STABILITY, RESONANCE AND LYAPUNOV INEQUALITIES FOR PERIODIC CONSERVATIVE SYSTEMS

ANTONIO CAÑADA AND SALVADOR VILLEGAS

Abstract. This paper is devoted to the study of Lyapunov type inequalities for periodic conservative systems. The main results are derived from a previous analysis which relates the best Lyapunov constants to some especial (constrained or unconstrained) minimization problems. We provide some new results on the existence and uniqueness of solutions of nonlinear resonant and periodic systems. Finally, we present some new conditions which guarantee the stable boundedness of linear periodic conservative systems.

1. Introduction

The classical Lyapunov criterion on the stability of Hill's equation

$$u''(t) + q(t)u(t) = 0, \; t \in \mathbb{R}, \tag{1.1}$$

with $q(\cdot)$ a $T-$periodic function, says that if

$$q \in L^1(0,T), \; \int_0^T q(t) \, dt > 0, \; \int_0^T q^+(t) \, dt \le \frac{4}{T}, \tag{1.2}$$

then (1.1) is stable (in the sense of Lyapunov, i.e. any solution $u(\cdot)$ of (1.1) satisfies $\sup_{t\in\mathbb{R}} \, (|u(t)| + |\dot{u}(t)|) < \infty$) ([14]). Here $q^+(t) = \max\{q(t), 0\}$ denotes the positive part of the function q.

Condition (1.2) has been generalized in several ways. In particular, in [17], the authors provide optimal stability criteria by using L^p norms of q^+, $1 \le p \le \infty$. In the proof, one of the main ideas is a useful relation between the eigenvalues of (1.1) associated to periodic and antiperiodic boundary conditions and those associated to Dirichlet boundary conditions (see Theorem 4.3 in [16]).

Despite its undoubted interest, there are not many studies on the stability properties for systems of equations

$$u''(t) + Q(t)u(t) = 0, \; t \in \mathbb{R}, \tag{1.3}$$

2000 *Mathematics Subject Classification.* 34B05, 34B15, 34C10, 34C15.

Key words and phrases. Lyapunov inequalities, periodic boundary value problems, resonance, stability, conservative systems.

The authors have been supported by the Ministry of Education and Science of Spain (MTM2008.00988).

where the matrix function $Q(\cdot)$ is $T-$periodic. A notable contribution was provided by Krein in [10]. In this work, the author assumes that $Q(\cdot) \in \Lambda$, where Λ is defined as

[Λ] *The set of real $n \times n$ symmetric matrix valued function $Q(\cdot)$, with continuous and $T-$periodic element functions $q_{ij}(t),\ 1 \leq i, j \leq n$, such that (1.3) has not nontrivial constant solutions and*

$$\int_0^T \langle Q(t)k, k\rangle \ dt \geq 0,\ \forall\ k \in \mathbb{R}^n.$$

Krein proved that in this case, all solutions of the system (1.3) are stably bounded (see Section 4 for the precise definition of this property) if $\lambda_1 > 1$, where λ_1 is the smallest positive eigenvalue of the eigenvalue problem

$$u''(t) + \mu Q(t)u(t) = 0,\ t \in \mathbb{R},\ u(0) + u(T) = u'(0) + u'(T) = 0. \tag{1.4}$$

The Lyapunov conditions (1.2) and those given in [17] for scalar equations, imply $\lambda_1 > 1$, but for systems of equations, and assuming $Q(\cdot) \in \Lambda$, it is not easy to give sufficient conditions to ensure the property $\lambda_1 > 1$. In fact, to the best of our knowledge, we do not know a similar result to Theorem 4.3 in [16], for the case of systems of equations.

In [6] the authors establish sufficient conditions for having $\lambda_1 > 1$ which involve L^1 restrictions on the spectral radius of some appropriate matrices which are calculated by using the matrix $Q(t)$. It is easy to check that, even in the scalar case, these conditions are independent from classical L^1-Lyapunov criterion (1.2).

In Section 4 we present some new conditions which allow to prove that $\lambda_1 > 1$. These conditions are given in terms of the L^p norm of appropriate functions $b_{ii}(t),\ 1 \leq i \leq n$, related to (1.3) through the inequality $Q(t) \leq B(t),\ \forall\ t \in [0, T]$, where $B(t)$ is a diagonal matrix with entries given by $b_{ii}(t),\ 1 \leq i \leq n$. These sufficient conditions are optimal in the sense explained in Remark 5 below. Here, the relation $C \leq D$ between $n \times n$ symmetric matrices means that $D - C$ is positive semi-definite.

Our main result in Section 4 is derived from a fundamental relation between the best L^p Lyapunov constant and the minimum of some especial constrained minimization problems. This relation is proved in Section 2. Think that the conditions (1.2) are resonant conditions, in the sense that the real number zero is the first eigenvalue of the periodic eigenvalue problem

$$u''(t) + \mu u(t) = 0,\ u(0) - u(T) = u'(0) - u'(T) = 0 \tag{1.5}$$

so that, these constrained minimization problems arises in a natural way. For other boundary conditions such as Dirichlet or antiperiodic boundary ones, the minimization problems associated to best Lyapunov constants are unconstrained minimization problems (see [15], [18]). Motivated by a completely different problem (an isoperimetric inequality known as Wulff theorem, of interest in crystallography), the authors studied in [7] (see also [8]) similar variational problems but, in our opinion, the relation between these minimization problems and L^p Lyapunov constants for periodic boundary conditions, is established for the first time in Section 2 of the present paper (see [3], for the case of Neumann boundary conditions).

Another important application of Lyapunov inequalities is the study of nonlinear resonant problems. If $G : \mathbb{R}^n \to \mathbb{R}$ is a C^2-mapping and A and B are real symmetric $n \times n$ matrices with respective eigenvalues $a_1 \leq \ldots \leq a_n$ and $b_1 \leq \ldots \leq b_n$ satisfying

$$\begin{aligned} & A \leq G''(u) \leq B, \ \forall \ u \in \mathbb{R}^n, \\ & 0 < a_i \leq b_i < \tfrac{4\pi^2}{T^2}, \ 1 \leq i \leq n, \end{aligned} \tag{1.6}$$

then, for each continuous and $T-$periodic function $h : \mathbb{R} \to \mathbb{R}^n$, the periodic problem

$$u''(t) + G'(u(t)) = h(t), \ t \in \ (0,T), \ u(0) - u(T) = u'(0) - u'(T) = 0, \tag{1.7}$$

has a unique solution (see [1], [2] and [12]). This last result is also true by using more general restrictions than (1.6) which involves higher eigenvalues of (1.5) (think that 0 and $\frac{4\pi^2}{T^2}$ are the first two eigenvalues of the eigenvalue problem (1.5)). The mentioned results only allow a weak interaction between the nonlinear term $G'(u)$ and the spectrum of the linear part (1.5) in the following sense: by using the variational characterization of the eigenvalues of a real symmetric matrix, it may be easily deduced that (1.6) imply that the eigenvalues $g_1(u) \leq \cdots \leq g_n(u)$ of the matrix $G''(u)$, satisfy

$$0 < a_i \leq g_i(u) \leq b_i < \frac{4\pi^2}{T^2}, \ \forall \ u \in \mathbb{R}^n, \ 1 \leq i \leq n, \tag{1.8}$$

and consequently, (1.6), which is a L^∞ restriction, may be seen as a non-resonant hypothesis. In Section 3 we provide for each p, with $1 \leq p \leq \infty$, L^p restrictions for boundary value problem (1.7) to have a unique solution. These are optimal in the sense shown in Remark 4 below. They are given in terms of the L^p norm of appropriate functions $b_{ii}(t), \ 1 \leq i \leq n$, related to (1.7) through the inequality $A(t) \leq G''(u) \leq B(t), \ \forall \ t \in [0,T]$, where $B(t)$ is a diagonal matrix with entries given by $b_{ii}(t), \ 1 \leq i \leq n$ and $A(t)$ is a convenient symmetric matrix which belongs to Λ and consequently, this avoids the resonance at the eigenvalue 0. Since our conditions are given in terms of L^p norms, we allow to the functions $g_i(u)$ to cross an arbitrary number of eigenvalues as long as certain L^p norms are controlled.

2. Preliminary results on scalar Lyapunov inequalities and minimization problems

This section will be concerned with some preliminary results on Lyapunov inequalities for the periodic boundary value problem

$$u''(t) + a(t)u(t) = 0,\ t \in (0,T),\ u(0) - u(T) = u'(0) - u'(T) = 0, \tag{2.1}$$

and the antiperiodic boundary value problem

$$u''(t) + a(t)u(t) = 0,\ t \in (0,T),\ u(0) + u(T) = u'(0) + u'(T) = 0, \tag{2.2}$$

where, from now on, we assume that $a \in L_T(\mathbb{R},\mathbb{R})$, the set of T-periodic functions $a : \mathbb{R} \to \mathbb{R}$ such that $a|_{[0,T]} \in L^1(0,T)$.

If we define the sets

(2.3)

$$\Lambda^{per} = \{a \in L_T(\mathbb{R},\mathbb{R})\backslash\{0\} : \int_0^T a(t)\, dt \geq 0 \text{ and (2.1) has nontrivial solutions }\}$$

$$\Lambda^{ant} = \{a \in L_T(\mathbb{R},\mathbb{R}) :\ \text{(2.2) has nontrivial solutions }\}, \tag{2.4}$$

let us observe that the positive eigenvalues of the eigenvalue problem

$$u''(t) + \lambda u(t) = 0,\ t \in (0,T),\ u(0) - u(T) = u'(0) - u'(T) = 0, \tag{2.5}$$

and the eigenvalues of the eigenvalue problem

$$u''(t) + \lambda u(t) = 0,\ t \in (0,T),\ u(0) + u(T) = u'(0) + u'(T) = 0, \tag{2.6}$$

belong, respectively, to Λ^{per} and Λ^{ant}. Therefore, for each p with $1 \leq p \leq \infty$, we can define, respectively, the L^p-Lyapunov constants β_p^{per} and β_p^{ant} for the periodic and the antiperiodic problem, as the real numbers

$$\beta_p^{per} \equiv \inf_{a \in \Lambda^{per} \bigcap L^p(0,T)} \|a^+\|_p,\quad \beta_p^{ant} \equiv \inf_{a \in \Lambda^{ant} \bigcap L^p(0,T)} \|a^+\|_p, \tag{2.7}$$

where

(2.8)

$$\|a^+\|_p = \left(\int_0^T |a^+(t)|^p\ dt \right)^{1/p}, 1 \leq p < \infty;\quad \|a^+\|_\infty = sup\ ess\ \ a^+.$$

An explicit expression for the constants β_p^{per} and β_p^{ant}, as a function of p and T, has been obtained in [18] (see also [3], [4] and [15] for the case of Neumann, mixed and Dirichlet boundary conditions, respectively).

A key point to extend the mentioned previous results on scalar problems to systems of equations, is the characterization of the L^p-Lyapunov constant as a minimum of a convenient minimization scalar problem, where only some appropriate subsets of the space $H^1(0,T)$ are used (here $H^1(0,T)$ is the usual Sobolev space). For the Dirichlet problem this was done by Talenti ([15]) and for the Neumann problem this was done by the authors in [3]. In the next two lemmas, we treat, respectively, with the periodic and the antiperiodic problem. In the proof, only those innovative details are shown.

Lemma 2.1. *If $1 \le p \le \infty$ is a given number, let us define the sets X_p^{per} and the functionals $I_p^{per} : X_p^{per} \setminus \{0\} \to \mathbb{R}$ as*

(2.9)
$$X_1^{per} = \{v \in H^1(0,T) : v(0) - v(T) = 0, \max_{t\in[0,T]} v(t) + \min_{t\in[0,T]} v(t) = 0\},$$

$$X_p^{per} = \left\{ v \in H^1(0,T) : v(0) - v(T) = 0, \int_0^T |v|^{\frac{2}{p-1}} v = 0 \right\}, \text{ if } 1 < p < \infty,$$

$$X_\infty^{per} = \{v \in H^1(0,T) : \ v(0) - v(T) = 0, \ \int_0^T v = 0\},$$

$$I_1^{per}(v) = \frac{\displaystyle\int_0^T v'^2}{\|v\|_\infty^2}, \quad I_p^{per}(v) = \frac{\displaystyle\int_0^T v'^2}{\left(\displaystyle\int_0^T |v|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}}, \text{ if } 1 < p < \infty, \quad I_\infty^{per}(v) = \frac{\displaystyle\int_0^T v'^2}{\displaystyle\int_0^T v^2}$$

Then, the L_p Lyapunov constant β_p^{per} defined in (2.7), satisfies

$$\beta_p^{per} = \min_{X_p^{per}\setminus\{0\}} I_p^{per}, \quad 1 \le p \le \infty. \tag{2.10}$$

Proof.

The case $p = 1$. It is very well known that $\beta_1^{per} = \frac{16}{T}$ ([9],[18]). Now, if $u \in X_1^{per} \setminus \{0\}$, then there exists $x_0 \in [0,T]$ such that $u(x_0) = 0$. Taking into account that u can be extended as a $T-$ periodic function to $\mathbb{R}$, if we define the function $v(x) = u(x + x_0)$, $\forall\ x \in \mathbb{R}$, then $v|_{[0,T]} \in X_1^{per} \setminus \{0\}$, $v(0) = v(T) = 0$ and $I_1^{per}(u) = I_1^{per}(v)$. In addition (if it is necessary, we can choose $-v$ instead of v), there exist $0 < x_1 < x_2 < x_3 < T$ such that

$$v(x_1) = \max_{[0,T]} v, \quad v(x_2) = 0, \quad v(x_3) = \min_{[0,T]} v.$$

If $x_0 = 0, x_4 = T$, it follows from the Cauchy-Schwarz inequality

$$\textstyle\int_0^T v'^2 = \displaystyle\sum_{i=0}^{3} \int_{x_i}^{x_{i+1}} v'^2 \ge \sum_{i=0}^{3} \frac{\left(\int_{x_i}^{x_{i+1}} |v'|\right)^2}{x_{i+1} - x_i} \ge$$

$$\sum_{i=0}^{3} \frac{\left(\int_{x_i}^{x_{i+1}} v'\right)^2}{x_{i+1} - x_i} = \sum_{i=0}^{3} \frac{(v(x_{i+1}) - v(x_i))^2}{x_{i+1} - x_i} = \tag{2.11}$$

$$\|v\|_\infty^2 \sum_{i=0}^{3} \frac{1}{x_{i+1} - x_i} \ge \frac{16}{T} \|v\|_\infty^2$$

Consequently

$$I_1^{per}(u) = \frac{\int_0^T u'^2}{\|u\|_\infty^2} = I_1^{per}(v) \geq \frac{16}{T}, \ \forall \ u \in X_1^{per} \setminus \{0\}. \tag{2.12}$$

On the other hand, the function $w \in X_1^{per} \setminus \{0\}$ defined as

$$w(x) = \begin{cases} x, \ \text{if } 0 \leq x \leq T/4, \\ -(x - \frac{T}{2}), \ \text{if } T/4 \leq x \leq 3T/4, \\ (x - T), \ \text{if } 3T/4 \leq x \leq T, \end{cases} \tag{2.13}$$

satisfies

$$\frac{\int_0^T w'^2}{\|w\|_\infty^2} = \frac{16}{T}$$

Consequently, the case $p = 1$ is proved.

The case $p = \infty$. It is very well known that $\beta_\infty^{per} = \frac{4\pi^2}{T^2}$, the first positive eigenvalue of the eigenvalue problem (2.5) (see [18]). From its variational characterization, we obtain

$$\beta_\infty^{per} = \min_{X_\infty^{per}\setminus\{0\}} \ I_\infty^{per}.$$

The case $1 < p < \infty$**.** Let us denote

$$m_p = \inf_{X_p^{per}\setminus\{0\}} \ I_p^{per}.$$

If $\{u_n\} \subset X_p^{per}\setminus\{0\}$ is a minimizing sequence, since the sequence $\{k_n u_n\}, \ k_n \neq 0$, is also a minimizing sequence, we can assume without loos of generality that $\displaystyle\int_0^T |u_n|^{\frac{2p}{p-1}} = 1$. Then $\left\{\displaystyle\int_0^T |u_n'^2|\right\}$ is also bounded. Moreover, for each u_n there is $x_n \in (0,T)$ such that $u_n(x_n) = 0$. Therefore, $\{u_n\}$ is bounded in $H^1(0,T)$. So, we can suppose, up to a subsequence, that $u_n \rightharpoonup u_0$ in $H^1(0,T)$ and $u_n \to u_0$ in $C[0,L]$ (with the uniform norm). The strong convergence in $C[0,L]$ gives us $u_0(0) - u_0(T) = 0$ and $\displaystyle\int_0^T |u_0|^{\frac{2p}{p-1}} = 1$. Therefore, $u_0 \in X_p^{per} \setminus \{0\}$. The weak convergence in $H^1(0,T)$ implies $I_p^{per}(u_0) \leq \liminf \ I_p^{per}(u_n) = m_p$. Then u_0 is a minimizer of I_p^{per} in $X_p^{per} \setminus \{0\}$.
Since $X_p^{per} = \{ \ u \in H^1(0,T) \ : \ u(0) - u(t) = 0, \ \varphi(u) = 0\}, \ \varphi(u) = \displaystyle\int_0^T |u|^{\frac{2}{p-1}} u$, if $u_0 \in X_p^{per} \setminus \{0\}$ is any minimizer of I_p^{per}, Lagrange multiplier Theorem implies that there is $\lambda \in \mathbb{R}$ such that

$$H'(u_0)(v) + \lambda\varphi'(u_0)(v) = 0, \ \forall v \in H^1(0,T) \text{ such that } v(0) - v(T) = 0.$$

Here $H : H^1(0,T) \to \mathbb{R}$ is defined by

$$H(u) = \int_0^T u'^2 - m_p \left(\int_0^T |u|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}$$

Also, since $u_0 \in X_p^{per}$ we have $H'(u_0)(1) = 0$. Moreover $H'(u_0)(v) = 0, \ \forall \ v \in H^1(0,T) \ : \ \ v(0) - v(t) = 0, \ \varphi'(u_0)(v) = 0$. Finally, as any $v \in H^1(0,T)$

satisfying $v(0)-v(T)=0$, may be written in the form $v=\alpha+z,\ \alpha\in\mathbb{R}$, and $z\in H^1(0,T)$ satisfying $z(0)-z(T)=0,\ \varphi'(u_0)(z)=0$, we conclude $H'(u_0)(v)=0,\ \forall\, v\in H^1(0,T)$, such that $v(0)-v(T)=0$. This implies that u_0 satisfies the problem

$$\begin{array}{c} u_0''(x)+A_p(u_0)|u_0(x)|^{\frac{2}{p-1}}u_0(x)=0,\ x\in(0,T), \\ u_0(0)-u_0(T)=0,\ u_0'(0)-u_0'(T)=0, \end{array} \tag{2.14}$$

where

$$A_p(u_0)=m_p\left(\int_0^T |u_0|^{\frac{2p}{p-1}}\right)^{\frac{-1}{p}} \tag{2.15}$$

If one has an exact knowledge about the number and distribution of the zeros of the functions u_0 and u_0', the Euler equation (2.14) can be integrated (see [3], Lemma 2.7). In our case, it is not restrictive to assume $u_0(0)=u_0(T)=0$ (see the previous case $p=1$). Then, if we denote the zeros of u_0 in $[0,T]$ by $0=x_0<x_2<\ldots<x_{2n}=T$ and the zeros of u_0' in $(0,T)$ by $x_1<x_3<\ldots<x_{2n-1}$, we obtain

$$m_p=\frac{4n^2I^2p}{T^{2-\frac{1}{p}}(p-1)^{1-\frac{1}{p}}(2p-1)^{1/p}}, \tag{2.16}$$

where $I=\displaystyle\int_0^1 \frac{ds}{\left(1-s^{\frac{2p}{p-1}}\right)^{1/2}}$.

The novelty here is that, for the periodic boundary value problem (2.14), $n\geq 2$ (see, again, the previous case $p=1$), while for the Neumann and Dirichlet problem $n\geq 1$.

The conclusion is that

$$m_p=\frac{16I^2p}{T^{2-\frac{1}{p}}(p-1)^{1-\frac{1}{p}}(2p-1)^{1/p}}, \tag{2.17}$$

that is, four times the corresponding L^p-Lyapunov constant for the Dirichlet and the Neumann problem. Finally, in [18] it is shown that this is, exactly, the L^p-Lyapunov constant for the periodic problem. Consequently, $m_p=\beta_p^{per},\ 1<p<\infty$. □

Remark 1. It is easily deduced from the previous discussion that the set Λ^{per} in (2.3) can be replaced by

$$\Lambda^{per}=\{a\in L_T(\mathbb{R},\mathbb{R}):\int_0^T a(t)\ dt>0 \text{ and (2.1) has nontrivial solutions }\}$$

Also, if $u\in X_1^{per}\setminus\{0\}$ is such that $I_1^{per}(u)=\frac{16}{T}$, then all the inequalities of (2.11) transforms into equalities. In particular $x_{i+1}-x_i=\frac{T}{4},\ 0\leq i\leq 3$, and, again, the Cauchy-Schwartz inequality (equality in this case) implies that the function v' is constant in each interval $[x_i,x_{i+1}],0\leq i\leq 3$. We deduce that there exists a nontrivial constant c and $x_0\in[0,T]$ such that $u(\cdot)=cw(\cdot+x_0)$, where w is given in (2.13).

Remark 2. Motivated by a completely different problem (an isoperimetric inequality known as Wulff theorem, of interest in crystallography), the authors studied in [7] similar variational problems (see also [8] for more general minimization problems). In our opinion, these variational problems are not related with Lyapunov inequalities in [7]. To the best of our knowledge, this was shown in [15] for Dirichlet boundary conditions and in [3] for Neumann boundary conditions. Since 0 is the first eigenvalue for Neumann and periodic boundary conditions, it is necessary to impose an additional restriction to the definition of the spaces $X_p,\ 1 \leq p \leq \infty$ in the case of Neumann and periodic conditions. This is not necessary in the case of Dirichlet or antiperiodic boundary ones (see the next lemma).

Lemma 2.2. *If $1 \leq p \leq \infty$ is a given number, let us define the sets X_p^{ant} and the functional $I_p^{ant} : X_p^{ant} \setminus \{0\} \to \mathbb{R}$, as*

(2.18)
$$X_p^{ant} = \left\{ v \in H^1(0,T) : v(0) + v(T) = 0 \right\},\ 1 \leq p \leq \infty,$$

$$I_1^{ant}(v) = \frac{\displaystyle\int_0^T v'^2}{\|v\|_\infty^2}, I_p^{ant}(v) = \frac{\displaystyle\int_0^T v'^2}{\left(\displaystyle\int_0^T |v|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}},\ \textit{if}\ \ 1 < p < \infty, \quad I_\infty^{ant}(v) = \frac{\displaystyle\int_0^T v'^2}{\displaystyle\int_0^T v^2}$$

Then, the L_p Lyapunov constant β_p^{ant} defined in (2.7), satisfies

$$\beta_p^{ant} = \min_{X_p^{ant}\setminus\{0\}}\ I_p^{ant}, \quad 1 \leq p \leq \infty. \tag{2.19}$$

Proof. **The case** $p = 1$. It is very well known that $\beta_1^{ant} = \frac{4}{T}$ ([18]). Now if $u \in X_1^{ant} \setminus \{0\}$, let us define the function $\tilde{u} : [0, 2T] \to \mathbb{R}$, as

$$\tilde{u}(x) = \begin{cases} u(x), \text{ if } 0 \leq x \leq T, \\ -u(x-T), \text{ if } T \leq x \leq 2T. \end{cases}$$

It is easily checked that $\tilde{u} \in H^1(0, 2T) \setminus \{0\},\ \tilde{u}(0) = \tilde{u}(2T),\ \max_{[0,2T]}\ \tilde{u} + \min_{[0,2T]}\ \tilde{u} = 0$. It is deduced, from the first part of Lemma 2.1, that

$$\frac{2\int_0^T u'^2}{\|u\|^2_{L^\infty(0,T)}} = \frac{\int_0^{2T} \tilde{u}'^2}{\|\tilde{u}\|^2_{L^\infty(0,2T)}} \geq \frac{16}{2T}.$$

and consequently,

$$\frac{\int_0^T u'^2}{\|u\|^2_{L^\infty(0,T)}} \geq \frac{4}{T},\ \forall\ u \in X_1^{ant} \setminus \{0\}.$$

Also, the function $v : [0, T] \to \mathbb{R}$, defined as

$$v(x) = \frac{T}{2} - x, \tag{2.20}$$

belongs to $X_1^{ant}\backslash\{0\}$ and $\dfrac{\int_0^T (v')^2}{\|v\|^2_{L^\infty(0,T)}} = \dfrac{4}{T}$. As a consequence, $\displaystyle\min_{X_1^{ant}\backslash\{0\}} I_1^{ant} = \frac{4}{T}$.

The case $p = \infty$. It is very well known that $\beta_\infty^{ant} = \frac{\pi^2}{T^2}$, the first eigenvalue of the eigenvalue problem (2.6) (see [18]). From its variational characterization, we obtain

$$\beta_\infty^{ant} = \min_{X_\infty^{ant}\backslash\{0\}} I_\infty^{ant}.$$

The case $1 < p < \infty$**.** Let us denote

$$M_p = \inf_{X_p^{ant}\backslash\{0\}} I_p^{ant}.$$

If $\{u_n\} \subset X_p^{ant}\backslash\{0\}$ is a minimizing sequence, since the sequence $\{k_n u_n\},\ k_n \neq 0$, is also a minimizing sequence, we can assume without loos of generality that $\displaystyle\int_0^T |u_n|^{\frac{2p}{p-1}} = 1$. Then $\left\{\displaystyle\int_0^T |u_n'^2|\right\}$ is also bounded. Moreover, for each u_n there is $x_n \in [0,T]$ such that $u_n(x_n) = 0$. Therefore, $\{u_n\}$ is bounded in $H^1(0,T)$. So, we can suppose, up to a subsequence, that $u_n \rightharpoonup u_0$ in $H^1(0,T)$ and $u_n \to u_0$ in $C[0,L]$ (with the uniform norm). The strong convergence in $C[0,L]$ gives us $u_0(0) + u_0(T) = 0$. Therefore, $u_0 \in X_p^{ant} \setminus \{0\}$. The weak convergence in $H^1(0,T)$ implies $I_p^{ant}(u_0) \leq \liminf\ I_p^{ant}(u_n) = M_p$. Then u_0 is a minimizer. Therefore,

$$H'(u_0)(v) = 0,\ \forall v \in H^1(0,T) \text{ such that } v(0) + v(T) = 0.$$

Here $H : H^1(0,T) \to \mathbb{R}$ is defined by

$$H(u) = \int_0^T u'^2 - M_p \left(\int_0^T |u|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}$$

This implies that u_0 satisfies the problem

$$\begin{gathered} u_0''(x) + A_p(u_0)|u_0(x)|^{\frac{2}{p-1}} u_0(x) = 0,\ x \in (0,T), \\ u_0(0) + u_0(T) = 0,\ u_0'(0) + u_0'(T) = 0, \end{gathered} \tag{2.21}$$

where

$$A_p(u_0) = M_p \left(\int_0^T |u_0|^{\frac{2p}{p-1}}\right)^{\frac{-1}{p}} \tag{2.22}$$

Since the function $a(x) \equiv A_p(u_0)|u_0(x)|^{\frac{2}{p-1}}$ satisfies $a(0) = a(T)$, it is not restrictive to assume that, additionally, $u_0(0) = u_0(T) = 0$. Then, we deduce from Lemma 2.7 in [3] that

$$M_p = \frac{4n^2 I^2 p}{T^{2-\frac{1}{p}}(p-1)^{1-\frac{1}{p}}(2p-1)^{1/p}}, \tag{2.23}$$

where $I = \displaystyle\int_0^1 \frac{ds}{\left(1 - s^{\frac{2p}{p-1}}\right)^{1/2}}$ and $n \in \mathbb{N}$ is such that we denote the zeros of u_0 in $[0,T]$ by $0 = x_0 < x_2 < \ldots < x_{2n} = T$ and the zeros of u_0' in $(0,T)$ by $x_1 < x_3 < \ldots < x_{2n-1}$. The novelty here, with respect to the periodic boundary value problem, is that $n \geq 1$. The conclusion is that

$$M_p = \frac{m_p}{4} \ \text{ if } 1 < p < \infty. \tag{2.24}$$

Finally, it is known that $\beta_p^{ant} = \frac{\beta_p^{per}}{4}$, if $1 < p < \infty$ (see [18]). The Lemma is proved. $\square$

Remark 3. We must remark that, if $w \in X_1^{ant} \setminus \{0\}$ is such that $I_1^{ant}(w) = \frac{4}{T}$, then there exists a nontrivial constant c and $x_0 \in [0,T]$ such that $w(x) = c(\frac{T}{2} - |x - x_0|), \ \forall \ x \in [0,T]$.

3. Resonant nonlinear periodic systems

In this section we consider systems of equations of the type (3.2) below, which models the Newtonian equation of motion of a mechanical system subject to conservative internal forces and periodic external forces. The main result is the following.

Theorem 3.1. *Let $G : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \ (t,u) \to G(t,u)$, be a continuous function, $T-$ periodic with respect to the variable t and satisfying:*

1. *$u \to G(t,u)$ is of class $C^2(\mathbb{R}^n, \mathbb{R})$, for every $t \in \mathbb{R}$.*
2. *There exist continuous and $T-$periodic matrix functions $A(\cdot)$, $B(\cdot)$, with $A(t)$ symmetric, $B(t)$ diagonal with entries $b_{ii}(t)$, and $p_i \in [1,\infty]$ $1 \leq i \leq n$, such that*

$$\left.\begin{array}{c} A(t) \leq G_{uu}(t,u) \leq B(t), \ \forall (t,u) \in \mathbb{R}^{n+1}, \\ \\ \int_0^T \langle A(t)k, k\rangle \ dt > 0, \ \forall \ k \in \mathbb{R}^n \setminus \{0\}, \\ \\ \|b_{ii}^+\|_{p_i} < \beta_{p_i}^{per}, \ \textit{if } p_i \in (1,\infty], \quad \|b_{ii}^+\|_{p_i} \leq \beta_{p_i}^{per}, \ \textit{if } p_i = 1. \end{array}\right\} \tag{3.1}$$

Then the boundary value problem

$$u''(t) + G_u(t,u(t)) = 0, \ t \in \mathbb{R}, \ u(0) - u(T) = u'(0) - u'(T) = 0, \tag{3.2}$$

has a unique solution.

Proof. It is based on two steps. In the first one we prove the uniqueness property. This suggests the way to prove existence of solutions.

1.- Uniqueness of solutions of (3.2).

Let us denote by $H_T^1(0,T)$ the subset of $T-$periodic functions of the Sobolev space $H^1(0,T)$. Then, if $v \in (H_T^1(0,T))^n$ and $w \in (H_T^1(0,T))^n$ are two solutions of (3.2), the function $u = v - w$ is a solution of the problem

$$u''(t) + C(t)u(t) = 0, \ t \in \mathbb{R}, \ u(0) - u(T) = u'(0) - u'(T) = 0, \tag{3.3}$$

where $C(t) = \displaystyle\int_0^1 G_{uu}(t, w(t) + \theta u(t))\ d\theta$ (see [11], p. 103, for the mean value theorem for the vectorial function $G_u(t,u)$). In addition,

$$A(t) \leq C(t) \leq B(t),\ \forall\ t \in \mathbb{R} \tag{3.4}$$

and

$$\int_0^T \langle u'(t), z'(t)\rangle \ = \int_0^T \langle C(t)u(t), z(t)\rangle,\ \forall\ z \in (H_T^1(0,T))^n.$$

In particular, we have

$$\begin{aligned} \int_0^T \langle u'(t), u'(t)\rangle \ &= \int_0^T \langle C(t)u(t), u(t)\rangle, \\ \int_0^T \langle C(t)u(t), k\rangle \ = \int_0^T \langle C(t)k, u(t)\rangle \ &= 0,\ \forall\ k \in \mathbb{R}^n. \end{aligned} \tag{3.5}$$

Therefore, for each $k \in \mathbb{R}^n$, we have

$$\int_0^T \langle (u(t)+k)', (u(t)+k)'\rangle \ = \int_0^T \langle u'(t), u'(t)\rangle \ =$$

$$\int_0^T \langle C(t)u(t), u(t)\rangle \ \leq \int_0^T \langle C(t)u(t), u(t)\rangle \ + \int_0^T \langle C(t)u(t), k\rangle \ +$$

$$\int_0^T \langle C(t)k, u(t)\rangle \ + \int_0^T \langle C(t)k, k\rangle \ =$$

$$\int_0^T \langle C(t)(u(t)+k), u(t)+k\rangle \ \leq \int_0^T \langle B(t)(u(t)+k), u(t)+k\rangle.$$

If $u = (u_i)$, then for each $i, 1 \leq i \leq n$, we choose $k_i \in \mathbb{R}$ satisfying $u_i + k_i \in X_{p_i}^{per}$, the set defined in Lemma 2.1. By using previous inequality, Lemma 2.1 and Hölder inequality, we obtain

$$\begin{aligned} \sum_{i=1}^n \beta_{p_i}^{per} \|(u_i+k_i)^2\|_{\frac{p_i}{p_i-1}} &\leq \sum_{i=1}^n \int_0^T (u_i(t)+k_i)'^2 \leq \\ \sum_{i=1}^n \int_0^T b_{ii}^+(t)(u_i(t)+k_i)^2 &\leq \sum_{i=1}^n \|b_{ii}^+\|_{p_i} \|(u_i+k_i)^2\|_{\frac{p_i}{p_i-1}}, \end{aligned} \tag{3.6}$$

where

$$\tfrac{p_i}{p_i-1} = \infty, \quad \text{if } p_i = 1$$

$$\tfrac{p_i}{p_i-1} = 1, \quad \text{if } p_i = \infty.$$

Therefore we have

$$\sum_{i=1}^n (\beta_{p_i}^{per} - \|b_{ii}^+\|_{p_i}) \|(u_i+k_i)^2\|_{\frac{p_i}{p_i-1}} \leq 0. \tag{3.7}$$

Now, (3.7) implies that, necessarily, $u \equiv 0$ (and as a consequence, $v \equiv w$). To see this, if u is a nontrivial function, then the function $u + k$ is also a nontrivial function. In fact, if $u + k \equiv 0$, we deduce that (3.3) has the nontrivial and constant solution $-k$ which imply

$$0 = \int_0^T \langle C(t)k, k\rangle \; dt \; \geq \int_0^T \langle A(t)k, k\rangle \; dt.$$

This is a contradiction with (3.1).

Then, if $u + k$ is a nontrivial function, some component, say $u_j + k_j$, is nontrivial.

If $p_j \in (1, \infty]$, then $(\beta_{p_j}^{per} - \|b_{jj}^+\|_{p_j})\|(u_j + k_j)^2\|_{\frac{p_j}{p_j-1}}$ is strictly positive and from (3.1), all the other summands in (3.7) are nonnegative. This is a contradiction with (3.7).

If $p_j = 1$, we must take into account that β_1^{per} is only attained in nontrivial functions of the form $y(t) = cw(t + x_0)$, where $w(t)$ is given in (2.13), c is a nontrivial constant and $x_0 \in [0, T]$ (see Remark 1 above). Any function y of this type do not belong to $C^1([0, T], \mathbb{R})$. Since any solution $u \in (H_T^1(0, T))^n$ of (3.3) belongs to $C^1([0, T], \mathbb{R})$, we have

$$\beta_{p_j}\|(u_j + k_j)^2\|_{\frac{p_j}{p_j-1}} < \int_0^T (u_j(x) + k_j)'^2.$$

This implies that the inequality (3.7) is strict and this is a contradiction with (3.1).

2.- Existence of solutions of (3.2).

First, we write (3.2) in the equivalent form

$$\left.\begin{array}{c} u''(t) + D(t, u(t))u(t) + G_u(t, 0) = 0, \; t \in \mathbb{R}, \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{array}\right\} \tag{3.8}$$

where the function $D : \mathbb{R}\times\mathbb{R}^n \to \mathcal{M}(\mathbb{R})$ is defined by $D(t, z) = \displaystyle\int_0^1 G_{uu}(t, \theta z) \, d\theta$. Here $\mathcal{M}(\mathbb{R})$ denotes the set of real $n \times n$ matrices.

If $C_T(\mathbb{R}, \mathbb{R})$ is the set of real $T-$periodic and continuos functions defined in $\mathbb{R}$, let us denote $X = (C_T(\mathbb{R}, \mathbb{R}))^n$ with the uniform norm (if $y(\cdot) = (y^1(\cdot), \cdots, y^n(\cdot)) \in X$, then $\|y\|_X = \displaystyle\sum_{k=1}^n \|y^k(\cdot)\|_\infty$). Since

$$A(t) \leq D(t, z) \leq B(t), \; \forall \; (t, z) \in \mathbb{R} \times \mathbb{R}^n, \tag{3.9}$$

we can define the operator $H : X \to X$, by $Hy = u_y$, being u_y the unique solution of the linear problem

$$\left.\begin{array}{c} u''(t) + D(t, y(t))u(t) + G_u(t, 0) = 0, \; t \in \mathbb{R}, \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{array}\right\} \tag{3.10}$$

In fact, (3.10) is a nonhomogeneous linear problem such that the corresponding homogeneous one has only the trivial solution (as in the previous step on uniqueness).

We will show that H is completely continuous and that $H(X)$ is bounded. The Schauder's fixed point theorem provides a fixed point for H which is a solution of (3.2).

The fact that H is completely continuous is a consequence of the Arzela-Ascoli Theorem. It remains to prove that $H(X)$ is bounded. Suppose, contrary to our claim, that $H(X)$ is not bounded. In this case, there would exist a sequence $\{y_n\} \subset X$ such that $\|u_{y_n}\|_X \to \infty$. From (3.9), and passing to a subsequence if necessary, we may assume that, for each $1 \leq i, j \leq n$, the sequence of functions $\{D_{ij}(\cdot, y_n(\cdot))\}$ is weakly convergent in $L^p(\Omega)$ to a function $E_{ij}(\cdot)$ and such that if $E(t) = (E_{ij}(t))$, then $A(t) \leq E(t) \leq B(t)$, a.e. in $\mathbb{R}$, ([13], page 157).

If $z_n \equiv \dfrac{u_{y_n}}{\|u_{y_n}\|_X}$, passing to a subsequence if necessary, we may assume that $z_n \to z_0$ strongly in X, where z_0 is a nonzero vectorial function satisfying

$$\left.\begin{array}{c} z_0''(t) + E(t)z_0(t) = 0, \ t \in \mathbb{R} \\ z_0(0) - z_0(T) = z_0'(0) - z_0'(T) = 0. \end{array}\right\} \tag{3.11}$$

But, $A(t) \leq E(t) \leq B(t), \ \forall \ t \in \mathbb{R}$ and, as in the first step on uniqueness, this implies that the unique solution of (3.11) is the trivial one. This is a contradiction with the fact that $\|z_0\|_X = 1$ and, as a consequence, $H(X)$ is bounded. □

Remark 4. Previous Theorem is optimal in the following sense: for any given positive numbers $\gamma_i, \ 1 \leq i \leq n$, such that at least one of them, say γ_j, satisfies

$$\gamma_j > \beta_{p_j}^{per}, \ \text{for some } p_j \in [1, \infty], \tag{3.12}$$

there exists a diagonal $n \times n$ matrix $A(\cdot)$ with continuous and $T-$periodic entries $a_{ii}(t), \ 1 \leq i \leq n$, satisfying $\|a_{ii}^+\|_{p_i} < \gamma_i, \ 1 \leq i \leq n, \ \int_0^T \langle A(t)k, k\rangle \ dt > 0, \ \forall \ k \in \mathbb{R}^n \setminus \{0\}$ and a continuous and $T-$periodic function $h : \mathbb{R} \to \mathbb{R}^n$, such that the boundary value problem

$$u''(t) + A(t)u(t) = h(t), \ t \in (0, T), \ u(0) - u(T) = u'(0) - u'(T) = 0, \tag{3.13}$$

has not solution.

To see this, if γ_j satisfies (3.12), then there exists some continuous and $T-$periodic function $a(t)$, with $\int_0^T a(t) \ dt > 0$, and $\|a^+\|_{p_j} < \gamma_j$, such that the scalar problem

$$w''(t) + a(t)w(t) = 0, \ t \in (0, T), \quad w(0) - w(T) = w'(0) - w'(T) = 0,$$

has nontrivial solutions (see the definition of $\beta_{p_j}^{per}$ in (2.7) and Remark 1). If w_j is one of these nontrivial solutions, and we choose

$$\begin{gathered} a_{jj}(t) = a(t), \quad a_{ii}(t) = \delta \in \mathbb{R}^+, \text{ if } i \neq j, \\ h_j \equiv w_j, \; h_j \equiv 0, \text{ if } i \neq j, \end{gathered}$$

with δ sufficiently small, then (3.13) has not solution.

Example 1. Now, we show an example which can not be studied by using the results proved by Ahmad and Lazer in [1] and [12], respectively, and Brown and Lin in [2]. In fact, in the next example, we allow to the eigenvalues of the matrix $G_{uu}(t,u)$ in the boundary value problem (3.2), to cross an arbitrary number of eigenvalues of (2.5).

To begin with the example, let $H : \mathbb{R}^n \to \mathbb{R}, \; u \to H(u)$ be a given function of class $C^2(\mathbb{R}^n, \mathbb{R})$ such that

(1) There exist a real constant symmetric $n \times n$ matrix A and a real constant diagonal matrix B,with respective eigenvalues

$$\begin{gathered} a_1 \leq a_2 \leq \ldots \leq a_n, \\ b_1 \leq b_2 \leq \ldots \leq b_n, \end{gathered}$$

satisfying

(2) $A \leq H_{uu}(u) \leq B, \; \forall \, u \in \mathbb{R}^n$,

(3) $0 < a_k \leq b_k < 1, \; 1 \leq k \leq n.$

Then for each continuous and $2\pi-$ periodic function $h : \mathbb{R} \to \mathbb{R}^n$, the boundary value problem

$$u''(t) + H_u(u(t)) = h(t), \; t \in \mathbb{R}, \; u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0, \tag{3.14}$$

has a unique solution. In fact, this is a particular case of more general results proved in [1] and [12] which involve higher eigenvalues of the eigenvalue problem (2.5).

Now, if $m : \mathbb{R} \to \mathbb{R}$, is a given continuous and $2\pi-$periodic function such that for some $p_i \in [1, \infty], \; 1 \leq i \leq n$,

$$\begin{gathered} m(t) \geq 0, \; \forall \, t \in \mathbb{R} \text{ and } m \text{ is not identically zero.} \\ \|m\|_{p_i} < \frac{\beta_{p_i}^{per}}{b_i}, \text{ if } p_i \in (1, \infty], \quad \|m\|_{p_i} \leq \frac{\beta_{p_i}^{per}}{b_i}, \text{ if } p_i = 1, \end{gathered} \tag{3.15}$$

then for each continuous and $2\pi-$ periodic function $h : \mathbb{R} \to \mathbb{R}^n$, the boundary value problem

$$u''(t) + m(t) H_u(u(t)) = h(t), \; t \in \mathbb{R}, \; u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0, \tag{3.16}$$

has a unique solution.

If in (3.15) we choose $p_i \neq \infty$, for some $1 \leq i \leq n$, then it is clear that the eigenvalues of the matrix $m(t)H_{uu}(u)$ in the boundary value problem (3.16), can cross an arbitrary number of eigenvalues of (2.5).

4. Stability for linear periodic systems

In this section we present some new conditions which allow to prove that a given periodic linear and conservative system is stably bounded. More precisely, we consider systems of the type

$$u''(t) + P(t)u(t) = 0, \ t \in \mathbb{R}, \tag{4.1}$$

where from now on we assume that the matrix function $P(\cdot) \in \Lambda$ (Λ was defined in the introduction).

The system (4.1) is said to be stably bounded ([10]) if there exists $\varepsilon(P) \in \mathbb{R}^+$, such that all solutions of the system

$$u''(t) + Q(t)u(t) = 0, \ t \in \mathbb{R}, \tag{4.2}$$

are bounded for all matrix function $Q(\cdot) \in \Lambda$, satisfying

$$\max_{1 \leq i,j \leq n} \int_0^T |p_{ij}(t) - q_{ij}(t)| \ dt < \varepsilon.$$

In [10], Krein proved that all solutions of the system (4.1) are stably bounded if $\lambda_1 > 1$, where λ_1 is the smallest positive eigenvalue of the eigenvalue problem

$$u''(t) + \mu P(t)u(t) = 0, \ t \in \mathbb{R}, \ u(0) + u(T) = u'(0) + u'(T) = 0. \tag{4.3}$$

Moreover, the eigenvalue λ_1 has a variational characterization given by

$$\frac{1}{\lambda_1} = \max_{y \in G_T} \int_0^T \langle P(t)y(t), y(t) \rangle \ dt, \tag{4.4}$$

where

$$G_T = \{y \in H^1(0,T) : y(0) + y(T) = 0, \ \sum_{i=1}^n \int_0^T (y_i'(t))^2 \ dt = 1\}. \tag{4.5}$$

Based on these previous results, we can prove the following theorem.

Theorem 4.1. *Let $P(\cdot) \in \Lambda$ be such that there exist a diagonal matrix $B(t)$ with continuous and $T-$periodic entries $b_{ii}(t)$, and $p_i \in [1, \infty]$, $1 \leq i \leq n$, satisfying*

$$\begin{gathered} P(t) \leq B(t), \ \forall \ t \in \mathbb{R}, \\ \|b_{ii}^+\|_{p_i} < \beta_{p_i}^{ant}, \ \text{if } p_i \in (1, \infty], \quad \|b_{ii}^+\|_{p_i} \leq \beta_{p_i}^{ant}, \ \text{if } p_i = 1. \end{gathered} \tag{4.6}$$

Then, the system (4.1) is stably bounded.

Proof. Let $y \in G_T$. Then by using the Lemma 2.2, we have

$$\int_0^T \langle P(t)y(t), y(t)\rangle \ dt \leq \int_0^T \langle B(t)y(t), y(t)\rangle \ dt \leq$$

$$\sum_{i=1}^n \int_0^T b_{ii}(t)(y_i(t))^2(t) \ dt \leq \sum_{i=1}^n \|b_{ii}^+(t)\|_{p_i} \|y_i^2\|_{\frac{p_i}{p_i-1}} \leq \tag{4.7}$$

$$\leq \sum_{i=1}^n \beta_{p_i}^{ant} \|y_i^2\|_{\frac{p_i}{p_i-1}} \leq \sum_{i=1}^n \int_0^T (y_i'(t))^2 \ dt = 1, \ \forall \ y \in G_T$$

where

$$\tfrac{p_i}{p_i-1} = \infty, \quad \text{if } p_i = 1$$

$$\tfrac{p_i}{p_i-1} = 1, \quad \text{if } p_i = \infty.$$

At this point, we claim

$$\frac{1}{\lambda_1} < 1. \tag{4.8}$$

In fact, (4.7) implies $\dfrac{1}{\lambda_1} \leq 1$. Now, if $\lambda_1 = 1$, let us choose $y(\cdot)$ as any nontrivial eigenfunction associated to $\mu = 1$ in (4.3), i.e.,

$$y''(t) + P(t)y(t) = 0, \ t \in \mathbb{R}, \ y(0) + y(T) = y'(0) + y'(T) = 0. \tag{4.9}$$

Then some component, say y_j, is nontrivial. If $p_j \in (1, \infty]$, then $(\beta_{p_j}^{ant} - \|b_{jj}^+\|_{p_j})\|y_j^2\|_{\frac{p_j}{p_j-1}} > 0$ and $(\beta_{p_i}^{ant} - \|b_{ii}^+\|_{p_i})\|y_i^2\|_{\frac{p_i}{p_i-1}} \geq 0, \ \forall \ i \neq j$, so that we have a strict inequality in (4.7). This is a contradiction with (4.9). If $p_j = 1$, we use the Remark 3. Since $y_j \in C^1[0, T]$, either $x_0 = 0$ or $x_0 = T$. In any case the function w of Remark 3 satisfies $w'(0) + w'(T) \neq 0$. Then we have $\beta_{p_j}^{ant} \|y_j^2\|_{\frac{p_j}{p_j-1}} < \int_0^T (y_j'(t))^2 \ dt$. In this case we have again a strict inequality in (4.7), which is a contradiction with (4.9). □

Remark 5. Previous Theorem is optimal in the following sense. For any given positive numbers $\gamma_i, \ 1 \leq i \leq n$, such that at least one of them, say γ_j, satisfies

$$\gamma_j > \beta_{p_j}^{ant}, \ \text{for some } p_j \in [1, \infty], \tag{4.10}$$

there exists a diagonal $n \times n$ matrix $P(\cdot) \in \Lambda$ with entries $p_{ii}(t), \ 1 \leq i \leq n$, satisfying $\|p_{ii}^+\|_{p_i} < \gamma_i, \ 1 \leq i \leq n$ and such that the system (4.1) is not stable.

To see this, if γ_j satisfies (4.10), then there exists some continuous and $T-$periodic function $p(t)$, not identically zero, with $\int_0^T p(t) \ dt \geq 0$, and $\|p^+\|_{p_j} < \gamma_j$, such that the scalar problem

$$w''(t) + p(t)w(t) = 0,$$

is not stable (see Theorem 1 in [17]). If we choose

$$p_{jj}(t) = p(t), \quad p_{ii}(t) = \delta \in \mathbb{R}^+, \text{ if } i \neq j,$$

with δ sufficiently small, then (4.1) is unstable.

Remark 6. The property of stable boundedness for the solutions of systems like (4.1) have been considered in [6]. The authors assume L^1 restrictions on the spectral radius of some appropriate matrices which are calculated by using the matrix $P(t)$. It is easy to check that, even in the scalar case, these conditions are independent from classical L^1- Lyapunov inequality and therefore, they are also independent from our results in this paper.

Example 2. Next we show a two dimensional system where we provide sufficient conditions, which may be checked directly by using the elements p_{ij} of the matrix $P(t)$, to assure that all hypotheses of the previous Theorem are fulfilled. The example is based on a similar one, shown by the authors in [5], in the study of Lyapunov inequalities for elliptic systems.

Let the matrix $P(t)$ be given by

$$P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{12}(t) & p_{22}(t) \end{pmatrix} \tag{4.11}$$

where

[H1]
$$\begin{gathered} p_{ij} \in C_T(\mathbb{R},\mathbb{R}), \ 1 \leq i,j \leq 2, \\ p_{11}(t) \geq 0, \ p_{22}(t) \geq 0, \ \det \ P(t) \geq 0, \ \forall \ t \in \mathbb{R}, \\ \det \ P(t) \neq 0, \ \textit{for some } t \in \mathbb{R}. \end{gathered}$$

$C_T(\mathbb{R},\mathbb{R})$ denotes the set of real, continuous and $T-$periodic functions defined in $\mathbb{R}$. In addition, let us assume that there exist $p_1, p_2 \in (1,\infty]$ such that

$$\|p_{11}\|_{p_1} < \beta_{p_1}^{ant}, \quad \|p_{22} + \frac{p_{12}^2}{\beta_{p_1}^{ant} - \|p_{11}\|_{p_1}}\|_{p_2} < \beta_{p_2}^{ant}. \tag{4.12}$$

Then (4.1) is stably bounded.

In fact, it is trivial to see that [**H1**] implies that the eigenvalues of the matrix $P(t)$ are both nonnegative, which implies that $P(t)$ is positive semi-definite. Also, since $\det \ P(t) \neq 0$, for some $t \in \mathbb{R}$, (4.1) has not nontrivial constant solutions. Therefore, $P(\cdot) \in \Lambda$, the set defined in the Introduction. Moreover, it is easy to check that for a given diagonal matrix $B(t)$, with continuous entries $b_{ii}(t), \ 1 \leq i \leq 2$, the relation

$$P(t) \leq B(t), \ \forall \ t \in \mathbb{R} \tag{4.13}$$

is satisfied if and only if, $\forall\ t \in \mathbb{R}$ we have

$$b_{11}(t) \geq p_{11}(t),\ b_{22}(t) \geq p_{22}(t), \\ (b_{11}(t) - p_{11}(t))(b_{22}(t) - p_{22}(t)) \geq p_{12}^2(t). \tag{4.14}$$

In our case, if we choose

$$b_{11}(t) = p_{11}(t) + \gamma,\ b_{22}(t) = p_{22}(t) + \frac{p_{12}^2(t)}{\gamma} \tag{4.15}$$

where γ is any constant such that

$$0 < \gamma < \beta_{p_1}^{ant} - \|p_{11}\|_{p_1}, \\ \left(\tfrac{1}{\gamma} - \tfrac{1}{\beta_{p_1} - \|p_{11}\|_{p_1}}\right) \|p_{12}^2\|_{p_2} < \beta_{p_2}^{ant} - \|p_{22} + \frac{p_{12}^2}{\beta_{p_1}^{ant} - \|p_{11}\|_{p_1}}\|_{p_2}, \tag{4.16}$$

then all conditions of Theorem 4.1 are fulfilled and consequently (4.1) is stably bounded.

Remark 7. Let us observe that we deduce from (4.12)

$$\|p_{11}\|_{p_1} < \beta_{p_1}^{ant}, \quad \|p_{22}\|_{p_2} < \beta_{p_2}^{ant}. \tag{4.17}$$

As a consequence, the uncoupled system

$$v''(t) + R(t)v(t) = 0,\ t \in \mathbb{R}, \tag{4.18}$$

where

$$R(t) = \begin{pmatrix} p_{11}(t) & 0 \\ 0 & p_{22}(t) \end{pmatrix} \tag{4.19}$$

is stably bounded.

Therefore, by using the definition of stably bounded system, (4.1) is stably bounded for any continuous and $T-$periodic function p_{12} with sufficiently small L_1- norm. However, (4.12) does not imply, necessarily, that the L_1-norm of the function p_{12} is necessarily small.

References

[1] S. Ahmad, *An existence theorem for periodically perturbed conservative systems,* Michigan Math. J., 20, (1973), 385-392.

[2] K.J. Brown and S.S. Lin, *Periodically perturbed conservative systems and a global inverse function theorem,* Nonlinear Anal., 4, (1980), 193-201.

[3] A. Cañada, J.A. Montero and S. Villegas, *Liapunov-type inequalities and Neumann boundary value problems at resonance*, MIA, Mathematical Inequalities and Applications, 8 (2005), 459-475.

[4] A. Cañada and S. Villegas, *Optimal Lyapunov inequalities for disfocality and Neumann boundary conditions using L^p norms,* Discrete and Continuous Dynamical Systems, series A, 20, (2008), 877-888.

[5] A. Cañada and S. Villegas, *Matrix Lyapunov inequalities for ordinary and elliptic partial differential equations,* Submitted for publication. arXiv:0906.1093

[6] S. Clark and D. Hinton, *Positive eigenvalues of second order boundary value problems and a theorem of M.G. Krein,* Proc. Amer. Math. Soc., 130, (2002), 3005-3015.

[7] B. Dacarogna,W. Gangbo and N. Subía, *Sur une généralisation de l'inégalité de Wirtinger,* Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 9, (1992), 29-50.

[8] G. Croce and B. Dacorogna, *On a generalized Wirtinger inequality,* Discrete and Continuous Dynamical Systems, 9, (2003), 1329-1341.
[9] W. Huaizhong and L. Yong, Existence and uniqueness of periodic solutions for Duffing equations across many points of resonance, J. Diff. Eqns., 108, (1994), 152-169.
[10] M. G. Krein, *On tests for stable boundedness of solutions of periodic canonical systems,* Amer. Math. Soc. Transl. Ser. (2), 120, (1983), 71-110.
[11] S. Lang, *Analysis.II,* Addison-Wesley, Reading, Mass., 1969.
[12] A.C. Lazer, *Application of a lemma on bilinear forms to a problem in nonlinear oscillations,* Proc. Amer. Math. Soc., 33, (1972), 89-94.
[13] E.B. Lee and L. Markus, *Foundations of Optimal Control Theory,* Wiley, New York, 1967.
[14] W. Magnus and S. Winkler, *Hill's Equation,* Dover, 1979.
[15] G. Talenti, *Best constant in Sobolev inequality,* Ann. Mat. Pura Appl., 110 (1976), 353-372.
[16] M. Zhang, *The rotation number approach to eigenvalues of the one-dimensional* $p-$*Laplacian,* J. London Math. Soc., 64, (2001), 125-143.
[17] M. Zhang and W. Li, A Lyapunov-type stability criterion using L^α norms, Proc. Amer. Math. Soc., 130, (2002), 3325-3333.
[18] M. Zhang, *Certain classes of potentials for p-Laplacian to be non-degenerate,* Math. Nachr., 278, (2005), 1823-1836.

Departamento de Análisis Matemático, Universidad de Granada, 18071 Granada, Spain.
E-mail address: acanada@ugr.es, svillega@ugr.es