10}t_7)). \end{aligned}$$

Since $\sum_{t \in \mathbb{F}_q} \phi(t) = 0$, to obtain non-zero values, it forces $a_8t_6 + a_9t_7 = 0$ and $a_8t_5 + a_{10}t_7 = 0$. Hence, $\frac{t_5}{a_{10}} = \frac{t_6}{a_9} = \frac{t_7}{a_8}$, and $\prod_{i=5}^7 x_i(t_i) = x_{567}(\frac{t_7}{a_8}) \in S_{567}$. By Lemma 2.2 (a), we have

$$\begin{aligned} \eta_2^U(\prod_{i=5}^7 x_i(t_i)) &= \delta_{a_8t_5, a_{10}t_7} \delta_{a_8t_6, a_9t_7} \frac{q^3}{2} \eta_2(x_{567}(\frac{t_7}{a_8})) \\ &= \delta_{a_8t_5, a_{10}t_7} \delta_{a_8t_6, a_9t_7} \frac{q^3}{2} \phi(At_0 \frac{t_7}{a_8}). \end{aligned}$$

Therefore, $\eta_2^U(\prod_{i=1}^{10} x_i(t_i)) = \frac{q^3}{2} \phi(b_{124} \frac{t_1}{a_{10}} + At_0 \frac{t_7}{a_8} + \sum_{i=8}^{10} a_i t_i)$ if $\prod_{i=1}^{10} x_i(t_i) \in F_{124}S_{567}X_8X_9X_{10} = Z$, as stated in the theorem.

Now we compute $\eta_2^U(x_3 \prod_{i=5}^7 x_i)$ with $x_3 \in F_3^\times = \{x_3(t_0^\phi)\}$ where $t_0^\phi = \frac{(t_0)_\phi}{A}$. Since $I_U(\eta_2) = K_2 \trianglelefteq U$ and SX_1X_2 is a representative set of U/K_2 , $({}^x\eta_2)^U = \eta_2^U \in Irr(U)$ for all $x \in SX_1X_2$. For each $x_2(s) \in X_2$, we have

$${}^{x_2(s)}\eta_2(x_5(t)) = \eta_2(x_5(t)x_8(ts)) = \phi(b_5t + a_8ts) = \phi(t(b_5 + a_8s)).$$

So instead of choosing $s = \frac{b_5}{a_8}$, we suppose that η_2 has $b_5 = 0$, i.e. $\eta_2(x_5) = 1$ for all $x_5 \in X_5$. It is easy to check that $t_0, \eta_2|_{F_{124}} = \phi_{b_{124}}$ and $\eta_2|_{F_3} = \phi_{b_3}$ are invariant under this conjugate action.

We have $[x_3(t_3)x_5(t_5)x_6(t_6)x_7(t_7), x_1(s_1)x_2(s_2)] = x_3(t_3)x_5(t_5 + t_3s_1)x_6(t_6 + t_3s_2)x_7(t_7)x_8(t_3s_1s_2 + t_5s_2 + t_6s_1)x_9(t_7s_1)x_{10}(t_7s_2)$.

Therefore,

$$\begin{aligned} & \eta_2^U(x_3(t_3)x_5(t_5)x_6(t_6)x_7(t_7)) = \sum_{x \in SX_1X_2} {}^x\eta_2(x_3(t_3)x_5(t_5)x_6(t_6)x_7(t_7)) \\ &= \frac{q}{2} \sum_{x \in X_1X_2} {}^x\eta_2(x_3(t_3)x_5(t_5)x_6(t_6)x_7(t_7)) \\ &= \frac{q}{2} \sum_{s_1, s_2} \eta_2(x_3(t_3)x_5(t_5 + t_3s_1)x_6(t_6 + t_3s_2)x_7(t_7)x_8(t_3s_1s_2 + t_5s_2 + t_6s_1)x_9(t_7s_1)x_{10}(t_7s_2)) \\ &= \frac{q}{2} \eta_2(x_3(t_3) \prod_{i=5}^7 x_i(t_i)) \sum_{s_1, s_2} \phi(b_6t_3s_2 + a_8(t_3s_1s_2 + t_5s_2 + t_6s_1) + a_9t_7s_1 + a_{10}t_7s_2) \\ &= \frac{q}{2} \eta_2(x_3(t_3)x_6(t_6)x_7(t_7)) \sum_{s_1, s_2} \phi(s_1(a_8t_3s_2 + a_8t_6 + a_9t_7) + s_2(b_6t_3 + a_{10}t_7 + a_8t_5)). \end{aligned}$$

Set $C(t_5, t_6, t_7) = \sum_{s_1, s_2} \phi(s_1(a_8t_3s_2 + a_8t_6 + a_9t_7) + s_2(b_6t_3 + a_{10}t_7 + a_8t_5))$.

We have

$$\begin{aligned} C(t_5, t_6, 0) &= \sum_{s_1, s_2} \phi(s_1(t_3s_2 + t_6)a_8 + s_2(b_6t_3 + a_8t_5)) \\ &= q \sum_{s_2 = \frac{t_6}{t_3}} \phi(\frac{t_6}{t_3}(b_6t_3 + a_8t_5)) \\ &= q\phi(b_6t_6 + \frac{a_8t_5t_6}{t_3}). \end{aligned}$$

Therefore, we get

$$\begin{aligned} \eta_2^U(x_3(t_3)x_5(t_5)x_6(t_6)) &= \frac{q^2}{2} \eta_2(x_3(t_3)x_6(t_6)) \phi(b_6t_6 + \frac{a_8t_5t_6}{t_3}) \\ &= \frac{q^2}{2} \eta_2(x_3(t_3)) \phi(\frac{a_8t_5t_6}{t_3}). \end{aligned}$$

Since $x_5(t_5)x_6(t_6)x_7(t_7) = x_5(t_5 + \frac{a_{10}t_7}{a_8})x_6(t_6 + \frac{a_9t_7}{a_8})x_{567}(\frac{t_7}{a_{10}})$, where $x_{567}(\frac{t_7}{a_8}) = x_5(a_{10} \frac{t_7}{a_8})x_6(a_9 \frac{t_7}{a_8})x_7(a_8 \frac{t_7}{a_8}) \in S_{124} \subset Z(\eta_2^U)$ and $\eta_2(x_{567}(\frac{t_7}{a_8})) = \phi_{At_0}(\frac{t_7}{a_8})$, we have

$$\begin{aligned}
\eta_2^U(x_3(t_3)x_5(t_5)x_6(t_6)x_7(t_7)) &= \phi_{At_0}(\frac{t_7}{a_8})\eta_2^U(x_3(t_3)x_5(t_5 + \frac{a_{10}t_7}{a_8})x_6(t_6 + \frac{a_9t_7}{a_8})) \\
&= \phi(At_0\frac{t_7}{a_8})\frac{q^2}{2}\eta_2(x_3(t_3))\phi(\frac{a_8}{t_3}(t_5 + \frac{a_{10}t_7}{a_8})(t_6 + \frac{a_9t_7}{a_8})) \\
&= \frac{q^2}{2}\phi(b_3t_3 + At_0\frac{t_7}{a_8} + \frac{a_8^2a_9a_{10}}{(t_0)_\phi}(t_5 + \frac{a_{10}t_7}{a_8})(t_6 + \frac{a_9t_7}{a_8})) \\
&= \frac{q^2}{2}\phi(b_3t_0^\phi + At_0\frac{t_7}{a_8} + \frac{A^2}{(t_0)_\phi}(\frac{t_5}{a_{10}} + \frac{t_7}{a_8})(\frac{t_6}{a_9} + \frac{t_7}{a_8})). \quad \square
\end{aligned}$$

5.4. Proof of Sylow 3-subgroups of $E_6(3^f)$.

5.4.1. *Proof of Lemma 3.2.* Set $\lambda = \lambda_{b_8, b_9, b_{10}}^{b_{12}, b_{13}, b_{14}, b_{15}, b_{16}}$ for the whole proof.

(a) Recall $H_3 = \prod_{i=12}^{16} X_i \trianglelefteq H$ is elementary abelian and $H_4H_3 \trianglelefteq U$. First, we show that $R_3 = \{x \in H_3 : |\lambda^U(x)| = \lambda^U(1)\}$ and $\lambda^U(r_3(t)) = q^8\phi_{B_3}(t)$ for all $r_3(t) \in R_3$.

Since λ is linear and $\lambda(x) \in \mathbb{C}$ for all $x \in H$, by the induction formula, we have $|\lambda^U(x)| = \lambda^U(1)$ iff ${}^y\lambda(x) = \lambda(x)$ for all $y \in TX_4$ which is a transversal of H in U . Since ${}^y\lambda(x) = \lambda(x)$ iff $\lambda([x, y]) = 1$, we are going to find all $x \in H_3$ such that $\lambda([x, y]) = 1$ for all $y \in TX_4$. It is clear that $[X_i, X_4] = \{1\} = [X_i, X_7X_{11}]$ for all $12 \leq i \leq 16$. Here, we write $\prod_{j=1}^6 x_j(u_j) \in T$ with $u_4 = 0$, it suffices to check for all $y = \prod_{j=1}^6 x_j(u_j) \in T$. For $t_i, u_j \in \mathbb{F}_q$, we have

$$\begin{aligned}
&[\prod_{i=12}^{16} x_i(t_i), \prod_{j=1}^6 x_j(u_j)] = \\
&[x_{12}(t_{12}), x_2(u_2)] [x_{15}(t_{15}), x_2(u_2)] [x_{16}(t_{16}), x_2(u_2)] [x_{13}(t_{13}), x_1(u_1)] \\
&[x_{15}(t_{15}), x_1(u_1)] [x_{14}(t_{14}), x_3(u_3)] [x_{16}(t_{16}), x_3(u_3)] [x_{12}(t_{12}), x_5(u_5)] \\
&[x_{13}(t_{13}), x_5(u_5)] [x_{14}(t_{14}), x_6(u_6)] [x_{15}(t_{15}), x_6(u_6)] \\
&= x_{17}(-t_{12}u_2)x_{19}(-t_{15}u_2)x_{20}(-t_{16}u_2)x_{17}(-t_{13}u_1)x_{18}(-t_{15}u_1)x_{19}(-t_{14}u_3) \\
&x_{21}(-t_{16}u_3)x_{18}(t_{12}u_5)x_{19}(t_{13}u_5)x_{20}(t_{14}u_6)x_{21}(t_{15}u_6)
\end{aligned}$$

Since $\lambda(x_i(t)) = \phi(t)$, $17 \leq i \leq 21$, for all $u_j \in \mathbb{F}_q$ it forces

$$(-t_{12} - t_{15} - t_{16})u_2 + (-t_{13} - t_{15})u_1 + (-t_{14} - t_{16})u_3 + (t_{12} + t_{13})u_5 + (t_{14} + t_{15})u_6 = 0.$$

So we have a system with variables t_i :

$$\begin{cases} -t_{12} - t_{15} - t_{16} = 0 \\ -t_{13} - t_{15} = 0 \\ -t_{14} - t_{16} = 0 \\ t_{12} + t_{13} = 0 \\ t_{14} + t_{15} = 0 \end{cases}$$

Since $\gcd(q, 3) = 3$, we have $t_{12} = t_{16} = t_{15}, t_{13} = t_{14} = -t_{15}$ for all $t_{15} = t \in \mathbb{F}_q$. So $x \in H_3$ satisfies $|\lambda^U(x)| = \lambda^U(1)$ iff $x = x_{12}(t)x_{13}(-t)x_{14}(-t)x_{15}(t)x_{16}(t) = r_3(t) \in R_3$ for $t \in \mathbb{F}_q$, i.e. $R_3 = \{x \in H_3 : |\lambda^U(x)| = \lambda^U(1)\}$.

By the above computation, to show that $\lambda^U|_{R_3} = \lambda^U(1)\phi_{B_3}$, it is enough to check that $\lambda(r_3(t)) = \phi_{B_3}(t)$. For each $r_3(t) \in R_3$, we have

$$\lambda(r_3(t)) = \phi(t(b_{12} - b_{13} - b_{14} + b_{15} + b_{16})) = \phi_{B_3}(t).$$

Now we show that $S_1 = \text{Stab}_T(\lambda|_{H_4H_3})$. Since $[H_4, T] = [H_3, X_7X_{11}] = \{1\}$, it suffices to find $y \in X_2X_1X_3X_5X_6$ such that $\lambda([x, y]) = 1$ for all $x \in H_3$. Using the above computation of $[\prod_{i=12}^{16} x_i(t_i), \prod_{j=1}^6 x_j(u_j)]$, for all $t_j \in \mathbb{F}_q$ it forces

$$(-u_2 + u_5)t_{12} + (-u_1 + u_5)t_{13} + (-u_3 + u_6)t_{14} + (u_1 + u_2 - u_6)t_{15} + (-u_2 - u_3)t_{16} = 0.$$

So we have a system with variables u_j :

$$\left\{ \begin{array}{l} -u_2 + u_5 = 0 \\ -u_1 + u_5 = 0 \\ -u_3 + u_6 = 0 \\ -u_1 - u_2 + u_6 = 0 \\ -u_2 - u_3 = 0. \end{array} \right.$$

Since $\gcd(q, 3) = 3$, we have $u_1 = u_5 = u_2, u_3 = u_6 = -u_2$ for all $u_2 = t \in \mathbb{F}_q$. So $\prod_{j=1}^6 x_j(u_j) = x_2(t)x_1(t)x_3(-t)x_5(t)x_6(-t) = s_1(t) \in S_1$.

(b) Since $H = Z(U)H_3X_8X_9X_{10}$, to find $Stab_T(\lambda) \subset Stab_T(\lambda|_{H_4H_3}) = S_1$, by (a), it is enough to find $s_1 \in S_1$ such that ${}^{s_1}\lambda(x_i) = \lambda(x_i)$ for $i = 8, 9, 10$. Again, for each $x_i(t_i) \in X_i$, $i = 8, 9, 10$ and $s_1 = s_1(t, r, s) \in S_1$, we compute $[x_i, s_1]$

$$\begin{aligned} [x_8(t_8), s_1] &= x_{20}(t_8s)x_{17}(-t_8r)x_{14}(t_8t)x_{20}(-t_8t^2)x_{13}(t_8t)x_{19}(t_8t^2). \\ [x_9(t_9), s_1] &= x_{21}(t_9s)x_{15}(t_9t)x_{21}(-t_9t^2)x_{12}(-t_9t)x_{18}(-t_9t^2)x_{13}(-t_9t)x_{19}(-t_9t^2)x_{17}(t_9t^2). \\ [x_{10}(t_{10}), s_1] &= x_{18}(-t_{10}r)x_{16}(-t_{10}t)x_{15}(t_{10}t)x_{21}(-t_{10}t^2)x_{14}(-t_{10}t)x_{20}(t_{10}t^2)x_{19}(-t_{10}t^2) \end{aligned}$$

Since $\lambda(x_i(t)) = \phi(b_i t)$, $12 \leq i \leq 16$ and $\lambda(x_i(t)) = \phi(t)$, $17 \leq i \leq 21$, from $\lambda([x_9(t_9), s_1]) = \lambda([x_{10}(t_{10}), s_1]) = 1$ for all $t_9, t_{10} \in \mathbb{F}_q$, we have

$$s = 2t^2 + b_{12}t + b_{13}t - b_{15}t \text{ and } r = 2t^2 - b_{14}t + b_{15}t - b_{16}t.$$

From $\lambda([x_8(t_8), s_1]) = 1$ for all $t_8 \in \mathbb{F}_q$, we have $s - r + b_{14}t + b_{13}t = 0$. Therefore, $s_1 \in Stab_T(\lambda)$ iff r, s as above and

$$\begin{aligned} s - r + b_{14}t + b_{13}t &= 2t^2 + b_{12}t + b_{13}t - b_{15}t - (2t^2 - b_{14}t + b_{15}t - b_{16}t) + b_{14}t + b_{13}t \\ &= t(b_{12} - b_{13} - b_{14} + b_{15} + b_{16}) \\ &= tB_3 = 0. \end{aligned}$$

Therefore, if $B_3 \neq 0$, then $Stab_T(\lambda) = \{1\}$.

(c) By (a), T/S_1 acts faithfully on the set of all extensions of $\lambda|_{H_4}$ to H_4H_3 with the same B_3 . Since $|T/S_1| = q^4 = |H_3/R_3|$, this action is transitive. Therefore, with $B_3 = 0$, there exists $x \in T$ such that $x\lambda = \lambda_{b'_8, b'_9, b'_{10}}^{0,0,0,0,0}$ for some $b'_8, b'_9, b'_{10} \in \mathbb{F}_q$.

Now set $\lambda = \lambda_{b_8, b_9, b_{10}}^{0,0,0,0,0}$, and $\overline{H_3}$ is the normal closure of H_3 in HX_4S_1 . To show that $\overline{H_3} \subset \ker(\lambda^{HX_4S_1}) \trianglelefteq HX_4S_1$, it suffices to show that $H_3 \subset \ker(\lambda^{HX_4S_1})$. By (a) $Stab_{TX_4}(\lambda|_{H_4H_3}) = S_1X_4$ which is a transversal of H in HX_4S_1 , the claim holds by the induction formula and $H_3 \subset \ker(\lambda)$.

By Lemma 1.5 for $G = U$ with $N = M = HX_4S_1$, $X = X_1X_3X_5X_6$, $Y = \overline{H_3}$ and $Z = H_4$, the induction map from $Irr(HX_4S_1/\overline{H_3}, \lambda)$ to $Irr(U, \lambda)$ is bijective. Since $\overline{H_3} \subset \lambda^{HX_4S_1}$, $Irr(HX_4S_1/\overline{H_3}, \lambda) = Irr(HX_4S_1, \lambda)$. \square

5.4.2. *Proof of Lemma 3.3.* Recall $R_2 = \{r_2(t) := x_8(-t)x_9(t)x_{10}(t) : t \in \mathbb{F}_q\} \leq H_2 = X_8X_9X_{10}$ and $\lambda = \lambda_{b_8, b_9, b_{10}}^{0,0,0,0,0}$. By Lemma 3.2 (c), it suffices to work with the quotient group $HX_4S_1/\overline{H_3}$.

(a) The fact $S_2 = Stab_{S_1}(\lambda)$ comes directly from Lemma 3.2 (b) with $B_3 = 0$. Since X_4S_1 is a transversal of H in HX_4S_1 and $[H_2, X_4] = \{1\}$, to show $R_2 = \{x \in H_2 : |\lambda^{HX_4S_1}(x)| = \lambda^{HX_4S_1}(1)\}$ we are going to find all $x \in H_2$ such that $\lambda([x, y]) = 1$ for all $y \in S_1$. Since $H \trianglelefteq HX_4S_1$ is abelian, using the computation in Lemma 3.2 (b), for $s_1(t, r, s) \in S_1$ and $x_8(t_8)x_9(t_9)x_{10}(t_{10}) \in H_2$ we have

$$\begin{aligned} &[x_8(t_8)x_9(t_9)x_{10}(t_{10}), s_1(t, r, s)] \\ &= [x_8(t_8), s_1(t, r, s)][x_9(t_9), s_1(t, r, s)][x_{10}(t_{10}), s_1(t, r, s)] \\ &= x_{20}(t_8s)x_{17}(-t_8r)x_{20}(-t_8t^2)x_{19}(t_8t^2)x_{21}(t_9s)x_{21}(-t_9t^2)x_{18}(-t_9t^2)x_{19}(-t_9t^2) \\ & \quad x_{17}(t_9t^2)x_{18}(-t_{10}r)x_{21}(-t_{10}t^2)x_{20}(t_{10}t^2)x_{19}(-t_{10}t^2) \end{aligned}$$

Therefore, with $\lambda|_{X_i} = \phi$ for all $17 \leq i \leq 21$, for all $t, r, s \in \mathbb{F}_q$ we need

$$(t_8 + t_9)s - (t_8 + t_{10})r + (t_9 - t_{10})t^2 = 0$$

So $t_9 = t_{10} = u$ and $t_8 = -u$ for all $u \in \mathbb{F}_q$, i.e. $x = r_2(u) \in R_2$.

To show that $\lambda^{HX_4S_1}|_{R_2} = \lambda^{HX_4S_1}(1)\phi_{B_2}$, it is enough to check that $\lambda(r_2(t)) = \phi_{B_2}(t)$. For each $r_2(t) \in R_2$ we have

$$\lambda(r_2(t)) = \phi(t(-b_8 + b_9 + b_{10})) = \phi_{B_2}(t).$$

(b) Suppose that $B_2 \notin \{c^2 : c \in \mathbb{F}_q^\times\}$. Let η be an extension of λ to HX_4 . By (a) that $S_2 = \text{Stab}_{S_1}(\lambda)$, hence $\text{Stab}_{S_1}(\eta|_H) = S_2$. Since S_1 is a transversal of HX_4 in HX_4S_1 , to find $\text{Stab}_{S_1}(\eta)$, it is enough to find all $s_2(t) \in S_2$ such that $\eta([x_4, s_2(t)]) = 1$ for all $x_4 \in X_4$. For each $s_2(t) \in S_2$, we have

$$[x_4(t_4), s_2(t)] = \begin{matrix} x_{10}(t_4t)x_9(t_4t)x_{21}(2t_4t^3)x_{21}(-t_4t^3)x_8(-t_4t) \\ x_{20}(-2t_4t^3)x_{17}(2t_4t^3)x_{20}(t_4t^3)x_{19}(-t_4t^3). \end{matrix}$$

Since $\eta(x_i(t)) = \phi(b_it)$, $8 \leq i \leq 10$ and $\eta(x_i(t)) = \phi(t)$, $17 \leq i \leq 21$, for all $t_4 \in \mathbb{F}_q$, $\eta([x_4(t_4), s_3(t)]) = 1$ forces

$$t_4(t^3 - B_2t) = t_4(t^3 - B_2t) \in \ker\phi.$$

Since $t_4(t^3 - B_2t) \in \ker\phi$ for all $t_4 \in \mathbb{F}_q$, we have $0 = t^3 - B_2t = t(t^2 - B_2)$. Since $B_2 \notin \{c^2 : c \in \mathbb{F}_q^\times\}$, the equation $t(t^2 - B_2) = 0$ only has trivial solution $t = 0$ in \mathbb{F}_q . Therefore, $s_2(t) = 1$, i.e. $\text{Stab}_{S_1}(\eta) = \{1\}$. Hence, $I_{HX_4S_1}(\eta) = HX_4$.

(c) Suppose $B_2 = c^2 \in \mathbb{F}_q^\times$ and let η be an extension of λ to HX_4 . Using the computation in (b), we continue with the analysis for the solutions of t to obtain $t_4t(t^2 - B_2) \in \ker\phi$ for all $t_4 \in \mathbb{F}_q$. So it forces $t(t^2 - B_2) = 0$. This equation has 3 solutions $\{0, \pm c\}$. Hence, $\text{Stab}_{S_1}(\eta) = \{1, s_2(\pm c)\} = F_2$. So $I_{HX_4S_1}(\eta) = HX_4F_2$. By the above argument, $[HX_4F_2, HX_4F_2] \subset \ker(\eta)$, hence η extends to $I_{HX_4S_1}(\eta)$.

To show that λ extends to HF_4S_2 , we check $[HF_4S_2, HF_4S_2] \subset \ker(\lambda)$. With the same argument, it is enough to check that $[s_2(t), x_4(t_4)] \in \ker\lambda$. By the computation in (b), we need $t_4(t^3 - B_2t) = t_4(t^3 - c^2t) \in \ker\phi$ for all $t \in \mathbb{F}_q$. By Proposition 1.3, since $t_4 \in \{0, \pm c_\phi\}$, the claim holds.

Let λ_1, λ_2 be two extensions of λ to HX_4F_2 , and γ an extension of λ to HF_4S_2 . Since the degree of all irreducible constituents of $\lambda^{HX_4S_1}$ is $\frac{q^3}{3}$, we have $\lambda_1^{HX_4S_1}, \lambda_2^{HX_4S_1}, \gamma^{HX_4S_1} \in \text{Irr}(HX_4S_1, \lambda)$.

Choose $1 \in S \subset S_1$ as a representative set of the double coset $HF_4S_2 \backslash HX_4S_1 / HX_4F_2$. Since $HF_4S_2 \cap HX_4F_2 = HF_4F_2$ and $HX_4F_2 \trianglelefteq HX_4S_1$, by Mackey formula,

$$\begin{aligned} (\lambda_1^{HX_4S_1}, \gamma^{HX_4S_1}) &= \sum_{s \in S} ({}^s\lambda_1|_{(HX_4F_2) \cap HF_4S_2}, \gamma|_{(HX_4F_2) \cap HF_4S_2}) \\ &= \sum_{s \in S} ({}^s\lambda_1|_{HF_4F_2}, \gamma|_{HF_4F_2}) \end{aligned}$$

For each $s \in S$, if ${}^s\lambda_1|_{HF_4F_2} = \gamma|_{HF_4F_2}$, then ${}^s\lambda_1|_H = \gamma|_H$. Since both are extensions of λ , we have ${}^s\lambda = \lambda$, i.e. $s \in \text{Stab}_{S_1}(\lambda) = S_2$. There is unique $1 \in S \cap S_2$ since S is a representative set of $HF_4S_2 \backslash HX_4S_1 / HX_4F_2$. So $(\lambda_1^{HX_4S_1}, \gamma^{HX_4S_1}) = (\lambda_1|_{HF_4F_2}, \gamma|_{HF_4F_2}) = 1$ iff $\lambda_1|_{F_i} = \gamma|_{F_i}, i \in \{2, 4\}$.

Therefore, $\lambda_1^{HX_4S_1} = \gamma^{HX_4S_1} = \lambda_2^{HX_4S_1}$ iff $\lambda_1|_{F_i} = \lambda_2|_{F_i}, i \in \{2, 4\}$. \square

5.5. Proofs of Sylow 5-subgroups of $E_8(5^f)$.

5.5.1. *Proof of Lemma 4.2.* (a) First we find all $x \in H_5$ such that $|\lambda^U(x)| = \lambda^U(1)$. Since TX_5 is a transversal of H in U , $[H_5, X_5] = \{1\} = [H_5, T_k]$ for all $k \geq 2$, and ${}^y\lambda(x) = \lambda(x)$ iff $\lambda([x, y]) = 1$, it suffices to find all $x \in H_5$ such that $\lambda([y, x]) = 1$ where $y \in T_1$. For each $y = \prod_{i=1}^8 x_i(u_i) \in T_1$ with $u_5 = 0$, and $x = \prod_{j=30}^{36} x_j(v_j) \in H_5$, to write shortly for the decomposition we write $x_i = x_i(-)$ and plug in the parameters in $(-)$ latter, we have

$$\begin{aligned} & \left[\prod_{j=30}^{36} x_j(v_j), \prod_{i=1}^8 x_i(u_i) \right] = [x_{30}, x_4] [x_{30}, x_6] [x_{31}, x_2] [x_{31}, x_7] [x_{32}, x_1] [x_{32}, x_6] [x_{33}, x_1] \\ & [x_{33}, x_4] [x_{33}, x_7] [x_{34}, x_3] [x_{34}, x_8] [x_{35}, x_2] [x_{35}, x_1] [x_{35}, x_8] [x_{36}, x_2] [x_{36}, x_3] \\ & = x_{37}(-v_{30}u_4)x_{38}(v_{30}u_6)x_{38}(-v_{31}u_2)x_{39}(v_{31}u_7)x_{37}(-v_{32}u_1)x_{40}(v_{32}u_6)x_{38}(-v_{33}u_1) \\ & x_{40}(-v_{33}u_4)x_{41}(v_{33}u_7)x_{41}(-v_{34}u_3)x_{42}(v_{34}u_8)x_{41}(-v_{35}u_2)x_{39}(-v_{35}u_1)x_{43}(v_{35}u_8) \\ & x_{42}(-v_{36}u_2)x_{43}(-v_{36}u_3). \end{aligned}$$

Since $\lambda|_{X_i} = \phi$ for all $i \in [37..43]$, for all s_j we need $(-v_{31} - v_{35} - v_{36})u_2 + (-v_{32} - v_{33} - v_{35})u_1 + (-v_{34} - v_{36})u_3 + (-v_{30} - v_{33})u_4 + (v_{30} + v_{32})u_6 + (v_{31} + v_{33})u_7 + (v_{34} + v_{35})u_8 = 0$.

Therefore, we obtain a system with variables v_i as follows.

$$\left\{ \begin{array}{l} -v_{31} - v_{35} - v_{36} = 0 \\ -v_{32} - v_{33} - v_{35} = 0 \\ -v_{34} - v_{36} = 0 \\ -v_{30} - v_{33} = 0 \\ v_{30} + v_{32} = 0 \\ v_{31} + v_{33} = 0 \\ v_{34} + v_{35} = 0. \end{array} \right.$$

Since $\gcd(q, 5) = 5$, $(v_{30}, v_{31}, v_{32}, v_{33}, v_{34}, v_{35}, v_{36}) = (v, v, -v, -v, -2v, 2v, 2v)$ for all $v \in \mathbb{F}_q$. Hence, $x = r_5(v) \in R_5$, i.e. $R_5 = \{x \in H_5 : |\lambda^U(x)| = \lambda^U(1)\}$.

To show that $\lambda^U|_{R_5} = \lambda^U(1)\phi_{B_5}$, it suffices to check that $\lambda(r_5(v)) = \phi_{B_5}(v)$. For each $r_5(v) \in R_5$, we have

$$\lambda(r_5(v)) = \phi(v(b_{30} + b_{31} - b_{32} - b_{33} - 2b_{34} + 2b_{35} + 2b_{36})) = \phi_{B_5}(v).$$

To show that $S_1 = \text{Stab}_T(\lambda|_{H_6H_5})$, we find all $y \in T$ such that $\lambda([x, y]) = 1$ for all $x \in H_6H_5$. Since $H_6 = Z(U)$ and $[H_5, T_k] = \{1\}$ for all $k \geq 2$, it is enough to find $y \in T_1$ such that $\lambda([x, y]) = 1$ for all $x \in H_5$. Using the above computation of $\left[\prod_{j=30}^{36} x_j(v_j), \prod_{i=1}^8 x_i(u_i) \right]$, we find u_i such that for all v_j :

$$(-u_4 + u_6)v_{30} + (-u_2 + u_7)v_{31} + (-u_1 + u_6)v_{32} + (-u_1 - u_4 + u_7)v_{33} + (-u_3 + u_8)v_{34} + (-u_2 - u_1 + u_8)v_{35} + (-u_2 - u_3)v_{36} = 0.$$

Therefore, we obtain a system with variables u_i as follows.

$$\left\{ \begin{array}{l} -u_4 + u_6 = 0 \\ -u_2 + u_7 = 0 \\ -u_1 + u_6 = 0 \\ -u_1 - u_4 + u_7 = 0 \\ -u_3 + u_8 = 0 \\ -u_2 - u_1 + u_8 = 0 \\ -u_2 - u_3 = 0. \end{array} \right.$$

Since $\gcd(q, 5) = 5$, we have $(u_2, u_1, u_3, u_4, u_6, u_7, u_8) = (2u, u, -2u, u, u, 2u, -2u)$ for all $u \in \mathbb{F}_q$. So $y = l_1(u) \in L_1$, i.e. $S_1 = \text{Stab}_T(\lambda|_{H_6H_5})$.

(b) Suppose $B_5 \neq 0$. To show that $\text{Stab}_T(\lambda) = \{1\}$, we are going to show that $\text{Stab}_{S_1}(\lambda|_{H_6H_5H_4}) = T_3T_4$, $\text{Stab}_{T_3T_4}(\lambda|_{H_6H_5H_4H_3}) = T_4$ and $\text{Stab}_{T_4}(\lambda) = \{1\}$.

First, we show that $Stab_{S_1}(\lambda|_{H_6H_5H_4}) = T_3T_4$. By the root heights, it is clear that $[H_6H_5H_4, T_3T_4] = \{1\}$, hence, $T_3T_4 \subset Stab_{S_1}(\lambda|_{H_6H_5H_4})$. It suffices to show that $Stab_{L_1T_2}(\lambda|_{H_6H_5H_4}) = \{1\}$, i.e. there is no nontrivial $y \in L_1T_2$ such that $\lambda([h, y]) = 1$ for all $h \in H_5H_4$. For each $y = \prod_{i=1}^{15} x_i(u_i) \in L_1T_2$ (with $u_5 = u_{12} = u_{13} = 0$ and $\prod_{i=1}^8 x_i(u_i) = l_1(u)$), and $h = \prod_{j=24}^{36} x_j(v_j) \in H_5H_4$, we have

$$\begin{aligned} & \left[\prod_{j=24}^{36} x_j(v_j), \prod_{i=1}^{15} x_i(u_i) \right] = [x_{24}, x_{10}][x_{24}, x_{14}][x_{25}, x_{14}][x_{26}, x_{15}][x_{26}, x_9][x_{26}, x_{11}] \\ & [x_{27}, x_{15}][x_{27}, x_{10}][x_{29}, x_{11}][x_{24}, x_2][x_{24}, x_2][x_{24}, x_2][x_{24}, x_2][x_{24}, x_6][x_{24}, x_6][x_{24}, x_6][x_{24}, x_6][x_{24}, x_6][x_{24}, x_6] \\ & [x_{25}, x_1][x_{25}, x_1][x_{25}, x_1][x_{25}, x_1][x_{25}, x_1][x_{25}, x_1][x_{25}, x_1][x_{25}, x_1][x_{25}, x_1][x_{25}, x_1][x_{25}, x_1][x_{25}, x_1][x_{25}, x_1][x_{25}, x_1][x_{25}, x_1] \\ & [x_{26}, x_3][x_{26}, x_3][x_{26}, x_3][x_{26}, x_3][x_{26}, x_3][x_{26}, x_3][x_{26}, x_3][x_{26}, x_3][x_{26}, x_3][x_{26}, x_3][x_{26}, x_3][x_{26}, x_3][x_{26}, x_3][x_{26}, x_3][x_{26}, x_3] \\ & [x_{27}, x_2][x_{27}, x_2][x_{27}, x_2][x_{27}, x_2][x_{27}, x_2][x_{27}, x_2][x_{27}, x_2][x_{27}, x_2][x_{27}, x_2][x_{27}, x_2][x_{27}, x_2][x_{27}, x_2][x_{27}, x_2][x_{27}, x_2][x_{27}, x_2] \\ & [x_{28}, x_2][x_{28}, x_2][x_{28}, x_2][x_{28}, x_2][x_{28}, x_2][x_{28}, x_2][x_{28}, x_2][x_{28}, x_2][x_{28}, x_2][x_{28}, x_2][x_{28}, x_2][x_{28}, x_2][x_{28}, x_2][x_{28}, x_2][x_{28}, x_2] \\ & = x_{37}(v_{24}u_{10})x_{39}(v_{24}u_{14})x_{41}(v_{25}u_{14})x_{42}(v_{26}u_{15})x_{38}(-v_{26}u_9)x_{40}(v_{26}u_{11})x_{43}(v_{27}u_{15}) \\ & x_{40}(v_{27}u_{10})x_{39}(-v_{28}u_9)x_{42}(-v_{29}u_{10})x_{43}(-v_{29}u_{11})x_{30}(-2v_{24}u)x_{38}(-2v_{24}u^2)x_{37}(2v_{24}u^2) \\ & x_{31}(v_{24}u)x_{39}(v_{24}2u^2)x_{30}(-v_{25}u)x_{37}(v_{25}u^2)x_{38}(-v_{25}u^2)x_{32}(-v_{25}u)x_{40}(-v_{25}u^2)x_{33}(v_{25}u) \\ & x_{41}(2v_{25}u^2)x_{33}(2v_{26}u)x_{40}(-2v_{26}u^2)x_{41}(4v_{26}u^2)x_{34}(2v_{26}u)x_{42}(-4v_{26}u^2)x_{33}(-2v_{27}u) \\ & x_{38}(2v_{27}u^2)x_{40}(2v_{27}u^2)x_{41}(-4v_{27}u^2)x_{31}(-v_{27}u)x_{39}(-2v_{27}u^2)x_{35}(v_{27}2u)x_{43}(-4v_{27}u^2) \\ & x_{34}(-2v_{28}u)x_{41}(-4v_{28}u^2)x_{42}(4v_{28}u^2)x_{35}(2v_{28}u)x_{43}(-4v_{28}u^2)x_{36}(-v_{28}2u)x_{36}(-v_{29}u). \end{aligned}$$

Since $\lambda|_{X_i} = \phi$ for all $i \in [37..43]$ and $\lambda|_{X_i} = \phi_{b_i}$ for the others, after evaluating the above with λ to get 1, for all v_j , we need

$$\begin{aligned} & v_{24}(u_{10} + u_{14} - 2b_{30}u + b_{31}u + 2u^2) + v_{25}(u_{14} - b_{30}u - b_{32}u + b_{33}u + u^2) + v_{26}(u_{15} - u_9 + u_{11} + 2b_{33}u - 2u^2 + 2b_{34}u) \\ & + v_{27}(u_{15} + u_{10} - 2b_{33}u - b_{31}u + 2b_{35}u - u^2) + v_{28}(-u_9 - 2b_{34}u + 2b_{35}u + u^2 - 2b_{36}u) + v_{29}(-u_{10} - u_{11} - b_{36}u) = 0. \end{aligned}$$

Hence, we have a system with variables u_i and u :

$$\left\{ \begin{array}{l} u_{10} + u_{14} - 2b_{30}u + b_{31}u + 2u^2 = 0 \\ u_{14} - b_{30}u - b_{32}u + b_{33}u + u^2 = 0 \\ u_{15} - u_9 + u_{11} + 2b_{33}u - 2u^2 + 2b_{34}u = 0 \\ u_{15} + u_{10} - 2b_{33}u - b_{31}u + 2b_{35}u - u^2 = 0 \\ -u_9 - 2b_{34}u + 2b_{35}u + u^2 - 2b_{36}u = 0 \\ -u_{10} - u_{11} - b_{36}u = 0. \end{array} \right.$$

It is equivalent to:

$$(*) \left\{ \begin{array}{l} u_9 = u^2 + (3b_{34} + 2b_{35} + 3b_{36})u, \\ u_{10} = -u^2 + (b_{30} - b_{31} - b_{32} + b_{33})u, \\ u_{11} = u^2 + (-b_{30} + b_{31} + b_{32} - b_{33} - b_{36})u, \\ u_{14} = -u^2 + (b_{30} + b_{32} - b_{33})u, \\ u_{15} = 2u^2 + (-b_{30} + 2b_{31} + b_{32} + b_{33} + 3b_{35})u, \\ (b_{30} + b_{31} - b_{32} - b_{33} - 2b_{34} + 2b_{35} + 2b_{36})u = 0. \end{array} \right.$$

The last equation is actually $B_5u = 0$. Since $B_5 \neq 0$, we have $u = 0$ and $u_9 = u_{10} = u_{11} = u_{14} = u_{15} = 0$, i.e. $Stab_{L_1T_2}(\lambda|_{H_6H_5H_4}) = Stab_{T_1T_2}(\lambda|_{H_6H_5H_4}) = \{1\}$.

Thus T_1T_2 acts faithfully on the set of all extensions of $\lambda|_{H_6}$ to $H_6H_5H_4$ with the same $B_5 \neq 0$, which is invariant under the action of T , i.e. $B_5(\lambda) = B_5(x\lambda)$ for all $x \in T$. Since $|H_5H_4/R_5| = q^{12} = |T_1T_2|$, this action is transitive. Therefore, we choose $\lambda|_{X_i} = \phi$ for all $i \in [37..43]$, $\lambda|_{X_{36}} = \phi_{B_5/2}$, and $\lambda|_{X_i} = 1_{X_i}$ for the others $X_i \subset H_5H_4$. By the root heights, we have $R_5 \prod_{i=37}^{43} X_i \subset Z(HX_5T_4T_3)$, $HX_5T_4T_3 \trianglelefteq U$ and $H_4 \prod_{i=30}^{35} X_i \trianglelefteq HX_5T_4T_3$. By Lemma 1.5 for $G = U$ with $N = M = HX_5T_3T_4$, $X = T_1T_2$, $Z = H_6R_5$ and $Y = H_4 \prod_{i=30}^{35} X_i$, the induction map from $Irr(HX_5T_4T_3/Y, \lambda)$ to $Irr(U, \lambda)$ is bijective. Since $X_5T_4T_3 = Stab_{X_5T}(\lambda|_{H_6H_5H_4})$

is a transversal of H in $HX_5T_4T_3$, we have $\lambda^{HX_5T_4T_3}|_Y = [HX_5T_4T_3 : H]\lambda|_Y = |X_5T_4T_3|1_Y$. Hence, $Irr(HX_5T_4T_3/Y, \lambda) = Irr(HX_5T_4T_3, \lambda)$.

Now we find $Stab_{T_4T_3}(\lambda|_{H_6H_5H_4H_3})$. Since $[H_6H_5H_4H_3, T_3T_4] = [H_3, T_3]$, we find $y \in T_3$ such that $\lambda([x, y]) = 1$ for all $x \in H_3$. For each $y = \prod_{j=16,17,22} x_j(u_j) \in T_3$ and $x = \prod_{i=18}^{21} x_i(v_i) \in H_3$ we have

$$\begin{aligned} [x, y] &= [x_{18}, x_{16}][x_{18}, x_{22}][x_{19}, x_{22}][x_{20}, x_{17}][x_{20}, x_{16}][x_{21}, x_{17}] \\ &= x_{37}(v_{18}u_{16})x_{42}(v_{18}u_{22})x_{43}(v_{19}u_{22})x_{40}(-v_{20}u_{17})x_{39}(-v_{21}u_{16})x_{41}(-v_{21}u_{17}). \end{aligned}$$

Since $\lambda|_{X_i} = \phi$ for all $i \in [37..43]$, for all v_i we need

$$v_{18}(u_{16} + u_{22}) + v_{19}u_{22} - v_{20}u_{17} + v_{21}(-u_{17} - u_{16}) = 0.$$

The only solution is $(u_{16}, u_{17}, u_{22}) = (0, 0, 0)$, i.e. $Stab_{T_4T_3}(\lambda|_{H_6H_5H_4H_3}) = T_4$.

Next, we find $Stab_{T_4}(\lambda)$. Since $[H, T_4] = [H_2, T_4]$, we find $y \in T_4$ such that $\lambda([x, y]) = 1$ for all $x \in H_2$. For each $y = x_{23}(u_{23}) \in T_4$ and $x = x_{12}(v_{12})x_{13}(v_{13}) \in H_2$ we have

$$[x_{12}(v_{12})x_{13}(v_{13}), x_{23}(u_{23})] = [x_{12}, x_{23}][x_{13}, x_{23}] = x_{37}(-v_{12}u_{23})x_{38}(-v_{13}u_{23}).$$

Evaluate with λ , for all v_i we need $(-v_{12} - v_{13})u_{23} = 0$. Therefore, the only solution is $u_{23} = 0$, i.e. $Stab_{T_4}(\lambda) = \{1\}$. So we finish the proof of $Stab_T(\lambda) = \{1\}$.

Let η, η' be two extensions of $\lambda|_{H_6H_5H_4}$ to HX_5 . By the bijection of the induction map from $Irr(HX_5T_4T_3, \lambda)$ to $Irr(U, \lambda)$, it suffices to show that $\eta^{HX_5T_4T_3} = \eta'^{HX_5T_4T_3}$ iff $\eta|_{R_j} = \eta'|_{R_j}$ for $j = 2, 3$ and $\eta|_{X_5} = \eta'|_{X_5}$. By the Mackey formula for the double coset $HX_5 \backslash HX_5T_4T_3 / HX_5 = HX_5T_4T_3 / HX_5$ represented by T_4T_3 we have

$$(\eta^{HX_5T_4T_3}, \eta'^{HX_5T_4T_3}) = \sum_{y \in T_4T_3} (\eta^y, \eta'^y)$$

Since $[X_5, T_3T_4] \subset H_4 \prod_{i=30}^{35} X_i \subset \ker(\lambda)$, we have ${}^y\eta|_{X_5} = \eta|_{X_5}$. Therefore, the restrictions to X_5 of both η, η' are clear for the proof. To show for the restrictions to R_k with $k = 2, 3$, we are going to prove that $R_2R_3 = \{x \in H_2H_3 : |\lambda^{HX_5T_4T_3}(x)| = \lambda^{HX_5T_4T_3}(1)\}$ and $T_4T_3 = Stab_{T_4T_3}(\lambda|_{R_2R_3})$. Then by $Stab_{T_4T_3}(\lambda) = \{1\}$ and $|T_4T_3| = q^4 = |H_3H_2/R_2R_3|$, the claim holds.

By the above computations of $[H_3, T_3]$ and $[H_2, T_4]$ we find all $x \in H_2H_3$ such that $\lambda([x, y]) = 1$ for all $y \in T_4T_3$. For $y = x_{16}(u_{16})x_{17}(u_{17})x_{22}(u_{22})x_{23}(u_{23}) \in T_3T_4$ and $x = x_{12}(v_{12})x_{13}(v_{13}) \prod_{i=18}^{21} x_i(v_i) \in H_2H_3$, we solve for v_i in the following.

$$u_{16}(v_{18} - v_{21}) + u_{17}(-v_{20} - v_{21}) + u_{22}(v_{18} + v_{19}) + u_{23}(-v_{12} - v_{13}) = 0$$

We have a system with variables v_i :

$$\begin{cases} v_{18} - v_{21} &= 0 \\ -v_{20} - v_{21} &= 0 \\ v_{18} + v_{19} &= 0 \\ -v_{12} - v_{13} &= 0. \end{cases}$$

We obtain solutions $(v_{18}, v_{19}, v_{20}, v_{21}) = (v, -v, -v, v)$ and $(v_{12}, v_{13}) = (s, -s)$ for all $v, s \in \mathbb{F}_q$. Therefore, $x \in R_2R_3$. Hence, ${}^y\lambda|_{R_2R_3} = \lambda|_{R_2R_3}$ for all $y \in T_4T_3$.

(c) Suppose that $B_5 = 0$. By (a), T/S_1 acts faithfully on the set of all extensions of $\lambda|_{H_6}$ to H_6H_5 with the same B_5 . Since $|H_5|/|R_5| = q^6 = |T/S_1|$, this action is transitive. Hence, there exists $x \in T$ such that ${}^x\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5$. Let λ be this linear. So $S_1X_5 = Stab_{TX_5}(\lambda|_{H_6H_5})$, a transversal of H in HX_5S_1 , and $\lambda^{HX_5S_1}|_{H_5} = \lambda^{HX_5S_1}(1)\lambda|_{H_5}$, i.e. $H_5 \subset \ker(\lambda^{HX_5S_1})$ and so is its normal closure $\overline{H_5}$ in HX_5S_1 .

By Lemma 1.5 with $G = U$, $N = M = HX_5S_1$, $X = \prod_{i=1}^4 X_iX_6X_7$, $Z = H_6$ and $Y = \overline{H_5}$, the induction map from $\text{Irr}(HX_5S_1/\overline{H_5}, \lambda)$ to $\text{Irr}(U, \lambda)$ is bijective. Since $\overline{H_5} \subset \ker(\lambda^{HX_5S_1})$, we have $\text{Irr}(HX_5S_1/\overline{H_5}, \lambda) = \text{Irr}(HX_5S_1, \lambda)$. \square

5.5.2. *Proof of Lemma 4.3.* Recall that λ is a linear character of H such that $\lambda|_{X_i} = \phi$ for all $X_i \subset H_6$, $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5$, and $\lambda|_{X_i} = \phi_{b_i}$ for the others $X_i \subset H_4H_3H_2$ where $b_i \in \mathbb{F}_q$. By Lemma 4.2 (c), we work with the quotient group $HX_5S_1/\overline{H_5}$. Abusing the notation of root groups, we call them root groups in the quotient group.

(a) By computation (*) in Lemma 4.2 (b) with $B_5 = 0$, $S_2 = \text{Stab}_{S_1}(\lambda|_{H_6H_5H_4})$. Now we show that $R_4 = \{x \in H_4 : |\lambda^{HX_5S_1}(x)| = \lambda^{HX_5S_1}(1)\}$. For each $l_1y_2y_3y_4 \in L_1T_2T_3T_4 = S_1$ and $h_4 \in H_4$, we have $[h_4, l_1y_2y_3y_4] = [h_4, l_1y_2]$. Hence, we are going to find all $h_4 \in H_4$ such that $\lambda([h_4, l_1y_2]) = 1$ for all $l_1y_2 \in L_1T_2$. Using the computation of $[\prod_{j=24}^{36} x_j(v_j), \prod_{i=1}^{15} x_i(u_i)]$ in Lemma 4.2 (b) with $b_j = 0$ for $j \in [30..36]$, we solve for v_i in the following equation:

$$u_9(-v_{26} - v_{28}) + u_{10}(v_{24} + v_{27} - v_{29}) + u_{11}(v_{26} - v_{29}) + u_{14}(v_{24} + v_{25}) + u_{15}(v_{26} + v_{27}) + u^2(2v_{24} + v_{25} - 2v_{26} - v_{27} + v_{28}) = 0.$$

So we have a system with variables v_i :

$$\left\{ \begin{array}{l} -v_{26} - v_{28} = 0 \\ v_{24} + v_{27} - v_{29} = 0 \\ v_{26} - v_{29} = 0 \\ v_{24} + v_{25} = 0 \\ v_{26} + v_{27} = 0 \\ 2v_{24} + v_{25} - 2v_{26} - v_{27} + v_{28} = 0 \end{array} \right.$$

We obtain the solution $(v_{24}, v_{25}, v_{26}, v_{27}, v_{28}, v_{29}) = (2v, -2v, v, -v, -v, v)$ for all $v \in \mathbb{F}_q$, i.e. $\lambda([h_4, l_1y_2]) = 1$ for all $l_1y_2 \in L_1T_2$ iff $h_4 = r_4(v) \in R_4$.

It is clear that $\lambda^{HX_5S_1}(r_4(v)) = \lambda^{HX_5S_1}(1)\phi_{B_4}(v)$ for all $r_4(v) \in R_4$ by checking directly that $\lambda(r_4(v)) = \phi_{B_4}(v)$.

(b) Suppose that $B_4 \neq 0$. Since $\text{Stab}_T(\lambda|_{H_6H_5H_4}) = S_2 = L_2T_3T_4$, we are going to show that $\text{Stab}_{S_2}(\lambda|_{H_6H_5H_4H_3}) = T_4$, and then, $\text{Stab}_{T_4}(\lambda) = 1$ is done by using the same argument in Lemma 4.2 (b). It means that we find all $y \in S_2$ such that $\lambda([x, y]) = 1$ for all $x \in H_3$ since $\lambda([H_6H_5H_4, S_2]) = \{1\}$.

It is clear that $T_4 \subset \text{Stab}_{S_2}(\lambda|_{H_6H_5H_4H_3})$. So by (a) and $|H_3| = q^4 = |L_2T_3|$, it suffices to show that L_2T_3 acts faithfully on the set of all extensions of $\lambda|_{H_6H_5H_4}$ to $H_6H_5H_4H_3$, i.e. $\text{Stab}_{L_2T_3}(\lambda|_{H_6H_5H_4H_3}) = \{1\}$. By the root heights and H is abelian, $[H_3, L_2T_3] = [H_3, T_3][H_3, L_2]$, where $[H_3, T_3]$ is computed in Lemma 4.2 (b). Since we work with $HX_5S_1/\overline{H_5}$, for each $x = \prod_{i=18}^{21} x_i(v_i) \in H_3$ and $y = \prod_{j=1}^{11} x_j(u_j) \prod_{j=14}^{17} x_j(u_j)x_{22}(u_{22}) \in L_2T_3$, we have

$$\begin{aligned} [x, y] &= [x, x_{16}x_{17}x_{22}][x_{18}, x_3][[[x_{18}, x_3], x_4], x_6][[[x_{18}, x_3], x_6], x_7][[x_{18}, x_3], x_{14}][x_{18}, x_6] \\ &[[[x_{18}, x_6], x_7], x_8][[x_{18}, x_6], x_{15}][[x_{18}, x_6], x_{11}][[[x_{18}, x_6], x_9], x_2][[[x_{19}, x_2], x_1], x_4] \\ &[[[x_{19}, x_2], x_1], x_6][[[x_{19}, x_2], x_4], x_6][[[x_{19}, x_2], x_6], x_7][[x_{19}, x_2], x_{14}][x_{19}, x_1][[[x_{19}, x_1], x_6], x_7] \\ &[[x_{19}, x_1], x_{10}][[x_{19}, x_1], x_{14}][x_{19}, x_6][[[x_{19}, x_6], x_7], x_8][[x_{19}, x_6], x_{10}][[x_{19}, x_6], x_{15}][x_{20}, x_2] \\ &[[x_{20}, x_2], x_3], x_4][[[x_{20}, x_2], x_3], x_7][[[x_{20}, x_2], x_7], x_8][[x_{20}, x_2], x_{15}][[x_{20}, x_2], x_{11}] \\ &[[x_{20}, x_2], x_9][x_{20}, x_3][[[x_{20}, x_3], x_7], x_8][[x_{20}, x_3], x_{10}][[x_{20}, x_3], x_{15}][x_{20}, x_7][[x_{20}, x_7], x_9] \\ &[x_{21}, x_4][[x_{21}, x_4], x_9][x_{21}, x_8][[x_{21}, x_8], x_{10}][[x_{21}, x_8], x_{11}] \\ &= x_{37}(v_{18}u_{16})x_{42}(v_{18}u_{22})x_{43}(v_{19}u_{22})x_{40}(-v_{20}u_{17})x_{39}(-v_{21}u_{16})x_{41}(-v_{21}u_{17}) \\ &x_{25}(2v_{18}u)x_{40}(-2v_{18}u^3)x_{41}(4v_{18}u^3)x_{41}(-2v_{18}u^3)x_{26}(v_{18}u)x_{42}(-4v_{18}u^3)x_{42}(2v_{18}u^3) \end{aligned}$$

$$\begin{aligned}
& x_{40}(v_{18}u^3)x_{38}(-v_{18}u^3)x_{25}(-2v_{19}u)x_{37}(-2v_{19}u^3)x_{38}(2v_{19}u^3)x_{40}(2v_{19}u^3)x_{41}(-4v_{19}u^3) \\
& x_{41}(2v_{19}u^3)x_{24}(-v_{19}u)x_{39}(-2v_{19}u^3)x_{37}(v_{19}u^3)x_{39}(v_{19}u^3)x_{27}(v_{19}u)x_{43}(-4v_{19}u^3) \\
& x_{40}(-v_{19}u^3)x_{43}(2v_{19}u^3)x_{26}(-2v_{20}u)x_{40}(4v_{20}u^3)x_{41}(-8v_{20}u^3)x_{42}(8v_{20}u^3)x_{42}(-4v_{20}u^3) \\
& x_{40}(-2v_{20}u^3)x_{38}(2v_{20}u^3)x_{27}(2v_{20}u)x_{43}(-8v_{20}u^3)x_{40}(-2v_{20}u^3)x_{43}(4v_{20}u^3)x_{28}(2v_{20}u) \\
& x_{39}(-2v_{20}u^3)x_{28}(-v_{21}u)x_{39}(v_{21}u^3)x_{29}(-2v_{21}u)x_{42}(-2v_{21}u^3)x_{43}(2v_{21}u^3).
\end{aligned}$$

Evaluating with λ to get 1, we need the following equation true for all v_i :

$$\begin{aligned}
& v_{18}(u_{16} + u_{22} + 2b_{25}u + b_{26}u - 2u^3) + v_{19}(u_{22} - b_{24}u - 2b_{25}u + b_{27}u - 3u^3) + v_{20}(-u_{17} - \\
& 2b_{26}u + 2b_{27}u + 2b_{28}u - 3u^3) + v_{21}(-u_{16} - u_{17} - b_{28}u - 2b_{29}u + u^3) = 0.
\end{aligned}$$

So we have a system with variables u_j and u :

$$\begin{cases}
u_{16} + u_{22} + 2b_{25}u + b_{26}u - 2u^3 = 0, \\
u_{22} - b_{24}u - 2b_{25}u + b_{27}u - 3u^3 = 0, \\
-u_{17} - 2b_{26}u + 2b_{27}u + 2b_{28}u - 3u^3 = 0, \\
-u_{16} - u_{17} - b_{28}u - 2b_{29}u + u^3 = 0.
\end{cases}$$

It is equivalent to:

$$\begin{cases}
u_{22} = 3u^3 + (b_{24} + 2b_{25} - b_{27})u, \\
u_{17} = 2u^3 + (3b_{26} + 2b_{27} + 2b_{28})u, \\
u_{16} = 4u^3 + (2b_{26} - 2b_{27} - 3b_{28})u, \\
(2b_{24} - 2b_{25} + b_{26} - b_{27} - b_{28} + b_{29})u = 0.
\end{cases}$$

The last equation in the system is actually $B_4u = 0$. Since $B_4 \neq 0$, the only solution of this system is $(u_{16}, u_{17}, u_{22}) = (0, 0, 0)$, i.e. $Stab_{L_2T_3}(\lambda|_{H_6H_5H_4H_3}) = \{1\}$. Hence, $Stab_{S_2}(\lambda|_{H_6H_5H_4H_3}) = T_4$ and $Stab_{S_1}(\lambda) = \{1\}$.

The above argument also proves that $L_1T_2T_3$ acts transitively on the set of all extensions of $\lambda|_{H_6H_5H_4}$ to $H_6H_5H_4H_3$ with the same $B_4 \neq 0$. The number of these extensions is $|H_4H_3|/|R_4|$. Therefore, there exists $x \in L_1T_2T_3$ such that ${}^x\lambda|_{X_i} = \phi$ for all $X_i \subset H_6$, ${}^x\lambda|_{X_{29}} = \phi_{B_4}$, ${}^x\lambda|_{X_i} = 1_{X_i}$ for the others $X_i \subset H_5H_4H_3$. Let λ be this linear character. By Lemma 1.5 with $G = HX_5S_1$, $N = M = HX_5T_4$, $X = L_1T_2T_3$, $Z = H_6R_4$, $Y = H_3 \prod_{i=24}^{28} X_i$, the induction map from $Irr(HX_5T_4/Y, \lambda)$ to $Irr(HX_5S_1, \lambda)$ is bijective. Let η, η' be two extensions of $\lambda|_{H_6H_5H_4H_3}$ to HX_5 . We have $\eta^{HX_5S_2}, \eta'^{HX_5S_2} \in Irr(HX_5S_2/Y, \lambda)$. Using the same argument in Lemma 4.2 (b), we obtain $(\eta^{HX_5S_2}, \eta'^{HX_5S_2}) = 1$ iff $\eta|_{R_2} = \eta'|_{R_2}$ and $\eta|_{X_5} = \eta'|_{X_5}$.

(c) Suppose that $B_4 = 0$. By (a), S_1/S_2 acts faithfully on the set of all extensions of $\lambda|_{H_6H_5}$ to $H_6H_5H_4$ with the same B_4 . Since $|S_1/S_2| = q^5 = |H_4/R_4|$, this action is transitive. Hence, with $B_4 = 0$, there exists $x \in S_1$ such that ${}^x\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5H_4$. Let λ be this linear character. Since $S_2X_5 = Stab_{S_1X_5}(\lambda|_{H_6H_5H_4})$ is a transversal of H in HX_5S_2 , we have $\lambda^{HX_5S_2}|_{H_4} = [HX_5S_2 : H]\lambda|_{H_4} = |X_5S_2|1_{H_4}$. So $H_4 \subset \ker(\lambda^{HX_5S_2})$. By Lemma 1.5 for $G = HX_5S_1$ with $N = M = HX_5S_2$, $X = T_2$, $Y = H_4$ and $Z = H_6$, the induction map from $Irr(HX_5S_2/\overline{H_5H_4}, \lambda)$ to $Irr(HX_5S_1, \lambda)$ is bijective where $\overline{H_5H_4}$ is the normal closure of H_5H_4 in HX_5S_2 . Since $H_5H_4 \subset \ker(\lambda^{HX_5S_2})$, we have $Irr(HX_5S_2/\overline{H_5H_4}, \lambda) = Irr(HX_5S_2, \lambda)$. \square

5.5.3. *Proof of Lemma 4.4.* Recall that λ is a linear character of H such that $\lambda|_{X_i} = \phi$ for all $X_i \subset H_6$, $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5H_4$, and $\lambda|_{X_i} = \phi_{b_i}$ for the others $X_i \subset H_3H_2$ where $b_i \in \mathbb{F}_q$. By Lemma 4.3 (c), we work with the quotient group $HX_5S_2/\overline{H_5H_4}$. Abusing the notation of root groups, we continue to call them root groups in the quotient group.

(a) By the computation in Lemma 4.3 (b) with $B_4 = 0$, it is clear that $S_3 = Stab_{S_2}(\lambda|_{H_6H_5H_4H_3})$. Now we show that $R_3 = \{x \in H_3 : |\lambda^{HX_5S_2}(x)| = \lambda^{HX_5S_2}(1)\}$.

Since X_5S_2 is a transversal of H in HX_5S_2 , we are going to find $x \in H_3$ such that $\lambda([x, y]) = 1$ for all $y \in S_2$. Since $[H_3, X_5] = \{1\} = [H_3, T_4]$, it is enough to work with $x \in H_3$ and $y \in S_2T_3$. For each $x = \prod_{i=18}^{21} x_i(v_i) \in H_3$ and $y = \prod_{j=1}^{11} x_j(u_j) \prod_{j=14}^{17} x_j(u_j)x_{22}(u_{22}) \in S_2T_3$, by the computation in Lemma 4.3, we find $(v_i)_{i \in [18..21]}$ satisfying for all u_j and u in the following equation:

$$u_{16}(v_{18} - v_{21}) + u_{17}(-v_{20} - v_{21}) + u_{22}(v_{18} + v_{19}) + u^3(-2v_{18} - 3v_{19} - 3v_{20} + v_{21}) = 0.$$

We have a system with variables v_i :

$$\begin{cases} v_{18} - v_{21} = 0, \\ -v_{20} - v_{21} = 0, \\ v_{18} + v_{19} = 0, \\ -2v_{18} - 3v_{19} - 3v_{20} + v_{21} = 0. \end{cases}$$

Its solutions are $(v_{18}, v_{19}, v_{20}, v_{21}) = (u, -u, -u, u)$ for all $u \in \mathbb{F}_q$, i.e. $x = r_3(u) \in R_3$. Now to show that $\lambda^{HX_5S_2}|_{R_3} = [HX_5S_2 : H]\phi_{B_3}$, it is enough to check $\lambda(r_3(t)) = \phi_{B_3}(t)$ which is clear.

(b) Suppose that $B_3 \neq 0$. By (a) we have $Stab_{S_2}(\lambda|_{H_6H_5H_4H_3}) = S_3 = L_3T_4$. To show that $Stab_{S_2}(\lambda) = \{1\}$, since $|L_3T_4| = q^2 = |H_2|$, we show that L_3T_4 acts faithfully on the set of all extensions of $\lambda|_{H_6H_5H_4H_3}$ to H , i.e. proving that there is no nontrivial $y \in L_3T_4$ such that $\lambda([x, y]) = 1$ for all $x \in H_2$.

By the root heights, $[H_2, L_3T_4] = [H_2, T_4][H_2, L_3]$, where $[H_2, T_4]$ is computed in Lemma 4.2(b). For $x = x_{12}(v_{12})x_{13}(v_{13}) \in H_2$ and $y = l_3(u)x_{23}(u_{23}) \in L_3T_4$, we have

$$\begin{aligned} [x, y] &= [x, x_{23}][x, l_3] = [x_{12}, x_{23}][x_{12}, x_2][[[[x_{12}, x_2], x_3], x_4], x_6][[[[x_{12}, x_2], x_3], x_6], x_7] \\ &[[[[x_{12}, x_2], x_6], x_7], x_8][x_{12}, x_3][[[[x_{12}, x_3], x_6], x_7], x_8][x_{12}, x_6][[[x_{12}, x_2], x_3], x_{14}] \\ &[[[[x_{12}, x_2], x_6], x_9][[[[x_{12}, x_2], x_6], x_{11}][[[[x_{12}, x_2], x_6], x_{15}][[[[x_{12}, x_3], x_6], x_{10}][[[[x_{12}, x_3], x_6], x_{15}]] \\ &[[[x_{12}, x_6], x_7], x_9][x_{12}, x_2], x_{22}][x_{12}, x_2], x_{16}][x_{12}, x_3], x_{22}][x_{12}, x_6], x_{17}][x_{12}, x_9], x_{10}] \\ &[[x_{12}, x_9], x_{14}][x_{13}, x_{23}][x_{13}, x_4][x_{13}, x_7][[[[x_{13}, x_4], x_7], x_9][[[[x_{12}, x_7], x_8], x_{10}] \\ &[[[x_{13}, x_7], x_8], x_{11}][[[[x_{13}, x_4], x_{17}][[[[x_{13}, x_7], x_{16}][[[[x_{13}, x_7], x_{17}][[[[x_{13}, x_{10}], x_{15}] \\ &[[[x_{13}, x_{10}], x_{11}][[[[x_{13}, x_{11}], x_{15}]] \\ &= x_{37}(-v_{12}u_{23})x_{18}(-2v_{12}u)x_{40}(4v_{12}u^4)x_{41}(-8v_{12}u^4)x_{42}(8v_{12}u^4)x_{19}(2v_{12}u)x_{43}(-8v_{12}u^4) \\ &x_{20}(v_{12}u)x_{41}(4v_{12}u^4)x_{38}(2v_{12}u^4)x_{40}(-2v_{12}u^4)x_{42}(-4v_{12}u^4)x_{40}(-2v_{12}u^4)x_{43}(4v_{12}u^4) \\ &x_{39}(-2v_{12}u^4)x_{42}(-6v_{12}u^4)x_{37}(-8v_{12}u^4)x_{43}(6v_{12}u^4)x_{40}(-2v_{12}u^4)x_{37}(v_{12}u^4)x_{39}(v_{12}u^4) \\ &x_{38}(-v_{13}u_{23})x_{20}(-v_{13}u)x_{21}(2v_{13}u)x_{39}(2v_{13}u^4)x_{42}(-4v_{13}u^4)x_{43}(4v_{13}u^4)x_{40}(2v_{13}u^4) \\ &x_{39}(-8v_{13}u^4)x_{41}(-4v_{13}u^4)x_{42}(2v_{13}u^4)x_{40}(v_{13}u^4)x_{43}(-2v_{13}u^4). \end{aligned}$$

Evaluating with λ to get 1, we obtain in the following equation:

$$v_{12}(-s_{23} - 2b_{18}u + b_{20}u + 2b_{19}u - 2u^4) + v_{13}(-u_{23} - b_{20}u + 2b_{21}u - 2u^4) = 0.$$

We have a system with variables u_j and u :

$$\begin{cases} -u_{23} - 2b_{18}u + b_{20}u + 2b_{19}u - 2u^4 = 0, \\ -u_{23} - b_{20}u + 2b_{21}u - 2u^4 = 0. \end{cases}$$

It is equivalent to:

$$\begin{cases} u_{23} = 3u^4 + (-2b_{18} + b_{20} + 2b_{19})u, \\ (b_{18} - b_{20} - b_{19} + b_{21})u = 0. \end{cases}$$

The last equation is actually $B_3u = 0$. Since $B_3 \neq 0$, the only solution is $(u_{23}, u) = (0, 0)$, i.e. $Stab_{L_3T_4}(\lambda) = \{1\}$ or L_3T_4 acts faithfully on the set of all extensions of $\lambda|_{H_6H_5H_4H_3}$ to H . Hence, we also get $Stab_{S_2}(\lambda) = \{1\}$.

Therefore, there exists $x \in L_3T_4$ such that ${}^x\lambda|_{X_i} = \phi$ for all $X_i \subset H_6$, ${}^x\lambda|_{X_{21}} = \phi_{B_3}$, ${}^x\lambda|_{X_i} = 1_{X_i}$ for the others $X_i \subset H_5H_4H_3$. Let λ be this linear. By Lemma 1.5 with $G = HX_5S_2$, $N = M = HX_5$, $X = S_2$, $Y = \prod_{i=18}^{20} X_i$ and $Z = H_6X_{21}$,

the induction map from $Irr(HX_5/Y, \lambda)$ to $Irr(HX_5S_2, \lambda)$ is bijective. Using the same technique in Lemma 4.2 (c), the rest statement holds.

(c) Suppose $B_3 = 0$. By (a), S_2/S_3 acts faithfully on the set of all extensions of $\lambda|_{H_6H_5H_4}$ to $H_6H_5H_4H_3$ with the same B_3 . Since $|S_2/S_3| = q^3 = |H_3/R_3|$, this action is transitive. Hence, there exists $x \in S_2$ such that ${}^x\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5H_4H_3$. Let λ be this linear. Since X_5S_3 is a transversal of H in HX_5S_3 and $S_3 = \text{Stab}_{S_2}(\lambda|_{H_6H_5H_4H_3})$, we have $\lambda^{HX_5S_3}|_{H_5H_4H_3} = \lambda^{HX_5S_3}(1)\lambda|_{H_5H_4H_3}$. Therefore, $H_5H_4H_3 \subset \ker(\lambda^{HX_5S_3})$, so is its normal closure $\overline{H_5H_4H_3}$ in HX_5S_3 .

By Lemma 1.5 with $G = HX_5S_2$, $N = M = HX_5S_3$, $X = T_3$, $Y = H_3$, $Z = H_6$, the induction map from $Irr(HX_5S_3/Y, \lambda)$ to $Irr(HX_5S_2, \lambda)$. Since $Y \subset \ker(\lambda^{HX_5S_3})$, we have $Irr(HX_5S_3/Y, \lambda) = Irr(HX_5S_3, \lambda)$.

5.5.4. *Proof of Lemma 4.5.* Recall that λ is a linear character of H such that $\lambda|_{X_i} = \phi$ for all $X_i \subset H_6 = Z(U)$, $\lambda|_{X_i} = 1_{X_i}$ for all $X_i \subset H_5H_4H_3$, and $\lambda|_{X_i} = \phi_{b_i}$ for the others $X_i \subset H_2$ where $b_i \in \mathbb{F}_q$. By Lemma 4.3 (c), we work with the quotient group $HX_5S_3/\overline{H_5H_4H_3}$. Abusing the notation of root groups, we continue to call them root group in the quotient group.

(a) By the computation in Lemma 4.4 (b) with $B_3 = 0$, $S_4 = \text{Stab}_{S_3}(\lambda)$. Now we show that $R_2 = \{x \in H_2 : |\lambda^{HX_5S_3}(x)| = \lambda^{HX_5S_3}(1)\}$. Since X_5S_3 is a transversal of H in HX_5S_3 , we are going to find $x \in H_2$ such that $\lambda([x, y]) = 1$ for all $y \in S_3$. Since $[H_2, X_5] = \{1\}$, it is enough to work with $x \in H_2$ and $y \in S_4$. For each $x = \prod_{i=12}^{13} x_i(v_i) \in H_2$ and $y = l_3(u)x_{23}(u_{23}) \in S_3T_4$, by the computation in Lemma 4.4 (b), we find (v_{12}, v_{13}) satisfying for all u_{23} and u in the following equation:

$$u_{23}(-v_{12} - v_{13}) + 2u^4(-v_{12} - v_{13}) = 0.$$

So $(v_{12}, v_{13}) = (v, -v)$ for all $v \in \mathbb{F}_q$, i.e. $x = r_2(v)$. Since $\lambda(r_2(v)) = \phi_{B_2}(v)$ for all $r_2(v) \in R_2$, we have $\lambda^{HX_5S_3}|_{H_2} = [HX_5S_3 : H]\phi_{B_2}$.

(b) Suppose that $B_2 \in \mathbb{F}_q - \{c^4 : c \in \mathbb{F}_q^\times\}$. Let η be an extension of λ to HX_5 . Since $S_4 = \text{Stab}_{S_3}(\lambda)$, to get $I_{HX_5S_3}(\eta) = HX_5$, we show that S_4 acts transitively on the set of all extensions of λ to HX_5 . Hence, we find all $l_4 \in S_4$ such that $\lambda([hx_5, l_4]) = 1$ for all $h \in H$ and $x_5 \in X_5$. Since $S_4 = \text{Stab}_{S_3}(\lambda)$, we have $\lambda([h, l_4]) = 1$ for all $h \in H, l_4 \in S_4$. Thus we compute $[x_5, l_4]$. Since we work with $HX_5S_3/\overline{H_5H_4H_3}$, for each $x_5(v_5) \in X_5$ and $l_4(u) \in S_4$, we have

$$\begin{aligned} [x_5(v_5), l_4(u)] &= [x_5, x_4][[[x_5, x_4], x_9], x_{10}][[[x_5, x_4], x_9], x_{14}][[[x_5, x_4], x_6], x_7], x_9] \\ &[[[x_5, x_4], x_6], x_{17}][[x_5, x_4], x_{23}][x_5, x_6][[[x_5, x_6], x_{10}], x_{11}][[[x_5, x_6], x_{10}], x_{15}] \\ &[[[x_5, x_6], x_{11}], x_{15}][[[[x_5, x_6], x_7], x_8], x_{10}][[[[x_5, x_6], x_7], x_8], x_{11}][[[x_5, x_6], x_7], x_{16}] \\ &[[[x_5, x_6], x_7], x_{17}][[x_5, x_6], x_{23}][[x_5, x_{14}], x_{16}][[x_5, x_{14}], x_{17}][[x_5, x_{11}], x_{22}] \\ &[[x_5, x_{10}], x_{16}][[x_5, x_{10}], x_{22}] \\ &= x_{12}(-v_5u)x_{37}(-v_5u^5)x_{39}(-v_5u^5)x_{39}(2v_5u^5)x_{40}(2v_5u^5)x_{37}(3v_5u^5)x_{13}(v_5u)x_{40}(v_5u^5) \\ &x_{42}(2v_5u^5)x_{43}(-2v_5u^5)x_{42}(-4v_5u^5)x_{43}(4v_5u^5)x_{39}(-8v_5u^5)x_{41}(-4v_5u^5)x_{38}(-3v_5u^5) \\ &x_{39}(4v_5u^5)x_{41}(2v_5u^5)x_{43}(-3v_5u^5)x_{42}(3v_5u^5)x_{37}(4v_5u^5) \end{aligned}$$

Evaluating with λ to get 1, for all v_5 we need

$$v_5(-(b_{12} - b_{13})u + u^5) \in \ker(\phi),$$

which is $v_5(u^5 - B_2u) \in \ker(\phi)$ for all v_5 . Hence, we solve for $u : u(u^4 - B_2) = 0$. Since $B_2 \in \mathbb{F}_q - \{c^4 : c \in \mathbb{F}_q^\times\}$, this equation only has one trivial solution $u = 0$, i.e. $\text{Stab}_{S_4}(\eta) = \{1\}$, or $I_{HX_5S_3}(\eta) = HX_5$.

(c) Suppose that $B_2 = c^4 \in \mathbb{F}_q^\times$. Let η be an extension of λ to HX_5 . Continue the computation in (b), the equation $u(u^4 - B_2) = 0$ has 5 solutions $u \in \{ac : a \in \mathbb{F}_5\}$, i.e. $l_4(u) \in F_4$. Hence, $I_{HX_5S_3}(\eta) = HX_5F_4$. Since $[HX_5, F_4] \subset \ker(\eta)$, η extends to HX_5F_4 , i.e. λ extends to HX_5F_4 .

Since $S_4 = \text{Stab}_{S_3}(\lambda) \cong \mathbb{F}_q$, we have $[H, S_4] \subset \ker(\lambda)$. So λ extends to $HS_4 \trianglelefteq HX_5S_3$. Let λ' be an extension of λ to HS_4 . We find $I_{HX_5S_3}(\lambda')$. Since $\text{Stab}_{X_5S_3}(\lambda') \subset \text{Stab}_{X_5S_3}(\lambda'|_H) = X_5S_4$, it is enough to find all $x_5 \in X_5$ such that $\lambda'([x_5, hl_4]) = 1$ for all $hl_4 \in HS_4$. Since HX_5 is abelian, we have $[x_5, hl_4] = [x_5, l_4]$. For each $x_5(v_5) \in X_5$ and $l_4(u) \in S_4$, by the computation in (b), we need

$$v_5(u^5 - B_2u) \in \ker(\phi), \text{ for all } u \in \mathbb{F}_q.$$

By Proposition 1.3, there are 5 solutions $v_5 \in \{ac_\phi : a \in \mathbb{F}_5\}$, i.e. $x_5(v_5) \in F_5$. Hence, $I_{HX_5S_3}(\lambda') = HF_5S_4$. Since $[F_5, S_4] \subset \ker(\lambda')$, λ' extends to HF_5S_4 , i.e. λ extends to HF_5S_4 .

Let λ_1, λ_2 be two extensions of λ to HX_5F_4 , and γ an extension of λ to HF_5S_3 . Since the degree of all irreducible constituents of $\lambda^{HX_5S_3}$ is $\frac{q^2}{5}$, we have $\lambda_1^{HX_5S_3}, \lambda_2^{HX_5S_3}, \gamma^{HX_5S_3} \in \text{Irr}(HX_5S_3, \lambda)$.

Choose $1 \in S \subset S_3$ as a representative set of the double coset $HF_5S_4 \backslash HX_5S_3 / HX_5F_4$. Since $HF_5S_4 \cap HX_5F_4 = HF_5F_4$ and $HX_5F_4 \trianglelefteq HX_5S_3$, by Mackey formula,

$$\begin{aligned} (\lambda_1^{HX_5S_3}, \gamma^{HX_5S_3}) &= \sum_{s \in S} ({}^s\lambda_1|_{(HX_5F_4) \cap HF_5S_4}, \gamma|_{(HX_5F_4) \cap HF_5S_4}) \\ &= \sum_{s \in S} ({}^s\lambda_1|_{HF_5F_4}, \gamma|_{HF_5F_4}) \end{aligned}$$

For each $s \in S$, if ${}^s\lambda_1|_{HF_5F_4} = \gamma|_{HF_5F_4}$, then ${}^s\lambda_1|_H = \gamma|_H$. Since both are extensions of λ , we have ${}^s\lambda = \lambda$, i.e. $s \in \text{Stab}_{S_3}(\lambda) = S_4$. There is unique $1 \in S \cap S_4$ since S is a representative set of $HF_5S_4 \backslash HX_5S_3 / HX_5F_4$. So $(\lambda_1^{HX_5S_3}, \gamma^{HX_5S_3}) = (\lambda_1|_{HF_5F_4}, \gamma|_{HF_5F_4}) = 1$ iff $\lambda_1|_{F_i} = \gamma|_{F_i}, i \in \{4, 5\}$.

Therefore, $\lambda_1^{HX_5S_3} = \gamma^{HX_5S_3} = \lambda_2^{HX_5S_3}$ iff $\lambda_1|_{F_i} = \lambda_2|_{F_i}, i \in \{4, 5\}$. \square

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