# On eigen-structures for pseudo-Anosov maps

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**Abstract:** We investigate various structures associated with the hyperbolic Markov and homological spectra of a pseudo-Anosov map $\phi$ on a surface. Each unstable eigenvalue of the action of $\phi$ on first cohomolgy yields an eigen-cocycle that is transverse and holonomy invariant to the stable foliation $\mathcal{F}^s$ of $\phi$. Each unstable eigenvalue $\mu$ of a Markov transition matrix for $\phi$ yields a holonomy invariant additive function $G$ on transverse arcs to $\mathcal{F}^s$ with $\phi^* G = \mu G$. Except when $\mu$ is the dilation of $\phi$, these transverse arc functions do not yield measures, but rather holonomy invariant eigen-distributions which are dual to Hölder functions. Stable homological and Markov eigenvalues yield analogous transverse structures to the unstable foliation of $\phi$. The main tool for working with the homological spectrum is the Franks-Shub Theorem which holds for a general manifold and map. For the Markov spectrum we use the correspondence of the leaf space of stable foliation with a one-sided subshift of finite type. This identification allows the symbolic analog of a transverse arc function to be defined, analyzed, and applied.

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## Preface to arxiv posted version

The contents of this paper will eventually be included in a monograph. It therefore contains more expository material and redundancy than is usual for a journal paper.

# 1 Introduction

One of the striking features of the theory of surface automorphisms, developed by Nielsen, Thurston, and many others, is the occurrence of linear and piecewise linear (PL) structures in situations which at first glance seem highly nonlinear. Examples include the PL-parameterization of closed curves and measured laminations on surfaces, the PL-action of the mapping class group in this parameterization, the affine structure of pseudo-Anosov homeomorphisms, and the Markov transition matrix which codes pseudo-Anosov dynamics. There is also a surprising amount of information which is sometimes obtainable from the action of an automorphism on first homology.

When there is a linear action the main objects of interest are often eigenvalues and eigenvectors. Within surface theory, pseudo-Anosov homeomorphisms play a central role, and the eigenvalue of greatest import for pseudo-Anosov maps is the dilation. It is denoted $\lambda$ and occurs in many circumstances: it is the Peron-Fröbenius eigenvalue of the Markov transition matrix, the spectral radius of the action on first homology when the invariant foliations are oriented, the exponential growth rate of the action of the pseudo-Anosov map on the fundamental group, and the spectral radius of the induced action on closed curves. Its eigenvector is used to construct the transverse measures to the invariant foliations of the pseudo-Anosov map and is reflected in the fundamental property of these measures usually written as $\phi_* \,\mathcal{F}^u = \lambda\, \mathcal{F}^u$ or $\phi_*\, m^u = \lambda\, m^u$.

Given the importance of this Peron-Fröbenius eigenvalue and eigenvector, it is natural to study the meaning and uses of the rest of various spectra and their associated eigenvectors. Described roughly, the main results here show that the other eigenvalues which are off the unit circle give rise to semi-conjugacies from a covering space to a linear map as well as to additive functions defined on transverse arcs to the invariant foliations. Since it is well known that pseudo-Anosov foliations are uniquely ergodic, these set functions cannot extend to measures, but they are regular enough to define holonomy invariant eigen-distributions in the sense of continuous linear functionals on a space of Hölder functions.

The paper begins with topological constructions based on results of Franks ([Fra70]) and Shub ([Shu78]). These constructions work on any manifold $M$ for any continuous map $f$. To describe them, assume that the action $f^*$ on first cohomology $H^1(M;\mathbb{C})$ has an expanding eigenvector $|\mu| > 1$ with eigen-class $c \in H^1(M;\mathbb{C})$, and so $f^*c = \mu c$. This says that for any cocycle $\zeta \in c$, we have $f^*\zeta = \mu\zeta + d\chi$ for some function $\chi$. A natural question is when is there an actual eigen-cocycle, i.e. a cocycle $\zeta$ with $f^*\zeta = \mu\zeta$?

Perhaps the simplest situation in which such an eigen-cocycle exists is when $f$ is smooth and there is a closed one-form $\omega$ with $f^*\omega = \mu\,\omega$. This is the case for pseudo-Anosov maps with orientable foliations where $\mu$ is the dilation $\lambda$ and the kernel of $\omega$ is tangent to the stable foliation. However, having an eigen-one-form is a very strong property and cannot be expected to hold in any generality. To get a general result we need to extend the space of closed one-forms to include enough cocycles so that each eigen-class actually contains an eigen-cocycle. Perhaps the simplest formulation of this extension uses what is called a path cocycle below. A closed one-form can be used to assign numbers to paths in a homologically invariant fashion, and we adopt this property as the definition of a path cocycle. In this language the Franks-Shub Theorem says that for an unstable eigenvalue $\mu$ there is always a path cocycle $F$ with $f^*F = \mu F$.

The first formulation and proof we give of the Franks-Shub Theorem in Theorem 3.3 is not in terms of path cocycles, but rather in terms of $c$-maps. These are maps from the universal free Abelian cover of $M$ into $\mathbb{R}$ or $\mathbb{C}$ which transform under the deck group as dictated by the cohomology class $c$. When such a map $\tilde{\alpha}$ represents an eigen-cohomology class it satisfies

$$\tilde{\alpha}\tilde{f} = \mu\tilde{\alpha} \tag{1.1}$$

for some lift $\tilde{f}$ and is thus a dynamical semiconjugacy from $\tilde{f}$ to a linear map. Since they are functions on a manifold, $c$-maps are often technically a bit easier to work with than cocycles and, in addition, they are useful for dynamical applications. The $c$-map formulation of the Franks-Shub Theorem is closer in spirit to Franks [Fra70] while the path cocycle version is closer to Shub [Shu78].

In §3.4 we note that when the level sets of an eigen-$c$-map are projected back to the base manifold $M$ they form an $f$-invariant decomposition of $M$. The corresponding eigen-path cocycle then describes the expansion by the factor $\mu$ "transverse" to the decomposition under the action by $f$. Since the sets of this decomposition can be quite wild, we cannot make much progress at this level of generality. From §5 to the end of the paper we focus on pseudo-Anosov homeomorphism, $\phi$, acting on a compact surface, $M$. These pseudo-Anosov maps are characterized by a pair of transverse foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ each equipped with a transverse measure that is expanded or contracted by a factor of $\lambda$ under the action by $\phi$. When these foliations are orientable, the stable foliation $\mathcal{F}^s$ is the decomposition determined by the eigen-$c$-map with eigenvalue $\lambda$, and the corresponding path cocycle is the transverse measure. By using $\phi^{-1}$ one obtains a decomposition and eigen-cocycle corresponding to the unstable foliation and its transverse measure. The next step is to note that any other eigen-path cocycle for an eigenvalue $|\mu| > 1$ is also transverse and holomomy invariant to $\mathcal{F}^s$. In Proposition 6.2, we show that the collection of transverse cocycles to $\mathcal{F}^s$ is isomorphic to the unstable subspace of $\phi^*$ acting on $H^1(M, \mathbb{C})$. In particular, each cohomology class in that space contains exactly one transverse cocycle.

In §7 we introduce the class of transverse arc functions (taf) and show that when the foliations are orientable, they correspond to transverse cocycles. In the case of non-orientable foliations there is no such connection and in §8 we begin the use of symbolic methods which work in both the non-orientable and orientable cases. The symbolic constructions are based on the standard correspondence between the leaf space of the stable foliation and the one-sided subshift of finite type $\Lambda_A^+$ generated by the transition matrix of the pseudo-Anosov map, $A$. Our main objects are the symbolic transverse arc functions (staf) on $\Lambda_A^+$. These are additive functions defined on the collection of cylinder sets which satisfy a coherence condition which ensures their correspondence to taf for $\phi$. Fact 8.2 says that the linear space of staf is naturally identified with the eventual image of $A$ and so is spanned by the eigen-staf with non-zero eigenvalues. Theorem 8.6 says that each staf yields an element of the continuous dual of the appropriate class of Hölder functions on $\Lambda_A^+$, but only the eigen-staf of the Peron-Fröbenius eigenvalue of $A$ extends to a signed or complex Borel measure on $\Lambda_A^+$.

The next step is to connect taf and symbolic taf in Theorem 9.1. While there is a symbolic taf for each eigenvalue $\mu$ of the transition matrix, only the unstable ones, $|\mu| > 1$, yield tafs to the stable foliation. The reason for this is roughly that under the correspondence of $\Lambda_A^+$ with the leaf space of $\mathcal{F}^s$, the Cantor set $\Lambda_A^+$ is collapsed into an arc $\Gamma$ transverse to $\mathcal{F}^s$. While a symbolic taf need only assign finite values to cylinder sets, a taf must assign a finite value for all transverse arcs. The collapses of cylinder sets from $\Lambda_A^+$ form a rather small subset in the collection of all arcs in $\Gamma$, and when a symbolic taf is not unstable, its push forward to $\Gamma$ would assign infinite values to any arc outside this small subset.

Using the correspondence of $\Lambda_A^+$ with $\mathcal{F}^s$ in conjunction with Theorem 8.6, we have in Theorem 11.3 that each taf yields an element of the continuous dual of the appropriate class of Hölder functions on transversals to $\mathcal{F}^s$, but only the eigen-staf of the Peron-Fröbenius eigenvalue of $A$ yields a measure, namely, the standard (and only) transverse measure to $\mathcal{F}^s$.

In the last few sections we study eigen-$c$-maps in more detail and show in Theorem 12.6 that for eigenvalues $1 < |\mu| < \lambda$, these $c$-maps are nowhere locally of bounded variation and nowhere differentiable as well as Hölder with exponent $\log(|\mu|)/\log(\lambda)$, but not Hölder for any larger exponents. The main idea is that as a consequence of the eigen-property (1.1), if we restrict the $c$-map to a lifted unstable leaf we get a function $f$ that everywhere satisfies $f(\lambda t) = \mu f(t)$. Since $|\mu| < \lambda$ this means that $f$ has to "fold up" everywhere leading to the low regularity. This result is illustrated in §12.3 with an example of a pseudo-Anosov map $\psi$ on a genus two surface for which the eigen-$c$-maps patch together in pairs yielding semiconjugies from $\psi$ to two toral automorphisms, the first with the same entropy as $\psi$ and the second with lesser entropy. The first semiconjugacy is a branched cover while the second semiconjugacy is nowhere differentiable and the preimage of a typical point is a Cantor Set.

There is a fair amount of literature associated with various aspects of this paper. While we have strived to keep the paper self-sufficient, we will describe at least part of this literature because of its importance for inspiration, reference, and further developments.

The first application of the Franks-Shub Theorem to the study of pseudo-Anosov maps was by Fathi in [Fat88]. That paper was the source of many ideas developed here. We also note that Robertson in [Rob07] proves a version of the Franks-Shub theorem concerning eigen-currents under homologically expanding smooth maps.

When a pseudo-Anosov foliation is orientable, one can consider the flow along the leaves. The transverse holonomy invariant structures developed here using the action of the pseudo-Anosov map are invariant under the flow. In [For97] Forni developed a deep and general theory of invariant distributions to a flow on a surface. He applied this to the study of the Teichmüller geodesic flow in [For02] (cf. [For06]). The main ideas of relevance here are roughly as follows. A pseudo-Anosov foliation is a periodic orbit under the Teichmüller flow. The second component of the Zorich-Kontsevich cocycle of the return map is essentially the action of the pseudo-Anosov mapping class on the first cohomology of the surface. Forni shows that the eigenvalues of this linear action yield holonomy (or flow) invariant distributions, or more precisely, basic currents of the foliation.

Next note that in [Bon97a] and [Bon97b] Bonahon develops the theory of transverse Hölder distributions to geodesic laminations on surfaces. There are many points of contact and significant differences between this theory and that of transverse structures to pseudo-Anosov invariant foliations. First, the lamination theory holds for any lamination while here we just consider the foliations associated with pseudo-Anosov maps. On the other hand, our main interest and methodology is the induced action of the pseudo-Anosov map on various linear structures while Bonahon's papers consider geometric and analytic aspects of the laminations without the action of a mapping class. Most fundamentally, the map which collapses a pseudo-Anosov invariant lamination to a foliation is not Hölder since any smooth transverse arc always intersects the lamination in a Cantor set of Hausdorff dimension zero. Thus the various manifestations of Hölder regularity in the two theories are different. For example, for a pseudo-Anosov stable foliation only the unstable eigenvalues of the transition matrix yields transverse arc functions (see Remark 11.1). The analogous structure for a lamination defined in [Bon97b] need only assign a finite value to arcs with endpoints in the complement of the lamination and thus stable eigenvalues also generate a transverse structure.

A pseudo-Anosov homeomorphism on a surface is hyperbolic at all but finitely many points and our use of symbolic dynamics puts us squarely within the classical theory of hyperbolic dynamics. Since we are concerned with the spectrum of Markov transition matrices, in some cases we are dealing with the simplest special case of the vast and deep theory of transfer operators. We do not describe this is any detail here, but refer the reader to Baladi's excellent book [Bal00]. Of direct relevance however is the paper [Rue87] in which Ruelle defines Gibbs distributions dual to Hölder functions on a subshift of finite type. Haydn in [Hay90] shows that Gibbs distributions are always eigen with respect to the action of the dual of the transfer operator. In Theorem 8.6 we show that the distributions constructed from eigen-stafs have this property.

A sequel to this paper will consider these connections in more detail as well as applications of this paper to the dynamics, statistics and geometry of of pseudo-Anosov maps.

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## 2 Preliminaries

### 2.1 Linear algebra

To set terminology we begin with some standard notions. For a square matrix $A$, recall that $\mu$ is called an *eigenvalue* if $\ker(A - \mu I) \neq 0$, and $v$ is called a corresponding *eigenvector* if $v \in \ker(A - \mu I)$ and $v$ is called a corresponding *generalized eigenvector* if $v \in \ker(A - \mu I)^k$ for some $k > 1$. Note that under these definitions an eigenvector is not a generalized eigenvector. The *generalized eigenspace* of $\mu$ is the subspace consisting of all $\mu$'s eigenvectors and generalized eigenvectors in addition to the zero vector. An *eigenchain* for $\mu$ of length $k \geq 1$ is a set of vectors $v_1, \dots, v_k$ with $v_1$ an eigenvector, each $v_i$ with $i > 1$ a generalized eigenvector with $(A - \mu I)v_{i+1} = v_i$ for all $i$. The generalized eigenspace of $\mu$ always has a basis consisting of the union of eigenchains; each eigenchain is the basis of one of the Jordan blocks corresponding to $\mu$.

Let $V$ be a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ and $T : V \to V$ a linear transformation. An eigenvalue $\mu$ of $T$ is called unstable, central, stable, and nilpotent, respectively, if $|\mu| > 1, |\mu| = 1, 0 < |\mu| < 1$, and $\mu = 0$. The *unstable subspace* of $T$, denoted $Un(T, V)$ is the direct sum of all the generalized eigenspaces of $T$ associated with unstable eigenvalues. The *central*, *stable*, and *nilpotent* subspaces are denoted by $Cen(T, V)$, $Stab(T, V)$, and $Nil(T, V)$, respectively, are associated with central, stable, and nilpotent eigenvalues. The direct sum decomposition, $V = Un(T, V) \oplus Cen(T, V) \oplus Stab(T, V) \oplus Nil(T, V)$, is preserved by $T$, and $T$ restricted to a factor is denoted $T^{Un}$, etc. The *non-nilpotent* subspace, denoted $NonN(T, V)$, is the direct sum of all the generalized eigenspaces connected with nonzero eigenvalues, and so $NonN(T, V) = Un(T, V) \oplus Cen(T, V) \oplus Stab(T, V)$. The non-nilpotent subspace is the same as the eventual range of $T$, $NonN(T, V) = \cap_{n \in \mathbb{N}} T^n(V)$, and $T^{NonN}$ is always a self-isomorphism of $NonN(T, V)$. The *hyperbolic* subspace is the direct sum of the stable and unstable ones.

There will be a variety of linear objects discussed in this paper. The terminology "unstable" when applied to such objects always indicates that the object is contained in the unstable subspace of the linear transformation under discussion.

We will also need some of the results that go under the general rubric of the Peron-Fröbenius Theorem. A matrix $A$ with $A^n > 0$ for some $n > 0$ always has a simple eigenvalue of largest modulus, which is always real and it has a strictly positive eigenvector. Let the unit length, strictly positive eigenvectors from the left be $\vec{\ell}$ and the right $\vec{r}$. No other eigenvalues have strictly positive eigenvectors, and if $\vec{v}$ is any non-negative vector, then $A^n\vec{v}/\|A^n\vec{v}\| \to \vec{r}$ as $n \to \infty$. In addition,

$$A^n/\lambda^n \to P \tag{2.1}$$

as $n \to \infty$ where $P = \vec{r}\,\vec{\ell}$. Note that here $\vec{r}$ is treated as a column vector and $\vec{\ell}$ as a row vector.

For future use we record an easy but somewhat technical fact whose proof is a straightforward application of the Peron-Fröbenius theorem and the Jordan canonical form.

**Fact 2.1** *Assume that $A$ is a square matrix with $A^n > 0$ for some $n > 0$, and let $\lambda$ be its Peron-Fröbenius eigenvalue and $r > 0$ be such that $\lambda > r > |\mu|$ for any eigenvalue $\mu \neq \lambda$. Given a vector $v \in NonN(A)$ which is not an eigenvector for $\lambda$, there exist $C > 0$ and $N$ so that for any vector $w$ with $A^n w = v$ and $n > N$, we have $\|w\|_1 > C/r^n$.*

**Definition 2.2 (The field $\mathbb{F}$)** *In the sequel it will often be the case that the appropriate field for coefficients of homology/cohomology or for the range of a map or homomorphism will depend on whether an eigenvalue $\mu$ under consideration is real or complex. To avoid the awkwardness of the constant repetition of the phrase "where the field $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$ depending on whether $\mu$ is real or complex", we adopt the convention that the field is denoted $\mathbb{F}$ and has a value $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ as is appropriate in the given situation.*

### 2.2 First homology, cohomology and the universal Abelian covering space

Let $M$ be a smooth, connected, compact manifold of any dimension. Fix a base point $x_0 \in M$ and a set of generators of the fundamental group $\pi_1(M, x_0)$ whose Abelianizations give the basis of $H_1(M; \mathbb{Z})$. The *universal Adelina covering space* (also called the homology cover) is the largest covering space of $M$ whose automorphism (or deck) group is Abelian. Thus, this covering space, which is denoted $\pi : \tilde{M} \to M$ (or just $\tilde{M}$) here, satisfies $\pi_*(\pi_1(\tilde{M})) = [\pi_1(M, x_0), \pi_1(M, x_0)]$, the commutator subgroup. To obtain a metric on $\tilde{M}$, we fix a metric on $M$, and lift it to $\tilde{M}$ yielding an equivariant, topological metric we denote $\tilde{d}$. The usual universal cover of $M$ which is the cover with deck group equal to $\pi_1(M)$ will only rarely be used here and is denoted $\hat{M}$.

For simplicity of exposition we assume that $H_1(M;\mathbb{Z})$ is torsion-free and has rank $d > 0$. We leave to the reader the minor changes needed for the case of torsion. Thus the deck group of $\tilde{M} \to M$ is $\mathbb{Z}^d$ where $d$ is the first Betti number of $M$. For $\vec{n} \in \mathbb{Z}^d$, we let $\delta_{\vec{n}}$ denote the corresponding element of the deck group.

Recall that a pair of paths $\gamma_1$ and $\gamma_2$ in $M$ are said to be *homologous*, if the loop $\gamma_1 \# \gamma_2^{-1}$ is null-homologous. A important feature of the universal Abelian cover $\tilde{M}$ is that a loop $\Gamma \subset M$ lifts to a loop in $\tilde{M}$ if and only if $\Gamma$ is null homologous in $M$, or equivalently, two paths in $M$, $\gamma_1$ and $\gamma_2$, with the same endpoints lift to two paths in $\tilde{M}$, $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, with the same endpoints if and only if $\gamma_1$ and $\gamma_2$ are homologous in $M$.

For a continuous self-map $f : M \to M$, let $f_*$ and $f^*$ be the induced actions on $H_1(M)$ and $H^1(M)$. Any $f : M \to M$ lifts to the universal Abelian cover $\tilde{M}$. If $\tilde{f}$ is a lift of $f$ to $\tilde{M}$, a fundamental relation is

$$\tilde{f} \circ \delta_g = \delta_{f_*(g)} \circ \tilde{f}. \tag{2.2}$$

For the purposes of comparison with other cocycles discussed below we also recall standard terminology surrounding first de Rham cohomology, $H^1_{DR}(M;\mathbb{R})$. The vector space $H^1_{DR}(M;\mathbb{R})$ is composed of the cohomology classes of closed one-forms on $M$ with two closed forms $\omega_1$ and $\omega_2$ being called *cohomologous* if there is a smooth function $\chi : M \to \mathbb{R}$ so that the exact one-form $d\chi$ satisfies $\omega_1 = \omega_2 + d\chi$. For a smooth loop $\Lambda$ in $M$ with homology class $[\Lambda] \in H_1(M;\mathbb{Z})$ and a closed one-form $\omega \in c$, define $\Phi_c([\Lambda]) = \int_\Lambda \omega$. This definition is independent of the choices and the map $c \mapsto \Phi_c$ is an isomorphism from $H^1_{DR}(M;\mathbb{R})$ to $Hom(H_1(M;\mathbb{Z}),\mathbb{R})$. Here we will usually extend $\Phi_c$ to a functional $H_1(M;\mathbb{R}) \to \mathbb{R}$ and without comment treat elements of $H^1(M;\mathbb{R})$ as elements of the dual space of $H_1(M;\mathbb{R})$, or after a choice of generators, with linear maps $\mathbb{R}^d \to \mathbb{R}$. We will also consider de Rham cohomology with complex coefficients where all the same definitions and properties apply.

### 2.3 Hölder spaces

If $(X,\rho)$ is a metric space, the space of all continuous $f : X \to \mathbb{F}$ is denoted $C^0(X,\mathbb{F})$. This space is given the sup-metric

$$d_0(f,g) = \sup\{|f(x) - g(x)| : x \in X\}, \tag{2.3}$$

when the supremum is finite. For $f \in C^0(X,\mathbb{F})$ and $0 < \nu \leq 1$, define

$$|f|_\nu = \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{d(x_1,x_2)^\nu}. \tag{2.4}$$

If $|f|_\nu \leq C < \infty$, then $f$ is said to be $(\nu, C)$-*Hölder*. The space of all Hölder $f$ with given exponent $0 < \nu < 1$ is denoted $C^\nu(X,\mathbb{F})$ and is given the metric

$$d_\nu(f,g) = d_0(f,g) + |f-g|_\nu. \tag{2.5}$$

It is standard that with these metrics for all $0 < \nu < 1$ and for $X$ compact, $C^\nu(X,\mathbb{F})$ is complete and separable. As is conventional, functions which are $(1,\lambda)$-Hölder will be called

$\lambda$-*Lipschitz*. The space of all Lipschitz $f$ is denoted $C^{Lip}(X,\mathbb{F})$. The notation $C^1(X,\mathbb{F})$ is reserved for the space of all functions with continuous first derivatives.

### 2.4 Note on the terms cocycle and distribution

The terms cocycles, cohomologous, etc have different meanings in dynamics and algebraic topology. Here it will always be the latter usage unless there is an explicit comment to the contrary.

Also, the term distribution is often used in dynamics to refer to an invariant foliation or lamination, as in "the unstable distribution". Here distribution will always be in the sense of Schwartz as an element of a dual space, i.e. a linear functional.

## 3 $C$-Maps and the Franks-Shub Theorem

As motivation and illustration we start with the construction of a $c$-map in the simplest case of an eigen-object as was described in the introduction. So assume that $f : M \to M$ is smooth and there we have a closed one-form $\omega$ with $f^*\omega = \mu\omega$ with $\mu \in \mathbb{R}$. Lift (or pullback) the eigen-one-form $\omega$ to $\tilde{\omega}$ on the universal Abelian cover $\tilde{M}$ and after fixing a base point $\tilde{z}_0 \in \tilde{M}$, define $\sigma_\omega : \tilde{M} \to \mathbb{R}$ by

$$\sigma_\omega(\tilde{z}) = \int_{\tilde{\gamma}} \tilde{\omega}, \tag{3.1}$$

where $\gamma$ is any smooth path in $\tilde{M}$ connecting $\tilde{z}_0$ and $\tilde{z}$. Since $\omega$ and thus $\tilde{\omega}$ is closed, the definition is independent of the choice of path. In fact, $\tilde{\omega}$ is exact in $\tilde{M}$ with $\tilde{\omega} = d\sigma_\omega$. Informally, this happens because the universal Abelian cover $\tilde{M}$ is exactly the space that is constructed by unwrapping $M$ just enough to remove all the obstructions to a closed form being exact. Now lift the diffeomorphism $f : M \to M$ to $\tilde{f} : \tilde{M} \to \tilde{M}$ and for simplicity assume that the base point $\tilde{z}_0$ is a fixed point of $\tilde{f}$. Thus, as a consequence of $f^*\omega = \mu\omega$, we have that $\sigma_\omega \circ \tilde{f} = \mu\ \sigma_\omega$. In other words, $\sigma_\omega$ gives a semi-conjugacy from $\tilde{f}$ on $\tilde{M}$ to multiplication by $\mu$ on $\mathbb{R}$.

The map $\sigma_\omega$ is a special case of a *c-map* defined in the next subsection. In Theorem 3.3 we show that in the general situation of a continuous self-map of a manifold $M$, each eigenvalue $\mu$ with $|\mu| > 1$ has a corresponding eigen-$c$-map.

### 3.1 Definition of $c$-maps

The definition of a $c$-map is in terms of a given, specified cohomology class. In §4 below, we consider structures more suitable for general cohomology formulation.

Given a non-zero class $c \in H^1(M,\mathbb{F})$, a map $\sigma : \tilde{M} \to \mathbb{F}$ is called a *c-map* if

$$\sigma \circ \delta_{\vec{n}} = \sigma + \Phi_c(\vec{n}), \tag{3.2}$$

for all $\vec{n} \in \mathbb{Z}^d$ where the linear functional $\Phi_c : \mathbb{F}^d \to \mathbb{F}$ represents the class $c$. Two $c$-maps representing the same class $c$ are said to be *cohomologous*. It follows from (3.2) that $\sigma_1$ and $\sigma_2$ are cohomologous if and only if

$$\sigma_1 \circ \delta_{\vec{n}} - \sigma_2 \circ \delta_{\vec{n}} = \sigma_1 - \sigma_2, \tag{3.3}$$

for all $\vec{n} \in \mathbb{Z}^d$. In turn, (3.3) happens if and only if

$$\sigma_1 = \sigma_2 + \chi \circ \pi, \tag{3.4}$$

for some continuous $\chi : M \to \mathbb{F}$, where $\pi : \tilde{M} \to M$ is the covering map.

**Remark 3.1** For a torus the universal cover is the same as the universal Abelian cover which can be identified with the vector space $H_1(M; \mathbb{R})$. In most other cases the universal Abelian cover is not a vector space, but it is often useful to adapt the heuristic of $\tilde{M}$ being identified with $H_1(M; \mathbb{R})$. (They are in fact "coarsely equivalent" by a standard equivariant embedding of $\tilde{M}$ into $\mathbb{R}^d$, see, for example, [Boy09]). In the case at hand, a real cohomology class is actually a linear functional $H_1(M; \mathbb{R}) \to \mathbb{R}$, but we "identify" it with a $c$-map $\tilde{M} \to \mathbb{R}$.

For a cohomology class $c \in H^1(M, \mathbb{F})$, let $S_c^\nu$ denote the collection of $c$-maps $\sigma \in C^\nu(\tilde{M}, \mathbb{F})$. The metric on $S_c^\nu$ is that induced as a subspace of $C^\nu(\tilde{M}, \mathbb{F})$. Now if $\omega$ is a $C^\infty$, closed one-form with $c = [\omega]$ in $H^1_{DR}(M; \mathbb{F})$, then it is immediate that $\sigma_\omega$ defined by (3.1) is a $c$-map. Now for any other $c$-map $\sigma \in S_c^\nu$, by (3.4), there exists an $\chi \in C^\nu(M, \mathbb{F})$ with $\sigma = \sigma_\omega + \chi \circ \pi$. This yields an isometry of $S_c^\nu$ with $C^\nu(M, \mathbb{F})$ and so, in particular, $S_c^\nu$ is a complete, separable metric space

**Remark 3.2** As we just noted, any $c$-map, $\sigma$, can be written $\sigma = \sigma_\omega + \chi \circ \pi$ with $\sigma_\omega$ constructed from a closed one-form in the class of $\sigma$. This implies that for any equivariant metric $\tilde{d}$ on $\tilde{M}$ there are $\lambda, K > 0$ so that for all $\tilde{x}, \tilde{y} \in \tilde{M}$, $|\sigma(\tilde{x}) - \sigma(\tilde{y})| \leq \lambda\, \tilde{d}(\tilde{x}, \tilde{y}) - K$. This property is something expresses by saying that $\sigma$ is large scale Lipschitz.

### 3.2 The Franks-Shub Theorem

This section contains the Franks-Shub Theorem formulated in the language of $c$-maps. This is essentially the point of view in Franks [Fra70], but applied to one eigenvalue at a time. Shub in [Shu78] gave an equivalent theorem in the language of Alexander-Spanier cocycles.

**Theorem 3.3 (Franks, Shub)** *Assume that $f : M \to$ is a continuous map of the smooth, connected manifold $M$ with $H^1(M; \mathbb{Z})$ torsion-free and $\mu \in \mathbb{F}$ is an eigenvalue of $f^* : H^1(M; \mathbb{Z}) \to H^1(M; \mathbb{Z})$ with $|\mu| > 1$ and eigenchain $\{c_1, \ldots, c_k\} \subset H^1(M; \mathbb{F})$. For each lift $\tilde{f} : \tilde{M} \to \tilde{M}$ of $f$ to the universal Abelian cover $\tilde{M}$ there exists unique $c_i$-maps $\tilde{\alpha}_i : \tilde{M} \to \mathbb{F}$ for $i = 1, \ldots, k$ with*

$$\tilde{\alpha}_1 \circ \tilde{f} = \mu \tilde{\alpha}_1 \text{ and for } i > 1,\ \tilde{\alpha}_i \circ \tilde{f} = \mu \tilde{\alpha}_i + \tilde{\alpha}_{i-1}. \tag{3.5}$$

*Further, if $f$ is Lipschitz with constant $\lambda$, then $\tilde{\alpha}_i \in C^\nu(M, \mathbb{F})$ for all $0 \leq \nu < \log(|\mu|)/\log(\lambda)$.*

**Proof:** The proof is by induction on the index $i$ in the eigenchain. To prove the case $i = 1$, assume that $c = c_1$ is an eigen-class for $f^*$. For $\sigma \in S_c^\nu$, let

$$F(\sigma) = \frac{\sigma \circ \tilde{f}}{\mu}.$$

Now using (2.2) and (3.2)

$$\begin{aligned} F(\sigma) \circ \delta_{\vec{n}} &= \frac{\sigma \circ \tilde{f} \circ \delta_{\vec{n}}}{\mu} \\ &= \frac{\sigma \circ \delta_{f_*(\vec{n})} \circ \tilde{f}}{\mu} \\ &= \frac{\sigma \circ \tilde{f} + \Phi_c(f_*(\vec{n}))}{\mu} \\ &= \frac{\sigma \circ \tilde{f}}{\mu} + \Phi_c(\vec{n}), \end{aligned}$$

where in the last line we used $\Phi_c \circ f_* = f^*(\Phi_c) = \mu\Phi_c$, because $\Phi_c$ represents the eigen-class $c$. Thus $F : S_c^0 \to S_c^0$. Now if $\sigma$ is $(C, \nu)$-Hölder, using the fact that $\tilde{f}$ is $\lambda$-Lipschitz, we get that $F(\sigma)$ is $(C\lambda^\nu/\mu, \nu)$-Hölder. Thus $F : S_c^\nu \to S_c^\nu$, for $0 < \nu < 1$ as well.

When $|\mu| > 1$, it is obvious that $F : S_c^0 \to S_c^0$ is a contraction with constant $|\mu|^{-1}$. Thus $F$ has a fixed point $\tilde{\alpha}$ which is at least $C^0$ and satisfies (3.5). For $0 < \nu < 1$,

$$\begin{aligned} d_\nu(F(\sigma_1), F(\sigma_2) &= \sup_{\tilde{x} \in \tilde{M}} \left| \frac{\sigma_1 \circ \tilde{f}(\tilde{x}) - \sigma_2 \circ \tilde{f}(\tilde{x})}{\mu} \right| \\ &\quad + \sup_{\tilde{x}_1 \neq \tilde{x}_2} \left| \frac{(\sigma_1 \circ \tilde{f}(\tilde{x}_1) - \sigma_1 \circ \tilde{f}(\tilde{x}_2)) - (\sigma_2 \circ \tilde{f}(\tilde{x}_1) - \sigma_2 \circ \tilde{f}(\tilde{x}_2))}{\mu \tilde{d}(\tilde{x}_1, \tilde{x}_2)^\nu} \right| \\ &= \frac{d_0(\sigma_1, \sigma_2)}{\mu} \\ &\quad + \sup_{\tilde{x}_1 \neq \tilde{x}_2} \left| \left( \frac{(\sigma_1 - \sigma_2)(\tilde{f}(\tilde{x}_1)) - (\sigma_1 - \sigma_2)(\tilde{f}(\tilde{x}_2))}{\mu \tilde{d}(\tilde{f}(\tilde{x}_1), \tilde{f}(\tilde{x}_2)^\nu} \right) \left( \frac{\tilde{d}(\tilde{f}(\tilde{x}_1), \tilde{f}(\tilde{x}_2)^\nu}{\tilde{d}(\tilde{x}_1, \tilde{x}_2)^\nu} \right) \right| \\ &\leq \frac{d_0(\sigma_1, \sigma_2)}{|\mu|} + \frac{|\sigma_1 - \sigma_2|_\nu \lambda^\nu}{|\mu|} \\ &\leq \frac{d_\nu(\sigma_1, \sigma_2)}{|\mu| \lambda^{-\nu}}, \end{aligned}$$

Where in the last line we used the fact that when $f$ is $\lambda$-Lipschitz, then the existence of a $C^0$-$\tilde{\alpha}$ satisfying (3.5) implies that $\lambda \geq |\mu|$. Thus if $|\mu|\lambda^{-\nu} > 1$, i.e. when $\nu < \log|\mu|/\log\lambda$, $F$ is a contraction on the complete vector space $S_c^\nu$, and so it has a unique Hölder fixed point $\tilde{\alpha}$, finishing the proof for $i = 1$.

Assume now that we have proven the result for $c_i$. Let $S_{c_{i+1}}^\nu$ be all the $\nu$-Hölder $c$-maps that represent the class $c_{i+1}$, and for $\sigma \in S_{c_{i+1}}^\nu$, let

$$G(\sigma) = \frac{\sigma \circ \tilde{f} - \tilde{\alpha}_i}{\mu},$$

where $\tilde{\alpha}_i$ is the eigen-$c$-map representing $c_i$ given by the inductive hypothesis. It is easy to check that in fact $G : S_{c_{i+1}}^\nu \to S_{c_{i+1}}^\nu$ and is a contraction for all $0 \leq \nu < 1$, and so $G$ has a unique fixed point, yielding a generalized eigen-$c$-map $\tilde{\alpha}_{i+1}$ with

$$\tilde{\alpha}_{i+1} \circ \tilde{f} = \mu \tilde{\alpha}_{i+1} + \tilde{\alpha}_i, \tag{3.6}$$

finishing the induction. ■

### 3.3 Remarks on the Franks-Shub Theorem

If $f$ is a homeomorphism then by using $f^{-1}$, there is an eigen-$c$-map for all eigenvalues with $0 < |\mu| < 1$. Note that $\mu = 0$ does not occur when $f$ is a homeomorphism.

While we used a fixed point argument, the version of the theorem in Shub [Shu78] (and implicitly in Franks [Fra70]) is proved using a summation formula for $\tilde{\alpha}$. In the current context the argument goes like this. Given the eigen-class $c$, pick a close one-form $\omega$ in the class and construct $\sigma_\omega$ as in (3.1). Now as in the proof above, $(f^*\sigma_\omega) \circ \delta_{\vec{n}} = \mu\sigma_\omega + \Phi_{\mu c}\vec{n}$, and so $(f^*\sigma_\omega)$ is a $(\mu c)$-map. Now $\mu\sigma_\omega$ is also a $(\mu c)$-map, and so by (3.4), there exists an $\chi \in C^\nu(M, \mathbb{F})$ with $f^*\sigma_\omega = \mu\sigma_\omega + \chi \circ \pi$. Now define

$$\tilde{\alpha} := \sigma_\omega + \sum_{j=1}^{\infty} \frac{\chi \circ \pi \circ \tilde{f}^{j-1}}{\mu^j}, \tag{3.7}$$

which converges since $|\mu| > 1$, and $f^*\tilde{\alpha} = \mu\tilde{\alpha}$ by direct verification.

A summation formula as in 3.7 is familiar classically from Weierstrass nowhere differentiable functions and more recently in the theory of fractal functions. It is therefore not surprising that even for very smooth $f$, the eigen-cmaps are often of very low regularity. For the case of eigenvalues of a pseudo-Anosov map with $1 < |\mu| < \lambda$, Theorem 12.6 below shows that $\tilde{\alpha}$ is nowhere differentiable.

The summation formula also reveals the connection to well-known facts about solutions to hyperbolic *dynamical cocycle* equation. Specifically, continue to assume that we have an eigen-class containing the closed one-form $\omega$ and the corresponding $c$-map $\sigma_\omega$ satisfies $f^*\sigma_\omega = \mu\sigma_\omega + \chi \circ \pi$ for some $\chi \in C^r(M, \mathbb{F})$ when $f$ is $C^r$. Since the eigen-$c$-map we seek is in the same class, we may represent $\tilde{\alpha}$ as $\tilde{\alpha} = \sigma_\omega + h \circ \pi$ for some $h \in C^0(M, \mathbb{F})$. Plugging into $f^*\tilde{\alpha} = \mu\tilde{\alpha}$ yields an equation $(h \circ \pi) \circ \tilde{f} - \mu(h \circ \pi) = -\chi \circ \pi$. Thus projecting to the base the semi-conjugacy questions reduces to solving the hyperbolic dynamical cocycle equation

$$h \circ f - \mu h = -\chi, \tag{3.8}$$

for $h$ given $\chi$. It is well known that the solution is the summation in (3.7) projected to the base, and that even for very regular $\chi$ the corresponding $h$ is often no more regular than Hölder.

We also note that methods similar to Theorem 3.3 yield a semiconjugacy from the full universal cover of $M$ to linear expansion by $\mu$ on $\mathbb{F}$. However, the smaller the cover, the more information a semiconjugacy yields about the dynamics. For example, pseudo-Anosov maps lifted to their universal cover are dynamically uninteresting having at most one recurrent point, but they can be transitive in the universal Abelian cover ([Boy09]). The most useful information is semiconjugacy from the manifold itself.

Thus a natural question is when the semiconjugacy given by a single eigen-$c$-map or a collection of eigen-$c$-maps descends to a semiconjugacy defined on the manifold $M$ itself. This simplest case is when the eigenvalue $\mu$ of $f^*$ on $H^1(M;\mathbb{Z})$ is an integer $n$ in which case the eigen-$c$-map descends to a semiconjugacy from $(M,f)$ to $z \mapsto z^n$ on the circle. The simplest case of this is when $M$ is the circle and $f$ is a degree-$n$ map with $|n| > 1$ (*cf.* [Boy06]). Perhaps the most studied case is when the spectrum of $f^*$ is pure hyperbolic (no eigenvalues are on the unit circle) in which case the eigen-$c$-maps fit together and descend to a semiconjugacy into a torus of dimension equal to that of $H^1(M;\mathbb{R})$. This result is contain in Franks paper [Fra70] and was the origin of our investigations. Fathi in [Fat88] uses Franks' result to send a pseudo-Anosov map on a surface into a compact invariant subset of an Anosov toral automorphism. Whether this can be done injectively in a fascinating open question also studied in [Ban03] and [BK06]. Fathi in [Fat88] also gives conditions in terms of the splitting of the characteristic polynomial of $f^*$ which imply a semiconjugacy from $M$ to a torus of dimension equal to the degree of an irreducible factor.

### 3.4 Decompositions and transverse structure

Recall that a *decomposition* of a manifold is way to write it as a disjoint union of sets. Assume now that $\tilde{\alpha}$ is an eigen-$c$-map for $\tilde{f}$ with factor $\mu$ with $|\mu| > 1$. We get a decomposition of $\tilde{M}$ by closed sets in the usual fashion: for each $r \in \mathbb{F}$, let $\tilde{X}_r := \tilde{\alpha}^{-1}(r)$. It is clear that the decomposition is $\tilde{f}$-invariant with $\tilde{f}(\tilde{X}_r) = \tilde{X}_{\mu r}$.

To describe the corresponding invariant decomposition in the base manifold $M$, first note that it follows directly from definition of a $c$-map and (2.2) that $\delta_{\vec{n}}(\tilde{X}_r) = \tilde{X}_{r+\Phi_c(\vec{n})}$, and so $\delta_{\vec{n}}(\tilde{X}_r) = \tilde{X}_r$ for all $\vec{n} \in \ker(\Phi_c)$. Next, recalling that $\pi : \tilde{M} \to M$ is the projection, it is easy to check that two projected decomposition elements sets $\pi(\tilde{X}_r)$ and $\pi(\tilde{X}_s)$ intersect if and only if they coincide in which case $s = r + \Phi(n)$ for some $n \in \mathbb{Z}^d$.

Thus the collection of sets $\pi(\tilde{X}_r)$ form a decomposition of $M$, and this collection can be be indexed as $\{X_\eta\}$ with $\eta \in \mathbb{F}/\Phi(\mathbb{Z}^d)$. By construction, the decomposition $\{X_\eta\}$ is $f$-invariant. Specifically, for each $\eta \in \mathbb{F}/\Phi(\mathbb{Z}^d)$, $f(X_\eta) = X_{\eta'}$ where $\eta'$ is the class of $\mu\eta$. Note that this makes sense, for if $r, r' \in F$ and $r - r' = \Phi(\vec{n})$, then $\mu r - \mu r' = \mu(r - r') = \Phi(f_*(\vec{n}))$ and so multiplication by $\mu$ on $\mathbb{F}$ respects the equivalence relation and so descends to a map on the quotient $\mathbb{F}/\Phi(\mathbb{Z}^d)$.

If the eigen-cohomology class $c$ is irrational in the sense that $\mathbb{Z}^d \cap \ker(\Phi_c) = 0$, $\pi$ restricted to every $\tilde{X}_r$ is injective, and so the sets $X_\eta$ generally wrap around $M$ and are not closed and are not the level sets of a continuous function on $M$. Instead, the decomposition of $M$ is made up of level sets of a family of locally defined continuous functions. Shub calls this a translational H-striation in [Shu78] where he notes that "H" stands for Haeflinger. He also notes that if $f$ has a collection $\mu_i$ of expanding eigenvalues, and so in our language we have many eigen-$c$-maps $\tilde{\alpha}_{\mu_i}$ and their corresponding $f$-invariant decompositions, $\{X_\eta^{(i)}\}$, one may also form the intersection decomposition $\{\cap X_\eta^{(i)}\}$ and it will also be $f$-invariant. If $f$ is a homeomorphism one may also include the decompositions coming from eigenvalues with $|\mu| < 1$.

The generalized eigen-cocycles coming from generalized eigenvectors of $f^*$ also yield invariant decompositions. For example, assume that $\tilde{\alpha}$ and $\tilde{\alpha}'$ satisfy (3.6). For $r, s \in \mathbb{F}$, let $\tilde{W}_{r,s} = \tilde{\alpha}^{-1}(r) \cap (\tilde{\alpha}')^{-1}(s)$. Then this yields a decomposition of $\tilde{M}$ which satisfies $\tilde{f}(\tilde{W}_{r,s}) = \tilde{W}_{\mu r, \mu s + r}$, and pushes down to one on $M$ that is transversally stretched but "skewed" as well.

In general, the decomposition elements $\tilde{X}_r$ and $X_\nu$ can be topologically extremely complicated, and we do not pursue this level of generality. In the case of main interest here, a pseudo-Anosov map $\phi$, if $\phi$ has oriented foliations and $\lambda$ is the eigenvalue of $\phi^*$ with largest modulus, then the invariant decompositions corresponding to $\lambda$ and $\lambda^{-1}$ are the stable and unstable foliations of $\phi$. For an eigenvalue $\mu$ with $1 < |\mu| < \lambda$, we shall see that the decomposition elements $X_\mu$ are unions of stable leaves and are typically locally a Cantor set cross an interval (see Theorem 12.6). The intersection decomposition elements formed using the decompositions coming from both $\mu$ and $\mu^{-1}$ are thus typically Cantor sets. In the next section we consider the cocycle which in the general topological case plays the same role of the transverse measure of a pseudo-Anosov map.

## 4 Cocycles

While dynamically valuable and simple to work with analytically, $c$-maps are awkward in various ways. For example, we have seen that the corresponding eigen-objects depend on the choice of lift of $f$ to the universal Abelian cover. In this section we consider two types of cocycle, each being convenient in certain circumstances. Our purpose here is not to develop full cohomology theories, but rather just describe the cocycles useful in the sequel.

As described in the introduction, these various cocycles may be viewed as ways to extend or complete the collection of closed one-forms to a larger theory in which each eigen-cohomology class contains a eigen-cocycle. As such, we proceed each definition with the version of the cocycle associated to a closed one-form.

### 4.1 Path cocycles

A path cocycle is the generalization of the integral of a closed one-form over an path. If $\omega$ is a closed one-form on $M$, and $\gamma$ is a smooth, oriented path in $M$, define

$$F_\omega(\gamma) = \int_\gamma \omega. \tag{4.1}$$

If $\gamma$ is not smooth, let $F_\omega(\gamma) = F_\omega(\gamma')$, where $\gamma'$ is smooth, has the same endpoints as $\gamma$, and is $C^0$-close to $\gamma$. This $F_\omega$ will be additive on oriented paths with a common endpoint and further, its value on a path depends just on the homology class of the path. Put another way, since $\omega$ is closed, if $\gamma_1$ and $\gamma_2$ are homologous rel endpoints, then $F_\omega(\gamma_1) = F_\omega(\gamma_2)$. This implies that if $\Gamma \subset M$ is a closed curve, then $F_\omega(\Gamma)$ depends just on the homology class of $\Gamma$ and so $F$ induces a linear functional $H_1(M, \mathbb{Z}) \to \mathbb{R}$ and thus yields a cohomology class on $M$. If $\omega$ is an eigen-one-form, $f^*\omega = \lambda\omega$, then $F_\omega$ will be an eigen-path cocycle in the sense that $f^* F_\omega = \lambda F_\omega$. Thus $F_\omega$ represents an expanding transverse structure when $|\mu| > 1$.

To give the general definition, let $\mathcal{P} = C^0([0,1], M)$ with the sup metric. Note that this is the space of *oriented paths*. The terminology *arc* will be used here for corresponding non-oriented set and will be considered in §7 below. Given two paths $\gamma_1, \gamma_2 \in \mathcal{P}$, with

$\gamma_1(1) = \gamma_2(0)$, then $\gamma_1 \# \gamma_2$ represents the usual path sum. The essentials of the following definition come from [Fat88].

**Definition 4.1 (Path cocycle)** *An path cocycle over $\mathbb{F}$ is a continuous map $F : \mathcal{P} \to \mathbb{F}$ which is*

*(a) Additive: if $\gamma_1(1) = \gamma_2(0)$, then $F(\gamma_1 \# \gamma_2) = F(\gamma_1) + F(\gamma_2)$.*

*(b) Closed or Homology Invariant: if $\Gamma$ is a closed loop with $[\Gamma] = 0$ in $H_1(M;\mathbb{Z})$, then $F(\Gamma) = 0$. Equivalently, if $\gamma_1$ and $\gamma_2$ have the same endpoints and are homologous, then $F(\gamma_1) = F(\gamma_2)$.*

As a consequence of (b), the value of an path cocycle is independent of parameterization of the path $\gamma$ in the sense that if $\tau : [0,1] \to [0,1]$ is an orientation-preserving homeomorphism, then $F(\gamma \circ \tau) = F(\gamma)$. Also note that if $\tau$ is an orientation-reversing homeomorphism, then $F(\gamma \circ \tau) = -F(\gamma)$. In addition, the inclusion of the requirement that $F$ be continuous implies that if $\gamma_n$ is a sequence of paths converging to a constant function then $F(\gamma_n) \to 0$.

To define cohomologous path cocycles we require the appropriate notion of an exact path cocycle. For a continuous function $\chi : M \to \mathbb{F}$, let $\delta\chi : \mathcal{P} \to \mathbb{F}$ be defined by $\delta\chi(\gamma) = \chi(\gamma(1)) - \chi(\gamma(0))$. It is easy to check that $\delta\chi$ satisfies Definition 4.1. We say that the two path cocycles $F_1$ and $F_2$ are cohomologous if $F_1 = F_2 + \delta\chi$ for some continuous $\chi \in C^0(M, \mathbb{F})$.

### 4.2 Cover cocycles

Again we start with the construction of this cocycle from a given closed one-form $\omega$. Pick $\tilde{x}, \tilde{y} \in \tilde{M}$ and let

$$\beta_\omega(\tilde{x}, \tilde{y}) = \int_{\tilde{\gamma}} \tilde{\omega}, \tag{4.2}$$

where $\tilde{\gamma}$ is any smooth path in $\tilde{M}$ connecting $\tilde{x}$ and $\tilde{y}$ and $\tilde{\omega}$ is a pull back to $\tilde{M}$ of $\omega$. Since $\omega$ is closed, the definition is independent of the choice of path. Further, $\beta_\omega : \tilde{M} \times \tilde{M} \to \mathbb{F}$ is invariant under the diagonal action of $\mathbb{Z}^d$, $\beta_\omega(\delta_{\vec{n}}\tilde{x}, \delta_{\vec{n}}\tilde{y}) = \beta_\omega(\tilde{x}, \tilde{y})$ and is additive in the sense that $\beta_\omega(\tilde{x}, \tilde{y}) + \beta_\omega(\tilde{y}, \tilde{z}) = \beta_\omega(\tilde{x}, \tilde{z})$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{M}$. These last two properties are used the define a general such cocycle.

**Definition 4.2 (Cover cocycle)** *A cover cocycle over $\mathbb{F}$ is a continuous map $\beta : \tilde{M} \times \tilde{M} \to \mathbb{F}$ which is*

*(a) Equivariant under the diagonal action of the deck group: so $\beta(\delta_{\vec{n}}\tilde{x}, \delta_{\vec{n}}\tilde{y}) = \beta(\tilde{x}, \tilde{y})$, for all $\vec{n} \in H_1(M;\mathbb{Z})$ and all $\tilde{x}, \tilde{y} \in \tilde{M}$.*

*(b) Additive:*

$$\beta(\tilde{x}, \tilde{y}) + \beta(\tilde{y}, \tilde{z}) = \beta(\tilde{x}, \tilde{z}) \tag{4.3}$$

*for all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{M}$.*

Note that if as with Alexander-Spanier cocycles (see [Spa81]) we define the differential of the cover cocycle $\beta$ as $\delta\beta : \tilde{M} \times \tilde{M} \times \tilde{M} \to \mathbb{F}$ defined as

$$\delta\beta(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \beta(\tilde{x}_2, \tilde{x}_3) - \beta(\tilde{x}_1, \tilde{x}_3) + \beta(\tilde{x}_1, \tilde{x}_2),$$

then condition (4.3) just says that $\delta\beta = 0$, i.e. $\beta$ is closed or is a cocycle.

The appropriate notion of cohomologous cocycles requires a definition of an exact cocycle. For a continuous function $\tilde{\chi} : \tilde{M} \to \mathbb{F}$, define $d\tilde{\chi}(\tilde{x}, \tilde{y}) := \tilde{\chi}(\tilde{y}) - \tilde{\chi}(\tilde{x})$. If $\tilde{\chi}(\delta_{\vec{n}}\tilde{x}) = \tilde{\chi}(\tilde{x})$ for all $\vec{n} \in \mathbb{Z}^d$, or equivalently, if $\tilde{\chi} = \chi \circ \pi$ for some $\chi \in C^0(M; \mathbb{F})$, then $d\tilde{\chi}$ is a cover cocycle. We then say that two path cocycles $\beta_1$ and $\beta_2$ are cohomologous if and only if

$$\beta_2 = \beta_1 + \chi \circ \pi,$$

for some $\chi \in C^0(M; \mathbb{F})$.

### 4.3 Correspondences and isomorphism to de Rham theory

Not surprisingly there is a simple correspondence between path and cover cocycles, and this correspondence respects cohomology classes. The collection of these cohomology classes has a natural structure as a vector space over $\mathbb{F}$ which is isomorphic to usual first cohomology.

**Fact 4.3** *There is a natural bijection between path cocycles and cover cocycles, and the collection of their cohomology classes is isomorphic to first de Rham cohomology* $H^1_{DR}(M; \mathbb{F})$.

**Proof:** Given an path cocycle $F$, define $\beta : \tilde{M} \times \tilde{M} \to \mathbb{F}$ by $\beta(\tilde{x}, \tilde{y}) = F(\pi \circ \tilde{\gamma})$ where $\tilde{\gamma}$ is any path in $\tilde{M}$ connecting $\tilde{x}$ to $\tilde{y}$. If we choose a different path $\tilde{\gamma}'$ connecting $\tilde{x}$ to $\tilde{y}$, then $\pi \circ \tilde{\gamma}$ and $\pi \circ \tilde{\gamma}'$ are homologous in $M$, and so $F(\pi \circ \tilde{\gamma}') = F(\pi \circ \tilde{\gamma})$, and so $\beta$ is well-defined. The map $\beta$ is clearly equivariant under the diagonal action of the deck group of $\tilde{M}$, and is additive since $F$ is.

Conversely, given a cover cocycle $\beta$, for $\gamma$ a path in $M$, define $F(\gamma) = \beta(\tilde{\gamma}(0), \tilde{\gamma}(1))$ where $\tilde{\gamma}$ is any lift of $\gamma$ to $\tilde{M}$. Now $F$ is independent of the choose of lift $\tilde{\gamma}$ because $\beta$ is equivariant under the diagonal action of the deck group of $\tilde{M}$. If $\gamma$ and $\gamma'$ are homologous in $M$, then they lift to $\tilde{\gamma}$ and $\tilde{\gamma}'$ in $\tilde{M}$ with the same pair of endpoints, and so $F(\gamma) = F(\gamma')$. Finally, the additivity of $F$ follows from that of $\beta$.

Under the correspondences just delineated, it is easy to see that exact path cocycles correspond to exact cover cocycles and so it yields an isomorphism between the vector spaces of cohomology classes of the cocycles.

To prove the second statement of the fact, if $\omega$ is a closed one-form on $M$, and $\gamma \in \mathcal{P}$ is smooth, define the path cycle $F_\omega(\gamma)$ as in (4.1). Now if $\chi : M \to \mathbb{F}$ is smooth, then its usual exterior derivative is the closed one-form $d\chi$ which yields the path cocycle $F_{d\chi}$ which is the same the exact path cocycle $\delta\chi$ defined below Definition 4.1. Thus the map $\omega \mapsto F_\omega$ induces a map from $H^1_{DR}(M; \mathbb{F})$ to the vector space of cohomology classes of path cocycles which is clearly a homomorphism.

Finally, given a cohomology class of path cocycles over $\mathbb{F}$ we will associate it with a linear functional $\mathbb{F}^d \to \mathbb{F}$ and thus a class in $H^1_{DR}(M, \mathbb{F})$ by the usual identifications. Given an path cocycle $F$, define $\Phi_F : H^1(M; \mathbb{Z}) \to \mathbb{F}$ as $\Phi_F([\Lambda]) = F(\Lambda)$ where $\Lambda$ is a closed loop in $M$ and $[\Lambda]$ its class in $H_1(M, \mathbb{Z})$. Let $\Phi_F$ also denote the extension to a linear functional $\mathbb{F}^d \to \mathbb{F}$. By Definition 4.1(b), the definition of $\Phi_F$ is independent of choice of $\Lambda$ in its homology class. For an exact path cocycle $\delta\chi(\Lambda) = 0$ for any loop $\Lambda$, and so $\Phi_{\delta\chi} = 0$. Thus $\Phi_F$ depends only on the cohomology class of $F$. For a closed one-form $\omega$, and a smooth loop $\Lambda$, since

$F_\omega(\Lambda) = \int_\Lambda \omega$, $\Phi_{F_\omega}$ represents the cohomology class of $\omega$, which implies that homomorphisms are isomorphisms, finishing the proof. ■

In view of the equivalence of path and cover cocycles, we will often just call them "cocycles" with the type clear from the context.

### 4.4 Topological one-forms

There is yet another structure generalizing closed one-forms which will be of use here. Shub calls it a translational H-structure (with "H" for Haeflinger) [Shu78] and Farber, et. al in [FKLZ04a, FKLZ04b] call it a topological one-form. In the context of this paper these represent the analog for path cocycles of the Poincaré lemma, i.e. closed one-forms are locally exact.

**Definition 4.4 (Topological one-form)** *A translational H-structure or topological one-form consists of a collection of pairs* $(U_i, f_i)$ *where*

*(a)* $\{U_i\}$ *is a finite open cover of* $M$ *by open topological disks* $U_i$,

*(b) the continuous maps* $f_i : U_i \to \mathbb{F}$ *satisfy the overlap condition that whenever* $U_i \cap U_j \neq \emptyset$ *there exists constants* $r_{ij} \in \mathbb{F}$ *with* $f_i - f_j = r_{ij}$ *on each connected component of* $U_i \cap U_j$.

The papers [FKLZ04a, FKLZ04b] prove the existence of Lyapunov topological one-forms for certain flows or subflows, a result that is complementary or perhaps "orthogonal" to the existence of eigen-cocycles. The following is noted in [FKLZ04a].

**Fact 4.5** *A topological one-form yields a unique path cocycle. A path cocycle and a finite open cover of* $M$ *by topological disks gives a topological one-form.*

**Proof:** Assume we are given a topological one-form $\{(U_i, f_i)\}$. For each $i$ we define a local path cocycle $\delta f_i$ defined on a path with $\gamma([a,b]) \subset U_i$ by $\delta f_i(\gamma) = f_i(\gamma(b)) - f_i(\gamma(a))$. Now given a general path $\gamma : [0,1] \to M$, we may find a a subdivision $0 = t_0 < t_j < \cdots < t_n = 1$, so that for each $i$ there is a $i(j)$ with $\gamma([t_j, t_{j+1}]) \subset U_{i(j)}$. Let $\gamma_j$ be $\gamma$ restricted to $[t_j, t_{j+1}]$ and define $F(\gamma) = \sum \delta f_{i(j)}(\gamma_j)$. The overlap conditions on the topological one-form imply that $F$ is additive in the sense of Definition 4.1(a). In addition, a standard argument breaks a large homotopy into a sequence of smaller ones, each of which moves across just one overlap $U_i \cap U_j$ at a time. This yields that $F$ is homotopy invariant in the sense that when $\gamma$ and $\gamma'$ have the same endpoints and are homotopic, then $F(\gamma) = F(\gamma')$.

It remains to show that $F$ is homology invariant as in Definition 4.1(b). This requires a standard argument whose algebraic content is that a homomorphism from a group $G$ with Abelian image always factors through the Abelianization of $G$. In the current context, the simplest approach is to note that since $F$ is homotopy invariant, it defines a continuous map $\hat{\beta} : \hat{M} \times \hat{M} \to \mathbb{F}$ where $\hat{M}$ is the *universal cover* of $M$ by letting $\hat{\beta}(\hat{x}, \hat{y}) = F(\hat{\pi} \circ \tilde{\gamma})$ where $\tilde{\gamma}$ is any path connecting $\hat{x}$ to $\hat{y}$ and $\hat{\pi} : \hat{M} \to M$ is the cover. Further, $\hat{\beta}(g \cdot \hat{x}, g \cdot \hat{y}) = \hat{\beta}(\hat{x}, \hat{y})$, for all $g \in \pi_1(M)$ (treated as deck transformations of the universal cover) and $\hat{\beta}(\hat{x}, \hat{y}) + \hat{\beta}(\hat{y}, \hat{z}) = \hat{\beta}(\hat{x}, \hat{z})$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in \hat{M}$. It is easy to check that these properties imply that $\hat{\beta}(\hat{x}, (h^{-1}gh) \cdot \hat{y}) = \hat{\beta}(\hat{x}, g \cdot \hat{y})$ and $\hat{\beta}(\hat{x}, (h^{-1}g^{-1}hg) \cdot \hat{y}) = \hat{\beta}(\hat{x}, \hat{y})$.

It is now straightforward to use $\hat{\beta}$ to produce a cover cocycle on the product of the homology covers $\beta : \tilde{M} \times \tilde{M} \to \mathbb{F}$. Define $\beta(\tilde{x}, \tilde{y}) := \hat{\beta}(\hat{x}, \hat{y})$ where $\hat{x}$ and $\hat{y}$ satisfy $p(\hat{x}) = \tilde{x}$ and $p(\hat{y}) = \tilde{y}$ where $p : \hat{M} \to \tilde{M}$ is the projection. To show this definition is independent of these choices, we assume that $p(\hat{x}') = \tilde{x}$ and $p(\hat{y}') = \tilde{y}$. This implies there are $g_x, g_y \in [\pi_1(M), \pi_1(M)]$ with $g_x \cdot \hat{x} = \hat{x}'$ and $g_y \cdot \hat{y} = \hat{y}'$. Thus by the properties given at the end of the previous paragraph, $\hat{\beta}(\hat{x}', \hat{y}') = \hat{\beta}(\hat{x}, \hat{y})$, as required. It now follows easily that $\beta$ is a cover cocycle in the sense of Definition 4.2 on the universal Abelian cover $\tilde{M}$. Thus using the correspondence of path and cover cocycles, we see that $F$ is an path cocycle.

Now given an path cocycle $F$ and a finite open cover $\{U_i\}$ by simply connected open sets, for each $i$ pick a base point $x_i \in U_i$ and for $x \in U_i$ let $f_i(x) = F(\gamma_x)$ where $\gamma_x$ is any path connecting $x_i$ to $x$. It is easy to check that the resulting $\{(U_i, f_i)\}$ is a topological one-form. ■

### 4.5 Eigen-cocycles and eigen-$c$-maps

In this subsection we make explicit the connection between the cocycles and $c$-maps, define eigen-cocycles, and then restate Theorem 3.3 in terms of cocycles. It is easiest to connect $c$-maps with cover cocycles; one then obtains the corresponding cocycles using Fact 4.3 .

Given a $c$-map $\sigma$, then $\beta(\tilde{x}, \tilde{y}) = \sigma(\tilde{y}) - \sigma(\tilde{x})$. is a cover cocycle. Conversely, given a cover cocycle $\beta$, fix a base point $\tilde{x}_0$ and define

$$\sigma(\tilde{x}) = \beta(\tilde{x}_0, \tilde{x}).$$

Now for any $\vec{n} \in \mathbb{Z}^d$, $\sigma(\delta_{\vec{n}}\tilde{x}) = \beta(\tilde{x}_0, \delta_{\vec{n}}\tilde{x}) = \beta(\tilde{x}_0, \tilde{x}) + \beta(\tilde{x}, \delta_{\vec{n}}\tilde{x}) = \beta(\tilde{x}_0, \tilde{x}) + \Phi_\beta(\vec{n})$, with $\Phi_\beta$ the linear functional representing the cohomology class of $\beta$ as in §2.2. Thus, $\sigma$ is a $c$-map where $c$ is the cohomology class of $\beta$. Note that changing the base point adds a constant to $\sigma$.

To define eigen-cocycles we need the action of a map $f$. Given a continuous $f : M \to M$ for the action on cover cocycles, pick a lift $\tilde{f}$ of $f$ and let $f^*\beta(\tilde{x}, \tilde{y}) = \beta(\tilde{f}\tilde{x}, \tilde{f}\tilde{y})$. Since any other lift of $f$ can be written as $\delta_{\vec{n}}\tilde{f}$ for some $\vec{n} \in \mathbb{Z}^d$, the definition is independent of choice of the lift of $f$. In particular, it defines an action of $f$ itself on cover cocycles whereas it is a specific lift of $f$ that acts on $c$-maps. For an path cocycle $F$, let $f^*F(\gamma) = F(f \circ \gamma)$.

The cover cocycle $\beta$ is called *eigen with factor* $\mu$ if $f^*\beta = \mu\beta$, and the path cocycle $F$ is eigen if $f^*F = \mu F$. The correspondence of these eigen-cocycles to eigen-$c$-maps requires one more notion. A $c$-map $\tilde{\alpha}$ is called *almost eigen* for the lift $\tilde{f}$ if $\tilde{\alpha} \circ \tilde{f} = \mu\tilde{\alpha} + K$ for some $K \in \mathbb{F}$.

Given an almost eigen-$c$-map $\tilde{\alpha}$, it follows easily that $\beta(\tilde{x}, \tilde{y}) = \tilde{\alpha}(\tilde{y}) - \tilde{\alpha}(\tilde{x})$ is an eigen-cocycle. Conversely, given an eigen-cocycle $\beta$ with factor $\mu$, fix a base point $\tilde{x}_0$ and define the corresponding $c$-map as in Fact 4.3: $\tilde{\alpha}(\tilde{x}) = \beta(\tilde{x}_0, \tilde{x})$. Choose a lift $\tilde{f}$, and then

$$\begin{aligned}
\tilde{\alpha}(\tilde{f}\tilde{x}) &= \beta(\tilde{x}_0, \tilde{f}\tilde{x}) \\
&= \beta(\tilde{f}\tilde{f}^{-1}\tilde{x}_0, \tilde{f}\tilde{x}) \\
&= \mu\beta(\tilde{f}^{-1}\tilde{x}_0, \tilde{x}) \\
&= \mu(\beta(\tilde{f}^{-1}\tilde{x}_0, \tilde{x}_0) + \beta(\tilde{x}_0, \tilde{x})) \\
&= K + \mu\tilde{\alpha}(\tilde{x}),
\end{aligned}$$

where $K = \mu\beta(\tilde{f}^{-1}\tilde{x}_0, \tilde{x}_0) = \mu\beta(\tilde{f}^{-1}\tilde{x}_0, \tilde{f}^{-1}\tilde{f}\tilde{x}_0) = \beta(\tilde{x}_0, \tilde{f}\tilde{x}_0)$. Changing the choice of lift or base point changes the constant $K$.

We see then that an eigen-cocycle yields an almost eigen-$c$-map. The fact that one gets just an *almost* eigen-$c$-map is awkward as is the fact that the $c$-map is eigen for a lift and not for $f$ itself. At least the first awkwardness is easily remedied in the "generic" case. If $\tilde{\alpha}$ is an almost eigen-$c$-map with constant $K$ and factor $\mu \neq 1$, let $\hat{\alpha} = \tilde{\alpha} + \frac{K}{\mu - 1}$. We then have $\hat{\alpha} \circ \tilde{f} = \mu\hat{\alpha}$, and so an almost eigen-$c$-map yields an actual eigen-$c$-map. A factor $\mu = 1$ can only occur under very special circumstances and is considered in Lemma 4.7.

Thus using the obvious definition of an eigenchain of cocycles we have the following corollary to Franks-Shub Theorem 3.3

**Corollary 4.6** *If $f : M \to M$ is a continuous map of the smooth, connected manifold $M$ and $\mu \in \mathbb{F}$ is an eigenvalue of $f^* : H^1(M;\mathbb{Z}) \to H^1(M;\mathbb{Z})$ with $|\mu| > 1$ and eigenchain $\{c_1, \ldots, c_k\} \subset H^1(M;\mathbb{F})$, then $f$ has eigenchains of path cocycles and cover cocycles with factor $\mu$ which represents the classes $\{c_1, \ldots, c_k\}$.*

The next lemma collects a few simple results about the connection of dynamics and eigen-cocycle factors which we need in the sequel.

**Lemma 4.7** *Assume $f : M \to M$ has an eigen-cocycle or generalized eigen-cocycle $\beta$ with factor $\mu$, and let $\tilde{d}$ be an equivariant metric on $\tilde{M}$.*

- *(a) $f$ has an eigen-cocycle with factor $\mu = 1$ if and only if $f$ is semiconjugate to rigid rotation on the unit circle $S^1$.*
- *(b) If $\mu \geq 1$ and $\tilde{\alpha}$ is an eigen- or generalized eigen-$c$-map corresponding to $\beta$ and $\tilde{x}, \tilde{y} \in \tilde{M}$ are such that $\tilde{d}(\tilde{f}^n\tilde{x}, \tilde{f}^n\tilde{y}) \to 0$ as $n \to \infty$, then $\tilde{\alpha}(\tilde{x}) = \tilde{\alpha}(\tilde{y})$. If $\mu > 1$ and $\tilde{d}(\tilde{f}^n\tilde{x}, \tilde{f}^n\tilde{y})$ is bounded as $n \to \infty$, then $\tilde{\alpha}(\tilde{x}) = \tilde{\alpha}(\tilde{y})$.*
- *(c) If $f$ is a homeomorphism and $\mu \leq 1$ and $\tilde{\alpha}$ is an eigen- or generalized eigen-$c$-map corresponding to $\beta$ and $\tilde{x}, \tilde{y} \in \tilde{M}$ are such that $\tilde{d}(\tilde{f}^n\tilde{x}, \tilde{f}^n\tilde{y}) \to 0$ as $n \to -\infty$, then $\tilde{\alpha}(\tilde{x}) = \tilde{\alpha}(\tilde{y})$. If $\mu < 1$ and $\tilde{d}(\tilde{f}^n\tilde{x}, \tilde{f}^n\tilde{y})$ is bounded as $n \to -\infty$, then $\tilde{\alpha}(\tilde{x}) = \tilde{\alpha}(\tilde{y})$.*

**Proof:** Assume that $f$ has an eigen-cocycle $\beta$ with factor $\mu = 1$. This means that $f^*$ acting on $H^1(M;\mathbb{Z})$ has an eigenvalue of 1 and so we may find an integral eigen-class $c$ which is represented by a linear $\Phi_c : \mathbb{Z}^d \to \mathbb{Z}$. Now if $\tilde{\alpha} : \tilde{M} \to \mathbb{R}$ is an almost eigen-$c$-map constructed from $\beta$ as in Corollary 4.6, then since $\tilde{\alpha}(\delta_{\vec{n}}\tilde{x}) = \tilde{\alpha}(\tilde{x}) + \Phi_c(\vec{n})$, $\tilde{\alpha}$ descend to a map $\overline{\alpha} : M \to S^1$. Since $\tilde{\alpha}\tilde{f} = \tilde{\alpha} + K$, $\overline{\alpha}$ gives a semiconjugacy between $f$ and rigid rotation by $K$ on the circle $S^1$. Conversely, a semiconjugacy $\overline{\beta} : M \to S^1$ between $f$ and rigid rotation will lift to an almost eigen-$c$-map $\tilde{\alpha} : \tilde{M} \to \mathbb{R}$ with factor 1 which in turn yields the desired eigen-cocycle, finishing the proof of (a).

We prove (b) under the assumption that $\tilde{\alpha}$ is an eigen-$c$-map. It is an easy induction on the eigenchain entries to then get the result for generalized eigen-$c$-map. By the semiconjugacy we have $|\mu|^n |\tilde{\alpha}(\tilde{x}) - \tilde{\alpha}(\tilde{y}))| = |\tilde{\alpha}\tilde{f}^n\tilde{x} - \tilde{\alpha}\tilde{f}^n\tilde{y}|$. Now if $\tilde{d}(\tilde{f}^n\tilde{x}, \tilde{f}^n\tilde{y}) \to 0$ as $n \to \infty$, then by continuity of $\tilde{\alpha}$, $|\tilde{\alpha}\tilde{f}^n\tilde{x} - \tilde{\alpha}\tilde{f}^n\tilde{y}| \to 0$, and since $|\mu| \geq 1$, the only possibility is $\tilde{\alpha}(\tilde{x}) = \tilde{\alpha}(\tilde{y})$. For the second sentence of (b), note that if $\tilde{d}(\tilde{f}^n\tilde{x}, \tilde{f}^n\tilde{y})$ is bounded, then since $\tilde{\alpha}$ is large scale Lipschitz (Remark 3.2), $|\tilde{\alpha}\tilde{f}^n\tilde{x} - \tilde{\alpha}\tilde{f}^n\tilde{y}|$ is also bounded and so if $|\mu| > 1$, we have

$\tilde{\alpha}(\tilde{x}) = \tilde{\alpha}(\tilde{y})$ again. For the proof of (c), use the same argument with $\tilde{f}^{-1}$. ■

**Remark 4.8** If $f^*$ has an eigenvalue with $|\mu| > 1$, then using the corresponding eigen-$c$-map one gets restrictions on the dynamics of any lift of $f$ to the homology cover $\tilde{M}$. For example, no lift $\tilde{f}$ to $\tilde{M}$ has a dense orbit. This was proved in [Boy09] using different methods. Also, if $\tilde{f}$ has a local product structure in the sense of hyperbolic dynamics, then $\tilde{f}$ has no eigen-$c$-maps with factor $|\mu| = 1$. Note that this certainly does not exclude $f^*$ having eigenvalues of modulus one.

**Remark 4.9** As in (3.7) there is a summation formula for eigen-cocycles. Given an eigen-class $c \in H^1(M; \mathbb{F})$ with eigenvalue $\mu$, pick a cover cocycle $\beta$ which represents the class. Since $f^*c = \mu c$, using the isomorphism between cover cocycles and usual first homology, $f^*\beta = \mu\beta + d\tilde{\chi}$, for some $\tilde{\chi} = \chi \circ \pi$ with $\chi \in C^0(M; \mathbb{F})$. If $|\mu| > 1$,

$$\beta_\mu := \beta + \sum_{j=1}^{\infty} \frac{d\tilde{\chi} \circ \tilde{f}^{j-1}}{\mu^j},$$

converges by Weierstrass $M$-test, and $f^*\beta_\mu = \mu\beta_\mu$ by direct verification. A similar sum can obviously be given for eigen-path cocycles.

# 5 PseudoAnosov maps

For the balance of this paper we will focus on pseudo-Anosov homeomorphisms of compact surfaces. See [Thu88, FLP91, CB88, Boy94] for more information and details.

## 5.1 Foliations, measures and metrics

A pseudo-Anosov homeomorphism $\phi$ is characterized by a pair of transverse measured foliations, one termed stable, $\mathcal{F}^s$, and one unstable, $\mathcal{F}^u$. The foliations are allowed to have a finite number of singularities or prongs of a controlled type and the foliation near a boundary component looks like a blown up prong. In this paper we will also include the case of pseudo-Anosov maps relative to a finite set in the class of pseudo-Anosov maps. These so-called rel pseudo-Anosov maps are allowed to have a finite number of one-prongs at specified points and the isotopy class of such a $\phi$ is always considered relative to these points.

Let $P$ be the collection of singular points of the foliations. For $x \in M - P$, let $L^s(x)$ be the leaf of $\mathcal{F}^s$ which contains $x$. If $x \in P$, then by convention $L^s(x) = \{x\}$. If $x$ is in a leaf that is associated with a singularity, then by $L^s(x)$ we mean the half-infinite leaf which"begins" at the singularity. In this case the *extended leaf* containing $x$ is the union of all the leaves associated with the singularity.

A basic fact is that every infinite and half-infinite leaf of a pseudo-Anosov invariant foliation is dense in the surface. The lifts of the foliations to the universal Abelian cover $\tilde{M}$ are denoted $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$. In the cover the leaf or half-infinite leaf containing $\tilde{x}$ is denoted $\tilde{L}^s(\tilde{x})$ or $\tilde{L}^u(\tilde{x})$.

A pseudo-Anosov map always has a Markov partition with a Peron-Fröbenius transition matrix. The Peron-Fröbenius eigenvalue of this matrix is usually denoted $\lambda$ and called the *dilation, expansion factor or stretch factor*. As described in the introduction, $\lambda$ arises in many different contexts in the study of pseudo-Anosov maps.

Each of the foliations carries a holonomy invariant transverse measure constructed from the Peron-Fröbenius eigenvalue, $\lambda$, and eigenvector of the transition matrix of a Markov partition (*cf.* Remark 9.2). An arc $\gamma$ that is transverse to $\mathcal{F}^s$ is assigned a measure $m^u(\gamma)$ which satisfies $m^u(\phi(\gamma)) = \lambda m_u(\gamma)$, and when $\gamma$ is transverse to $\mathcal{F}^u$ one has $m^s(\phi(\gamma)) = \lambda^{-1} m^s(\gamma)$. The reader is cautioned that there is a fair amount of diversity in the literature in the assigning of the labels "stable" and "unstable" to structures associated with a pseudo-Anosov map.

Using the transverse measures one constructs a topological metric $d_\phi$ based on the arc metric which is defined on short arcs transverse to both foliations by $\sqrt{(m^u)^2 + (m^s)^2}$. It is common to express these metrics in terms of a flat structure with conic singularities.

The lift of the metric $d_\phi$ to the universal Abelian cover $\tilde{M}$ is denoted $\tilde{d}_\phi$. A useful property is that $\tilde{x}$ and $\tilde{y}$ are in the same extended leaf of $\tilde{\mathcal{F}}^s$ if and only if $\tilde{d}_\phi(\tilde{\phi}^n(\tilde{x}), \tilde{\phi}^n(\tilde{y})) \to 0$ as $n \to \infty$.

Again for simplicity of exposition we restrict to the case of orientation preserving pseudo-Anosov maps on orientable surfaces.

### 5.2 Rectangles and Markov partitions

At this point we fix once and for all a given pseudo-Anosov map $\phi$ with its pair of transverse measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$. A *rectangle*, $R$, is a topological disk whose boundary consists of 4 segments which are alternately arcs in leafs of $\mathcal{F}^s$ and $\mathcal{F}^u$. These arcs are called the stable and unstable edges of $R$, and if we are considering an indexed set of rectangles (eg. a Markov partition) $\{R_i\}$, then the edges are denoted $E^s_{i,1}, E^u_{i,1}, E^s_{i,2}$, and $E^u_{i,2}$ . We also allow a singularity to be a common endpoint of two of these arcs, i.e. a rectangle can have a singularity at a "corner". The interior of rectangle is always a chart for both foliations, i.e. there is a homeomorphism $Int(R) \to (0,1)^2$ which takes $\mathcal{F}^s \cap Int(R)$ to the vertical foliation of the open unit square and $\mathcal{F}^u \cap Int(R)$ to the horizontal.

An *oriented rectangle*, $R$, is a rectangle with the additional data of a homeomorphism $h : R \to [0,1]^2$ so that $h$ restricted to $Int(R)$ gives a chart for both foliations as just described. Pulling back allows us to speak of the top, bottom, left and right edges of $R$, clockwise rotation about the boundary of $R$, etc.

**Definition 5.1 (Cover by rectangles)** *A cover of $M$ by rectangles is a finite collection of rectangles $\{R_i\}$ with*

*(a)* $\cup R_i = M$,

*(b) When $i \neq j$, $Int(R_i) \cap Int(R_j) = \emptyset$, and if for some $i \neq j$, $Fr(R_i) \cap Fr(R_j)$ is nonempty, then it is connected and is contained in exactly one edge of each of $R_i$ and $R_j$.*

Note that this implies that all singularities are in the boundary of some $R_i$.

A Markov partition for a pseudo-Anosov is a special cover by rectangles which has nice dynamical properties. A construction of covers by rectangles and a Markov partition is given in [FLP91], exposé 10.

We may assume that for our given pseudo-Anosov map $\phi$ we have chosen a Markov partition $\{R_i\}$ for $i = 1, \ldots, d$ that is fine enough so that each $\phi(R_j) \cap R_i$ contains at most one connected component. Thus its transition matrix $A$ is a $d \times d$ matrix with entries of either 0 or 1 with $A_{ij} = 1$ if and only if $\phi(R_j) \cap R_i \neq \emptyset$.

As is usual, the matrix $A$ generates a two-sided subshift of finite type $\Lambda_A$ and there is a semi-conjugacy $(\Lambda_A, \sigma) \to (M^2, \phi)$ given by the "address map". We shall mainly be concerned here with the one-sided shift space built from $A$ as described in §8 below. For a pseudo-Anosov map the matrix $A$ always satisfies $A^n > 0$ for all $n$ larger than some $N$ which implies that the associated shifts are topologically transitive and that the Peron-Fröbenius theorem holds for $A$.

### 5.3 The pseudo-Anosov spectrum

In this subsection we survey what is known about the structure and uses of the pseudo-Anosov spectrum. Most of this material is not used in the sequel, but it provides a valuable context.

We use the term pseudo-Anosov spectrum to encompass both the spectrum of the action of the pseudo-Anosov map on first homology and the spectrum of Markov transition matrices. When a distinction is needed, the first is called the *homological spectrum* and the latter the *Markov spectrum.* When $\phi$ has orientable foliations a theorem of Rykken ([Ryk99]) implies that the hyperbolic portions of these spectra agree (*cf.* Remark 10.2). In the general case they each may contain hyperbolic eigenvalues not contained in the other. While one may always lift a pseudo-Anosov to a branched cover with orientable foliations, it is usually not very straightforward to track the influence of the lift on the spectra, and so in most cases it is necessary to consider the two spectra separately.

We first discuss the homological spectrum. A homeomorphism $f$ of a closed surface preserves the homological intersection form of closed curves. In a standard basis this form is the standard symplectic form and so the matrix of $f_*$ on $H_1(M; \mathbb{Z})$ is symplectic and so has a palindromic characteristic polynomial. Since boundary components of the surface are permuted by a homeomorphism, their presence only contributes roots of unity to the homological spectrum and so the homological characteristic polynomial as a whole is always palindromic. This implies that if $\mu$ is an eigenvalue, then so are $\mu^{-1}, \overline{\mu}$ and $\overline{\mu}^{-1}$. In particular, since characteristic polynomials are monic, elements of the homological spectrum are algebraic units.

In terms of the associated eigenvectors on homology/cohomology, there are a number of well-known interpretations of an *oriented* measured foliation on a surface as a cycle or cocycle. For example, the measured foliation generates a geometric current as in Ruelle and Sullivan [RS75]. Almost equivalently, one may flow along the foliation and the transverse measure induces a flow-invariant ergodic measure. This measure can be assigned a Schwartzman asymptotic cycle ([Sch57]) which then represents the oriented measured foliation. In addition, as noted in the introduction, there is a closed one-form whose kernel is tangent to the foliation and integration of the form along transverse arcs yields the transverse measure. Finally, a oriented measured foliation can be represented as a weighted oriented train track which can be interpreted as a real homology chain.

Now given a pseudo-Anosov map, $\phi$, with orientable foliations, we may identify, say $\mathcal{F}^u$ with a cycle. Under the action of $\phi$, this cycle scales by a factor of $\lambda$ if $\phi$ preserves the orientation of the foliations and $-\lambda$ if it reverses it. Thus the eigenvector corresponding to $\pm\lambda$ is concretely represented by the cycle of $\mathcal{F}^u$. On implication is that when $\phi$ has orientable foliations, the spectral radius of the action on first homology is $\lambda$. The converse is also true, but less well known, see [BB07] for a proof. Proposition 6.2 below gives concrete realizations as eigen-cocycles for the eigenvectors of the rest of the hyperbolic homological spectrum.

To discuss the Markov spectrum we must start by pointing out that it has not yet been properly defined since each pseudo-Anosov map has infinitely many different Markov partitions with each having its own spectrum. We therefore define the Markov spectrum as the intersection of all the spectra of all these Markov partitions. Since all Markov partitions give rise to subshifts of finite type with essentially the same dynamics, results from symbolic dynamics (see [Kit98]) may be used to show that all the Markov partitions yield the same hyperbolic spectrum (in addition to any Galois conjugates of hyperbolic elements).

Birman et al ([JB10]) show that the characteristic polynomial of any Markov matrix for a pseudo-Anosov map must be palindromic or anti-palindromic with perhaps an additional factor of $x^n$, and so as with the homological spectrum, all elements of the Markov spectrum are algebraic units. It is worth noting that individual factors of the characteristic polynomial over $\mathbb{Z}$ do not in themselves have to be palindromic. In particular, $\lambda$ does not have to be a reciprocal algebraic unit, i.e. $\lambda$ and $\lambda^{-1}$ do not have to share the same minimal polynomial. In simple examples the case of reciprocal $\lambda$ is most common; see, for example, [AF91] and [Fri85] for non-reciprocal examples.

Since the pseudo-Anosov map's dynamics are coded by the corresponding subshift of finite type, the results from symbolic dynamics concerning the spectra of transition matrices apply after a few provisos to pseudo-Anosov maps. Let us fix a Markov partition with matrix $A$ which has spectrum $\lambda > |\mu_2| \geq \cdots \geq |\mu_{d-2}| > \lambda^{-1}$.

A simple result in symbolic dynamics says that $\text{trace}(A^n)$ counts the number of fixed points of the $n^{th}$ iterate of the shift. Since $\text{trace}(A^n) = \lambda^n + |\mu_2|^n + \cdots + \lambda^{-n}$, one has that the primary exponential growth rate of periodic points in $\Lambda_A^+$ is $\lambda$ with the rest of the spectrum providing correction terms. The construction of the Markov partitions for a pseudo-Anosov map guarantees that there are only a finite number of periodic orbits of $\phi$ which are multiply-coded by the symbolic model, and so there is a constant $C$ (which depends on the Markov partition) with $|\,\text{trace}(A^n) - \#\,\text{Fix}(\phi^n)| < C$ for all $n$.

We also recall that $\phi$ acts on the weights carried by its forward invariant train track $\tau$ via $A^T$. Thus the spectrum of $A$ describes how laminations carried $\tau$ converge to the unstable lamination of $\phi$ and to other subspaces in the generalized eigen-flag of $A$. After some work to handle laminations near the unstable lamination of $\phi$ but not carried by $\tau$, this implies that for the dynamics induced by $\phi$ on the boundary of Teichmüller space, orbits converge (in the Thurston metric) to the fixed point corresponding to $\mathcal{F}^u$ at a slowest rate of $|\mu_2/\lambda|^n$.

A final note on the pseudo-Anosov spectrum: it is certainly possible that a given pseudo-Anosov map has only one unstable and one stable eigenvalue in its spectrum. In this case either $\lambda$ is quadratic or else is a Salem number, and $\phi$'s foliations lack all the additional transverse structures described here. It would be very interesting to understand what this implies about the geometry and dynamics of $\phi$.

## 6 Transverse cocycles for pseudo-Anosov maps

Given a pseudo-Anosov map and its invariant foliations, we define a special kind of path cocycle, called a *transverse cocycle*, which is adapted to to the stable foliation $\mathcal{F}^s$ in the sense that the cocycle only sees the part of paths which are transverse to $\mathcal{F}^s$. These cocycles are static objects connected with the foliations, but the main result of this section connects them to the dynamics by showing that the collection of all transverse cocycles is a vector space which is spanned by the expanding eigen-cocycles.

**Definition 6.1 (Transverse cocycle)** *An path cocycle $F$ is said to be transverse to $\mathcal{F}^s$, if $F(\gamma) = 0$ for all paths $\gamma$ whose image is contained in leaves of $\mathcal{F}^s$. A c-map $\sigma$ is said to be transverse to $\mathcal{F}^s$ if $\tilde{x} \in \tilde{L}(\tilde{y})$ implies that $\sigma(\tilde{x}) = \sigma(\tilde{y})$.*

It will be usually be the case that there is a fixed pseudo-Anosov map and stable foliation under consideration in which case $F$ and $\sigma$ are just called a transverse path cocycles and $c$-maps. By the additivity of path cocycles, if two homotopic paths $\gamma_1, \gamma_2$ have their initial points on the same leaf of $\mathcal{F}^s$ and their final points on the same leaf of $\mathcal{F}^s$, then $F(\gamma_1) = F(\gamma_2)$. Thus transverse path cocycles are said to be holonomy invariant. Also note that the natural correspondence of path cocycles and $c$-maps given in §4.5 respects the property of being transverse.

Let $\mathcal{TC}(\mathcal{F}^s)$ be the collection of transverse cocycles to $\mathcal{F}^s$ with values in $\mathbb{F}$; it is immediate that $\mathcal{TC}(\mathcal{F}^s)$ is a vector space over $\mathbb{F}$. The next fact says that this space is isomorphic to the unstable subspace of $\phi^*$ acting on $H^1(M; \mathbb{F})$. It is worth emphasizing that the elements of $\mathcal{TC}(\mathcal{F}^s)$ are individual cocycles, not their cohomology classes. Thus, in particular, the following fact says there is a unique transverse cocycle in each unstable cohomology class.

**Proposition 6.2** *With the various structures as defined above we have the commutative diagram:*

$$\begin{array}{ccc} \mathcal{TC}(\mathcal{F}^s) & \xrightarrow{\cong} & Un(\phi^*, H^1(M;\mathbb{F})) \\ {\scriptstyle \phi^*}\downarrow & & \downarrow{\scriptstyle \phi^*} \\ \mathcal{TC}(\mathcal{F}^s) & \xrightarrow{\cong} & Un(\phi^*, H^1(M;\mathbb{F})) \end{array}$$

*This implies that the eigen-objects and their factors correspond.*

**Proof:** The result will follow after we prove two observations. The first observation that a generalized eigen-cocycle is transverse to $\mathcal{F}^s$ if and only if its factor satisfies $|\mu| > 1$. We shall prove the observation for eigen-cocycles and leave the small adjustments necessary for generalized eigen-cocycles to the reader.

Recall from §4.5 that if $F$ is an eigen-cocycle with factor $\mu$, after fixing a lift $\tilde{\phi}$ and a base point, $F$ corresponds to an almost eigen-$c$-map $\sigma$ with factor $\mu$. Now if $|\mu| > 1$, Lemma 4.7(b) shows that $\sigma$ and thus $F$ are transverse to $\mathcal{F}^s$. On the other hand, if $|\mu| \geq 1$, Lemma 4.7(c) says $\sigma$ and thus $F$ are transverse to $\mathcal{F}^u$. Thus if we assume that $F$ is also transverse to $\mathcal{F}^u$, $\sigma$ is constant on both stable and unstable leaves in $\tilde{M}$. Using the local structure of the foliations (either a product or near a singularity) it follows easily that $\sigma$ is locally constant and so since $\tilde{M}$ is connected, $\sigma$ is constant, and so its corresponding cocycle is $F = 0$. Thus we see that there are no nontrivial transverse eigen-cocycles with factor $|\mu| \leq 1$, completing the proof of the first observation.

The second observation is that when a cohomology class contains a transverse cocycle, the cocycle is unique. To prove this note that by definition when two $c$-maps $\sigma_1$ and $\sigma_2$ are cohomologous, then $\sigma_1 = \sigma_2 + \chi \circ \pi$ for some $\chi \in C^0(M, \mathbb{F})$. If $\sigma_1$ and $\sigma_2$ are both transverse $c$-maps, then so is $\chi \circ \pi$. This implies that $\chi$ is constant on leaves of $\mathcal{F}^s$, and so $\chi$ is constant since every leaf of $\mathcal{F}^s$ is dense in $M$. Thus $\sigma_1$ and $\sigma_2$ differ by a constant and so the correspond to the same cocycle.

To construct the isomorphisms in the theorem statement, first it is immediate that $(\phi^*)^{Un}$ is a self-isomorphism of $Un(\phi^*, H^1(M; \mathbb{F}))$. Since $\phi$ and $\phi^{-1}$ preserve leaves of $\mathcal{F}^s$, the induced map $\phi^*$ is also an isomorphism of $\mathcal{TC}(\mathcal{F}^s)$. By the second observation, $\mathcal{TC}(\mathcal{F}^s)$ is a finite dimensional vector space. Thus the generalized eigenvectors of $\phi^*$ acting on cocycles in $\mathcal{TC}(\mathcal{F}^s)$ and those of $\phi^*$ acting on $Un(\phi^*, H^1(M; \mathbb{F}))$ give bases for those spaces. Using Theorem 3.3 and the first observation above we obtain a bijection between these collections of generalized eigenvectors and thus an isomorphism between the spaces $\mathcal{TC}(\mathcal{F}^s)$ and $Un(\phi^*, H^1(M; \mathbb{F}))$. The commutativity of the diagram is immediate from the construction of the isomorphism. ■

**Remark 6.3** Note that the proof shows that pseudo-Anosov have no eigen-cocycles with factors $|\mu| = 1$. Also note that by using $\phi^{-1}$ we have that the vector space of cocycles transverse to the unstable foliation $\mathcal{F}^u$ is isomorphic to $Stab(\phi^*, H^1(M; \mathbb{F}))$.

For what follows we also need a local version of a transverse cocycle.

**Definition 6.4 (Local transverse cocycle)** *Given a pseudo-Anosov map $\phi$, a local transverse cocycle for $\phi$ is a cover of $M$ by rectangles $\{R_1, \ldots, R_d\}$ defined as in Definition 5.1 in addition to a family of continuous functions $f_{i,j} : E^u_{i,j} \to \mathbb{F}$ for $i = 1, \ldots, d$ and $j = 1, 2$ with the properties that*

*(a) For each $i$, $f_{i,1}$ and $f_{i,2}$ are holonomy invariant in $R_i$, i.e. if $g_i$ is the homeomorphism that takes $E^u_{i,1}$ to $E^u_{i,2}$ by sliding along leaves, then $f_{i,1} = f_{i,2} \circ g_i$.*

*(b) When $E^u_{i,j} \cap E^u_{i',j'} \neq \emptyset$, it is connected and $f_{i,j} - f_{i',j'}$ is constant on the intersection.*

The crucial features here are that the rectangles do not have to be oriented and the functions $f_{1,j}$ are just defined on unstable edges, and they agree up to a constant on overlapping of unstable edges.

**Fact 6.5** *A local transverse cocycle gives rise to a unique transverse cocycle, and a transverse cocycle and a cover of $M$ by rectangles yields a unique local transverse cocycle.*

**Proof:** We first formally define the operation of "collapsing down stable leaves to an unstable edge". Specifically, given a rectangle $R_i$ define $h_i : R_i \to E^u_{i,1}$ so that $h_i(z)$ is the point where the stable leaf through $z$ hits $E^u_{i,1}$, or formally, $h_i(z) = (L^u(z) \cap R_i) \cap E^u_{i,1}$.

Now given local transverse cocycle $f_{i,j} : E^u_{i,j} \to \mathbb{F}$ based on the cover by rectangles $\{R_1, \dots, R_d\}$, for $i = 1, \dots, d$, define $f_i : R_i \to \mathbb{F}$ as $f_i(z) = f_{i,1} \circ h_i$. Now we fix a rectangle $R_i$ and let $R_{i_1}, \dots, R_{i_m}$ be the rectangles whose frontiers intersect that of $R_i$. For each of these rectangles we may find a constant $c_{i_n}$ so that $f_{i_n} + c_{i_n}$ agrees with $f_i$ on the intersection of the respective frontiers. Now enlarge $R_i$ slightly to an open $R_i'$ and define $f_i' : R_i' \to \mathbb{F}$ so that $f_i' = f_i$ on $R_i$ and $f_i' = f_{i_n} + c_{i_n}$ on each $R_{i_n} \cap R_i'$. After doing this construction for $i = 1, \dots, d$, it is then easy to check that the family $(R_i', f_i')$ is a topological one-form as defined in Definition 4.4. By Fact 4.5 it gives a path cocycle and by construction it is a transverse cocycle.

Now given a transverse cocycle $F$, for each unstable edge $E^u_{i,2}$ fix a endpoint $p_i$ and define $f_{i,2} : E^u_{i,2} \to \mathbb{F}$ by $f_{i,2}(x) = F([p_i, x])$ where $[p_i, x]$ is the path in $E^u_{i,2}$ from $p_i$ to $x$. Now define $f_{i,1} = f_{i,2} \circ g_i$, with $g_i$ as in Definition 6.4. By the holonomy invariance of $F$ it follows that the family of functions $f_{i,j}$ defines a local transverse cocycle on the given cover by rectangles. ■

# 7 Transverse arc functions for pseudo-Anosov maps

In this section, §8, and §9 we consider transverse structures for pseudo-Anosov maps which depend on the foliations and symbolic dynamics and are not associated with cohomology. In Theorem 10.1 we show that when the foliations are oriented, all the various structures agree.

A *transverse arc function (taf)* is a geometric version of a transverse cocycle which may also be viewed as a generalization of the transverse measure to the pseudo-Anosov invariant foliation. The transverse measure is unique and it is usually constructed using the Peron-Fröbenius eigenvalue/vector of the transition matrix. In §9 below we show how tafs arise from any other expanding eigenvalues/vectors, and in Theorem 11.3 we show that these other tafs yield not transverse measures, but rather distributions in the sense of elements of the continuous dual to a space of test functions which in this case just need to be Hölder.

## 7.1 Definitions and basic properties

By an arc we mean the *image* of an embedding $\gamma : [a, b] \to M$, for $a < b$. Given an arc $\Gamma$, any embedding $\gamma : [a, b] \to M$ with $\Gamma = \gamma([a, b])$ is called a *parameterization* of $\Gamma$, and we sometimes also write $\Gamma = \mathrm{im}(\tau)$, with im meaning "image". Thus, to be specific, an *arc* is a closed subset of $M$ and a *path* is a parameterization of an arc. An arc carries no intrinsic orientation, while a path is naturally oriented.

Given a pseudo-Anosov map $\phi$ with stable foliation $\mathcal{F}^s$, let $\mathcal{I}$ be the collection of smooth arcs which are transverse to $\mathcal{F}^s$ in $M - P$, where recall that $P$ is the collection of singularities. We also allow arcs in $\mathcal{I}$ to have their endpoints at a singularity of $\mathcal{F}^s$. The collection $\mathcal{I}$ is always given the Hausdorff topology (but is not closed under it).

Informally, two arcs $\Gamma_0, \Gamma_1 \in \mathcal{I}$ are *holonomic* on $\mathcal{F}^s$ if one can slide one to the other along leaves of $\mathcal{F}^s$. More formally, they are holonomic if there is a family of parameterizations $\gamma : [0, 1] \times [0, 1] \to M$ written as $\gamma_s(t)$ with

(a) $\mathrm{im}(\gamma_0) = \Gamma_0$ and $\mathrm{im}(\gamma_1) = \Gamma_1$,

(b) $\mathrm{im}(\gamma_s) \in \mathcal{I}$ for all $s \in [0, 1]$,

(c) $\gamma_s(t) \in L^s(\gamma_0(t))$ for all $s, t \in [0, 1]$.

**Definition 7.1 (Transverse arc function)** *Given a pseudo-Anosov map, a transverse arc function (taf) to the stable foliation* $\mathcal{F}^s$ *is a continuous map* $G : \mathcal{I} \to \mathbb{F}$ *which is*

(a) *Holonomy invariant,*

(b) *Internally additive: If* $\Gamma \in \mathcal{I}$ *and* $\gamma : [a, b] \to M$ *is a parameterization of* $\Gamma$*, then for all* $a < p < b$, $G(\gamma([a, b])) = G(\gamma([a, p]) + G(\gamma([p, b])$.

As with transverse cocycles, it will be useful to have local version of a taf; the definition requires a cover by *oriented* rectangles.

**Definition 7.2 (Local transverse arc function)** *A local taf is a cover of* $M$ *by oriented rectangles* $\{R_1, \dots, R_n\}$ *coupled with a family of continuous functions* $f_{i,j} : E^u_{i,j} \to \mathbb{F}$ *for* $i = 1, \dots, n$ *and* $j = 1, 2$ *with the properties that*

(a) *For each* $i$, $f_{i,1}$ *and* $f_{i,2}$ *are holonomy invariant, i.e. if* $g_i$ *is the homeomorphism that takes* $E^u_{i,1}$ *to* $E^u_{i,2}$ *by sliding along leaves, then* $f_{i,1} = f_{i,2} \circ g_i$.

(b) *When* $E^u_{i,j} \cap E^u_{i',j'} \neq \emptyset$*, it is connected and* $f_{i,j} - \epsilon(i, j, i', j') f_{i',j'}$ *is constant on the intersection, where* $\epsilon(i, j, i', j') = 1$ *if the orientations of* $E^u_{i,j}$ *and* $E^u_{i',j'}$ *agree, and* $-1$ *otherwise.*

We then have the analog of Fact 6.5 which simply says that we have given a proper local version of a taf. We omit the straightforward proof.

**Fact 7.3** *A local taf gives rise to a unique taf. A taf and a cover of* $M$ *by oriented rectangles yields a unique local taf.*

**Remark 7.4** Let $\mathcal{I}'$ denote $\mathcal{I}$ union the collection of points in $M$, and so formally $\mathcal{I}' = \mathcal{I} \cup M$. We also give $\mathcal{I}'$ the Hausdorff topology. Fact 7.3 implies that we may extend $G$ to a continuous map $G'$ of $\mathcal{I}'$ with $G'$ having the value of zero on any point. Informally this says that a taf has no atoms.

We define the pull back of a taf $G$ under the pseudo-Anosov map $\phi$ by $(\phi^* G)(\Gamma) = G(\phi(\Gamma))$. Since $\phi$ is smooth away from its singularities and $\phi$ preserves the foliations, the image under $\phi$ of an element of $\mathcal{I}$ is always in $\mathcal{I}$, and thus the pull-back under $\phi$ of taf to $\mathcal{F}^s$ is also a taf to $\mathcal{F}^s$. We say that $G$ is an *eigen-taf* for $(\phi, \mathcal{F}^s)$ with factor $\mu \in \mathbb{F}$ if $\phi^* G = \mu G$.

Recall that given a pseudo-Anosov map on $M$, if the foliations are not oriented, there is a unique two-fold branched cover $p : \overline{M} \to M$ called the *orientation cover*. It is the smallest branched cover in which the lift of $\mathcal{F}^s$ is oriented (see, for example, [BB07] for more details). Since the branch points of orientation cover are always singularities, transverse cocycles and tafs to $\mathcal{F}^s$ on $M$ pull back to transverse cocycles and tafs to $\overline{\mathcal{F}}^s$ on $\overline{M}$ with the obvious definitions.

### 7.2 Transverse arc functions and transverse cocycles

The next proposition gives the equivalence of transverse arc functions with transverse cocycles when the foliation $\mathcal{F}^s$ is oriented. Thus in this case using Fact 6.2 there is an eigen-taf for every eigenvalue $\mu$ of $\phi^*$ acting on $H^1(M; \mathbb{Z})$ with $|\mu| > 1$.

**Proposition 7.5** *Let $\phi$ be a pseudo-Anosov map on an orientable surface $M$ such that the stable foliation $\mathcal{F}^s$ is orientable. There are natural isomorphisms which make the following commute.*

$$\begin{array}{ccccc}
\mathcal{TC}(\mathcal{F}^s) & \xrightarrow{\cong} & Un(\phi^*, H^1(M;\mathbb{F})) & \xrightarrow{\cong} & \mathcal{TAF} \\
\phi^* \downarrow & & \downarrow \phi^* & & \downarrow \phi^* \\
\mathcal{TC}(\mathcal{F}^s) & \xrightarrow{\cong} & Un(\phi^*, H^1(M;\mathbb{F})) & \xrightarrow{\cong} & \mathcal{TAF}
\end{array}$$

*Under the isomorphism eigen- and generalized eigen-cocycles correspond to eigen- and generalized eigen-tafs with the same factors. In particular, all eigen-tafs have factors $|\mu| > 1$.*

**Proof:** The first isomorphism is given by Proposition 6.2. For the second the construction is local. When $\mathcal{F}^s$ is orientable, one may assign a coherent family of orientations to any cover by rectangles, i.e. the family has the property that when $E^u_{i,j} \cap E^u_{i',j'} \neq \emptyset$, the orientations of $E^u_{i,j}$ and $E^u_{i',j'}$ always agree. Thus when $\mathcal{F}^s$ is orientable, a local taf and a local transverse cocycle are the same object, and so the result follows from Fact 6.5 and Fact 7.3. The last two statements of the proposition follow from the commutativity of the diagram and the fact that the horizontal maps are all isomorphisms. ■

We have the following corollary which also holds for pseudo-Anosov with nonorientable foliations.

**Corollary 7.6** *If $\phi$ is a pseudo-Anosov map, then all eigen-tafs to $\mathcal{F}^s$ have factors $|\mu| > 1$ and all eigen-cocycles have factors $|\mu| \neq 1$.*

**Proof:** For a general pseudo-Anosov, if there was eigen-taf to $\mathcal{F}^s$ with factor $|\mu| \leq 1$, then we can pull it back to the orientation double cover and get a contradiction to Proposition 7.5. The second statement follows from Remark 4.8, since a pseudo-Anosov map has a local product structure at all but finitely many points. ■

## 8 Symbolic transverse arc functions

As is often the case in dynamics, symbolic methods simplify and clarify certain technical issues. In this section we define the symbolic analog of a transverse arc function which is called a *symbolic transverse arc function* and use it in Theorem 9.1 to characterize the collection of taf's and eigen-taf's.

The construction makes use of a standard technique in hyperbolic dynamics which uses a one-sided subshift of finite type to model the stable foliations of an Axiom A diffeomorphism (see, for example, [BM77]). There are two basic but closely related approaches to doing this. The first is to form a quotient space by identifying sequences with the same future; the quotient is then essentially the leaf space of the foliation. The second, which we adopt here, is to identify each length one cylinder set with the leaf space in the corresponding Markov rectangle.

The construction of an staf uses the one-sided subshift of finite type $\Lambda_A^+$ built from the $\{0,1\}$-transition matrix $A$. We shall assume that the symbol set is $S := \{1, 2, \ldots, d\}$. An allowable transition is a pair $(i, j)$ with $A_{ij} = 1$. The notation $i \to j$ means that $(i, j)$ is allowable. Thus the shift space is defined as

$$\Lambda_A^+ = \{\underline{s} \in S^{\mathbb{N}} : s_k \to s_{k+1} \text{ for all } k \in \mathbb{N}\}.$$

An *allowable block* for the subshift is a finite list of allowable transitions. The collection of allowable blocks for a given subshift is denoted $\mathcal{B}(A)$. The number of symbols in a block is its *length* and is denoted $\ell(b)$. For an allowable block $b$, $[b]$ is the corresponding cylinder set starting at the zero$^{th}$ place,

$$[b] = \{\underline{s} \in \Lambda_+ : s_j = b_j \text{ for } j = 0, \dots, \ell(b) - 1\}. \tag{8.1}$$

Note that in this paper all cylinder sets start at the zero$^{th}$ place unless otherwise noted. In what follows that matrix $A$ is usually fixed, and so we will often suppress the dependence on $A$.

**Definition 8.1 (Symbolic transverse arc function)** *Assume that $(\Lambda^+, \sigma)$ is a one-sided subshift of finite type on $d$-symbols with $\{0, 1\}$-transition matrix $A$. Let $K$ be an $\mathbb{F}$-valued function on the collection of allowable blocks $\mathcal{B}(A)$, so $K : \mathcal{B}(A) \to \mathbb{F}$. The set function $K$ is called a symbolic transverse arc function (staf) if it is*

*(a) Additive: For all allowable blocks $s_0 s_1 \dots s_{n-1}\, j$,*

$$K(s_0 s_1 \dots s_{n-1}\, j) = \sum_{j \to k} K(s_0 s_1 \dots s_{n-1}\, j\, k). \tag{8.2}$$

*(b) Coherent: For all allowable blocks $s_0 \dots s_n\, j$ and $s_0' \dots s_n'\, j$,*

$$K(s_0 \dots s_n\, j) = K(s_0' \dots s_n'\, j). \tag{8.3}$$

*If there exist constants $C > 0$ and $r > 0$ so that for all $b \in \mathcal{B}$,*

$$|K(b)| < C r^{-\ell(b)}, \tag{8.4}$$

*then $K$ is said to have a $(C, r)$-exponential bound or just an $r$-exponential bound.*

In most of the symbolic dynamics literature in a bound such as (8.4) it is required that $r > 1$. This will often be the case here, but for the definition of staf's of exponential bound we are just requiring $r > 0$.

Let $\mathcal{A}(A)$ be the algebra generated by the cylinder sets of $\Lambda_A^+$. It is easy to check that $\mathcal{A}$ is all finite disjoint unions of cylinder sets of $\Lambda_A^+$. Condition Definition 8.1(a) says that $K$ yields a finitely-additive function on $\mathcal{A}$. Conversely, any finitely additive map $\mathcal{A} \to F$ yields an additive $K : \mathcal{B} \to \mathbb{F}$. Thus a staf is a coherent, finitely additive set function on $\mathcal{A}$.

The smallest $\sigma$-algebra containing $\mathcal{A}$ is the Borel sets. In Theorem 8.6(a) below, we see that for a mixing $(\Lambda_+, \sigma)$ the only staf which yields a Borel measure will be eigen-staf corresponding to the Perron-Frobenius eigenvector of $A$. On the other hand, Theorem 8.6(b) shows that any staf yields a distribution in sense of an element of the dual space of a class of Hölder functions on $\Lambda_A^+$.

We will also see in Fact 8.5 below that every staf has an exponential bound. The "transverse" in the nomenclature " symbolic transverse arc function" comes from the condition in Definition 8.1(b) which ensures that if $A$ is the transition matrix of a pseudo-Anosov map $\phi$, then $K$ yields a transverse structure to the stable foliation. In Fact 9.1 we show that only staf with an exponential bound with $r > 1$ correspond to taf to $\mathcal{F}^s$.

The collection of all symbolic transverse arc functions to the one-sided subshift determined by $A$ is denoted $\mathcal{STAF}(A)$. Before getting to more measure theoretic type results we give a simple alternative description of $\mathcal{STAF}(A)$. Given the $d \times d$-matrix $A$, a *thread* of $A$ is an infinite list of vectors $(\vec{v}_0, \vec{v}_1, \dots)$ with each $\vec{v}_j \in \mathbb{F}^d$, so that

$$\vec{v}_n = A\vec{v}_{n+1}, \tag{8.5}$$

for all $n \in \mathbb{N}$. The collection of threads of $A$ is the inverse limit of $\mathbb{F}^d$ for which $A$ is all the one-step transition maps and is denoted $\varprojlim(\mathbb{F}^d, A)$.

Now $\mathcal{STAF}(A)$ is also clearly a vector space over $\mathbb{F}$. For $K \in \mathcal{STAF}(A)$, by coherence, for any allowable block $b = s_0 \dots s_{n-1}j$, the value $K(b)$ depends only on the last symbol in the block $j$. Thus for each $n \in \mathbb{N}$ we can define $\mathbf{K}^{(n)}$ as the vector constructed from the values of $K$ on length-$n$ cylinder sets with the $j^{th}$ component of $\mathbf{K}^{(n)}$ being $(\mathbf{K}^{(n)})_j = K(s_0 \dots s_{n-1}j)$, for any allowable block $s_0 \dots s_{n-1}j$. Thus $K \in \mathcal{STAF}(A)$ yields a list of vectors $\mathbf{K} := (\mathbf{K}_0, \mathbf{K}_1, \dots)$. Since the transition matrix $A$ is a $\{0, 1\}$-matrix, the additivity condition in Definition 8.1(a) translates as $\mathbf{K}^{(n)} = A\mathbf{K}^{(n+1)}$ for all $n \in \mathbb{N}$, and so $\mathbf{K}$ is a thread.

**Fact 8.2** *The assignment $K \mapsto \mathbf{K}$ just described is a vector space isomorphism from $\mathcal{STAF}(A)$ to $\varprojlim(\mathbb{F}^d, A)$. Further, $\varprojlim(\mathbb{F}^d, A)$ is isomorphic to $NonN(A, \mathbb{F}^d)$, the non-nilpotent subspace of $A$ acting on $\mathbb{F}^d$.*

**Proof:** It is obvious that $K \mapsto \mathbf{K}$ is vector space monomorphism. To see that it is surjective, note that it follows from (8.5) that $\mathbf{K}^{(k)} = A^n\mathbf{K}_{(k+n)}$ for all $n \in \mathbb{N}$, and thus each $\mathbf{K}^{(k)}$ is contained in the eventual image of $A$, namely $\cap_{n\in\mathbb{N}} A^n(\mathbb{F}^d)$. Using the Jordan form of $A$ it is easily seen that the eventual image is exactly $NonN(A, \mathbb{F}^d)$. Since $A$ restricted to $NonN(A, \mathbb{F}^d)$ is invertible, each $\mathbf{K}_0 \in NonN(A, \mathbb{F}^d)$ yields a unique thread $\mathbf{K}$, completing the proof. ■

### 8.1 Eigen-staf

There is a natural action of $A$ on threads, namely,

$$A^*(\mathbf{K}) = (A\mathbf{K}_0, A\mathbf{K}_1, A\mathbf{K}_2, \dots) = (A\mathbf{K}_0, \mathbf{K}_0, \mathbf{K}_1, \dots). \tag{8.6}$$

using (8.5). It is clear that $A^*$ is a vector space self-isomorphism of $\varprojlim(\mathbb{F}^d, A)$.

We next define the induced co-action of the left shift $\sigma$ on the space of staf, $\mathcal{STAF}(A)$. It will correspond to the action of $A$ on threads under the isomorphism of Fact 8.2.

**Definition 8.3 (Co-action of $\sigma$ on threads)** *For $K \in \mathcal{STAF}(A)$ define $\sigma^*K$ on blocks as*

(a) *For each block $b$ of length greater than one, $(\sigma^* K)(b) = K(\sigma(b))$,*
(b) *For each symbol $j$ (a block of length one),*

$$\sigma^* K(j) = \sum_{j \to k} K(k).$$

It is also clear that $\sigma^*$ is a vector space self-isomorphism of $\mathcal{STAF}(A)$.

If we let $K$ denote the action on cylinder sets instead of blocks, $\sigma^* K$ is the pull back of $K$ under $\sigma$, $(\sigma^* K)([b]) = K(\sigma([b]))$. Note that in contrast to what is usual for measures, we are pulling back not pushing forward. This makes sense since the image of a cylinder set is always the finite union of cylinder sets. However, it is also important to note that a cylinder set here are always based at the "decimal point". Thus while $\sigma^* K$ is a staf and so trivially extends to the algebra $\mathcal{A}$ of finite unions of cylinder sets, this extension does *not* satisfy $\sigma^* K = K \circ \sigma$. As a simple example, by definition of the extension to $\mathcal{A}$, $(\sigma^* K)([ik] \uplus [jk]) = \sigma^* K([ik]) + \sigma^* K([jk]) = K([k]) + K([k])$, but $K(\sigma([ik] \uplus [jk])) = K([k])$.

From Definition 8.3 it is easy to confirm the following.

**Fact 8.4** *If $A$ is a $\{0,1\}$-matrix defining a subshift of finite type and $\mathcal{STAF}(A)$ the space of symbolic transverse arc functions is defined as above, then there are natural isomorphisms which make the following commute.*

$$\begin{array}{ccccc}
NonN(A, \mathbb{F}^d) & \xrightarrow{\cong} & \varprojlim(\mathbb{F}^d, A) & \xrightarrow{\cong} & \mathcal{STAF}(A) \\
{\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A^*} & & \downarrow{\scriptstyle \sigma_*} \\
NonN(A, \mathbb{F}^d) & \xrightarrow{\cong} & \varprojlim(\mathbb{F}^d, A) & \xrightarrow{\cong} & \mathcal{STAF}(A)
\end{array}$$

*Since every map in the diagram is an isomorphism the eigen- and generalized eigen-objects of the vertical maps correspond. In particular, an eigen-staf $K_\mu$ with factor $\mu$ corresponds to a thread $v^{(0)}, v^{(1)}, \dots$ with $v^{(0)}$ a right eigenvector of $A$ with eigenvalue $\mu$, and $v^{(n)} = \mu^{-n} v^{(0)}$. Thus $K_\mu(s_0 \dots s_{n-1}\, j) = \mu^{-n} (v^{(0)})_j$. Similarly, generalized eigen-staf correspond to generalized eigenvectors of $A$.*

### 8.2 Staf with exponential bound

In the sequel we will require the additional information about the role of exponential bounds contained in the next fact.

**Fact 8.5** *Let $A$ be a $\{0,1\}$-matrix defining a subshift of finite type which is irreducible and aperiodic, or equivalently, there is an $N$ so that $A^n > 0$ for all $n > N$.*
(a) *If $K$ is an eigen- or generalized eigen-staf with eigenvalue $\mu$, then it has exponential bound with $r = |\mu|$.*
(b) *Given a generalized eigenbasis $\{K_i\}$ for $\sigma^*$ acting on $\mathcal{STAF}$, if $K \in \mathcal{STAF}$ is written*

$$K = \sum c_i K_{\mu_i}, \tag{8.7}$$

*then $K$ has an exponential bound with $r$ equal to the minimum of the $|\mu_i|$ such that $c_i \neq 0$ in expression (8.7), and $\mu_i$ is the eigenvalue corresponding to $K_i$.*

(c) *$K \in \mathcal{STAF}$ has an exponential bound with $r > 1$ if and only if for any sequence of allowable blocks $b_n$ with $b_n \to \underline{s}$ for some sequence $\underline{s}$, we have $K(b_n) \to 0$.*

**Proof:** The proofs of (a) and (b) are straightforward linear algebra using the observation that a thread $\mathbf{K}$ corresponds to a staf $K$ with an $r$-exponential bound if and only if for all $n > 0$, $\|\mathbf{K}^{(n)}\| \leq C'/r^n$ for some constant $C'$ and vector norm $\|\cdot\|$.

For (c) assume that $K$ is not unstable. From Fact 8.2 this means that $\mathbf{K}^{(0)} \notin Un(A, \mathbb{F}^d)$. Letting $B$ be the inverse of $A$ restricted to the non-nilpotent subspace of $A$, we then have that $\mathbf{K}^{(0)} \notin Stab(B, NonN(A, \mathbb{F}^d))$ and since $B\mathbf{K}^{(n)} = \mathbf{K}^{(n+1)}$ we have that $\mathbf{K}^{(n)} \not\to 0$ as $n \to \infty$. Now since $A$ is irreducible by hypothesis, let $N$ be such that $n > N$ implies that $A^n > 0$. Since $\mathbf{K}^{(n)} \not\to 0$, we may find a symbol (or component) $a$ and a subsequence $n_i \to \infty$ with $n_{i+1} - n_i > N$ for all $i$ with $(\mathbf{K}^{n_i})_a \not\to 0$ as $i \to \infty$. For each $j > 0$ since $n_{i+1} - n_i > N$, we may find a block $b_j = s_0 \dots s_{n_j}$ with $s_{n_i} = a$ for $i = 1, \dots, j$. By construction, $K(b_j) = (\mathbf{K}^{n_j})_a \not\to 0$, but $[b_j] \to \underline{s} = s_0 s_1 \dots$. The other implication in (c) is trivial. ■

### 8.3 Functions with exponential bound

As is common in symbolic dynamics we will use an exponential decay condition on the variation as an alternative description of the Hölder condition. A function $f : \Lambda^+ \to \mathbb{F}$ is said have an $r$-exponential bound if there is a constant $C > 0$ so that for all allowable blocks $b$,

$$\max\{|f(\underline{s}) - f(\underline{s}')| : \underline{s}, \underline{s}' \in [b]\} \leq C r^{-\ell(b)}$$

The collection of all $f$ with an $r$-exponential decay bound is denoted $\mathcal{E}^r(\Lambda_+, \mathbb{F})$. If $|f|_\infty$ is the usual sup-norm and

$$|f|_r = \sup_{b \in \mathcal{B}} \sup_{\underline{s}, \underline{s}' \in [b]} |f(\underline{s}) - f(\underline{s}')| r^{\ell(b)},$$

then $\|f\|_r = |f|_\infty + |f|_r$ is a norm making $\mathcal{E}^r(\Lambda^+, \mathbb{F})$ into a Banach space when $r > 1$.

There is a certain fluidity in the notion of Hölder functions on a shift because there is a family of natural metrics which all give the same topology, namely, for $\theta > 1$ let

$$d_\theta(\underline{t}, \underline{s}) = \sum_{i=0}^{\infty} \frac{1 - \delta(t_i, s_i)}{\theta^i},$$

where $\delta(t_i, t_s)$ is the Kronecker delta. As one varies the metrics it will vary Hölder exponents, whereas the exponential decay description is metric independent.

The main observation needed to go back and forth from Hölder bounds to exponential bounds is that there exist constants $k_1, k_2 > 0$

$$\frac{k_1}{r^{\ell(b)}} \leq \operatorname{diam}_\theta([b]) \leq \frac{k_2}{r^{\ell(b)}} \tag{8.8}$$

for all blocks $b \in \mathcal{B}$, where $\operatorname{diam}_\theta([b])$ is the diameter of the set $[b]$ in the metric $d_\theta$. This implies when $r > \theta > 1$ that $f \in \mathcal{E}^r(\Lambda^+, \mathbb{F})$ if and only if $f \in C^\nu(\Lambda^+)$ for $\nu = \log(r)/\log(\theta)$, and further the identity map $\mathcal{E}^r(\Lambda^+, \mathbb{F}) \to C^\nu(\Lambda^+)$ is continuous in the given norms.

### 8.4 Connections to other standard measures and distributions.

In this subsection we make a few remarks on the relation of staf to other standard elementary constructions in the ergodic theory of subshifts of finite type. To easily discuss the connection of staf's to some standard measures on shifts we need to change notation a bit. Let $\vec{r}_\mu$ and $\vec{\ell}_\mu$ be right (column) and left (row) eigenvectors of $A$ corresponding to an eigenvalue $\mu$ normalized so that $\vec{r}_\mu \vec{\ell}_\mu = 1$. Recall from above that $\mathcal{A}$ is the algebra generated by the cylinder sets. In this notation the function $K_\mu : \mathcal{A} \to \mathbb{F}$ induced by $K_\mu([s_0 \ldots s_n]) = \mu^{-n}(\vec{r}_\mu)_{s_n}$ is the eigen-staf with eigenvalue $\mu$ given in Fact 8.4.

On the other hand, $J_\mu : \mathcal{A} \to \mathbb{F}$ induced by $J_\mu([s_0 \ldots s_n]) = \mu^{-n}(\vec{\ell}_\mu)_{s_0}(\vec{r}_\mu)_{s_n}$ will not be a staf, but is $\sigma$-invariant in the usual sense that $J_\mu(\sigma^{-1}(Y)) = J_\mu(Y)$ for all $Y \in \mathcal{A}$. These $J_\mu$ will generate Hölder distributions in the sense of Theorem 8.6(b) below, and as in that Theorem, only $J_\lambda$ where $\lambda$ is the Peron-Fröbenius eigenvalue of $A$ will extend to an invariant measure on the Borel $\sigma$-algebra of $\Lambda_A^+$. This invariant measure is usually called the *Parry measure* and is the measure of maximal entropy for $(\Lambda_A^+, \sigma)$.

These $J_\mu$ can also be extended to invariant distributions on the full-shift $\Lambda_A$. Letting $[s_0 \ldots s_n]_j$ denote the cylinder set beginning at place $j$, so $[s_0 \ldots s_n]_j = \{\underline{t} \in \Sigma_A : t_{j+i} = s_i \text{ for } i = 0, \ldots, n\}$, we get an $\sigma$-invariant distribution on the full shift generated by $\hat{J}_\mu([s_0 \ldots s_n]_j) = \mu^{-n}(\vec{\ell}_\mu)_{s_0}(\vec{r}_\mu)_{s_n}$. Once again $\hat{J}_\lambda$ extends to an invariant measure also called the Parry measure.

We also remark that $\hat{J}_\mu$ is the product of the staf $K_\mu$ on the positive one-sided shift $\Lambda_A^+$ and an analogous $K_\mu^-$ on the negative one-sided shift $\Lambda_A^-$ in the following sense. For a cylinder set $b^- = [s_{-n} \ldots s_{-1} s_0]$ in the negative one sided shift, let $K_\mu^-(b^-) = \mu^{-n}(\vec{\ell}_\mu)_{s_{-n}}$, and so $K_\mu^-$ is eigen under the right shift on $\Lambda_A^-$. Given a "rectangle block" in the full shift, $b_r = [s_{-n} \ldots s_{-1} s_0 \ldots s_n]$, we have that $\hat{J}_\mu(b_r) = K_\mu(\pi_+ b_r) K_\mu^-(\pi_- b_r)$, where $\pi_+ : \Lambda_A \to \Lambda_A^+$ and $\pi_- : \Lambda_A \to \Lambda_A^-$ are the projections.

### 8.5 Ruelle's transfer operator

In this subsection we comment on the connection of the action $\sigma^*$ on staf and Ruelle's transfer operator. See [Bal00] for more information on the transfer operator and related structures.

The (unweighted) transfer operator $\mathcal{L}$ acts on bounded functions $f : \Lambda_A^+ \to \mathbb{F}$ by

$$\mathcal{L} f(\underline{s}) = \sum_{\underline{t} : \sigma(\underline{t}) = \underline{s}} f(\underline{t}) = \sum_{j \to s_0} f(j\underline{s}). \tag{8.9}$$

It is standard and simple to verify that for each $r > 1$, if $f \in \mathcal{E}^r(\Lambda^+, \mathbb{F})$, then $\mathcal{L} f \in \mathcal{E}^r(\Lambda^+, \mathbb{F})$, and $\mathcal{L} : \mathcal{E}^r(\Lambda^+, \mathbb{F}) \to \mathcal{E}^r(\Lambda^+, \mathbb{F})$ is a continuous, linear operator. The dual of $\mathcal{L}$ acting on a functional $L \in \mathcal{E}^r(\Lambda^+, \mathbb{F})^*$ is $\mathcal{L}^* L(f) = L(\mathcal{L} f)$. A simple computation yields that the action of $\mathcal{L}$ on indicator functions of cylinder sets is $\mathcal{L} \mathbb{1}_{[b]} = \mathbb{1}_{\sigma([b])}$ when $\ell(b) > 1$, and for length one cylinder sets $\mathcal{L} \mathbb{1}_{[j]} = \mathbb{1}_X$, where $X = \uplus_{j \to k} [k]$.

Now given a staf $K$, denote its corresponding functional on indicator functions of cylinder sets by $\hat{K}$, and so $\hat{K}(\mathbb{1}_{[b]}) = K(b)$. By definition the dual of $\mathcal{L}$ acts on this functional by $\mathcal{L}^* \hat{K}(\mathbb{1}_{[b]}) := \hat{K}(\mathcal{L} \mathbb{1}_{[b]})$ and another simple computation shows that

$$\mathcal{L}^* \hat{K}(\mathbb{1}_{[b]}) = (\sigma^* K)(b), \tag{8.10}$$

or put another way, $\mathcal{L}^*\hat{K} = \widehat{\sigma^* K}$. Thus the co-action of $\sigma$ on stafs is basically the same as the dual action of the unweighted transfer operator.

This is to be expected. The co-action $\sigma^*$ was defined to mimic the action of a pseudo-Anosov map on tafs. A standard technique in hyperbolic dynamics is to collapse down stable manifolds and use the transfer operator of the resulting expanding dynamical systems to find interesting measures. Symbolically, this is often accomplished using the one-sided shift to represent the leaf space of the stable foliation as is done here.

In the case of pseudo-Anosov maps, after collapsing down stable manifolds one gets a branched one-manifold (the train track) on which the pseudo-Anosov map induces an expanding map. It would be interesting the apply the transfer operator theory directly to this system.

### 8.6 Staf as measures and distributions

In this subsection we show that all staf give rise to functionals dual to a class of functions with exponential bound, but only the Peron-Fröbenius staf gives a functional dual to the space of all continuous functions, i.e. gives a signed measure.

The construction of a distribution from a staf is straightforward. We do it in an elementary manner: for a staf $K$ we define "$\int f\ dK$" by imitating the usual construction of the Riemann integral but we use the partition of $\Lambda_A^+$ into blocks of length $n$ at the $n^{th}$ stage. Since the staf $K$ has an exponential bound, convergence is obtained by restricting to functions $f$ with an appropriate exponential bound.

To describe why only the Peron-Fröbenius staf gives a measure, we first recall a definition. Let $\mathcal{A}$ be an algebra of subsets of a set $X$ and $F : \mathcal{A} \to \mathbb{F}$ a finitely additive set function. Given a finite partition $\mathcal{P} = \{X_1, \ldots, X_n\}$ of $X$ into disjoint subsets (often written $X = \uplus X_n$) with each $X_n \in \mathcal{A}$, the *variation* of $F$ on $\mathcal{P}$ is $\mathrm{var}(F, \mathcal{P}) = \sum |F(X_n)|$ and the *total variation* of $F$ on $X$ is $\mathrm{var}(F, X) = \sup\{\mathrm{var}(F, \mathcal{P})\}$ with the sup over all finite partitions of $X$. It is a standard fact (see, for example, Theorem 6.4 in [Rud87]) that if $F$ extends to a measure on the $\sigma$-algebra generated by $\mathcal{A}$, then $F$ has bounded variation. More specifically, if $F$ does not have bounded variation, then $F$ is not countably additive in the sense that there are a sequence of disjoint sets $Y_n \in \mathcal{A}$ with $\sum F(Y_n)$ diverging.

Let us say that $A^N > 0$ for $n > N$ and for simplicity that $A$ is invertible. There are two main ideas behind the fact that only the Peron-Fröbenius eigenvectors give actual measures. Using the Peron-Fröbenius Theorem, the only vectors $\mathbf{v}$ with $A^{-n}\mathbf{v} > 0$ for all $n > 0$ are Peron-Fröbenius eigenvectors. Thus by the correspondence of Fact 8.4, any staf other than the Peron-Fröbenius one assigns both positive and negative values to various cylinder sets. The second idea is that number of cylinder sets which end in a given symbol grows like $\lambda^n$ whereas the $K$-value of each shrinks by at most $1/|\mu|^n$, where $\mu$ is the eigenvalue of second largest modulus. Thus the $K$-value of the collection of length $n$ cylinder sets which end in a given symbol grows like $\pm\lambda^n/|\mu|^n \to \infty$.

As noted in the introduction, Ruelle in [Rue87] defines Gibbs distributions dual to Hölder functions on a subshift of finite type and Haydn [Hay90] shows that Gibbs distributions are always eigen with respect to the action of the transfer operator. In the previous subsection we saw that by treating a staf as a functional on the indicator functions of blocks that the action of $\sigma^*$ on stafs is the same as that of the dual of the transfer operator $\mathcal{L}^*$. After showing that "$\int \mathcal{L}f \, dK = \int f \, d(\sigma^* K)$", we have that eigen-stafs induce eigen-distributions.

**Theorem 8.6** *Assume that $A$ is irreducible and aperiodic with Peron-Fröbenius eigenvalue $\lambda$, and let $\mathcal{A}$ be the algebra generated by the cylinder sets of $\Lambda^+$. Assume $K \in \mathcal{STAF}$ and $\mathbf{K}$ is the corresponding thread with $\mathbf{K}^{(0)} \in NonN(A)$ given by Fact 8.4*

*(a) $K$ has bounded variation as a set function if and only if $\mathbf{K}^{(0)}$ is a Peron-Fröbenius eigenvector. In particular, the only $K \in \mathcal{STAF}$ which extend to a Borel measure on $\Lambda^+$ are those constructed from the Peron-Fröbenius eigenvector as in Fact 8.4*

*(b) If $K \in \mathcal{STAF}(A)$ has an $r_2$-exponential bound with $0 < r_2 \le \lambda$, it defines a continuous, linear functional $L_K$ on $\mathcal{E}^{r_1}(\Lambda^+, \mathbb{F})$ for all $r_1 > \lambda/r_2$.*

*(c) If $K$ is an eigen-staf under the action of $\sigma^*$, then the functional $L_K$ is an eigen-distribution under the action of the dual transfer operator $\mathcal{L}^*$.*

**Proof:** The argument that the Peron-Fröbenius staf $K_\lambda$ extends to a measure is standard in symbolic dynamics: since $K_\lambda$ viewed as a finitely additive set function on $\mathcal{A}$ has positive values on all cylinder sets, and there no countable unions of cylinder sets in $\mathcal{A}$, the Caratheodory extension theorem gives that $K_\lambda$ extends to a Borel measure.

Now assume that $K$ is such that $\mathbf{K}^{(0)}$ is not a Peron-Fröbenius eigenvector. Thus by Fact 2.1, there is a $C > 0$, $N$ and $0 < r < \lambda$ so that $n > N$ implies that

$$\|\mathbf{K}^{(n)}\|_1 > C/r^n. \tag{8.11}$$

For each $n > 0$ we define the partition $\mathcal{P}^{(n)}$ of $\Lambda_+$ as having elements

$$P_j^{(n)} := \{\underline{s} \in \Lambda_+ : s_n = j\}$$

for $j = 1, \ldots, d$. If we let $N_j^{(n)}$ denote the number of disjoint blocks of length $(n+1)$ making up $P_j^{(n)}$, then $N_j^{(n)}$ is exactly the number of ways to make $n$ allowable transitions and end with $j$, and so $N_j^{(n)}$ is counted by the sum of the entries in the $j^{th}$ column of $A^n$, or

$$N_j^{(n)} = \sum_{i=1}^{d} (A^n)_{ij}.$$

The Peron-Fröbenius Theorem as in (2.1) shows that there is a constant $C_1 > 0$ so that

$$N_j^{(n)} > C_1 \lambda^n \tag{8.12}$$

for all $j$ and $n > N$.

Since each block of length $n+1$ that ends with $j$ has $K$-value equal to $\mathbf{K}_j^{(n)}$, we have $K(P_j^{(n)}) = N_j^{(n)} \mathbf{K}_j^{(n)}$. Thus by (8.11) and (8.12),

$$\operatorname{var}(K, \mathcal{P}^{(n)}) = \sum_{j=1}^{d} |K(P_j^{(n)})| \ge \frac{C C_1 \lambda^n}{r^n} \to \infty, \tag{8.13}$$

as $n \to \infty$, showing that $K$ is not of bounded variation on $\mathcal{A}$. As noted above the theorem, this implies that $K$ cannot be extended to a signed or complex Borel measure.

Now to prove (b), assume that $K$ has a $(c_2, r_2)$-exponential bound. Let $b_{n,1}, b_{n,2}, \ldots, b_{n,N(n)}$ be an enumeration of the allowable length $n$ blocks. Note that it follows from (2.1), there exists a $c_3$ with $N(n) < c_3\lambda^n$. Given $f$ has an $(c_1, r_1)$ exponential bound, we may assume that the $c_1$ is optimal in the sense that $c_1 = |f|_{r_1}$. Now for each $n \in \mathbb{N}$ pick elements $\underline{x}_{n,i} \in b_{n,i}$ and define

$$S_n = \sum_{i=1}^{N(n)} f(\underline{x}_{n,i})K(b_{n,i}). \tag{8.14}$$

We shall show that the sequence $S_n$ is Cauchy.

Fix a length $n$-block $b_{i,n}$ and label the length $(n+1)$ blocks it contains as $b_{i_1,n+1}, \ldots, b_{i_k,n+1}$, so $b_{i,n} = \uplus_{j=1}^{k} b_{i_j,n+1}$ as a disjoint union. Using the finite additivity of $K$,

$$\begin{aligned} |f(x_{i,n})K(b_{i,n}) - \sum_{j=1}^{k} f(x_{i_j,n+1})K(b_{i_j,n+1})| &= |\sum_{j=1}^{k}(f(x_{i,n}) - f(x_{i_j,n+1})|K(b_{i_j,n+1})| \\ &\leq kc_1 r_1^{-n} c_2 r_2^{-(n+1)}. \end{aligned} \tag{8.15}$$

Now certainly $k \leq d$ the total number of symbols, and the total number of blocks of length $n$ is $N(n) < c_3\lambda^n$. Thus adding (8.15) over the length $n$ blocks

$$|S_n - S_{n+1}| \leq dc_1c_2c_3r_2^{-1}(\frac{\lambda}{r_1r_2})^n. \tag{8.16}$$

Since by assumption, $\frac{\lambda}{r_1r_2} < 1$, the sequence $S_n$ is Cauchy and we define

$$L_K(f) = \lim_{n\to\infty} S_n.$$

To see that this definition is independent of the choice of evaluation points, note that if $\underline{x}'_{n,i} \in b_{n,i}$ is another choice of points,

$$\begin{aligned} |S_n - S'_n| &\leq \sum_{i=1}^{N(n)} c_1 r_1^{-n} c_2 r_2^{-n} \\ &\leq c_1c_2c_3(\frac{\lambda}{r_1r_2})^n \\ &\to 0, \end{aligned}$$

as $n \to \infty$ when $\frac{\lambda}{r_1r_2} < 1$.

Now $L_K$ is obviously linear. To see that it is continuous, note that $|L_K(f)| \leq |S_0| + |L_K(f) - S_0|$. Recalling that $c_1 = |f|_r$, the Cauchy estimate (8.16) and summing the tail of the geometric series shows that $|L_K(f) - S_0| \leq \hat{C}|f|_{r_1}$ with

$$\hat{C} = \frac{dc_2c_3}{r_2}\left(1 - \frac{\lambda}{r_1r_2})\right)^{-1},$$

independent of $f$. Since $|S_0| \le \|f\|_\infty K(\Lambda_+)$, we have that for some $C'$, $|L_K(f)| \le C'\|f\|_{r_1}$, and so $L_K$ is bounded and thus continuous.

To prove (c) we show that if $K$ has an $r_2$-exponential bond and $f \in \mathcal{E}^{r_1}(\Lambda^+, \mathbb{F})$, then

$$L_K(\mathcal{L}f) = L_{\sigma^* K}(f). \tag{8.17}$$

Thus if $\sigma^* K = \mu K$, since $L_{\mu K} = \mu L_K$, we have proved (c).

First note that for the indicator function of a cylinder set $\mathbb{1}_{[b]}$, the construction of $L_K$ yields $L_K(\mathbb{1}_{[b]}) = \hat{K}(\mathbb{1}_{[b]}) = K(b)$, with $\hat{K}$ as in §8.5. Thus using (8.10) we have that

$$L_K(\mathcal{L}\mathbb{1}_{[b]}) = (\sigma^* K)(b) \tag{8.18}$$

Continuing the notation as in (8.14), for each $n \in \mathbb{N}$ let

$$f_n = \sum_{i=1}^{N(n)} f(\underline{x}_{n,i})\mathbb{1}_{[b_{n,i}]}. \tag{8.19}$$

and so $L_K(f_n) = S_n$. Now above we defined $L_K(f)$ as $\lim S_n$, but here we need to note that each $f_n \in \mathcal{E}^{r_1}(\Lambda^+, \mathbb{F})$ and $f_n \to f$ in the norm of that space, and so now we have $L_k(f_n) \to L_K(f)$ by the continuity of $L_K$.

Using (8.18) and (8.19),

$$L_K(\mathcal{L}f_n) = \sum_{i=1}^{N(n)} f(\underline{x}_{n,i})\sigma^* K(b_{n,i})$$

and so $L_K(\mathcal{L}f_n) \to L_{\sigma^* K}(f)$. But since $\mathcal{L}$ is continuous, $L_K(\mathcal{L}f_n) \to L_K(\mathcal{L}f)$, and we have (8.17), finishing the proof of (c). ■

**Remark 8.7** It is worth noting that the eigen-staf given in Fact 8.4, do *not* give rise to eigen-distributions under the induced action of $\sigma$ on $\mathcal{E}^{r_1}(\Lambda^+, \mathbb{F})$. As a trivial example, for a constant function $c$, one has $L_K(c) = L_K(c \circ \sigma)$, for all staf $K$. The underlying reason is contained in the comment given after Definition 8.3: an eigen-staf $K$ is eigen for the induced action of $\sigma$ on the space of staf's, or as we have just seen, for the dual action of the transfer operator, and not for the action of $\sigma$ on $\Lambda_A^+$.

## 9 Transverse arc functions and symbolic transverse arc functions

In this section we connect the staf constructed from the transition matrix to tafs to the stable foliation $\mathcal{F}^s$ of a pseudo-Anosov map. In contrast to the connection of cocycles and tafs in Proposition 7.5, the correspondence of stafs and tafs does not require that the foliations be oriented.

We start with a few remarks on the task at hand in order to motivate the subsequent definitions and proof. A staf $K$ assigns numbers to allowable blocks of a one-sided subshift of finite type. When a shift is coding a pseudo-Anosov map, an allowable block $b$ of the one-sided shift corresponds naturally to certain arcs inside unstable leaves. More specifically, if $b =$

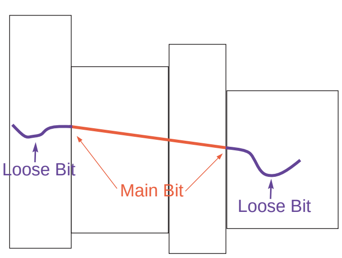
$s_0 s_1 \dots s_{n-1} s_n$ is an allowable block, let $S_b := \cap_{i=0}^n \phi^{-i} R_{s_i}$. Then $S_b$ is also a rectangle in the sense of §5.2 and any arc $\Gamma_b$ connecting its stable edges is said to represent $b$. Alternatively, $\Gamma \in \mathcal{I}$ represents $b$ if and only if $\phi^i(\Gamma) \subset R_{s_i}$ for all $i = 0, \dots, n$.

If a staf $K$ is to correspond to a taf $G$, then clearly for any arc $\Gamma$ representing $b$, $G(\Gamma)$ should be equal to $K(b)$. Now note that given a Markov rectangle $R_j$ and $n > 0$, $\phi^{-n}(R_j)$ is a long, thin band of stable leaves bounded by $\phi^{-n}(E_{j,1})$ and $\phi^{-n}(E_{j,2})$. Arcs from $\mathcal{I}$ connecting these two boundaries are all holonomic, and so all should be assigned the same value by a taf. Since $\phi^{-n}(R_j) = \cup S_b$ with the union over all allowable blocks $b$ of length $n$ that terminate in $j$, we see given a taf, a corresponding staf must assign the same value to all allowable blocks of the same length that end in the same symbol. This is exactly the coherence condition on a staf given in Definition 8.1.

As just noted, given a staf $K$, the first step in using it to construct a taf is clear, any arc representing an allowable block $b$ should be assigned the value $K(b)$. But what about other transverse arcs? Any arc $\Gamma \in \mathcal{I}$ can be written as the union of arcs representing allowable blocks $b_1, \dots$ and so $G(\Gamma) = \sum K(b_n)$ is the proper definition. However, it is difficult to see from the symbolic dynamics for a given arc $\Gamma \subset M$ what the constituent allowable blocks $b_1, \dots$ should be. What we do here is allow the pseudo-Anosov dynamics to organize the sum. Specifically, start with a transverse arc and take a high iterate which yields a very long image arc. We can then list the Markov rectangles it crosses. These yield subarcs which represent blocks. We then take higher and higher iterates and pass to the limit.

In constructing a taf, Fact 7.3 above says that it suffices to construct a local taf, and for this, it is only necessary to assign values to arcs $\Gamma \in \mathcal{I}$ which are contained in the unstable edges of Markov rectangles. Thus if we let $\mathcal{I}^u$ be the collection of arcs $\Gamma$ contained in unstable leaves, then a function $G : \mathcal{I}^u \to \mathbb{F}$ which satisfies Definition 7.1(a) and (b) will yield a unique taf.

We need a few more definitions before the main result. An arc $\Gamma \in \mathcal{I}^u$ is called a *loose bit* if $\Gamma$ is contained wholly inside a single Markov rectangle, but doesn't go all the way across. In other words, $\Gamma$ perhaps hits one of the unstable edges, but not both. For an arc $\Gamma \in \mathcal{I}^u$, its *loose bits* are the portions at each end of the arc that are loose bits. Formally, the loose bits of $\Gamma$ are the connected components of some $\Gamma \cap R_j$ which are loose bits. The *main bit* of $\Gamma$ is $\Gamma$ minus its loose bits, or equivalently, the part of $\Gamma$ that goes all the way across Markov rectangles. See Figure.




By Remark 7.4, we know a taf has no atoms. By Fact 8.5(c), a staf has no atoms only when it is unstable, i.e. has an exponential bound with $r > 1$. Equivalently, the corresponding thread's initial vector $\mathbf{K}^{(0)}$ should be in $Un(A, \mathbb{F}^d)$. We call such stafs *unstable* staf and denote the collection of them as $\mathcal{STAF}^u$.

**Theorem 9.1** *Let $\phi$ be a pseudo-Anosov map with a Markov partition giving rise to the $\{0,1\}$-transition matrix $A$. The collection of unstable symbolic transverse arc functions to the one-sided shift $\Lambda_A^+$ is naturally identified with the collection of all transverse arc functions to the foliation $\mathcal{F}^s$. The identification is a vector space isomorphism which makes the following* *diagram commute:*

$$\begin{array}{ccc} \mathcal{TAF} & \xrightarrow{\cong} & \mathcal{STAF}^u \\ {\scriptstyle\phi^*}\big\downarrow & & {\scriptstyle\sigma^*}\big\downarrow \\ \mathcal{TAF} & \xrightarrow{\cong} & \mathcal{STAF}^u. \end{array} \tag{9.1}$$

*As a consequence, under the identification eigen-staf and generalized eigen-staf with eigenvalues $|\mu| > 1$ correspond to the eigen-taf and generalized eigen-taf.*

**Proof:** Throughout the proof it will be convenient to identify a staf $K$ with its thread $\mathbf{K}$ as justified by Fact 8.2. We will frequently do this without further comment.

We first consider the map $\mathcal{STAF}^u \to \mathcal{TAF}$. As noted above, it suffices to define a taf $G$ on arcs from $\mathcal{I}^u$. Let $E^u$ denote the union of the unstable edges of Markov rectangles. For a $\Gamma \in \mathcal{I}^u$ with $\Gamma \cap E^u = \emptyset$, define $I(\Gamma) \in \mathbb{N}^d$ so that its $j^{th}$ component is the number of complete crossings of $\Gamma$ with the rectangle $R_j$ (so we don't count loose bits), and let $w(\Gamma) \in \mathbb{N}^d$ be

$$w(\Gamma) = I(\phi(\Gamma)) - A^T I(\Gamma). \tag{9.2}$$

Thus $w$ counts the crossings of $\phi(\Gamma)$ which are missed by the symbolic count, $A^T I(\Gamma)$. This is the same as the full crossings contributed by the image of the loose bits of $\Gamma$, or more precisely, if $\ell_1$ and $\ell_2$ are the loose bits of $\Gamma$, then $w(\Gamma) = I(\phi(\ell_1)) + I(\phi(\ell_2))$. Thus there is a constant $c_1 > 0$ with

$$\|w(\Gamma)\| \leq 2 \sup\{\|I(\phi(\ell))\| : \ell \text{ is a loose bit}\} < c_1 < \infty. \tag{9.3}$$

Now by the definition of a Markov partition (see, for example, [FLP91], page 201), $\phi^{-1}(E^u) \subset E^u$. This implies that any $\Gamma$ with $\Gamma \cap E^u = \emptyset$, satisfies $\phi^n(\Gamma) \cap E^u = \emptyset$, for all $n \geq 0$. Call these $\Gamma$ *well-coded*. For a well-coded $\Gamma$ and for $n \in \mathbb{N}$, let

$$G_n(\Gamma) = \langle \mathbf{K}^{(n)}, I(\phi^n(\Gamma))\rangle, \tag{9.4}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{F}^d$. By induction from (9.2),

$$I(\phi^n(\Gamma)) = (A^T)^n I(\Gamma) + \sum_{i=1}^{n} (A^T)^{n-i} w(\phi^{i-1}(\Gamma)), \tag{9.5}$$

and so

$$\begin{aligned} G_n(\Gamma) &= \langle \mathbf{K}^{(n)}, (A^T)^n I(\Gamma)\rangle + \sum_{i=1}^{n} \langle \mathbf{K}^{(n)}, (A^T)^{n-i} w(\phi^{i-1}(\Gamma))\rangle \\ &= \langle A^n \mathbf{K}^{(n)}, I(\Gamma)\rangle + \sum_{i=1}^{n} \langle A^{n-i} \mathbf{K}^{(n)}, w(\phi^{i-1}(\Gamma))\rangle \\ &= \langle \mathbf{K}^{(0)}, I(\Gamma)\rangle + \sum_{i=1}^{n} \langle \mathbf{K}^{(i)}, w(\phi^{i-1}(\Gamma))\rangle. \end{aligned} \tag{9.6}$$

Thus if $K$ has an exponential bound with constants $c_2$ and $r > 1$, using the Cauchy-Schwartz inequality, $G_n(\Gamma)$ is bounded by $\langle \mathbf{K}^{(0)}, I(\Gamma)\rangle + \sum_{i=1}^{n} c_1 c_2 r^{-n}$, and so $G_n(\Gamma)$ converges as $n \to \infty$ by the Weierstrass $M$-test. Thus we may define

$$G(\Gamma) = \lim_{n\to\infty} G_n(\Gamma). \tag{9.7}$$

It follows easily from the definition that $G$ is internally additive on well-coded arcs. We now show that it is holonomy invariant. Since $K$ has an exponential bound with $r > 1$, this will follow directly from (9.4) after we prove the following claim: If $\Gamma_0$ and $\Gamma_1$ are well-coded and holonomic, then there is a constant $C = C(\Gamma_0, \Gamma_1)$ with

$$\|I(\phi^n(\Gamma_0)) - I(\phi^n(\Gamma_1))\| \leq C, \tag{9.8}$$

for all $n > 0$.

To prove the claim, first assume that $\Gamma_1$ and $\Gamma_2$ are "close together" in the sense that there is a holonomy $H : [0,1] \times [0,1] \to M$ that is bijective. Then $\mathrm{im}(H)$ is a rectangle in the sense of §5.2. In particular, for all $n \in \mathbb{N}$, $\phi^n(\mathrm{im}(H))$ at most contains all the edges from $E^u$. This implies that (9.8) holds for $\Gamma_1$ and $\Gamma_2$ with $C$ equal to twice the number of Markov rectangles, or $C = 2d$. More generally, if $\Gamma_1$ and $\Gamma_2$ are such that their holonomy $H$ is at most $k$-to-one, then by decomposing $H$ into a collection of holonomies of arcs close together, we get that (9.8) holds with $C = 2kd$, completing the proof of the claim.

We next show that $G$ is continuous on well-coded arcs. For this, first note that from (9.6) and the fact that $K$ has an exponential bound with constants $c_2$ and $r$ we have that for any $\Gamma$ and $n > 0$, $|G_n(\Gamma) - G(\Gamma)| \leq c_1 c_2 r^{-n-1}/(1 - r^{-1})$, and thus using (9.4), for any $\Gamma_0$ and $\Gamma_1$ and $n > 0$,

$$|G(\Gamma_0) - G(\Gamma_1)| < 2c_1 c_2 r^{-n-1}/(1 - r^{-1}) + c_2 r^{-n} |I(\phi^n(\Gamma_0)) - I(\phi^n(\Gamma_1))|. \tag{9.9}$$

Since $G$ is holonomy invariant, it suffices to show continuity of $G$ for $\Gamma_0$ and $\Gamma_1$ in the same leaf of $\mathcal{F}^u$. By the continuity of $\phi$, for any $n > 0$ there is a $\delta > 0$ so that $d(\Gamma_0, \Gamma_1) < \delta$ in the Hausdorff topology implies that $|I(\phi^n(\Gamma_0)) - I(\phi^n(\Gamma_1))| < 2$, which coupled with (9.9) gives the continuity of $G$.

Finally, to complete the definition of $G$ on arcs contained in unstable leaves, for $\Gamma' \in E^u$, pick a well-coded arc $\Gamma$ with $\Gamma$ holonomic to $\Gamma'$, and let $G(\Gamma') = G(\Gamma)$. Since we have shown that the value of $G$ is holonomy invariant among well-coded arcs, $G$ is well-defined. The fact that this extension to all of $\mathcal{I}$ satisfies Definition 7.1 follows from its properties on well-coded arcs, completing the definition of the map $\mathcal{STAF}^u \to \mathcal{TAF}$.

Next we consider the map $\mathcal{TAF} \to \mathcal{STAF}^u$. For a given taf $G$ and an allowable block $b$, pick an arc $\Gamma$ that represents $b$ and define $K : \mathcal{B} \to \mathbb{F}$ by $K(b) = G(\Gamma)$. Because $G$ is additive and holonomy invariant, $K$ is additive and coherent, and thus is a staf. Finally, if $K$ was not unstable, then by Fact 8.5(c) there would be a sequence of allowable blocks $b_n$ with the sets $[b_n]$ converging to a point in the Hausdorff topology but $K(b_n) \not\to 0$. This would imply the existence of a nested collection of arcs $\Gamma_n \in \mathcal{I}^u$ which represent the $b_n$ and the sets $\Gamma_n$ converging to a point in the Hausdorff topology and $G(\Gamma_n) \not\to 0$. This violates the property of tafs given in Remark 7.4, and so $K \in \mathcal{STAF}^u$, as required.

It is clear that both of the maps $\mathcal{STAF}^u \to \mathcal{TAF}$ and $\mathcal{TAF} \to \mathcal{STAF}^u$ are vector space homomorphisms. We next show that they are isomorphisms by showing their compositions are the identity.

For $K \in \mathcal{STAF}^u$ denote the image under $\mathcal{STAF}^u \to \mathcal{TAF}$ as $G_K$. To show that $\mathcal{STAF}^u \to \mathcal{TAF} \to \mathcal{STAF}^u$ is the identity we must show that if $\Gamma$ represents the block $b \in \mathcal{B}$, then $G_K(\Gamma) = K(b)$. Assume that $b = [s_0 \dots a]$ and $\ell(b) = \ell$. This implies that $\phi^\ell(\Gamma)$ is contained in the Markov rectangle $R_a$ and in fact connects its stable edges. This, in turn, implies that $I(\phi^\ell(\Gamma))_a = 1$ for $i = 1$, and is zero otherwise. Further, for all $j \geq \ell$, $I(\phi^{j+1}(\Gamma)) = A^T I(\phi^j(\Gamma))$. Thus for all $n > \ell$, $I(\phi^n(\Gamma)) = (A^T)^{n-\ell} I(\phi_\ell(\Gamma))$, and so from (9.4),

$$\begin{aligned} G_n(\Gamma) &= \langle \mathbf{K}^{(n)}, (A^T)^{n-\ell} I(\phi^\ell(\Gamma))\rangle \\ &= \langle A^{n-\ell}\mathbf{K}^{(n)}, I(\phi^\ell(\Gamma))\rangle \\ &= \langle \mathbf{K}^{(\ell)}, I(\phi^\ell(\Gamma))\rangle \\ &= (\mathbf{K}^{(\ell)})_a. \end{aligned}$$

Finally, since $b$ terminates in $a$ and has length $\ell$, $K(b) = (\mathbf{K}^{(\ell)})_a = G_n(\Gamma) \to G_K(\Gamma)$.

For $G \in \mathcal{TAF}$ denote the corresponding element under $\mathcal{TAF} \to \mathcal{STAF}^u$ as $K_G$. To show that the composition $\mathcal{TAF} \to \mathcal{STAF}^u \to \mathcal{TAF}$ is the identity, using the result of the last paragraph we have that $G_{K_G}(\Gamma) = G(\Gamma)$ for any arc $\Gamma$ that represents a block $b$. It suffices then to prove the claim that a taf is uniquely determined by its value on arcs which represent allowable blocks. Since a taf is continuous by definition, this will follow once we show that finite unions of arcs which represent allowable blocks are dense in $\mathcal{I}^u$. Given an arc $\Gamma \in \mathcal{I}^u$, for any $n > 0$, if $J_i^{(n)}$ is a component of $\phi^n(\Gamma) - E^s$, then define $\Gamma_i^{(n)} := Cl(\phi^{-n}(J_i^{(n)})) \subset \Gamma$, and we have that $\Gamma_i^{(n)}$ represents a block in $\Lambda_A^+$. Since $\Gamma$ is contained in an unstable leaf, $\phi$ is uniformly expanding on $\Gamma$, and so $\cup_i \Gamma_i^{(n)} \to \Gamma$ in the Hausdorff topology as $n \to \infty$, proving the claim.

Finally, we show that the diagram (9.1) commutes. Given a $K \in \mathcal{STAF}^u$ with thread $\mathbf{K}$, since by definition $\phi^* G_K(\Gamma) = G_K(\phi(\Gamma))$ we have that

$$\begin{aligned} \phi^* G_K(\Gamma) &= \lim_{n\to\infty} \langle \mathbf{K}^{(n)}, I(\phi^{n+1}(\Gamma))\rangle \\ &= \lim_{n\to\infty} \langle A\mathbf{K}^{(n+1)}, I(\phi^{n+1}(\Gamma))\rangle. \end{aligned}$$

On the other hand, since $A^*\mathbf{K} = \{A\mathbf{K}_0, A\mathbf{K}_1, \dots\}$, using Fact 8.4 we have

$$G_{\sigma^* K}(\Gamma) = \lim_{n\to\infty} \langle A\mathbf{K}^{(n)}, I(\phi^n(\Gamma))\rangle,$$

and so $\phi^* G_K = G_{\sigma^* K}$. ■

**Remark 9.2** One of the standard construction of the transverse measure $m^u$ to the stable foliation $\mathcal{F}^s$ is essentially as the Peron-Fröbenius eigen-taf $G_\lambda$ which as just shown corresponds to the Peron-Fröbenius eigen-staf $K_\lambda$. In Theorem 11.3 we indicate the standard proof that $G_\lambda$ extends to a measure. We shall need this fact before the proof and note that its proof is in fact independent of its use (and in any event is standard).

In what follows it will be convenient to adopt the convention that the transverse measure $m^u$ is adapted to the transition matrix $A$ in the sense that it is constructed from a Peron-Fröbenius eigenvector $\vec{v}_\lambda$ which have been is normalized so that $\|\vec{v}_\lambda\|_1 = 1$. This implies, in particular, that the Markov rectangle $R_j$ has $m^u$-width equal to the $j^{th}$ component of $\vec{v}_\lambda$.

## 10 Putting it all together

For the reader's convenience we summarize the results of the last few sections concerning structures associated with the various spectra of pseudo-Anosov maps.

**Theorem 10.1** *Let $\phi$ be an orientation-preserving pseudo-Anosov map on an orientable surface with a Markov partition giving rise to the $\{0,1\}$-transition matrix $A$.*

*(a) The collection of symbolic transverse arc functions on $\Lambda_A^+$ is naturally identified with the non-nilpotent subspace of $A$, and under this identification eigen-staf and generalized eigen-staf correspond to eigenvectors and generalized eigenvectors of $A$.*

*(b) The collection of unstable staf is naturally identified with the collection of transverse arc functions to the foliation $\mathcal{F}^s$, and under this identification eigen-staf and generalized eigen-staf with eigenvalues $|\mu| > 1$ correspond to the eigen- and generalized eigen-transverse arc functions.*

*(c) When the foliation $\mathcal{F}^s$ is orientable, each of the eigen- and generalized eigen-objects in (b) is also identified with an eigen- and generalized eigen-transverse cocycle.*

*Diagrammatically when $\mathcal{F}^s$ is orientable we have*

$$\begin{array}{ccccccccc}
Un(\phi^*, H^1(M;\mathbb{F})) & \xrightarrow{\cong} & \mathcal{TC}(\mathcal{F}^s) & \xrightarrow{\cong} & \mathcal{TAF} & \xrightarrow{\cong} & \mathcal{STAF}^u & \xrightarrow{\cong} & Un(A, \mathbb{F}^d) \\
\phi^* \big\downarrow & & \phi^* \big\downarrow & & \phi^* \big\downarrow & & \big\downarrow \sigma^* & & \big\downarrow A \\
Un(\phi^*, H^1(M;\mathbb{F})) & \xrightarrow{\cong} & \mathcal{TC}(\mathcal{F}^s) & \xrightarrow{\cong} & \mathcal{TAF} & \xrightarrow{\cong} & \mathcal{STAF}^u & \xrightarrow{\cong} & Un(A, \mathbb{F}^d)
\end{array}$$

*When $\mathcal{F}^s$ is not orientable the isomorphism $\mathcal{TC}(\mathcal{F}^s) \cong \mathcal{TAF}$ does not exist, but the left portion and the right portion of the diagram still commutes.*

*Using the inverse $\phi^{-1}$ one has analogous results connecting staf of $A^T$ to structures transverse to the unstable foliation $\mathcal{F}^u$.*

**Remark 10.2** Theorem 7.5 implies the main portion of a theorem of Rykken [Ryk99], namely, in the case of orientable foliations the hyperbolic portions of the homological and Markov spectra of $\phi_*$ agree.

## 11 Transverse arc distributions

It is well-known that pseudo-Anosov foliations are uniquely ergodic, i.e. up to scalar multiple they have a unique holonomy invariant transverse measure, and this measure is usually built from the Peron-Fröbenius eigenvector of the transition matrix. Thus the corresponding tafs are the only ones that extend to measures. In light of the correspondence of tafs and stafs given in Theorem 9.1 and the result in Theorem 8.6(b), it is not surprising that all tafs yield distributions in the sense of an elements of the dual to a space of Hölder functions. These results could be develped directly, but given the symbolic results above it is technically easier to use the semi-conjugacy to "push them down" to the manifold. We first review the aspects of the semiconjugacy that are of relevance here.

Fix a Markov rectangle $R_j$ and a well-coded (in the sense given in the proof of Theorem 9.1) arc $\Gamma \in \mathcal{I}^u$ contained in $R_j$ which connects the two stable edges $E^s_{j,1}$ and $E^s_{j,2}$. Since $\phi$ uniformly expands $\Gamma$ in forward time, for each $\underline{s} \in [j] \subset \Gamma^+_A$ there is a unique point $x \in \Gamma$ with $\phi^n(x) \in R_{s_n}$ for all $n > 0$. Thus we maybe define $\omega : [j] \to \Gamma$ by the assignment of $\underline{s}$ to its corresponding $x$. The definition of a Markov partition and its corresponding subshift ensure that $\omega$ is onto. Now if for some $n > 0$, the forward iterate $\phi^n(x)$ intersects a stable boundary of a Markov rectangle, then there will be exactly two sequences $\underline{s}, \underline{s}'$ with $\omega(\underline{s}) = \omega(\underline{s}') = x$. Those $x$ whose forward orbits miss the unstable boundaries have a unique $\omega$-preimage.

It follows from the construction of $\omega$ that for any cylinder $[b] \subset [j]$, its image $\gamma_b := \omega([b]) \subset \Gamma$ is a compact arc representing $[b]$ in the sense given above Theorem 9.1. We call the collection of all such arcs $\mathcal{C}$, explicitly,

$$\mathcal{C} = \{\gamma \subset \Gamma : \gamma = \omega([b]) \text{ for some } [b] \subset [j]\}. \tag{11.1}$$

The collection of endpoints of arcs in $\mathcal{C}$ is exactly the set of $x \in \Gamma$ with non-unique $\omega$-preimage.

Given a staf $K$, we may define its push-forward onto $\mathcal{C}$ as $\omega_* K : \mathcal{C} \to \mathbb{F}$ via $\omega_* K(\gamma_b) = K(b)$. The proof of Theorem 9.1 says that $\omega_* K$ always extends to a taf on $\mathcal{F}^s$, and all taf are so obtained. For this reason if a staf $K$ corresponds to a taf $G$ in the isomorphism of Theorem 9.1, we will write $G = \omega_* K$. This is a slight abuse of notation because, as just described, endpoints of arcs $\gamma_b \in \mathcal{C}$ are double coded and so $\omega^{-1}(\gamma_b)$ is equal to $[b]$ union a pair of sequences not in $[b]$ whose $\omega$-images are the two endpoints of $\gamma_b$. In general, each closed subarc $\Gamma' \subset \Gamma$ has a preimage $\omega^{-1}(\Gamma')$ which can be written as the countable union of blocks $b_n$ and perhaps one or two additional sequences, with the latter only present when one or both endpoints of $\Gamma'$ is also an endpoint of an arc $\gamma_b \in \mathcal{C}$. When $\omega^{-1}(\Gamma') = \uplus [b_n]$ (with the possible addition of two extra sequences), it follows from the proof of Theorem 9.1 that $\omega_* K(\Gamma') = \sum K(b_n)$.

The connection of the Peron-Fröbenius taf $G_\lambda$ to the transverse measure $m^u$ (*cf.* Remark 9.2) gives that for any subarc $\Gamma' \subset \Gamma$, $G_\lambda(\Gamma')$ is the $d_\phi$-diameter of $\Gamma'$, i.e. $G_\lambda(\Gamma') = m^u(\Gamma')$. Since $K_\lambda$ has an exponential bound of $\lambda$ and $G_\lambda = \omega_*(K_\lambda)$, we have that $\omega$ has an exponential bound of $\lambda$ , and so by §8.3, $\omega$ is Lipschitz considered as a map $(\Lambda^+, d_\lambda) \to (\Gamma, d_\phi)$.

**Remark 11.1** The connection of stafs and tafs via the map $\omega$ make it clear why the stable and center staf on $\Lambda^+$ don't push forward under $\omega$ to taf for $\phi$. Specifically, let $x \in \Gamma$ be such that it is not an endpoint of an arc in $\mathcal{C}$. If $K$ is not unstable then any collection $\gamma_{b_n} \in \mathcal{B}$ with $x = \cap \gamma_{b_n}$ will have $|K(b_n)| \not\to 0$. Further, let $\Gamma' \subset \Gamma$ be a closed arc such that at least one of its endpoints is not also the endpoint of an arc from $\mathcal{C}$. Again, if $K$ is not unstable then although we may write $\Gamma' = \cup \gamma_{b_n}$ with the blocks $b_n$ disjoint, the $\sum K(b_n)$ won't converge for the simple reason that $|K(b_n)| \not\to 0$. On the other hand, given a unstable staf $K$, the exponential bound $|K(b_n)| \leq r^{-\ell(b_n)}$ for $r > 1$ will imply that the sum $\sum K(b_n)$ will converge if, for example, there is a uniform bound on the number of blocks of each length. This is the case when $w(\cup[b_n])$ is an arc, but not the case in the construction of Theorem 11.3(a) below based on that of Theorem 8.6(a).

In constructing measures and distributions from taf we focus attention on the taf restricted to a single arc $\Gamma$ in the unstable foliation. By holonomy invariance and density of leaves for pseudo-Anosov foliations this information will transfer to any other transverse arc. For concreteness and simplicity we fix $\Gamma$ as in the beginning of this section: $\Gamma$ is an arc in an unstable leaf connecting the two stable boundaries of a Markov box $R_j$. We also assume that $\Gamma$ is *not* one of the unstable boundary components.

**Definition 11.2 (Hölder transverse arc function)** *A taf $G$ is $(C, \nu)$-Hölder if*

$$|G(J)| \leq C(m^u(J))^\nu, \tag{11.2}$$

*for all subarcs $J \subset \Gamma$.*

The taf $G$ extends to a signed Borel measure if there is such measure which agrees with $G$ on all compact subarcs of $\Gamma$.

**Theorem 11.3** *Let $\phi$ be a pseudo-Anosov map with a Markov partition $\{R_i\}$ with associated subshift of finite type $\Lambda_A^+$. For a length one cylinder set $[j]$, let $\omega : [j] \to \Gamma$ be the coding map defined above where $\Gamma$ is an arc in an unstable leaf connecting the stable boundaries of the Markov rectangle $R_j$.*

*(a) If $G$ is a taf with $G = \omega_* K$ for a staf $K$ with an $r$-exponential bound with $1 < r \leq \lambda$, then $G$ is $(\log r)/(\log \lambda)$-Hölder*

*(b) The only taf which has bounded variation and thus extends to a signed Borel measure on transversals to $\mathcal{F}^s$ are the Peron-Fröbenius taf.*

*(c) If the taf $G$ is $\nu_2$-Hölder, it defines a continuous, linear functional on $C^{\nu_1}(\Gamma, \mathbb{F})$ for all $\nu_1 > 1 - \nu_2$.*

**Proof:** We use the notation in the proof of Theorem 9.1 . To prove (a), for the given subarc $J \subset \Gamma$ let $N$ be the smallest integer so that a portion of $\phi^N(J)$ goes all the way across some Markov box. The smallest $m^u$-width of any Markov box is the smallest component (which is always nonzero) of the normalized Peron-Fröbenius eigenvector (*cf.* Remark 9.2), and we call this smallest value $c_4$. We then have that $\lambda^N m^u(J) = m^u(\phi^N(J))) \geq c_4$. Now also note that by the choice of $N$, $w(\phi^n(J) = 0$ for all $n < N$. Thus since $I(J) = 0$, by (9.6) we have using Cauchy-Schwarz with $c_1$ and $c_2$ as defined in the proof of Theorem 9.1 and $\nu := (\log r)/(\log \lambda)$,

$$\begin{aligned} G(J) &= \sum_{i=N}^{\infty} \langle \mathbf{K}^{(i)}, w(\phi^{i-1}(\Gamma)) \rangle \leq \sum_{i=N}^{\infty} c_1 c_2 r^{-i} \\ &= r^{-N} c_1 c_2 (1 - r^{-1})^{-1} = (\lambda^{-N})^\nu c_1 c_2 (1 - r^{-1})^{-1} \\ &\leq (m^u(J))^\nu \frac{c_1 c_2}{c_4^\nu (1 - r^{-1})}, \end{aligned}$$

as required.

For part (b), using the sets $P_j^{(n)}$ in the proof of Theorem 8.6, for each $n > 0$ and $j = 1, \ldots, d$, let $\hat{P}_j^{(n)} = \omega(P_j^{(n)})$. Thus for each $n$, $\Gamma = \cup_j \hat{P}_j^{(n)}$. By Theorem 9.1, if $G$ is not a Peron-Fröbenius taf, then $G = \omega_* K$ for $K$ not a Peron-Fröbenius staf. Thus using (8.13), $\sum_j |G(\hat{P}_j^{(n)})| = \sum_j |K(P_j^{(n)})| \to \infty$ as $n \to \infty$. Now $G$ has the property that when a family of intervals $J_k$ converge down to a point (cf. Remark 7.4), then $G(J_k) \to 0$, and thus any extension measure can have no atoms. For each $n$, when $j \neq j'$, $\hat{P}_j^{(n)} \cap \hat{P}_{j'}^{(n)}$ is a finite set of points, and so we see that any countably additive extension of $G$ to the Borels would not be of bounded variation and thus not a measure.

For part (c), given $f : (\Gamma, d_\theta) \to \mathbb{F}$ that is $\nu_1$-Hölder, since as noted above, $\omega : (\Lambda^+, d_\lambda) \to (\Gamma, d_\phi)$ is Lipschitz, $f \circ \omega : (\Lambda_+, d_\lambda) \to \mathbb{F}$ is also $\nu_1$-Hölder. It then follows from the statement at the end of §8.3 that $f$ has an $r_1 := \lambda^{\nu_1}$ exponential decay bound. Now let $K$ be the staf with $G = \omega_* K$, then by part (a), $K$ has an $r_2 := \lambda^{\nu_2}$ exponential decay bound. Thus $\nu_1 + \nu_2 > 1$ implies $\lambda/(r_1 r_2) < 1$, and so we may define $L_G : C^{\nu_1}(\Gamma, \mathbb{F}) \to \mathbb{F}$ by $L_G(f) = L_K(f \circ \omega)$ with $L_K$ as in Theorem 8.6. It follows immediately that $L_G$ in linear and its continuity follows from that of $L_K$ in conjunction with the fact that $\omega$ is Lipschitz and the equivalence of the Hölder and exponential bound norms given at the end of §8.3. ∎

## 12 Regularity

In this section we investigate the regularity of the various transverse structures for pseudo-Anosov maps. Notions of regularity are easiest to formulate with $c$-maps. As noted in §6, an eigen-$c$-map $\tilde{\alpha}$ with factor $|\mu| > 1$ is constant on leaves of the lifted stable foliation $\tilde{\mathcal{F}}^s$ in $\tilde{M}$. Since the lifted unstable foliation $\tilde{\mathcal{F}}^u$ is transverse to $\tilde{\mathcal{F}}^s$, the regularity of $\tilde{\alpha}$ restricted to a leaf $\tilde{L}$ of $\tilde{\mathcal{F}}^u$ indicates the regularity of $\tilde{\alpha}$ as well as that of its associated path cocycle. Thus by parameterizing unstable leaves the analysis is simplified to studying the regularity of a map $f : \mathbb{R} \to \mathbb{R}$.

We shall also study the regularity of tafs and transverse cocycles using their restriction to unstable leaves. To be definite, fix an arc $\Gamma$ as in Theorem 11.3 which is a connected arc of an unstable leaf intersected with a Markov box. We parameterize $\Gamma$ by its $m^u$-arclength and identify points with their parameterization in $[0, a]$, where $a = m^u(\Gamma)$. Given a taf $G$, define the "cumulative distribution function" $H : [0, a] \to \mathbb{F}$ via $H(x) = G([0, x])$. Using the additivity of a taf, we can reconstruct $G$ from $H$ via the formula $G([c, d]) = H(d) - H(c)$ for $0 \leq c \leq d \leq a$. A similar construction may be done for a transverse cocycle. Since after lifting to the orientation double cover taf's and path cocycles correspond by Proposition 7.5, and this lifting does not change transverse structures for small intervals, we see that the regularity of taf and cocycle are the same. In addition, using the correspondence of $c$-maps and cocycles from §4.5, if we lift $\Gamma$ to $\tilde{\Gamma}$ the universal Abelian cover $\tilde{M}$ and for a path cocycle $F$ we define $\tilde{H}$ there analogously to $H$, we have that for $\tilde{x} \in \tilde{\Gamma}$ (again identifying a point in $\Gamma$ with its parameterization), $\tilde{H}(x) = \tilde{\alpha}(x) - \tilde{\alpha}(0)$. Thus taf, transverse cocycles and $c$-maps all have the same local regularity properties.

There are certain facts about regularity which follow from what we have done thus far. Firstly, we know from Theorem 11.3(b), that the only taf with bounded variation are the

Peron-Fröbenius ones. Thus we would expect that the corresponding non-Peron-Fröbenius $c$-maps would also not be BV, which we show below in Theorem 12.6.

We could also proceed more directly using the standard results from elementary measure theory which connect cumulative distribution functions like $H$ on intervals to measures. The result of relevance is that a set function $G$ defined on subarcs extends to a signed measure if and only if the corresponding cumulative distribution function $H$ has bounded variation. Thus from the unique ergodicity of the stable foliation we know immediately that $c$-maps and tafs are not BV.

We shall reprove this result below as part of the larger investigation into the properties of eigen-$c$-maps. It is also worth noting that once we know a taf on a small subarc in any unstable leaf, we essentially know it everywhere. This is a consequence of holonomy invariance coupled with the fact that every leaf of $\mathcal{F}^s$ is dense in the surface.

The point of view which motivates the next few sections is that the irregularity of eigen-$c$-maps associated with non-Peron-Fröbenius eigenvalues is the consequence of the scaling properties of the functions $f$ which are the $c$-maps restricted to unstable leaves. Specifically, in $\tilde{M}$ under the action of $\tilde{\phi}$ the parameterization of a leaf $\tilde{L}$ by arc length transforms by multiplication by $\lambda$, the dilation of the pseudo-Anosov map. Thus an eigen-$c$-map for eigenvalues $1 < |\mu| < \lambda$ yield functions $f$ with the scaling $f(\lambda t) = \mu f(t)$ everywhere. This implies using Lemma 12.3(c) that the associated eigen-$c$-map is Hölder but nowhere differentiable and not locally of bounded variation. Such scalings are an example of a more general principle which roughly states that semi-conjugacies from higher entropy to lower entropy systems must have fractal-like structure and thus the resulting low regularity. This is a basic principle which has many applications so it is worth pursuing in the fairly well understood situation considered here.

Many of the main ideas in this section were adapted from Fathi [Fat88].

### 12.1 Steep functions

As just described, the functions $\mathbb{R} \to \mathbb{R}$ which are the restrictions of an eigen-$c$-map to an unstable leaf will have the property that the action of $\tilde{\phi}$ rescales the parameterization by the dilation while rescaling the image by the eigenvalue $\mu$, or $f(\lambda t) = \mu f(t)$. When $|\mu| < \lambda$ this will imply that $f$ has a property called steepness at 0 and using the density of leaves, $f$ will be steep everywhere. Steepness is a kind of "anti-Hölder" condition.

**Definition 12.1 (Steep function)** *A continuous function $f : (a, b) \to \mathbb{C}$ is said to be $(C, \nu)$-steep from the right in the neighborhood $(c, d)$ at the point $p \in (c, d) \subset (a, b)$ if for all $p' \in (p, d)$,*

$$\max \{|f(t) - f(p)| : 0 < t - p \leq p' - p\} > C(p' - p)^{\nu}. \tag{12.1}$$

*The notion of $(C, \nu)$-steep from the left is defined using the obvious alterations. The function is called $(C, \nu)$-steep from both sides if it is steep from the left and right, and simply $(C, \nu)$-steep if is steep from the left or right for some neighborhood.*

The proof of next lemma is routine and we omit it.

**Lemma 12.2** *Assume that $f_n : (a, b) \to \mathbb{C}$ and $f_n \to f_0$ uniformly.*

*(a) If every $f_n$ is $(C,\nu)$-steep at $p \in (a,b)$ for the neighborhood $(c,d)$, then $f_0$ is also.*

*(b) If $f_0$ is $(C,\nu)$-steep at $p$ for the neighborhood $(c,d)$, then for each $\epsilon$ with $0 < \epsilon < C$ there is an $N$ so that $n > N$ implies that each $f_n$ is $(C-\epsilon,\nu)$-steep at $p$ for the neighborhood $(c-\epsilon, d-\epsilon)$.*

The next lemma shows that the scaling property of $f$ implies steepness as well as giving some of the consequences of being everywhere steep.

**Lemma 12.3** *Assume that $f : \mathbb{R} \to \mathbb{C}$ is continuous, and $\mu \in \mathbb{C}$, $\lambda \in \mathbb{R}$ satisfy $1 < |\mu| < \lambda$. Let $\nu = \log(|\mu|)/\log(\lambda)$.*

*(a) If $f$ satisfies*

$$f(\lambda t) = \mu f(t) \text{ or } f(-\lambda t) = \mu f(t) \tag{12.2}$$

*for all $t \in \mathbb{R}$ and $f$ is not identically zero, then there exists a $C > 0$ so that $f$ is $(C,\nu)$-steep at zero in both directions for all neighborhoods of zero.*

*(b) If $f$ is $(C,\nu)$-steep at zero for the neighborhood $(c,d)$, then for all $k \in \mathbb{Z}$ and $r \in \mathbb{R}$,*

$$\frac{f(\lambda^k t)}{\mu^k} + r \text{ and } \frac{f(-\lambda^k t)}{\mu^k} + r, \tag{12.3}$$

*are $(C,\nu)$-steep at zero for the neighborhoods $(\lambda^{-k}c, \lambda^{-k}d)$ and $(-\lambda^{-k}d, -\lambda^{-k}c)$, respectively.*

*(c) If $f$ is $(C,\nu)$-steep at every point $p \in (a,b)$, then on $(a,b)$ $f$ is nowhere differentiable, nowhere locally of bounded variation, nowhere locally injective and not Hölder for any exponent larger than $\nu$. Further, if $f$ is real-valued, there exists a dense, $G_\delta$-set $Z \subset f(a,b)$ so that $z \in Z$ implies that $f^{-1}(z)$ is a Cantor set, and if $f$ is complex-valued there exists a dense, $G_\delta$-set $R$ contained in the interval between the infimum and supremum of $|f(t)|$ for $t \in (a,b)$ so that $r \in R$ implies that the image $f(a,b)$ intersects the circle $|z| = r$ in a Cantor set.*

**Proof:** To prove (a), assume first that $f(\lambda t) = \mu f(t)$ which implies that $f(0) = 0$. On $(0,\infty)$ we may define the continuous function $g(t) := |f(t)|/t^\nu$ which then satisfies $g(\lambda t) = g(t)$. Since $f$ is not identically zero, $m := \max\{|g(t)| : t \in (1,\lambda]\} > 0$, and we let $t_0 \in (1,\lambda]$ be such that $|g(t_0)| = m$. Now given $p' > 0$, let $k$ be such that $\lambda^k t_0 \le p' < \lambda^{k+1} t_0$. Letting $C = m/|\mu|$, we have $|f(\lambda^k t_0)| = |\mu|^k t_0^\nu m = C(\lambda^{k+1}t_0)^\nu > C(p')^\nu$. This shows steepness to the right at zero; the proof of steepness to the left is similar. If $f$ is complex-valued, apply the same argument to $|f(t)|$.

Now if we assume that $f(-\lambda t) = \mu f(t)$, then $f(\lambda^2 t) = \mu^2 f(t)$, and since $\nu = \log(\mu^2)/\log(\lambda^2)$, the result follows by what we have just proved, completing (a).

The proof of (b) is easy. To prove (c), since $0 < \nu < 1$, the definition (12.1) implies that at any point $p \in \mathbb{R}$,

$$\limsup_{t \to p} \frac{|f(t) - f(p)|}{|t-p|} > \limsup_{t \to p} C|t-p|^{\nu-1} = \infty, \tag{12.4}$$

and so $f$ is nowhere differentiable. Further, (12.1) also implies that on any interval $[a,b]$ on which $f$ is $(C,\nu)$-steep at every point, the variation of $f$ on $[a,b]$ satisfies

$$Var(f;[a,b]) > C|b-a|^\nu.$$

Given any interval $[c,d]$ and an $n \in \mathbb{N}$, we define the subintervals

$$I_{i,n} = [c + i(d-c)/n, c + (i+1)(d-c)/n],$$

for $i = 0, 1, \dots n-1$. The variation of $f$ on $[c,d]$ satisfies

$$Var(f;[c,d]) \geq \sum_{i=1}^{n} Var(f; I_{i,n}) \geq Cn(\frac{d-c}{n})^\nu \to \infty,$$

as $n \to \infty$, showing that $f$ is nowhere locally BV.

Assume that $f$ is real-valued. Since $f$ is nowhere locally BV, it is certainly nowhere locally constant and since we are in dimension one this implies $f$ is light (i.e. point inverses are totally disconnected). It is also clearly nowhere locally injective, and so the assertion about the typical point inverse follows from a theorem of Blokh, et al [BOT06]. The assertion about complex-valued $f$ follow by applying the argument just given to $|f|$.

Finally, to prove the result about the Hölder exponent, first note that if $\nu' > \nu$ and for a given $[a,b]$ if the supremum in (12.1) is achieved at $t_0$, then

$$|f(t_0) - f(a)| > C|b-a|^\nu \geq C|t_0 - a|^\nu > C|t_0 - a|^{\nu'}.$$

Thus if $f$ were Hölder with exponent $\nu' > \nu$, the corresponding constant must be $C' > C$. But then chose an interval $[c,d]$ with

$$|d-c| < \left(\frac{C}{C'}\right)^{\frac{1}{\nu' - \nu}},$$

and then if the supremum on $[c,d]$ in (12.1) is achieved at $t_1$,

$$|f(t_1) - f(c)| > C|d-c|^\nu > C'|t_1 - c|^{\nu'},$$

a contradiction to the assumption that $f$ is $(C', \nu')$-Hölder. ■

### 12.2 Regularity of eigen-$c$-maps for pseudo-Anosov homeomorphisms

We now apply the results of the previous section to study the regularity of $c$-maps restricted to leaves of $\tilde{\mathcal{F}}^u$.

Recall from §5.1 that $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$ are the lifts of $\phi$'s foliations to the universal Abelian $\tilde{M}$. The leaves containing a point $\tilde{x} \in \tilde{M}$ are denoted $\tilde{L}^s(\tilde{x})$ and $\tilde{L}^u(\tilde{x})$. Leaves "terminating" in a singularities are called half-infinite. Further, the metric $\tilde{d}_\phi$ is derived from the transverse measures. In particular, arc length along an unstable leaf is induced by the measure $m^u$.

Now fix once and for all an orientation on all the leaves of $\tilde{\mathcal{F}}^s$. There is no assumption that these chosen orientations fit together in any kind of coherent fashion. For $\tilde{x}$ that is not a singularity of the lifted foliations, if $\tilde{L}^u(\tilde{x})$ is a half-infinite leaf, let $a(\tilde{x})$ be the $m^u$-distance in $\tilde{L}^u(\tilde{x})$ from $\tilde{x}$ to the singularity, and if $\tilde{L}^u(\tilde{x})$ is a regular leaf, let $a(\tilde{x}) = \infty$. Now define $s_{\tilde{x}} : (-a(\tilde{x}), \infty) \to \tilde{L}(\tilde{x})$ as the parameterization by $m^u$-arclength which agrees with the chosen orientation on $\tilde{L}^u(\tilde{x})$, and further, $s_{\tilde{x}}(0) = \tilde{x}$. Since arc length on unstable leaves is the measure $m^u$, we have

$$\tilde{\phi} \circ s_{\tilde{x}}(t) = s_{\tilde{\phi}(\tilde{x})}(\epsilon \lambda t), \tag{12.5}$$

for all $t \in \mathbb{R}$ and $\tilde{x} \in \tilde{M}$, with $\epsilon = 1$, if $\tilde{\phi}$ preserves the chosen orientations from $\tilde{L}^u(\tilde{x})$ to $\tilde{L}^u(\tilde{\phi}(\tilde{x}))$, and $\epsilon = -1$, if $\tilde{\phi}$ reverses these orientations. Also note that

$$s_{\delta_{\vec{n}}\tilde{x}} = \delta_{\vec{n}} s_{\tilde{x}}, \tag{12.6}$$

and if $\tilde{x}$ and $\tilde{x}'$ are on the same leaf and $\tilde{x}' = s_{\tilde{x}}(t_0)$, then

$$s_{\tilde{x}}(t) = s_{\tilde{x}'}(t - t_0). \tag{12.7}$$

**Lemma 12.4** *Assume $\phi : M \to M$ is a pseudo-Anosov map and $\mu$ is an unstable eigenvalue of $\phi^*$ on $H^1(M;\mathbb{Z})$. Fix a lift $\tilde{\phi}$ of $\phi$ to $\tilde{M}$ and let $\tilde{\alpha} : \tilde{M} \to \mathbb{F}$ be the eigen-c-map given by Theorem 3.3. For each $\tilde{x} \in \tilde{M}$ which is not a singularity, let $f_{\tilde{x}} : (-a(\tilde{x}), \infty) \to \mathbb{F}$ be given by $f_{\tilde{x}} = \tilde{\alpha} \circ s_{\tilde{x}}$ with $s_{\tilde{x}}$ defined as above. Then*

*(a) $f_{\delta_{\vec{n}}\tilde{x}} = f_{\tilde{x}} + \Phi(\vec{n})$ where $\Phi : \mathbb{Z}^d \to \mathbb{F}$ represents the eigen-cohomology class of $\mu$.*

*(b) $f_{\tilde{\phi}(\tilde{x})}(t) = \mu f_{\tilde{x}}(\epsilon t/\lambda)$ with $\epsilon = 1$, if $\tilde{\phi}$ preserves the chosen orientations from $\tilde{L}^u(\tilde{x})$ to $\tilde{L}^u(\tilde{\phi}(\tilde{x}))$, and $\epsilon = -1$, if $\tilde{\phi}$ reverses these orientations.*

*(c) If $\tilde{x}_j$ are not singular points and $\tilde{x}_j \to \tilde{x}$, then there are $\epsilon_j = \pm\,\mathrm{id}$ and a neighborhood of the origin $(c, d)$ so that $f_{\tilde{x}_j} \circ \epsilon_j \to f_{\tilde{x}}$ uniformly on $(c, d)$.*

**Proof:** Using (12.6) and fact that $\tilde{\alpha}$ is a $c$-map,

$$f_{\delta_{\vec{n}}\tilde{x}}(t) = \tilde{\alpha} \circ s_{\delta_{\vec{n}}\tilde{x}} = \tilde{\alpha} \circ \delta_{\vec{n}} s_{\tilde{x}} = \tilde{\alpha} \circ s_{\tilde{x}} + \Phi(n),$$

proving (a). Since $\tilde{\alpha}$ is a semiconjugacy, (12.5) yields

$$f_{\tilde{\phi}(\tilde{x})}(t) = \tilde{\alpha} \circ s_{\tilde{\phi}(\tilde{x})}(t) = \tilde{\alpha} \circ \tilde{\phi} \circ s_{\tilde{x}}(\epsilon t/\lambda) = \mu \tilde{\alpha} \circ s_{\tilde{x}}(\epsilon t/\lambda),$$

proving (b). Part (c) follows from the fact that as $\tilde{x}_j \to \tilde{x}$ their corresponding leaves converge smoothly. ■

If the eigenvalue $\mu$ is the Peron-Fröbenius eigenvalue $\lambda$, then $\phi$ has oriented foliations and so the eigen-$c$-map is essentially the Peron-Fröbenius eigen-taf. Thus as a consequence of the correspondence of the Peron-Fröbenius taf and the transverse measure $m^u$ (*cf.* Remark 9.2), when $\mu = \lambda$ each function $f$ is a translate of the identity or minus the identity. Thus we henceforth assume that $\mu \neq \lambda$.

**Lemma 12.5** *Assume $\phi : M \to M$ is a pseudo-Anosov map and $\mu$ is an unstable eigenvalue of $\phi^*$ on $H^1(M;\mathbb{Z})$ with $\mu$ not equal to the dilation $\lambda$. Fix a lift $\tilde{\phi}$ of $\phi$ to $\tilde{M}$ and let $\tilde{\alpha} : \tilde{M} \to \mathbb{F}$ be the eigen-c-map given by Theorem 3.3. Let $\nu = \log(|\mu|)/\log(\lambda)$ and for a nonsingular point $\tilde{x} \in \tilde{M}$, let the function $f_{\tilde{x}}$ be defined as in Lemma 12.4. There exists a $C > 0$ so that the maps $f_{\tilde{x}}$ are $(C, \nu)$-steep at every point in their domain.*

**Proof:** We first prove the result under the assumption that $\phi$ has an interior fixed point which is not a singularity and is not on the boundary and satisfies one more property given shortly. The general case will then follow easily.

Let $p \in M$ be the assumed fixed point of $\phi$. Pick a lift $\tilde{p}$ and let $\tilde{\phi}$ be the lift of $\phi$ to $\tilde{M}$ with $\tilde{\phi}(\tilde{p}) = \tilde{p}$. The additional assumption is that $\tilde{\phi}$ preserves the chosen orientation on $\tilde{L}^u(\tilde{x})$. From Lemma 12.4(b) we get $f_{\tilde{p}}(\lambda t) = \mu f_{\tilde{p}}(t)$, and so by Lemma 12.3(a), $f_{\tilde{p}}$ is $(C, \nu)$-steep at zero.

Let $y \in M$ be such that its forward $\phi$-orbit is dense in $M$ and its unstable leaf $L^u(y)$ is not associated with any singularity. Fix a lift $\tilde{y}$ of $y$. There exist $k_j \to \infty$ and $\vec{n}_j \in \mathbb{Z}^d$ with $\tilde{y}_j := \delta_{n_j} \circ \tilde{\phi}^{k_j}(\tilde{y}) \to \tilde{p}$, and so by Lemma 12.4(c), there are $\epsilon_j$ so that $f_{\tilde{y}_j} \circ \epsilon_j \to f_{\tilde{p}}$ uniformly in some neighborhood of zero. Thus by Lemma 12.4(a)(b),

$$f_j(t) := \mu^{k_j} f_{\tilde{y}} \left( \frac{\epsilon_j(t)}{\lambda^{k_j}} \right) + \Phi(n_j) \to f_{\tilde{p}},$$

uniformly in some neighborhood of zero, and so by Lemma 12.2(b), for all $m \in \mathbb{N}$ there exists a $J_m$, so that $j > J_m$ implies that $f_j$ is $(C - (1/m), \nu)$-steep at 0. Thus by Lemma 12.2(a), $f_{\tilde{y}}$ is $(C - (1/m), \nu)$-steep at zero for all $m$ and thus is $(C, \nu)$-steep at zero.

Now for any $\tilde{x}$, we also have $\ell_j \to \infty$ and $\vec{m}_j \in \mathbb{Z}^d$ with $\delta_{m_j} \circ \tilde{\phi}^{\ell_j}(\tilde{y}) \to \tilde{x}$, and so by the same argument as the previous paragraph, $f_{\tilde{x}}$ is $(C, \nu)$-steep at zero. To get that $f_{\tilde{x}}$ is also $(C, \nu)$-steep at other points in its domain, simply observe that by (12.7), $f_{\tilde{x}}(t) = f_{\tilde{x}'}(0)$ for $\tilde{x}' = s_{\tilde{x}}(t)$, finishing the proof under the assumption that $\phi$ has a nonsingular, interior fixed point.

The proof for the case when $\phi$ does not have an interior, nonsingular fixed point follows easily from the following observations. Since the periodic points of $\phi$ are dense in $M$, we can certainly find an interior point $p$ and an $M > 1$ with $\phi^M(p) = p$. We then pick a lift $\tilde{p}$, let $N = 2M$, and note there is a $\vec{n} \in \mathbb{Z}^d$ so that $\delta_{\vec{n}}\, \tilde{\phi}^N(\tilde{p}) = \tilde{p}$. Further, since $N$ is even, $\delta_{\vec{n}}\, \tilde{\phi}^N$ preserves the orientation on $\tilde{L}^u(\tilde{p})$. If $\tilde{\alpha}$ is the $c$-map for $\tilde{\phi}$ with factor $\mu$, then by definition, $\tilde{\alpha}\tilde{\phi} = \mu\tilde{\alpha}$. Letting $\tilde{\alpha}' := \tilde{\alpha} + \Phi_\mu(\vec{n})/(\mu^N - 1)$ and $\tilde{\psi} = \delta_{\vec{n}}\, \tilde{\phi}^N$, we have that $\tilde{\alpha}'\tilde{\psi} = \mu^N \tilde{\alpha}'$, and so by the uniqueness of the eigen-$c$-maps given in Theorem 3.3, we know that $\tilde{\alpha}'$ is the eigen-$c$-map for $\tilde{\psi}$ with factor $\mu^N$. We now apply the above arguments to functions $f'_{\tilde{x}}$ defined using $\tilde{\alpha}'$. First noticing that $\nu = \log(|\mu|)/\log(\lambda) = \log(|\mu^N|)/\log(\lambda^N)$, then noting that $\tilde{\alpha}'$ differs from $\tilde{\alpha}$ by at most a constant, we have the desired results for the original functions $f_{\tilde{x}}$ defined using $\tilde{\alpha}$. ■

Using Lemma 12.5 and Lemma 12.3 we get

**Theorem 12.6** *Assume $\phi : M \to M$ is a pseudo-Anosov map and $\mu$ is an eigenvalue of $\phi^*$ on $H^1(M; \mathbb{Z})$ with $1 < |\mu| < \lambda$, where $\lambda$ is the dilation of $\phi$. Fix a lift $\tilde{\phi}$ of $\phi$ to $\tilde{M}$ and let $\tilde{\alpha} : \tilde{M} \to \mathbb{F}$ be the eigen-c-map given by Theorem 3.3, and let $\nu = \log(|\mu|)/\log(\lambda)$.*

*The c-map $\tilde{\alpha}$ restricted to each leaf of the lifted unstable foliation $\tilde{\mathcal{F}}^u$ is $\nu$-Hölder, nowhere differentiable, nowhere locally of bounded variation, and nowhere locally injective. Further, $\tilde{\alpha}$ is not Hölder for any exponent larger than $\nu$. If $\mu$ is real, the generic point inverse of $\tilde{\alpha}$ restricted to an unstable leaf is a Cantor set. If $\mu$ is complex, then for a dense, $G_\delta$-set of $r$ values in $[0, \infty)$, the image in $\mathbb{C}$ of $\tilde{\alpha}$ restricted to an unstable leaf intersects the circle $|z| = r$ in a Cantor set.*

### 12.3 Example

We give an example due to Gavin Band which illustrates the main results. Let $M$ be a closed, genus-two surface and $\psi : M \to M$ a pseudo-Anosov map such that the characteristic polynomial of $\psi_*$ acting on $H_1(M, \mathbb{Z})$ splits over the integers into a pair of irreducible quadratic factors. Assume that the roots of the first factor are $\lambda_1$ and $\eta_1$ and those of the second are $\lambda_2$ and $\eta_2$. Note that of necessity $\eta_i = \lambda_i^{-1}$. Further, we assume that all roots are real, $\lambda_1 > \lambda_2 > 1$, and $\lambda_1$ is the dilation of $\psi$. These conditions imply that $\psi$ has orientable foliations. It is easy to build examples of this type using Rauzy induction.

Using Theorem 3.3 with $\psi$ and $\psi^{-1}$ we get four semi-conjugacies which we denote $\tilde{\alpha}_{\lambda_i}$ and $\tilde{\alpha}_{\eta_i}$ for $i = 1, 2$. As a consequence of a theorem of Fathi [Fat88], for both $i = 1$ and $i = 2$ the paired semi-conjugacies $(\tilde{\alpha}_{\lambda_i}, \tilde{\alpha}_{\eta_i}) : \tilde{M} \to \mathbb{R}^2$ descend to semiconjugacies $(M, \psi) \to (\mathbb{T}^2, \Phi_i)$, where $\Phi_i$ is a linear toral automomorphims with eigenvalues $\lambda_i$ and $\eta_i$. We call these semiconjugacies $\beta_1$ and $\beta_2$. Note that both semi-conjugacies of necessity take leaves of $\mathcal{F}^u$ and $\mathcal{F}^s$ to leaves of the unstable and stable foliations of the toral automorphisms.

The existence of $\beta_1$ also follows from Franks and Rykken [FR99], who show that $\beta_1$ is always a branched cover with branch points and their images singularities. Thus $\beta_1$ is smooth at all but finitely many points and the preimages ${\beta_1}^{-1}(x)$ are finite sets with a uniformly bounded cardinality.

On the other hand, as a consequence of Theorem 12.6, $\beta_2$ is nowhere differentiable, is nowhere locally BV, is $\nu$-Hölder for $\nu = \log(|\lambda_2|)/\log(\lambda_1)$, not Hölder for exponents greater than $\nu$, and for a dense, $G_\delta$-set $Z \subset \mathbb{T}^2$, $z \in Z$ implies that ${\beta_2}^{-1}(z)$ is a Cantor set.

This example is reminiscent of a theorem from symbolic dynamics which gives a dichotomy for the semiconjugacy $h$ between two transitive subshifts of finite type. The theorem says that the shifts have the same entropy if and only if $h$ is bounded to one and have different entropy if and only if the generic point inverse of $h$ is a Cantor set. This theorem is an easy extension of results in Chapter 4 of [Kit98] and could be applied with some work to the example using the symbolic models of $\psi$, $\Phi_1$ and $\Phi_2$.

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