

PINCHING ON OPEN MANIFOLDS

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ABSTRACT. We show that the 2-jet bundle of local Riemannian metrics on an arbitrary differentiable manifold admits a section which pointwise fulfills the curvature relation $\sec(g) = a$ for any $a \in \mathbb{R}$. It follows by Gromov's h-principle for open, invariant differential relations that every noncompact differentiable manifold carries arbitrarily pinched (incomplete) Riemannian metrics.

1. INTRODUCTION

A basic question in Riemannian geometry is what effect the existence of a Riemannian metric with particular curvature properties has on the topology of the underlying manifold. One usually considers complete metrics and obtains even on an elementary level rather strong restrictions. For instance, complete manifolds of negative sectional curvature are always aspherical and all nontrivial elements of the fundamental group have infinite order. Furthermore a closed manifold does not support two metrics of different signed sectional curvatures. We also mention the result of Gromoll and Meyer [3] that a complete open manifold of positive sectional curvature is diffeomorphic to some \mathbb{R}^m .

The situation changes completely if we skip the completeness assumption, which is automatically imposed on metrics on closed manifolds but is an extra condition on open manifolds. Gromov remarks in his thesis [4] the astonishing fact that every open manifold carries metrics with strict positive and negative sectional curvature.

In this paper we extend this result and prove the existence of arbitrarily pinched (incomplete) metrics on open manifolds.

Theorem 1.1. *Let M be an open manifold. Given $\delta > 0$ and $a \in \mathbb{R}$ there exists a Riemannian metric on M such that all sectional curvatures are in $(a - \delta, a + \delta)$.*

In other words the geometric significance of the sign of the curvature and even of curvature bounds depends on the completeness of the underlying metric. It turns out that the only question concerning curvature which involves geometry in the incomplete case is the existence of metrics of constant sectional curvature, since then topological obstructions are known. For example such manifolds have trivial Pontryagin classes (cf. Theorem 43 and Corollary 44 in [9]).

Theorem 1.1 is an application of a deep differential topological insight of M. Gromov who greatly generalized in his thesis [4] Smale-Hirsch-Phillips immersion-submersion theory (see [8], [6], [7]) by proving what is now called the h-principle for invariant, open differential relations on open manifolds.

Roughly speaking, a partial differential relation \mathcal{R} is any condition imposed on the partial derivatives of an unknown function. By substituting derivatives by new independent variables one gets an underlying algebraic relation. Obviously the existence of a formal solution, i. e. a solution of the corresponding algebraic relation, is a necessary condition for the solvability of \mathcal{R} . It turns out that under certain circumstances any formal solution of \mathcal{R} can be deformed into a genuine one.

In general one considers a smooth fibre bundle over the underlying manifold. A partial differential relation is a condition imposed on the r -jet bundle of local sections. Now we could try and construct a section of this r -jet bundle which pointwise fulfills the relation, i.e. a formal solution, and deform it to get a real solution. In other words, we reduce the problem to algebraic-topological obstruction theory. We refer to [2] and [5] for expositions of this technique.

In our case we study the bundle of symmetric bilinear forms the positive definite sections of which are Riemannian metrics on the underlying manifold. The observation that the sectional curvature only depends on the 2-jet of the metric allows us to translate the pinching problem into a curvature relation, see Definition 3.2. In Lemma 3.5 we will prove that on any manifold there are formal solutions with *constant* curvature. Then, as an application of Gromov's h-principle, Theorem 3.4 implies the existence of arbitrarily pinched metrics.

2. SOME BASIC FACTS CONCERNING JETS

Given a (C^∞ -)smooth fibre bundle $q : V \rightarrow M$ over a smooth manifold M of dimension m we identify two local smooth sections σ_1 and σ_2 defined in a neighbourhood of some point $p \in M$ if in local coordinates on M and V they have the same partial derivatives up to order r at p . An equivalence class $[(\sigma, p)] = j_p^r \sigma$ under this relation is called r -jet of σ at p . We denote by V^r the space of r -jets of local sections. A partial differential relation of order r is a subset $\mathcal{R} \subset V^r$.

In the following we consider the bundle $q : E \rightarrow M$ of the positive definite 2-forms on M , which is an open subbundle of the bundle $q : S^2T^*M \rightarrow M$ of the symmetric bilinear forms on M . Having chosen a chart neighbourhood $U \subset M$ we have trivialisations $q^{-1}(U) \cong U \times S(m)$ and $q^{-1}(U) \cong U \times P(m)$ where we denote by $S(m) \subset \mathbb{R}^{m \times m}$ the vectorspace ($\dim S(m) =: d$) of symmetric $m \times m$ matrices and by $P(m) \subset S(m)$ the open subset of positive definite $m \times m$ matrices.

We consider the projection map

$$\begin{aligned} q^2 : E^2 &\rightarrow M \\ j_p^2 g &\mapsto p. \end{aligned}$$

Let $p \in M$ and $j_p^2 g \in E^2$ a 2-jet represented by a local Riemannian metric g . Using a chart $(U, \Phi, x^1, \dots, x^m)$ around p , we obtain a local description

$$(g_{ij}^\Phi(\Phi(p)), \frac{\partial g_{ij}^\Phi}{\partial x^k}(\Phi(p)), \frac{\partial^2 g_{ij}^\Phi}{\partial x^l \partial x^k}(\Phi(p))) \in P(m)^2 := P(m) \times \mathbb{R}^{dm} \times \mathbb{R}^{\frac{dm(m+1)}{2}}$$

of $j_p^2 g$ taking symmetries of the partial derivatives into account. Vice versa, given $(g_{ij}^\Phi, g_{ijk}^\Phi, g_{ijkl}^\Phi) \in P(m)^2$ we choose the associated Taylor polynomial h_{ij} near $\Phi(p)$ such that

$$(h_{ij}(\Phi(p)), \frac{\partial h_{ij}}{\partial x^k}(\Phi(p)), \frac{\partial^2 h_{ij}}{\partial x^l \partial x^k}(\Phi(p))) = (g_{ij}^\Phi, g_{ijk}^\Phi, g_{ijkl}^\Phi).$$

As a consequence there is a 1-1 correspondence

$$(q^2)^{-1}(U) \xrightarrow{1-1} U \times P(m)^2.$$

Now let $(W, \Psi, y^1, \dots, y^m)$ be another chart defined near p . Keeping in mind that

$$g_{ij}^\Psi = \sum_{k,l} \frac{\partial x^k}{\partial y^i} \frac{\partial x^l}{\partial y^j} g_{kl}^\Phi$$

a change of coordinates on M obviously induces a linear transformation of $P(m)^2$. This allows us to give E^2 a canonical structure of a smooth manifold and we have proven

Lemma 2.1. *The space E^2 of 2-jets of local Riemannian metrics on M defines a smooth fibre bundle*

$$\begin{aligned} q^2 : E^2 &\rightarrow M \\ j_p^2 g &\mapsto p \end{aligned}$$

with linear transformations of the fibre $P(m)^2 = P(m) \times \mathbb{R}^{dm} \times \mathbb{R}^{\frac{dm(m+1)}{2}}$.

3. THE CURVATURE RELATION

Let $\tau = j_p^2 g$ be a 2-jet of a local Riemannian metric g at $p \in M$ and $(U, \Phi, x^1, \dots, x^m)$ a chart around p . We may assume that g is defined on U . In local coordinates τ is associated to

$$(\Phi(p), \tau_{ij}, \tau_{ijk}, \tau_{ijkl}) = (\Phi(p), g_{ij}^\Phi(\Phi(p)), \partial_k g_{ij}^\Phi(\Phi(p)), \partial_l \partial_k g_{ij}^\Phi(\Phi(p))).$$

On U the metric g induces the Levi-Civita connection ∇ given locally by means of the Christoffel symbols defined by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^m \Gamma_{ij}^k \cdot \frac{\partial}{\partial x^k}$$

or more explicitly

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^m (g^\Phi)^{lk} (\partial_i g_{jl}^\Phi + \partial_j g_{li}^\Phi - \partial_l g_{ij}^\Phi)$$

where $((g^\Phi)^{ij})$ is the inverse of the matrix (g_{ij}^Φ) . We note that the entries $(g^\Phi)^{ij}$ are rational functions in terms of the g_{ij}^Φ . It follows that the components R_{ijk_s} of the curvature tensor given by

$$g \left(R \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^s} \right) = g \left(\nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^s} \right)$$

at p are completely determined by τ and independent of the choice of the representative g . This implies that the sectional curvature

$$\text{sec}(V) = \frac{g(R(v, w)v, w)}{g(v, v) \cdot g(w, w) - g(v, w)^2}$$

of a plane $V \subset T_p M$ spanned by two linearly independent vectors $v, w \in T_p M$ only depends on τ . In other words, we have a well-defined notion of sectional curvature of 2-jets of local Riemannian metrics.

Definiton 3.1. *Let $\tau \in E^2$ be a 2-jet at $p \in M$ of a Riemannian metric g on a neighbourhood U of p and V a plane in $T_p M$. We define the sectional curvature $\text{sec}_\tau V$ as the sectional curvature $\text{sec}_{(U, g)} V$ with respect to the Levi-Civita connection induced by g on U .*

Definiton 3.2. *Given $a \in \mathbb{R}$ and $\delta \geq 0$ the curvature relation $\mathcal{R}_{\delta, a}$ is defined as the subset*

$$\mathcal{R}_{\delta, a} = \{\tau \in E^2 : \text{sec}_\tau V \in (a - \delta, a + \delta) \text{ for all planes } V \subset T_{q^2(\tau)} M\}$$

where we think of (a, a) as $\{a\}$ if $\delta = 0$ by abuse of notation.

We call a continuous section α of E^2 satisfying $\alpha(M) \subset \mathcal{R}_{\delta, a}$ a **formal** solution to the curvature relation $\mathcal{R}_{\delta, a}$. A formal solution α is **holonomic** if $j^2 g = \alpha$ for some section g of E . In other words, g is a Riemannian metric on M such that all sectional curvatures lie in $(a - \delta, a + \delta)$. We refer to [2] and [5] for these notions in the broader context of partial differential relations.

Let M_1 and M_2 be two differentiable manifolds of the same dimension. As above we define bundles $q_i : E_i \rightarrow M_i$ and $q_i^2 : E_i^2 \rightarrow M_i$ as well as curvature relations $\mathcal{R}_{\delta, a}^i$ with $i \in \{1, 2\}$. A local diffeomorphism $f : U_1 \rightarrow U_2$ between open subsets $U_1 \subset M_1$ and $U_2 \subset M_2$ induces in a canonical way a map

$$f_* : E_1^2|_{U_1} \rightarrow E_2^2|_{U_2}$$

as follows: Let $\tau = j_p^2 g \in E_1^2$. We may assume that its representative g is defined on U_1 . The push-forward $f_* g$ yields a metric on U_2 via

$$f_* g(v, w) = g(f^* v, f^* w)$$

for all $v, w \in T_{\tilde{p}} M_2$ and arbitrary $\tilde{p} \in U_2$. Now we define

$$f_* \tau := j_{f(p)}^2 f_* g.$$

One readily checks that $f_*\tau$ is well-defined. Suppose that $\tau = j_p^2 g \in \mathcal{R}_{\delta,a}^1$. Due to the fact that $f : (U_1, g) \rightarrow (U_2, f_*g)$ is an isometry we have

$$\sec_{f_*\tau} E(v, w) = \sec_\tau E(f^*v, f^*w)$$

where $E(v, w)$ is the plane spanned by two linearly independent vectors $v, w \in T_{f(p)}U_2$. In other words, $f_*\tau \in \mathcal{R}_{\delta,a}^2$. In this sense curvature relations are invariant under local diffeomorphisms.

Lemma 3.3. *The restriction $q^2 : \mathcal{R}_{\delta,a} \rightarrow M$ defines a subbundle of $q^2 : E^2 \rightarrow M$. This bundle is open if $\delta > 0$.*

Proof. Let $p \in M$ and (U, Φ) a chart of M near p . We obtain a local trivialisation $(q^2)^{-1}(U) \cong U \times P(m)^2$ of the bundle $q^2 : E^2 \rightarrow M$. Any 2-jet $\tau \in \mathcal{R}_{\delta,a}$ has a local representation

$$(p, \tau_{ij}, \tau_{ijk}, \tau_{ijkl}) \in U \times P(m)^2.$$

We note that the sectional curvature depends only on $(\tau_{ij}, \tau_{ijk}, \tau_{ijkl})$. Consequently, $\tilde{\tau} = (\tilde{p}, \tau_{ij}, \tau_{ijk}, \tau_{ijkl}) \in \mathcal{R}_{\delta,a}$ for any $\tilde{p} \in U$. We define $F_{\delta,a}$ as a subset of $P(m)^2$ in the following way: $(\tau_{ij}, \tau_{ijk}, \tau_{ijkl}) \in F_{\delta,a}$ if and only if for some (and hence any) $\tilde{p} \in U$ we have

$$\tilde{\tau} = (\tilde{p}, \tau_{ij}, \tau_{ijk}, \tau_{ijkl}) \in \mathcal{R}_{\delta,a}.$$

In other words,

$$(q^2)^{-1}(U) \cap \mathcal{R}_{\delta,a} \cong U \times F_{\delta,a}.$$

As a result, $q^2 : \mathcal{R}_{\delta,a} \rightarrow M$ is trivial over charts of M . A change of coordinates on M induces a linear transformation of $F_{\delta,a} \subset P(m)^2$, because the sectional curvature is independent of the choice of local coordinates.

Now assume $\delta > 0$. We identify $T_p M$ with \mathbb{R}^m and parametrize the set of all planes in $T_p M$ by the Stiefel manifold $V_{m,2}$ of the orthonormal 2-frames $(v, w) \in \mathbb{R}^m$. Furthermore, let $E(v, w)$ be the plane in \mathbb{R}^m spanned by two linearly independent vectors v and w . The function

$$\begin{aligned} \eta : P(m)^2 \times V_{m,2} &\rightarrow \mathbb{R} \\ (\sigma', (v, w)) &\mapsto \sec_{(p,\sigma)}(E(v, w)) \end{aligned}$$

is continuous and $V_{m,2}$ is compact. It follows that $F_{\delta,a}$ is an open subset of $P(m)^2$. \square

Let $\delta > 0$. We denote by $\Gamma\mathcal{R}_{\delta,a}$ the space of formal solutions equipped with the compact-open topology and write

$$\Gamma_{\mathcal{R}_{\delta,a}} E = \{g \in \Gamma^\infty(E) : j^2 g \in \mathcal{R}_{\delta,a}\}$$

for the space of smooth sections of E , i.e. Riemannian metrics on M , such that all sectional curvatures lie in $(a-\delta, a+\delta)$. The map $j^2 : \Gamma_{\mathcal{R}_{\delta,a}} E \rightarrow \Gamma\mathcal{R}_{\delta,a}$ induces the weak C^2 -topology on $\Gamma_{\mathcal{R}_{\delta,a}} E$.

So far we have discussed all technical notions we need to apply Gromov's general h-principle for open, invariant relations on open manifolds to our special case of curvature relations (cf. Theorem 3.12 in [1]):

Theorem 3.4. *Let M be an open manifold and $\mathcal{R}_{\delta,a}$ a curvature relation with $\delta > 0$. Then*

$$j^2 : \Gamma_{\mathcal{R}_{\delta,a}} E \rightarrow \Gamma \mathcal{R}_{\delta,a}$$

is a weak homotopy equivalence.

In other words, if we can prove that there exist formal solutions we are done. Surjectivity on π_0 then yields arbitrarily pinched Riemannian metrics.

Lemma 3.5. *The space $\Gamma \mathcal{R}_{\delta,a}$ of formal solutions is nonempty for any $a \in \mathbb{R}$ and $\delta \geq 0$.*

Proof. It suffices to show that $\mathcal{R}_{0,a} \subset \mathcal{R}_{\delta,a}$ admits formal solutions. We will prove that the fibre $F_{0,a} =: F_a$ smoothly deformation retracts to an arbitrary element of F_a .

Note that F_a is nonempty. In a neighbourhood of $0 \in \mathbb{R}^m$ we choose a Riemannian metric g with constant sectional curvature a . Then

$$\tau' = (\tau_{ij}, \tau_{ijk}, \tau_{ijkl}) = (g_{ij}(0), \partial_k g_{ij}(0), \partial_l \partial_k g_{ij}(0)) \in F_a.$$

Let τ be the Taylor polynomial which represents the 2-jet $(0, \tau') \in \mathbb{R}^m \times F_a$. We regard τ as a Riemannian metric defined in a neighbourhood of $0 \in \mathbb{R}^m$.

Now we choose a reference basis $B = \{b_1, \dots, b_m\}$ of \mathbb{R}^m . Applying the Gram-Schmidt procedure yields an orthonormal basis B_τ w.r.t. $\tau(0)$. We change to normal coordinates centered at 0 w.r.t. τ and B_τ , i.e. we identify $T_0 \mathbb{R}^m$ with \mathbb{R}^m via B_τ , and the exponential map induces a linear and invertible transformation

$$\begin{aligned} \tilde{L} : F_a &\rightarrow F_a \\ \tau' &\mapsto \tilde{\tau}' = (\delta_{ij}, 0, \tilde{\tau}_{ijkl}), \end{aligned}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Let $\psi' \in P(m)^2$ and ψ the Taylor polynomial associated to $(0, \psi') \in \mathbb{R}^m \times P(m)^2$. Like τ we think of ψ as a Riemannian metric defined near $0 \in \mathbb{R}^m$. Then

$$\psi_t = t \cdot \tau + (1 - t) \cdot \psi, \quad t \in [0, 1]$$

is a Riemannian metric in a neighbourhood of $0 \in \mathbb{R}^m$ with $\psi_0 = \psi$ und $\psi_1 = \tau$. We write

$$\psi'_t = ((\psi_t)_{ij}(0), \partial_k (\psi_t)_{ij}(0), \partial_l \partial_k (\psi_t)_{ij}(0)) \in P(m)^2.$$

The Gram-Schmidt procedure w.r.t. $\psi_t(0)$ transforms B into an orthonormal basis

$$B(t, \psi') = \{b_1(t, \psi'), \dots, b_m(t, \psi')\}$$

with respect to $\psi_t(0)$ such that the maps

$$\begin{aligned} [0, 1] \times P(m)^2 &\rightarrow \mathbb{R}^m \\ (t, \psi') &\mapsto b_i(t, \psi') \quad i = 1, \dots, m \end{aligned}$$

are smooth and $B(1, \psi') = B_\tau$. Changing to normal coordinates at 0 w.r.t. ψ_t and $B(t, \psi')$, we obtain a family

$$L(t, \psi') : P(m)^2 \rightarrow P(m)^2$$

of linear and invertible transformations with $L(1, \psi') = \tilde{L}$.

We claim that $L(t, \psi')$ depends smoothly on (t, ψ') (and so does $L(t, \psi')^{-1}$ by Kramer's rule).

We observe that $(t, \psi', x) \mapsto \psi_t(x)$ is smooth. Let $(t_0, \phi') \in [0, 1] \times P(m)^2$, then $\phi_{t_0}(0) \in P(m)$. We find a neighbourhood V of (t_0, ϕ') and a neighbourhood U of $0 \in \mathbb{R}^m$ such that $\psi_t(x) \in P(m)$ if $(t, \psi', x) \in V \times U$.

The Christoffel symbols associated to ψ_t depend smoothly on $(t, \psi') \in V$, i.e.

$$\begin{aligned} V \times U &\rightarrow \mathbb{R} \\ (t, \psi', x) &\mapsto \Gamma_{ij}^k(t, \psi', x) \end{aligned}$$

is smooth. Thus we obtain a system

$$\ddot{x}^k + \sum_{i,j=1}^m \Gamma_{ij}^k(t, \psi', x) \dot{x}^i \dot{x}^j = 0, \quad k = 1, \dots, m$$

of geodesic equations. Hence there exist neighbourhoods V' of (t_0, ϕ') and W of $0 \in T_0\mathbb{R}^m$ such that

$$\begin{aligned} V' \times W &\rightarrow \mathbb{R}^m \\ (t, \psi', v) &\mapsto \exp_0(t, \psi', v) \end{aligned}$$

is smooth and our claim follows.

We write $\tilde{\psi}' = L(0, \psi')(\psi')$ and define a smooth map

$$\begin{aligned} h : [0, 1] \times P(m)^2 &\rightarrow P(m)^2 \\ (s, \psi') &\mapsto s \cdot \tilde{\tau}' + (1 - s) \cdot \tilde{\psi}'. \end{aligned}$$

We set

$$\begin{aligned} G : [0, 1] \times P(m)^2 &\rightarrow P(m)^2 \\ (t, \psi') &\mapsto L(t, \psi')^{-1}(h(t, \psi')) \end{aligned}$$

and obtain a smooth map which satisfies

$$G(0, \psi') = L(0, \psi')^{-1}(h(0, \psi')) = L(0, \psi')^{-1}(\tilde{\psi}') = \psi'$$

and

$$G(1, \psi') = L(1, \psi')^{-1}(h(1, \psi')) = \tilde{L}^{-1}(\tilde{\tau}') = \tau'.$$

In case $\psi' = \tau'$ it follows that $\psi_t = \tau$ and $L(t, \tau') = \tilde{L}$ independent of $t \in [0, 1]$. Thus, $h(s, \tau') = \tilde{\tau}'$ for all $s \in [0, 1]$ and $G(t, \tau') = \tilde{L}^{-1}(\tilde{\tau}') = \tau'$ for all $t \in [0, 1]$. In other words, G is a deformation retraction of $P(m)^2$ to $\tau' \in F_a$.

We claim that the restriction $G|_{[0,1] \times F_a}$ is a deformation retraction of F_a to $\tau' \in F_a$.

Now assume $\psi' \in F_a$. It follows that $\tilde{\psi}' \in F_a$ is of the form $(\delta_{ij}, 0, \tilde{\psi}_{ijkl})$ as well as $\tilde{\tau}' = (\delta_{ij}, 0, \tilde{\tau}_{ijkl}) \in F_a$ and $h(s, \psi') = (\delta_{ij}, 0, s \cdot \tilde{\tau}_{ijkl} + (1-s) \cdot \tilde{\psi}_{ijkl})$.

Suppose we have a Riemannian metric g defined near $0 \in \mathbb{R}^m$ which satisfies $g_{ij}(0) = \delta_{ij}$ and $\partial_k g_{ij}(0) = 0$. Then an elementary calculation shows

$$R_{ijk_s}(0) = \frac{1}{2} (\partial_j \partial_k g_{si} - \partial_j \partial_s g_{ik} - \partial_i \partial_k g_{sj} + \partial_i \partial_s g_{jk})(0).$$

Thus we conclude $h(s, \psi') \in F_a$ for all $s \in [0, 1]$. Taking into account that the transformations $L(t, \psi')$ are induced by coordinate changes we have

$$G(t, \psi') = L(t, \psi')^{-1}(h(t, \psi')) \in F_a$$

for all $t \in [0, 1]$.

It follows that F_a is contractible and by elementary obstruction theory there exists a global section of $q^2 : \mathcal{R}_{0,a} \rightarrow M$, i.e. a formal solution. \square

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