Betti numbers of smooth Schubert varieties and the remarkable formula of Kostant, Macdonald Shapiro and Steinberg

Ersan Akyildiz
James B. Carrell

Abstract The purpose of this note is to give a refinement of the product formula proved in [1] for the Poincaré polynomial of a smooth Schubert variety in the flag variety of an algebraic group G over $\mathbb{C}$. This yields a factorization of the number of elements in a Bruhat interval $[e, w]$ in the Weyl group W of G provided the Schubert variety associated to w is smooth. This gives an elementary necessary condition for a Schubert variety in the flag variety to be smooth.

1. Introduction

Let G be a semisimple linear algebraic group over $\mathbb{C}$, B a Borel subgroup of G and $T \subset B$ a maximal torus. Let Φ denote the root system of the pair (G,T) and Φ^+ the set of positive roots determined by B. Let $\alpha_1, \dots, \alpha_\ell$ denote the basis of Φ associated to Φ^+, one recall the *height* of $\alpha = \sum k_i\alpha_i \in \Phi$ is defined to be $ht(\alpha) = \sum k_i$. Finally, let $W = N_G(T)/T$ is the Weyl group of (G,T).

A remarkable formula due to Kostant [10], Macdonald [12], Shapiro and Steinberg [13] says that

$$(1) \qquad \sum_{w\in W} t^{2\ell(w)} = \prod_{\alpha\in\Phi^+} \frac{1-t^{2\mathrm{ht}(\alpha)+2}}{1-t^{2\mathrm{ht}(\alpha)}} = \prod_{i=1}^{\ell}(1+t^2+\cdots+t^{2m_i}),$$

where $m_1, \dots, m_\ell$ are the exponents of G. The left hand side of (1) is a well known expression for the Poincaré polynomial $P(G/B,t)$ of the flag variety G/B. An equivalent formulation of this identity is that if the exponents of (G,T) are ordered so that $m_\ell \geq \cdots \geq m_1$, then the corresponding partition is dual to the partition $h_1 \geq \cdots \geq h_k$ of $|\Phi^+|$, where h_i is the number of roots of height i and $k = h-1$ is the height of the highest root, h being the Coxeter number.

In [1], the authors gave a proof of the first half of (1) using the fact that the cohomology algebra of G/B is the coordinate algebra of the fixed point scheme of a certain G_a action on G/B, which is the unipotent radical of a certain $\mathfrak{B}$-action as explained in Section 2 below. Consequently, one is able to deduce a generalization of (1) for each smooth Schubert variety in G/B. The purpose of this note is essentially to elaborate on this aspect of (1). Along the way, we will present an improved version of the product formula for the Poincaré polynomial of a so called $\mathfrak{B}$-regular variety.

Before stating our results, let us recall the notion of a Schubert variety in the G/B setting and some of the (associated) combinatorics of W. For each

$w \in W$, let wB denote the coset $n_w B \in G/B$, where $w = n_w T$. It is well known that the unique T-fixed point in the B-orbit BwB is wB. Furthermore, by appealing to the double coset (Bruhat) decomposition BWB of G, G/B is the union of these B-orbits as w varies over W. Furthermore, if v and w, are distinct elements of W, then $vB \neq wB$, so W and $(G/B)^T$ may be identified by putting $w = wB$. Thus, Bw denotes BwB. The Schubert variety in G/B associated to w is by definition the Zariski closure $X(w)$ of Bw. It is well known that $\dim X(w) = \ell(w)$, where $\ell(w)$ is the combinatorial length of w with respect to the above simple reflections $r_{\alpha_1}, \dots, r_{\alpha_\ell}$ associated to the simple roots defined above. The Bruhat-Chevalley order $\leq$ on W is the partial order on W defined by putting $x \leq w$ iff $X(x) \subset X(w)$. It coincides with the standard Bruhat order defined combinatorially in terms of the reflections. It is well known that the Bx for all $x \leq w$ give an affine paving of $X(w)$. Thus the Poincaré polynomial of $X(w)$ is given by the formula

$$P(X(w), t) = \sum_{x \leq w} t^{2\ell(x)}. \tag{2}$$

The first identity in (1) is a consequence of basic results about smooth projective varieties admitting a regular action. In particular, any smooth Schubert variety $X(w)$ in G/B being such a variety, one obtains an analogous product formula: namely,

$$P(X(w), t) = \prod_{\alpha \in \Phi(w)} \frac{1 - t^{2\mathrm{ht}(\alpha)+2}}{1 - t^{2\mathrm{ht}(\alpha)}}, \tag{3}$$

where $\Phi(w) = \{\alpha > 0 \mid r_\alpha \leq w\}$. It is well known (see [4] for example) that when $X(w)$ is smooth, $-\Phi(w)$ is the set of weights of the action of T on the Zariski tangent space $T_e(X(w))$ at the identity coset.

In particular, the Poincaré polynomial of a smooth Schubert variety admits a factorization

$$P(X(w), t) = \prod_{1 \leq i \leq r} (1 + t^2 + \cdots + t^{2k_i}) \tag{4}$$

for certain positive integers k_i. An explicit description of the k_i and their multiplicities is precisely stated below in Theorem 3.

2. Regular Actions and the Product Formula

The product formula (3) is a consequence of the fact that smooth Schubert varieties admit a regular action. To recall this notion, let $\mathfrak{B}$ denote the upper triangular Borel subgroup of $SL(2, \mathbb{C})$, and consider an algebraic action $\mathfrak{B} \circlearrowright X$ on a smooth complex projective variety X. When the unipotent radical $\mathfrak{U}$ of $\mathfrak{B}$ has a exactly one fixed point o on X, we call $\mathfrak{B} \circlearrowright X$ regular and say

X is a $\mathfrak{B}$-regular variety. The following assertions are are fundamental for our results. The first three are proved in [5] and the last in [2].

(1) If X is $\mathfrak{B}$-regular, the fixed point set of a maximal torus of $\mathfrak{B}$ is finite.

(2) If $\mathfrak{T}$ is the maximal torus on the diagonal of $\mathfrak{B}$ then $o \in X^{\mathfrak{T}}$.

(3) If $\lambda : \mathbb{C}^* \to \mathfrak{T}$ is the 1-psg $\lambda(s) = \mathrm{diag}[s, s^{-1}]$, then the Bialynicki-Birula cell

$$X_o = \{x \in X \mid \lim_{s\to\infty} \lambda(s) \cdot x = o\} \tag{5}$$

is a dense open subset of X. Consequently, the weights of λ on $T_o(X)$ are all negative.

(4) X_o is $\mathfrak{T}$-equivariantly isomorphic with the Zariski tangent space $T_o(X)$.

Let $a_1 > \cdots > a_k$ denote the distinct weights of λ on $T_o(X)$, and let M_{a_i} denote the weight space corresponding to a_i. Thus,

$$T_o(X) = M_{a_1} \oplus M_{a_2} \oplus \cdots \oplus M_{a_k}. \tag{6}$$

As o is also $\mathfrak{B}$-fixed, $T_o(X)$ is a Lie($\mathfrak{B}$)-module. Now Lie($\mathfrak{B}$) is generated by

$$h = \lambda_*(1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and since $[h, v] = 2v$, we have $hv(m) = 2v(m) + vh(m)$ for all $m \in T_o(X)$. Hence, $v(M_{a_i}) \subset M_{a_i+2}$. We will say that a regular variety is *homogeneous* when all nonzero elements of $\ker v$ have the same weight: equivalently, $\ker v \subset M_{a_1}$. Homogeneity has a number of nice consequences. For example, if $i > 1$, then v is injective on M_{a_i}, so the weight spaces have non-increasing dimension: $\dim M_{a_j} \le \dim M_{a_i}$ if $j > i$. The following result is proven in [1, Theorem 3].

Theorem 1. *Suppose X is a homogeneous regular variety. Then:*

(*a*) $\ker v = M_{a_1}$;

(*b*) *if* $i > j > 0$, $\dim M_{a_i} \le \dim M_{a_j}$;

(*c*) *the weights occur in a string: in fact,* $a_i = -2i$ *for each* $i = 1, \ldots, k$.

By the theory of regular varieties, the cohomology algebra $H^*(X, \mathbb{C})$ of a $\mathfrak{B}$-regular variety is isomorphic to the graded algebra $\mathbb{C}[X_o]/I$, where I is a homogeneous ideal (in the principal grading on $\mathbb{C}[X_o]$) with $\dim M_{a_i}$ generators of degree $-a_i$. Let $m_i = \dim M_{a_i}$. The *generalized Kostant-Macdonald formula* (7) follows from

Theorem 2. *Suppose X is a homogeneous $\mathfrak{B}$-regular variety. Then*

$$P(X, t) = \prod_{1 \le i \le k} \left(\frac{1 - t^{2i_i+2}}{1 - t^{2i}} \right)^{m_i}. \tag{7}$$

.

For the details, see [1]. Putting $d_i = \dim M_{-2i} - \dim M_{-2i-2}$ for $i = 1, \ldots k-1$ and $d_k = \dim M_{-2k}$ and applying Theorems 1 and 2, we get

Corollary 1. *Assuming X is as above,*

$$P(X,t) = \prod_{i=1}^{k} (1 + t^2 + \cdots + t^{2i})^{d_i}. \tag{8}$$

Thus $b_2(X) = \dim M_{-2}$, and there exists a partition $b_2(X) = \sum_{i=1}^{k} d_i$ such that the Euler number $\chi(X)$ is given by

$$\chi(X) = \prod_{i=1}^{k} (i+1)^{d_i}. \tag{9}$$

Moreover, the partition of $\dim X$ given by

$$k = \cdots = k > (k-1) = \cdots = (k-1) > \cdots > 1 = \cdots = 1, \tag{10}$$

where i is repeated d_i times, is dual to the partition

$$\dim M_{-2} \geq \dim M_{-4} \geq \cdots \geq \dim M_{-2k}. \tag{11}$$

Proof. The identity (9) follows immediately from (8). The second assertion is a well known number theoretic and combinatorial fact. See for example [9]. □

3. G/B as a homogeneous Regular Variety

Let $\mathfrak{g}$, $\mathfrak{b}$ and $\mathfrak{t}$ denote the Lie algebras of G, B and T, and recall that $T_e(G/B)$ is naturally isomorphic, as a $\mathfrak{b}$-module, to $\mathfrak{g}/\mathfrak{b}$. To see that G/B is $\mathfrak{B}$-regular, let $\alpha_1, \ldots, \alpha_\ell$ be the simple roots for the root system Φ of (G,T) determined by B, and select a $\mathfrak{t}$-weight vector $e_{\alpha_1} \in \mathfrak{b}$ for each simple root α_i. Put

$$e = \sum_{i=1}^{\ell} e_{\alpha_i},$$

and let h to be the unique element of $\mathfrak{t}$ such that $\alpha_i(h) = 2$ for each i. Note that $\alpha(h) = 2\mathrm{ht}(\alpha)$ for any $\alpha \in \Phi$. Let $\mathfrak{B}$ denote the solvable subgroup of B corresponding to the two dimensional solvable subalgebra $\mathbb{C}h \oplus \mathbb{C}e$. Then the action $\mathfrak{B} \circlearrowright G/B$, is regular with $o = e$ (under the identification of W with $(G/B)^T$). The h-weight subspaces of $T_e(G/B)$ are the

$$M_{-2i} = \mathrm{span}\{e_{-\alpha} \mid \alpha \in \Phi^+,\, \alpha(h) = -i\}$$

for $i = 1, \ldots, k$, k being the height of the longest root. Clearly $\ker e = M_{-2}$, so $\mathfrak{B} \circlearrowright G/B$ is also homogeneous. In particular, $d_i = h_i - h_{i+1}$ (i.e. the number of roots of height i less the number of roots of height $i+1$).

Notice in particular that Corollary 1 gives an interesting expression for the order of the Weyl group: namely,

$$|W| = \prod_{i=1}^{k} (i+1)^{d_i},$$

where k is the height of the highest root. For example, if $G = SL(n, \mathbb{C})$, then $W = S_n$, $k = n-1$ and each $d_i = 1$, so

$$P(SL(n,\mathbb{C})/B, t) = \prod_{i=1}^{n-1} (1 + t^2 + \cdots + t^{2i}),$$

and $|W| = n!$. These facts are, of course, well known.

Since Schubert varieties in G/B are B-stable, the smooth Schubert varieties $X(w)$ are $\mathfrak{B}$-regular and homogeneous. Each T-stable line in $T_e(X(w))$ has weight α for some $\alpha \in -\Phi(w)$, and thus

$$T_e(X(w)) = \bigoplus_{\alpha \in \Phi(w)} \mathbb{C} e_{-\alpha}.$$

This expression immediately yields the h-weight space decomposition. In fact

$$M_{-2i} = \operatorname{span}\{e_{-\alpha} \mid ht(\alpha) = i,\ r_\alpha \leq w\}.$$

Hence, if k_w is the height of the highest root (or roots) α such that $r_\alpha \leq w$, then the weights of h on $T_o(X(w))$ are $-2, -4, \ldots, -2k_w$.

Applying Theorem 1 in this setting gives

Theorem 3. *Let $X(w)$ be a smooth Schubert variety in G/B. Then $X(w)$ is a homogeneous $\mathfrak{B}$-regular variety such that the weights on $T_e(X(w))$ form the string of even integers, $-2 \geq -4 \geq \cdots \geq -2k_w$. Furthermore,*

$$\dim M_{-2i} = |\{e_{-\alpha} \mid ht(\alpha) = i,\ r_\alpha \leq w\}|.$$

In particular,

$$P(X(w), t) = \prod_{i=1}^{k_w} (1 + t^2 + \cdots + t^{2i})^{d_i}. \tag{12}$$

Consequently,

$$\chi(X(w)) = |[1, w]| = \prod_{i=1}^{k_w} (i+1)^{d_i}. \tag{13}$$

As far as we know, formulas (12) and (13) are new. The expression for $|[1, w]|$ provides a simple necessary condition for the smoothness of $X(w)$ which only requires knowing the height of each $\alpha \in \Phi(w)$ and the Euler number $|[1, w]|$ of $X(w)$. Furthermore the assertion on dual partitions of Corollary 1 generalizes to smooth Schubert varieties a well known fact about the heights of positive roots; namely that the partition of $\dim G/B = |\Phi^+|$ given by the exponents of G is dual to the partition $h_1 > h_2 > \cdots > h_k$, where $h_i = |\{\alpha > 0 \mid ht(\alpha) = i\}|$.

Corollary 2. *Suppose $X(w)$ is a smooth Schubert variety, and let $h_{w,i} = |\{\alpha > 0 \mid ht(\alpha) = i,\ r_\alpha \leq w\}|$. Then the partition $h_{w,1} \geq h_{w,2} \geq \cdots \geq h_{w,k}$ of $\ell(w)$, is dual to the partition to the partition (10) where each i, $1 \leq i \leq k$, is repeated d_i times.*

4. The formula in the rationally smooth case

Suppose $X(w)$ is a Schubert variety for which $P(X(w),t)$ has the form (8) but doesn't necessarily arise from a product formula as in (3). Then clearly $P(X(w),t)$ is palindromic, so $X(w)$ has to be rationally smooth, by the main result of [4]. On the other hand, one can ask if there also exist singular Schubert varieties for which (3) holds. The converse is answered by a result of Sara Billey [3]: If $X(w)$ is a rationally smooth Schubert variety in G/B, there exist not necessarily distinct positive integers $j_1, j_2, \ldots, j_k$ such that

$$P(X(w),t) = \prod_{1 \leq i \leq k} (1 + t^2 + \cdots + t^{2j_i}). \tag{14}$$

Perhaps as one should expect, the product formulas (14) and (3) needn't coincide, except in the smooth case. To see this, consider the following example.

Example 1. Let $\Phi(B_2)$ be the root system of type B_2 with two simple roots α and β where α is long and β is short, and positive roots α, β, $\alpha+\beta$ and $\alpha + 2\beta$. All Schubert varieties in B_2/B are rationally smooth, and the unique singular Schubert variety is $X(w)$, where $w = r_\beta r_{\alpha+2\beta} r_\beta$. The fact that $X(w)$ is singular follows easily since if it were smooth, the heights 1 and 3 of $\alpha,\ \beta,\ \alpha+2\beta$ would have to be consecutive integers, which obviously they aren't. The right hand side of product formula (3) is

$$\frac{(1+t)(1+t)(1-t^4)}{(1-t^3)},$$

which isn't even a polynomial. In this case, the factorization (14) of $P(X(w),t)$ is

$$P(X(w),t) = (1+t)(1+t+t^2).$$

In types ADE, (14) follows from (3) and Peterson's ADE Theorem [7] which says that if G is simply laced, every rationally smooth Schubert variety in G/B is smooth. The proof of (14) for types B and C is given in [3]. The case of G_2 was checked by hand, while the F_4 case was verified by computer.

The above comments lead naturally to the question of which rationally smooth Schubert varieties are smooth. The answer is quite easy to state. Let $TE(X(w))$ be the span of the tangent lines of the T-stable curves in $X(w)$ containing e: that is,

$$TE(X(w)) = \bigoplus_{\alpha \in \Phi(w)} \mathbb{C}e_{-\alpha}.$$

The following answer is given in (see [**?**]):

Theorem 4. *Suppose G doesn't contain G_2-factors and let $X(w)$ be a rationally smooth Schubert variety in G/B. Then $X(w)$ is smooth if and only if $TE(X(w))$ is a B-submodule of $T_e(X(w))$.*

In fact, Theorem 4 fails in type G_2: there exists a singular rationally smooth Schubert variety $X(w)$ in G_2/B such that $TE(X(w))$ is a B-submodule of $T_e(X(w))$. The Poincaré polynomial of this Schubert variety is also given by the product formula (3). To see this explicitly, let α and β denote respectively the long and short simple roots for G_2 corresponding to B, and let $r = r_\alpha$ and $s = r_\beta$ be the corresponding reflections. Let $w = srsrs$. Now $\ell(w) = 5$ and it is not hard to see that

$$\Phi(w) = \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta, \alpha + 3\beta\}.$$

It follows that $TE(X(w))$ is indeed a B-submodule of $T_e(X(w))$. However, it is well known that $X(w)$ is singular: for example; for example, see [11]. Curiously, (3) holds for $X(w)$ too. Indeed, the heights in $\Phi(w)$ are 1, 2, 3, and 4, while $d_1 = 1$, $d_2 = d_3 = 0$ and $d_4 = 1$. Thus the right hand side of (3) is

$$(1 + t^2)(1 + t^2 + t^4 + t^6 + t^8),$$

which is indeed the Poincaré polynomial of $X(w)$. Thus, there exist singular examples where the product formula makes sense and coincides with Billey's factorization.

It would be interesting to know whether or not Theorem 4 holds under the weaker condition that $TE(X(w))$ is only a $\mathfrak{B}$-submodule. We conclude with an example illustrating Theorem 3.

Example 2. Let $\alpha_1, \dots, \alpha_4$ denote the simple roots for the root system of D_4 labelled according to the usual labelling of the Dynkin diagram [9], and denote the corresponding reflections by 1, 2, 3 , 4. Consider the element $w = 2142132$ of $W(D_4)$ of length 7. By Goresky's extremely useful tables [8],

$$P(X(w), t) = 1 + 4t^2 + 9t^4 + 13t^6 + 13t^8 + 9t^{10} + 4t^{12} + t^{14},$$

so $X(w)$ is smooth since its Poincaré polynomial is palindromic. One easily checks that

$$\Phi(w) = \{1,\ 2,\ 3,\ 4,\ 212,\ 232,\ 242\},$$

so the heights are 1 and 2 with multiplicities 4 and 3 respectively. Thus, $d_1 = 1$ and $d_2 = 3$. Hence, Theorem 3 gives the expression

$$P(X(w), t) = (1 + t^2)(1 + t^2 + t^4)^3,$$

which agrees with the above expression.

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Ersan Akyildiz

Institute of Applied Mathematics

Middle East Technical University

Ankara 06531, Turkey

ersan@metu.edu.tr

James B. Carrell

Department of Mathematics

University of British Columbia

Vancouver, Canada V6T 1Z2

carrell@math.ubc.ca