CURVATURE INVARIANTS AND GENERALIZED CANONICAL OPERATOR MODELS

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Abstract. There are two models for contraction operators on reproducing kernel Hilbert spaces, that of Sz.-Nagy-Foias introduced in the sixties and the later one from the late seventies due to M. Cowen and the first author. In comparing the two models, this paper interprets the former as a quotient Hilbert module of vector-valued Hardy spaces. Alongside this resolution is a resolution of hermitian anti-holomorphic vector bundles for which the curvatures can be calculated. Moreover, one can obtain other models replacing the Hardy space by other Hilbert spaces of holomorphic functions on the unit disk such as the weighted Bergman spaces. Further, one can decide when such quotient modules are unitarily equivalent and, perhaps, similar. In particular, it seems that the results are independent of the building block Hilbert spaces of holomorphic functions used. The techniques involved are a blend of complex geometry and harmonic analysis. In many cases, questions about the quotient Hilbert modules are reduced to questions involving anti-holomorphic sub-bundles of trivial finite-dimensional bundles over the disk.

1. Introduction

One goal of operator theory is to obtain unitary invariants, ideally, in the context of a concrete model for the operators being studied. For a multiplication operator on a space of holomorphic functions on the unit disk $\mathbb{D}$, which happens to be contractive, there are two distinct approaches to models and their associated invariants, one due to Sz.-Nagy and Foias [10] and the other due to M. Cowen and the first author [1]. The starting point for this work was an attempt to compare the two sets of invariants and models obtained in these approaches. Although one could, in principle, work at the same level of generality as that in which these models are framed, we opt to consider some of the simplest possible cases in which the various phenomena possible present themselves, in order to make the relationships clearer. Extensions of these results to more general situations can proceed later, particularly those

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dictated by particular concrete applications. We discuss some possibilities for generalizations at the end of the paper.

For the Sz.-Nagy-Foias canonical model theory, the Hardy space, $H^2(\mathbb{D})$, of holomorphic functions on the unit disk $\mathbb{D}$ is central if one allows the functions to take values in some coefficient Hilbert space $\mathcal{E}$. In this case, we will now denote the space by $H^2(\mathbb{D}) \otimes \mathcal{E}$. One can view the canonical model Hilbert space (in the case of a $C_{\cdot 0}$ contraction T) as given by the quotient of $H^2(\mathbb{D}) \otimes \mathcal{E}_*$, for some Hilbert space $\mathcal{E}_*$, by the range of a map T_Θ defined to be multiplication by a bounded holomorphic operator-valued function, $\Theta(z) \in \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$, from $H^2(\mathbb{D}) \otimes \mathcal{E}$ to $H^2(\mathbb{D}) \otimes \mathcal{E}_*$. If one assumes that the multiplication operator associated with $\Theta(z)$ defines an isometry (or is inner), and satisfies a certain non-triviality assumption (pureness), then $\Theta(z)$ is the characteristic operator function for the operator T. Hence, $\Theta(z)$ provides a complete unitary invariant for the operator M_z defined to be the compression of multiplication by z to the quotient Hilbert space of $H^2(\mathbb{D}) \otimes \mathcal{E}_*$ by the range of $\Theta(z)$. In general, neither the operator T nor its adjoint T^* is in the $B_n(\mathbb{D})$ class of [1] but we are interested in the case in which the adjoint T^* is in $B_n(\mathbb{D})$ and we study the relation between its complex geometric invariants (see [1]) and $\Theta(z)$.

We use the language of Hilbert modules which we believe to be natural in this context. Also we consider "models" obtained as quotient Hilbert modules in which the Hardy module is replaced by other Hilbert modules of holomorphic functions on $\mathbb{D}$ such as the Bergman or weighted Bergman spaces. Once we make this change, the requirement that $\Theta(z)$ be inner becomes artificial and we require instead that the range of T_Θ is closed. In most cases we assume that some analogue of the corona condition holds.

After a preliminary Section 2 in which the above terminology is made precise, we introduce in Section 3 an illustrative family of examples of quotient Hilbert modules in which the role of the Hardy space is played by weighted Bergman spaces, which lie in the $B_1(\mathbb{D})$ class. We determine when two of these examples are unitarily equivalent by calculating the curvatures of the associative hermitian anti-holomorphic vector bundles. The proof is completed using a calculation involving harmonic analysis. In Section 4 we proceed to the more general case of these phenomena and again, determine when two such quotient Hilbert modules are unitarily equivalent. Here we represent the associated hermitian anti-holomorphic bundle as a twisted tensor product of the bundle for the basic Hilbert module by a line bundle determined by the multiplier used. A version of this representation was used earlier by Uchiyama [15] and Treil and the third author [9]. However, we observe that while the bundles obtained in the exact sequence of bundles are all pull-backs from an infinite dimensional Grassmanian, they are all actually the tensor product of a resolution of sub-bundles of finite rank, trivial bundles by the fixed bundle for the basic Hilbert module. Hence, all calculations and proofs can be carried out in this finite dimensional context.

In Section 5 we explore some similarity questions for quotient Hilbert modules drawing upon the research of two earlier groups. First, the similarity question in the Hardy space context was originally studied by Sz.-Nagy and Foias [10] and more recently by Treil and the third author [9]. In the latter work, similarity is shown to be equivalent to the existence of a bounded function whose Laplacian is related to the curvature. The second research, by a group of Chinese researchers (cf. [8]), shows that in the case of contractive Hilbert modules over $\mathbb{D}$, some results for similarity are independent of the particular basic Hilbert module. For example, a quotient Hilbert module defined by the Bergman module is similar to the Bergman module if and only if the same is true for the analogous quotient Hilbert module defined using the Hardy module. Our proof of this fact rests on the tensor product factorization mentioned above since the finite bundle involved does not depend on the basic Hilbert module used. In Section 6 we conclude with a number of remarks including several avenues, we believe, worthy of exploration. Finally, analogues of many of the results in this paper will be carried over to the several variables context in [7].

2. Preliminaries

We begin by providing precise definitions for the terminology used in the introduction, which will be needed in the paper. We start with the fundamental notion of a contractive Hilbert module.

Definition 2.1. *Let T be a linear operator on a Hilbert space $\mathcal{H}_T$. Then $\mathcal{H}_T$ is said to be a contractive Hilbert module over $\mathbb{C}[z]$ relative to T if the module action*

$$\mathbb{C}[z] \times \mathcal{H}_T, \qquad p \cdot h \mapsto p(T)h$$

defines bounded operators for $p \in \mathbb{C}[z]$ such that

$$\|p \cdot h\|_{\mathcal{H}_T} = \|p(T)h\|_{\mathcal{H}_T} \leq \|p\|_\infty \|h\|_{\mathcal{H}_T},$$

for all $h \in \mathcal{H}_T$, where $\|p\|_\infty$ is the supremum norm on $\mathbb{D}$.

If the Hilbert module $\mathcal{H}_T$ is contractive, then the operator M_z defined by multiplication by z is a contraction and $\mathcal{H}_T = \mathcal{H}_{M_z}$. Conversely, since one can extend the module action from $\mathbb{C}[z]$ to all of the disk algebra $A(\mathbb{D})$, using the von-Neumann inequality, a contraction operator gives rise to a contractive Hilbert module. Recall that $A(\mathbb{D})$ consists of the functions continuous on the closure of $\mathbb{D}$ that are holomorphic on $\mathbb{D}$.

Now we restrict our attention to a special class of Hilbert modules. Here $\mathbb{C}[z] \otimes_{alg} \mathbb{C}^n$ denotes the algebraic tensor product.

Definition 2.2. *A Hilbert module $\mathcal{R}$ over $A(\mathbb{D})$ is said to be a contractive quasi-free Hilbert module of multiplicity n, $1 \leq n < \infty$, if there is an inner product defined on $\mathbb{C}[z] \otimes_{alg} \mathbb{C}^n$ with $\mathcal{R} = \overline{\mathbb{C}[z] \otimes_{alg} \mathbb{C}^n}$ such that*

(i) the evaluation operator $\boldsymbol{ev}_w : \mathbb{C}[z] \otimes_{alg} \mathbb{C}^n \to \mathbb{C}^n$ defined by $\boldsymbol{ev}_w(f) = f(w)$ is bounded for all $w \in \mathbb{D}$,

(ii) the module multiplication M_z on $\mathcal{R}$ defined by $M_z f = zf$ is contractive, and

(iii) for all $\{f_i\}_{i=1}^\infty$ in $\mathbb{C}[z] \otimes_{alg} \mathbb{C}^n$, $\boldsymbol{ev}_w(f_i) = f_i(w) \to 0$ if and only if $f_i \to 0$ in $\mathcal{R}$.

One can identify a quasi-free Hilbert module $\mathcal{R}$ as a subspace of the space $\mathcal{O}(\mathbb{D}, \mathbb{C}^n)$ of holomorphic functions taking values in $\mathbb{C}^n$ so that $\{f(w_0) : f \in \mathcal{R}\} = \mathbb{C}^n$ for $w_0 \in \mathbb{D}$ and such that the module multiplication agrees with pointwise multiplication. On $\mathbb{D}$ it seems likely that this latter description characterizes the quasi-free Hilbert modules of finite multiplicity. An affirmative answer depends on the existence of n generators for $\mathcal{R}$.

The Hardy module $H^2(\mathbb{D})$, the Bergman module $L^2_a(\mathbb{D})$, and the weighted Bergman modules $L^{2,\alpha}_a(\mathbb{D})$ for all $\alpha > -1$ are all quasi-free Hilbert modules of multiplicity one over $A(\mathbb{D})$. Recall that $H^2(\mathbb{D})$ consists of the functions f holomorphic on $\mathbb{D}$ for which

$$\|f\|_2 = (\sum_{k=0}^{\infty} |a_k|^2)^{\frac{1}{2}} < \infty,$$

where $\sum_{k=0}^{\infty} a_k z^k$ is the Taylor series expansion of f. Similarly, for all α, $-1 < \alpha < \infty$, the weighted Bergman space $L^{2,\alpha}_a(\mathbb{D})$ consists of the holomorphic functions f on $\mathbb{D}$ for which

$$\|f\|_{2,\alpha} = (\frac{1}{2\pi}\int_{\mathbb{D}} |f(z)|^2 dA_\alpha)^{\frac{1}{2}} < \infty,$$

where dA_α is the weighted area measure $dA_\alpha = (1+\alpha)(1-|z|^2)^\alpha d\,A$ with dA the area measure on $\mathbb{D}$.

The assumptions in the definition of a quasi-free Hilbert module $\mathcal{R}$ assure one that $\mathcal{R}$ is a contractive and an analytic reproducing kernel Hilbert module. We briefly recall the definition of the latter.

Definition 2.3. *A function $K : \mathbb{D} \times \mathbb{D} \to \mathcal{L}(\mathcal{E})$ for a Hilbert space $\mathcal{E}$, is said to be a positive definite kernel if the operator $K(z, z)$ is positive and injective for all $z \in \mathbb{D}$ and*

$$\langle \sum_{i,j=1}^{p} K(z_i, z_j)\eta_j, \eta_i\rangle \geq 0,$$

for all $\eta_i \in \mathcal{E}$, $z_i \in \mathbb{D}$, $1 \leq i \leq p$ and for all $p \in \mathbb{N}$.

Given a positive definite kernel, one can construct the Hilbert space $\mathcal{H}_K$ of $\mathcal{E}$-valued functions which is defined to be the closure of

$$\text{span}\,\{K(\cdot, z)\eta : z \in \mathbb{D}, \eta \in \mathcal{E}\},$$

with the inner product

$$\langle K(\cdot, w)\eta,\, K(\cdot, z)\zeta\rangle_{\mathcal{H}_K} = \langle K(z, w)\eta, \zeta\rangle_{\mathcal{E}},$$

for all $z, w \in \mathbb{D}$ and $\eta, \zeta \in \mathcal{E}$. The evaluation of a function f in $\mathcal{H}_K$ at a point $z \in \mathbb{D}$ is given by the reproducing property so that

$$\langle f(z), \eta\rangle_{\mathcal{E}} = \langle f, K(\cdot, z)\eta\rangle_{\mathcal{H}_K},$$

for all $f \in \mathcal{H}_K, z \in \mathbb{D}$ and $\eta \in \mathcal{E}$. In particular, the evaluation operator $\boldsymbol{ev}_z : \mathcal{H}_K \to \mathcal{E}$ defined by $\boldsymbol{ev}_z(f) = f(z)$ is bounded for all $z \in \mathbb{D}$.

Conversely, given a Hilbert space $\mathcal{H}$ of holomorphic $\mathcal{E}$-valued functions on $\mathbb{D}$ with a bounded evaluation operator $\boldsymbol{ev}_z \in \mathcal{L}(\mathcal{H}, \mathcal{E})$ for each $z \in \mathbb{D}$, one can construct the reproducing kernel

$$\boldsymbol{ev}_z \circ \boldsymbol{ev}_w^* : \mathbb{D} \times \mathbb{D} \to \mathcal{L}(\mathcal{E}),$$

for all $z, w \in \mathbb{D}$ such that $\mathcal{H} = \mathcal{H}_{\boldsymbol{ev}_z \circ \boldsymbol{ev}_w^*}$. To ensure that $\boldsymbol{ev}_z \circ \boldsymbol{ev}_w^*$ is injective, the set of values of $f \in \mathcal{H}$ must equal $\mathcal{E}$ for $z \in \mathbb{D}$.

For $H^2(\mathbb{D})$, the kernel function is $K(z,w) = (1-z\bar{w})^{-1}$. For $L_a^{2,\alpha}(\mathbb{D})$, one can calculate the reproducing kernel function and show that

$$K(z,w) = (1-z\bar{w})^{-2-\alpha} = \sum_{k=0}^{\infty} \frac{\Gamma(k+2+\alpha)}{n!\Gamma(2+\alpha)}(z\bar{w})^k,$$

where Γ is the gamma function.

A reproducing kernel Hilbert space $\mathcal{H}_K$ is said to define a *contractive reproducing kernel Hilbert module* over $A(\mathbb{D})$ if the operator M_z is contractive. In other words, $\mathcal{H}_K$ is the Hilbert module with module multiplication obtained from the multiplication operator M_z. Moreover, one can show that $M_\varphi \mathcal{H}_K \subseteq \mathcal{H}_K$, where M_φ is the multiplication operator defined by $\varphi \in H^\infty(\mathbb{D})$ or, the multiplier algebra for $\mathcal{H}_K$ is $H^\infty(\mathbb{D})$. One can prove that if $\mathcal{H}_K$ is a reproducing kernel Hilbert module over $A(\mathbb{D})$, then

$$M_\varphi^* K(\cdot,z)\eta = K(\cdot,z)\varphi(z)^*\eta,$$

for all φ in $H^\infty(\mathbb{D})$ and $\eta \in \mathcal{E}$. In particular,

$$M_z^*(K(\cdot,z)\eta) = \bar{z}(K(\cdot,z)\eta),$$

for $\eta \in \mathcal{E}$ and $z \in \mathbb{D}$.

In [1] M. Cowen and the first author introduced a class of operators, $B_n(\mathbb{D})$, which includes M_z^* for the operator M_z defined above on a contractive reproducing Hilbert module with $\dim \mathcal{E} = n$. We recall this notion. (Note that T^* corresponds to the operator M_z defined on a Hilbert module.)

Definition 2.4. *For $1 \leq n < \infty$, the operator T on the Hilbert space $\mathcal{H}_T$ is in the class $B_n(\mathbb{D})$ if*

(i) ran $(T-w) = \mathcal{H}_T$ for all $w \in \mathbb{D}$,

(ii) dim ker$(T-w) = n$ for all $w \in \mathbb{D}$, and

(iii) $\vee_{w\in\mathbb{D}}$ker $(T-w) = \mathcal{H}_T$.

By a result of Shubin [12], one can show that for T on $\mathcal{H}$ in $B_n(\mathbb{D})$ there is a hermitian anti-holomorphic rank n vector bundle $E_{\mathcal{H}}^*$ over $\mathbb{D}$ defined as the pull-back of the anti-holomorphic map $w \mapsto \ker(T-w)$ from $\mathbb{D}$ to the Grassmannian $Gr(n,\mathcal{H})$ of n-dimensional subspaces of $\mathcal{H}_T$. (We use the notation $E_{\mathcal{H}}^*$ since this bundle is the dual of a natural hermitian holomorphic vector bundle $E_{\mathcal{H}}$.) As a consequence, there exists a frame $\{\psi_i\}_{i=1}^n$ of anti-holomorphic $\mathcal{H}$-valued functions on $\mathbb{D}$ such that

$$\vee_{i=1}^n \psi_i(w) = \ker(T-w) \subseteq \mathcal{H},$$

for $w \in \mathbb{D}$. In many cases, the existence of such a frame is sufficient for many applications of complex geometrical method to operator theory. Thus we will introduce a weaker version of $B_n(\mathbb{D})$ after we establish a fact to prepare for this.

Proposition 2.5. *Let $\{\varphi_i\}_{i=1}^n$ and $\{\tilde{\varphi}_i\}_{i=1}^n$ be anti-holomorphic functions, $\varphi_i : \mathbb{D} \to \mathcal{H}$ and $\tilde{\varphi}_i : \mathbb{D} \to \tilde{\mathcal{H}}$, for Hilbert spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$ with $T \in \mathcal{L}(\mathcal{H})$ and $\tilde{T} \in \mathcal{L}(\tilde{\mathcal{H}})$ satisfying*

(1) $T\varphi_i(w) = \bar{w}\varphi_i(w)$, $\tilde{T}\tilde{\varphi}_i(w) = \bar{w}\tilde{\varphi}_i(w)$, $i = 1,\ldots,n$, $w \in \mathbb{D}$ and

(2) $\overline{span}\{\varphi_i(w) : 1 \leq i \leq n, w \in \mathbb{D}\} = \mathcal{H}$ and $\overline{span}\{\tilde{\varphi}_i(w) : 1 \leq i \leq n, w \in \mathbb{D}\} = \tilde{\mathcal{H}}$.

Then there exists an anti-holomorphic partial isometry-valued function $V(w) : \mathcal{H} \to \tilde{\mathcal{H}}$ such that $\ker V(w) = \overline{span}\{\varphi_i(w) : 1 \leq i \leq n\}^{\perp}$ and $\operatorname{ran} V(w) = \overline{span}\{\tilde{\varphi}_i(w) : 1 \leq i \leq n\}$ if and only if there exists a unitary operator $V : \mathcal{H} \to \tilde{\mathcal{H}}$ such that $V\varphi_i(w) = V(w)\varphi_i(w)$ for $1 \leq i \leq n$ and $w \in \mathbb{D}$.

Proof. Same as in the proof of the rigidity theorem in [1] where the above hypotheses are restated in the language of bundles. ■

The first author would like to point out that the basic calculation, used to prove the rigidity theorem in [1], appeared earlier in the book by Polya [11]. This reference was pointed out to him by N. Nikolski.

Definition 2.6. *For $1 \leq n < \infty$, the operator T on $\mathcal{H}$ is in the class $B_n^w(\mathbb{D})$ or weak-$B_n(\mathbb{D})$ if there exist anti-holomorphic functions $\{\psi_i\}_{i=1}^n$ from $\mathbb{D}$ to $\mathcal{H}$ so that*
(i) $\vee_{i=1}^n \psi_i(w) \subseteq \ker(T - w)$ for $w \in \mathbb{D}$,
(ii) $\vee_{w\in\mathbb{D}} \vee_{i=1}^n \{\psi_i(w)\} = \mathcal{H}$,
(iii) $\{\psi_i(w)\}_{i=1}^n$ is linearly independent for $w \in \mathbb{D}$, and
(iv) $\dim \ker(T - w) = n$ for all $w \in \mathbb{D}$.

The class $B_n^w(\mathbb{D})$ is closely related to the class considered earlier by Uchiyama [15].

One can extend the rigidity theorem of [1] to the case of Hilbert modules in $B_n^w(\mathbb{D})$, since the tuple $\{\psi_i\}$ frames a rank n hermitian anti-holomorphic bundle.

If one assumes only that the dimension of $\ker(T - w)$ is finite and constant, without the requirement that $(T - w)^*$ is onto or has closed range, then it is not clear if these spaces form a bundle or that there exists a frame. However, as we will see, in many cases, such a frame can be shown to exist.

We continue this section with a brief discussion of some complex geometric notions. Since the anti-holomorphic vector bundle $E_{\mathcal{H}}^*$, for a Hilbert module $\mathcal{H} \in B_n(\mathbb{D})$ also has a hermitian structure, one can define the canonical Chern connection $\bigtriangledown$ on $E_{\mathcal{H}}^*$ along with its associated curvature two-form $\mathcal{K}_{E_{\mathcal{H}}^*}(z)$. For the $n = 1$ case, $E_{\mathcal{H}}^*$ is a line bundle and

$$\mathcal{K}_{E_{\mathcal{H}}^*}(z) = -\frac{1}{4} \bigtriangledown^2 \log \|\gamma_z\|^2 \, dz \wedge d\bar{z}, \;\; z \in \mathbb{D}, \tag{2.1}$$

where $\bigtriangledown^2 = 4\partial\bar{\partial}$ is the Laplacian and γ_z is an anti-holomorphic cross section of the bundle. In Section 4 we will use the formula for the curvature for the case in which the bundle $E_{\mathcal{H}}^*$ is not a line bundle.

Finally, we define the Sz.-Nagy-Foias model. Let T be a contraction operator on the Hilbert space $\mathcal{H}$ in the $C_{\cdot 0}$ class; that is, $T^{*n} \to 0$ in the strong operator topology. The Sz.-Nagy-Foias model for T has the form $H^2_{\mathcal{D}_*}(\mathbb{D})/\Theta H^2_{\mathcal{D}}(\mathbb{D})$, where $\mathcal{D}$ and $\mathcal{D}_*$ are coefficient Hilbert spaces and $\Theta(z) : \mathbb{D} \to \mathcal{L}(\mathcal{D}, \mathcal{D}_*)$ is a bounded holomorphic operator-valued function such that $\Theta(e^{it})$ is an isometry on $\partial\mathbb{D}$ a.e. In the next section we take up the case in which $\mathcal{D} = \mathbb{C}$ and $\mathcal{D}_* = \mathbb{C}^2$ in some detail. In Section 4 we extend some results to the case in which $\mathcal{D}$ and $\mathcal{D}_*$ are finite dimensional.

3. Family of examples

We consider a family of quotient Hilbert modules including and generalizing those of Sz.-Nagy-Foias in which the basic building blocks are the weighted Bergman spaces.

Definition 3.1. *A pair of functions* $\{\varphi_1, \varphi_2\}$ *in* $H^\infty(\mathbb{D})$ *is said to be a corona pair if it satisfies the corona condition*

$$|\varphi_1(z)|^2 + |\varphi_2(z)|^2 \geq \epsilon > 0,$$

for some ϵ *and all* $z \in \mathbb{D}$. *We denote by* Φ *the function* $\Phi = (\varphi_1, \varphi_2) : \mathbb{D} \to \mathbb{C}^2$ *or* $\Phi : \mathbb{D} \to \mathcal{L}(\mathbb{C}, \mathbb{C}^2)$ *defined by*

$$\Phi(z) = (\varphi_1(z), \varphi_2(z)),$$

for $z \in \mathbb{D}$.

We begin by considering the case for the Hardy module. Let $\{\varphi_1, \varphi_2\}$ be a corona pair and $\mathcal{H}_\Phi$ be the quotient Hilbert module given by

$$0 \longrightarrow H^2(\mathbb{D}) \otimes \mathbb{C} \xrightarrow{T_\Phi} H^2(\mathbb{D}) \otimes \mathbb{C}^2 \xrightarrow{\pi_\Phi} \mathcal{H}_\Phi \longrightarrow 0,$$

where the first map is T_Φ which is defined by $T_\Phi f = (\varphi_1 f, \varphi_2 f)$ and the second map, π_Φ, is the quotient Hilbert module map. The fact that Φ is a corona pair implies that the range of T_Φ is closed. Note that the $A(\mathbb{D})$-action on the quotient Hilbert module $\mathcal{H}_\Phi$ is the compression of the multiplication operator $M_z \otimes I_{\mathbb{C}^2}$ on $H^2(\mathbb{D}) \otimes \mathbb{C}^2$ to the co-submodule $(\operatorname{ran} T_\Phi)^\perp$, where M_z is the standard module multiplication on $H^2(\mathbb{D})$. Equivalently, the module multiplication operator on $\mathcal{H}_\Phi$ is $P_{\mathcal{H}_\Phi}(M_z \otimes I_{\mathbb{C}^2})|_{\mathcal{H}_\Phi}$. We denote this operator by N_z. In Section 4 we will show that $\mathcal{H}_\Phi$ is in $B_1(\mathbb{D})$ but first we demonstrate that it is in $B_1^w(\mathbb{D})$ which is sufficient for our purpose here.

Proposition 3.2. *Let* $\{\varphi_1, \varphi_2\}$ *be a corona pair and* $\{e_1, e_2\}$ *an orthonormal basis for* $\mathbb{C}^2$. *Then* $\gamma_w = k_w \otimes (\overline{\varphi_2(w)} e_1 - \overline{\varphi_1(w)} e_2) = \overline{\varphi_2(w)} k_w \otimes e_1 - \overline{\varphi_1(w)} k_w \otimes e_2$ *is a non-vanishing anti-holomorphic function from* $\mathbb{D}$ *to* $H^2(\mathbb{D}) \otimes \mathbb{C}^2$ *such that*

(1) $N_z^* \gamma_w = \bar{w} \gamma_w$ *for* $w \in \mathbb{D}$ *and*

(2) $\vee_{w \in \mathbb{D}} \gamma_w = \mathcal{H}_\Phi$,

where k_w *is a kernel function for* $H^2(\mathbb{D})$; *that is,* $M_z^* k_w = \bar{w} k_w$ *for* $w \in \mathbb{D}$.

Proof. Since φ_1, φ_2 are holomorphic and k_w is anti-holomorphic, the fact that $w \mapsto \gamma_w$ is anti-holomorphic follows. Furthermore, since $\{\varphi_1, \varphi_2\}$ is a corona pair, the functions have no common zero and hence $\gamma_w \neq \mathbf{0}$ for $w \in \mathbb{D}$. Now, for $f \in H^2(\mathbb{D})$, $T_\Phi f = \varphi_1 f \otimes e_1 + \varphi_2 f \otimes e_2$ and therefore for all $w \in \mathbb{D}$,

$$\begin{aligned} \langle T_\Phi f, \gamma_w \rangle &= \langle \varphi_1 f, k_w \rangle \langle e_1, \overline{\varphi_2(w)} e_1 \rangle - \langle \varphi_2 f, k_w \rangle \langle e_2, \overline{\varphi_1(w)} e_2 \rangle \\ &= \varphi_1(w) f(w) \varphi_2(w) - \varphi_2(w) f(w) \varphi_1(w) = 0. \end{aligned} \tag{3.1}$$

Hence, $\gamma_w \in (\operatorname{ran} T_\Phi)^\perp = \mathcal{H}_\Phi$. Moreover,

$$\begin{aligned} N_z^* \gamma_w = (M_z \otimes I_{\mathbb{C}^2})^* \gamma_w &= M_z^*(\overline{\varphi_2(w)} k_w) \otimes e_1 - M_z^*(\overline{\varphi_1(w)} k_w) \otimes e_2 \\ &= \overline{\varphi_2(w)} \bar{w} k_w \otimes e_1 - \overline{\varphi_1(w)} \bar{w} k_w \otimes e_2 \\ &= \bar{w} \gamma_w. \end{aligned}$$

Finally in order to show that (2) holds, it suffices to prove that for $G = g_1 \otimes e_1 + g_2 \otimes e_2 \in H^2(\mathbb{D}) \otimes \mathbb{C}^2$ such that $G \perp \vee_{w\in\mathbb{D}} \gamma_w$, we have $G \in \ker R_\Phi$ and that in turn implies $G \in \operatorname{ran} T_\Phi$. Here, $R_\Phi : H^2(\mathbb{D}) \otimes \mathbb{C}^2 \to H^2(\mathbb{D}) \otimes \mathbb{C}$ is the operator defined to be $R_\Phi(g_1 \otimes e_1 + g_2 \otimes e_2) = \varphi_2 g_1 - \varphi_1 g_2$. By the corona theorem there exist $\psi_1, \psi_2 \in H^\infty(\mathbb{D})$ such that $\varphi_1\psi_1 + \varphi_2\psi_2 = 1$ which implies the exactness of the Koszul complex for the pair $(M_{\varphi_1}, M_{\varphi_2})$. Hence, there exists an $f \in H^2(\mathbb{D})$ satisfying $\varphi_1 f = g_1$ and $\varphi_2 f = g_2$, so $G \in \ker R_\Phi$ is indeed in $\operatorname{ran} T_\Phi$ which concludes the proof. ■

In generalizations of this result in the following section we offer another proof using directly the formulation of the corona theorem in terms of the existence of a left inverse for the multiplier map Φ. The above proof has the advantage of showing in this case that $\mathcal{H}_\Phi$ is similar to $H^2(\mathbb{D})$. To see that, note that R_Φ defines a module isomorphism between $\mathcal{H}_\Phi$ and $H^2(\mathbb{D})$. We summarize this observation and a further consequence in the following corollary.

Corollary 3.3. *For a corona pair $\{\varphi_1, \varphi_2\}$ in $H^\infty(\mathbb{D})$, $\mathcal{H}_\Phi \in B_1(\mathbb{D})$ and is similar to $H^2(\mathbb{D})$.*

Before continuing, let us indicate a part of a more general argument why $\mathcal{H}_\Phi \in B_1(\mathbb{D})$. If one considers localization of the exact sequence

$$\cdots \to H^2(\mathbb{D}) \otimes \mathbb{C} \xrightarrow{T_\Phi} H^2(\mathbb{D}) \otimes \mathbb{C}^2 \xrightarrow{\pi_\Phi} \mathcal{H}_\Phi \to 0,$$

one can see that $\dim \ker (N_z - w)^* = 1$ for $w \in \mathbb{D}$. In particular, localizing the exact sequence at $w \in \mathbb{D}$ yields

$$H^2(\mathbb{D})/I_w \cdot H^2(\mathbb{D}) \longrightarrow (H^2(\mathbb{D}) \otimes \mathbb{C}^2)/I_w \cdot (H^2(\mathbb{D}) \otimes \mathbb{C}^2) \longrightarrow \mathcal{H}_\Phi/I_w \cdot \mathcal{H}_\Phi \longrightarrow 0,$$

or

$$\mathbb{C}_w \otimes \mathbb{C} \xrightarrow{I_{\mathbb{C}_w} \otimes \Phi(w)} \mathbb{C}_w \otimes \mathbb{C}^2 \xrightarrow{\pi_\Phi(w)} \mathcal{H}_\Phi/I_w \cdot \mathcal{H}_\Phi \longrightarrow 0.$$

(Recall that I_w is the maximal ideal, $I_w = \{p(z) \in \mathbb{C}[z] : p(w) = 0\}$ in $\mathbb{C}[z]$.) Since the range of $\Phi(w)$ is one dimensional, we see that $\dim \ker \pi_\Phi(w) = 1$ and hence $\dim \mathcal{H}_\Phi/I_w \cdot \mathcal{H}_\Phi = 1$ for $w \in \mathbb{D}$. To show $\mathcal{H}_\Phi \in B_1(\mathbb{D})$, we need to know that $N_z - w$ is onto for $w \in \mathbb{D}$. We postpone that argument till Section 4.

There is another interpretation of the preceding discussion, closely related to the work of Uchiyama [15].

Proposition 3.4. *Let $\{\varphi_1, \varphi_2\}$ be a corona pair and L be the quotient hermitian holomorphic line bundle defined so that*

$$0 \longrightarrow \mathbb{C} \xrightarrow{\Phi} \mathbb{C}^2 \longrightarrow L \longrightarrow 0.$$

Then

$$E^*_{\mathcal{H}_\Phi} \cong E^*_{H^2(\mathbb{D})} \otimes L^*,$$

as hermitian anti-holomorphic vector bundles over $\mathbb{D}$.

We will provide more details for this result in the next section in which we replace the Hardy module by a general quasi-free Hilbert module over $A(\mathbb{D})$.

A result of M. Cowen and the first author [1] can be reformulated to state that the curvature is a complete unitary invariant for Hilbert modules in $B_1(\mathbb{D})$. More precisely, two Hilbert modules $\mathcal{H}$ and $\tilde{\mathcal{H}}$ in $B_1(\mathbb{D})$ are unitarily equivalent if and only if

$$\mathcal{K}_{\mathcal{H}}(w) = \mathcal{K}_{\tilde{\mathcal{H}}}(w), \qquad w \in \mathbb{D}.$$

Based on Proposition 2.5, one can extend the result to Hilbert modules in $B_1^w(\mathbb{D})$. Using this generalization, we obtain the following theorem.

THEOREM 3.5. *Let $\{\varphi_1, \varphi_2\}$ and $\{\psi_1, \psi_2\}$ be two corona pairs in $H^\infty(\mathbb{D})$. Then the quotient Hilbert modules $\mathcal{H}_\Phi$ and $\mathcal{H}_\Psi$ are unitarily equivalent if and only if*

$$\nabla^2 log\,(|\varphi_1(z)|^2 + |\varphi_2(z)|^2) = \nabla^2 log\,(|\psi_1(z)|^2 + |\psi_2(z)|^2), \qquad \textit{for all}\, z \in \mathbb{D}.$$

Proof. The result follows easily once we can compute the norm of the section

$$\gamma_w(\cdot) = k_w(\cdot) \otimes (\overline{\varphi_2(w)}, -\overline{\varphi_1(w)}), \quad w \in \mathbb{D},$$

as

$$\|\gamma_w\|^2 = \|k_w\|^2 \|(\overline{\varphi_2(w)}, -\overline{\varphi_1(w)})\|^2 = \|k_w\|^2 (|\varphi_1(w)|^2 + |\varphi_2(w)|^2),$$

and hence, we get

$$\mathcal{K}_{\mathcal{H}_\Phi}(w) = \mathcal{K}_{H^2(\mathbb{D})}(w) - \frac{1}{4} \nabla^2 \log\,(|\varphi_1(w)|^2 + |\varphi_2(w)|^2). \tag{3.2}$$

We have the analogous formula for $\mathcal{K}_{\mathcal{H}_\Psi}$ and the result follows. ∎

In the next section, we show that formula (3.2) holds when $H^2(\mathbb{D})$ is replaced by other quasi-free Hilbert module of multiplicity one over $\mathbb{D}$. Here we extend the result replacing the Hardy module by a weighted Bergman module $L_a^{2,\alpha}(\mathbb{D})$ for $-1 < \alpha < \infty$. The calculations are all the same. In particular, one has

$$\cdots \longrightarrow L_a^{2,\alpha}(\mathbb{D}) \otimes \mathbb{C} \xrightarrow{T_\Phi} L_a^{2,\alpha}(\mathbb{D}) \otimes \mathbb{C}^2 \xrightarrow{\pi_\Phi} L_a^{2,\alpha}(\mathbb{D})_\Phi \longrightarrow 0,$$

and

$$\mathcal{K}_{L_a^{2,\alpha}(\mathbb{D})_\Phi}(w) = \mathcal{K}_{L_a^{2,\alpha}(\mathbb{D})}(w) - \frac{1}{4} \nabla^2 \log\,(|\varphi_1(w)|^2 + |\varphi_2(w)|^2). \tag{3.3}$$

REMARK 3.6. Observe that the above calculation shows that the hermitian anti-holomorphic line bundle corresponding to the quotient Hilbert module $\mathcal{H}_\Phi$ is the twisted vector bundle obtained from the bundle tensor product of the anti-hermitian holomorphic line bundle for $H^2(\mathbb{D})$ with the line bundle $\coprod_{w\in\mathbb{D}} \mathbb{C}^2/\Phi(w)\mathbb{C}$. This holds in general. More precisely, suppose the Hilbert module $\mathcal{H}_\theta$ is in $B_n(\mathbb{D})$, where $\theta \in H^\infty_{\mathcal{L}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$ and $\mathcal{E}$ and $\mathcal{E}_*$ are Hilbert spaces, T_θ has closed range, and $\mathcal{H}_\theta$ is the quotient Hilbert module

$$0 \to H^2(\mathbb{D}) \otimes \mathcal{E} \xrightarrow{T_\theta} H^2(\mathbb{D}) \otimes \mathcal{E}_* \to \mathcal{H}_\theta \to 0.$$

Then the hermitian anti-holomorphic vector bundle $E^*_{\mathcal{H}_\theta}$ for $\mathcal{H}_\theta$ is the bundle tensor product of $E^*_{H^2(\mathbb{D})}$ with the rank n bundle $\coprod_{w\in\mathbb{D}} \mathcal{E}_*/\overline{\mathrm{ran}\theta(w)}$. In particular, the latter bundle depends only on a family of finite dimensional objects.

In the following, we will discuss a specific case of a more general question in which one considers contractive Hilbert modules $\mathcal{H}, \tilde{\mathcal{H}} \in B_1^w(\mathbb{D})$, with corona pairs Φ and Ψ in $H^\infty(\mathbb{D})$. In particular, which quotient Hilbert modules of this form yield unitarily equivalent Hilbert modules

$$\mathcal{H}_\Phi \cong \tilde{\mathcal{H}}_\Psi?$$

We answer this question here in case $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are the Hardy or weighted Bergman modules. First, we recall a few basic facts.

Since the kernel function for $L_a^{2,\alpha}(\mathbb{D})$ is

$$\mathbb{D} \times \mathbb{D} \ni (z, w) \mapsto k_w^\alpha(z) = (1 - z\bar{w})^{-2-\alpha},$$

for all $z, w \in \mathbb{D}$, $-1 < \alpha < \infty$ and $L_a^{2,\alpha}(\mathbb{D}) \in B_1(\mathbb{D})$; it follows that

$$\mathcal{K}_{L_a^{2,\alpha}(\mathbb{D})}(w) = -\frac{\partial^2 \log \|k_w^\alpha\|^2}{\partial\omega\partial\bar{w}} = -\frac{2+\alpha}{(1-|w|^2)^2}.$$

and

$$\mathcal{K}_{H^2(\mathbb{D})}(w) = -\frac{1}{(1-|w|^2)^2}.$$

In particular, $L_a^{2,\alpha}(\mathbb{D}) \cong L_a^{2,\beta}(\mathbb{D})$ if and only if $\alpha = \beta$ for $-1 < \alpha, \beta < \infty$. Similarly, $L_a^{2,\alpha}(\mathbb{D}) \ncong H^2(\mathbb{D})$ for $-1 < \alpha < \infty$. We extend this result to quotient Hilbert modules defined for such Hilbert modules which is the main result in this section.

Theorem 3.7. *Let Φ and Ψ be two corona pairs in $H^\infty(\mathbb{D})$. Then the quotient Hilbert modules $L_a^{2,\alpha}(\mathbb{D})_\Phi$ and $L_a^{2,\beta}(\mathbb{D})_\Psi$ are unitarily equivalent if and only if $\alpha = \beta$ and*

$$\bigtriangledown^2 log\Big(\frac{|\varphi_1(w)|^2 + |\varphi_2(w)|^2}{|\psi_1(w)|^2 + |\psi_2(w)|^2}\Big) = 0.$$

Moreover, $H^2(\mathbb{D})_\Phi$ is not unitarily equivalent to $L_a^{2,\alpha}(\mathbb{D})_\Psi$ for any Φ and Ψ.

Proof. By equation (3.3), we get

$$\mathcal{K}_{L_a^{2,\alpha}(\mathbb{D})_\Phi}(w) = -\frac{2+\alpha}{(1-|w|^2)^2} - \frac{1}{4}\bigtriangledown^2 \log\left(|\varphi_1(w)|^2 + |\varphi_2(w)|^2\right),$$

and

$$\mathcal{K}_{L_a^{2,\alpha}(\mathbb{D})_\Psi}(w) = -\frac{2+\beta}{(1-|w|^2)^2} - \frac{1}{4}\bigtriangledown^2 \log\left(|\psi_1(w)|^2 + |\psi_2(w)|^2\right).$$

The sufficiency part of the theorem follows from the above formulas. To prove that the conditions are necessary, suppose $L_a^{2,\alpha}(\mathbb{D})_\Phi \cong L_a^{2,\beta}(\mathbb{D})_\Psi$. It is enough to prove that $\alpha = \beta$. Since the curvature is a complete unitary invariant, we have

$$\mathcal{K}_{L_a^{2,\alpha}(\mathbb{D})_\Phi}(w) = \mathcal{K}_{L_a^{2,\alpha}(\mathbb{D})_\Psi}(w),$$

and so

$$\frac{4(\alpha-\beta)}{(1-|w|^2)^2} = \bigtriangledown^2 \log\frac{|\psi_1(w)|^2 + |\psi_2(w)|^2}{|\varphi_1(w)|^2 + |\varphi_2(w)|^2}.$$

By the above equality, the result follows from the following lemma. This is because the fact that Φ and Ψ are corona pairs implies that the logarithm of the quotient is bounded. ∎

LEMMA 3.8. *There does not exist a bounded function* $f : \mathbb{D} \to (0, \infty)$ *such that*

$$\frac{1}{(1-|z|^2)^2} = \bigtriangledown^2 f(z). \tag{3.4}$$

Proof. Suppose f is a bounded function which solves equation (3.4). Since $\frac{1}{4} \bigtriangledown^2 [(|z|^2)^m] = \partial\bar{\partial}[(|z|^2)^m] = m^2(|z|^2)^{m-1}$ for all $m \in \mathbb{N}$, it follows that

$$\bigtriangledown^2 g(z) = \frac{1}{(1-|z|^2)^2},$$

where

$$g(z) = \frac{1}{4}\sum_{m=1}^{\infty} \frac{|z|^{2m}}{m} = -\frac{1}{4}\log(1-|z|^2),$$

and the series converges for all $z \in \mathbb{D}$. Consequently, the general solution of equation (3.4) is given by

$$f(z) = g(z) + h(z),$$

where $h(z)$ is a harmonic function. Moreover, by assumption

$$|g(z) + h(z)| \le M,$$

for some $M > 0$; that is,

$$-g(z) - M \le h(z) \le -g(z) + M.$$

This yields

$$\exp(h(z)) \le \exp(-g(z) + M) = (1-|z|^2)\exp(M).$$

Thus for $z = re^{i\theta}$ we have $\exp(h(re^{i\theta})) \le (1-r^2)\exp(M)$, so that $\exp(h(re^{i\theta})) \to 0$ uniformly as $r \to 1^-$. This implies that $\exp h(z) \equiv 0$ by the maximum principle since $\exp h(z) = |\exp(h(z) + i\tilde{h}(z))|$, where $\tilde{h}$ is a harmonic conjugate for h. This is a contradiction, and the proof is complete. ∎

A key idea for this proof was provided to the authors by E. Straube.

A natural question arises as to what happens to these results if we weaken the corona condition. We will discuss this situation in Section 6.

4. Curvature equalities for general $\mathcal{R}$

An interesting observation in the proof of Theorem 3.5 is that the difference of the curvature of $\mathcal{H}_\Phi$ and that of $H^2(\mathbb{D})$,

$$\mathcal{K}_{\mathcal{H}_\Phi}(w) - \mathcal{K}_{H^2(\mathbb{D})}(w) = -\frac{1}{4}\bigtriangledown^2 \log(|\varphi_1(w)|^2 + |\varphi_2(w)|^2)\, dz \wedge d\bar{z},$$

depends only on the multipliers φ_1 and φ_2 and not on $H^2(\mathbb{D})$. Moreover, the same is seen to be true when $H^2(\mathbb{D})$ is replaced by any of the family of weighted Bergman Hilbert modules. In this section, we prove that this phenomenon holds for a large class of quasi-free Hilbert modules.

THEOREM 4.1. *Suppose $\mathcal{R}$ is a contractive, reproducing kernel Hilbert module over $\mathbb{D}$ with kernel function $k : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ such that $\mathcal{R} \in B_1(\mathbb{D})$ and set $k_w = k(\cdot, w)$ for $w \in \mathbb{D}$. Assume that $\Phi = \{\varphi_1, \varphi_2\}$ is a corona pair in $H^\infty(\mathbb{D})$, $T_\Phi : \mathcal{R} \otimes \mathbb{C} \to \mathcal{R} \otimes \mathbb{C}^2$, and $\mathcal{R}_\Phi$ is the quotient Hilbert module defined so that $\mathcal{R}_\Phi = (\mathcal{R} \otimes \mathbb{C}^2)/ran\ T_\Phi$. Then*

(1) $\gamma_w = k_w \otimes (\overline{\varphi_2(w)} e_1 - \overline{\varphi_1(w)} e_2) = k_w \otimes det \begin{bmatrix} e_1 & \overline{\varphi_1(w)} \\ e_2 & \overline{\varphi_2(w)} \end{bmatrix}$ is anti-holomorphic and satisfies $N_z^* \gamma_w = \bar{w} \gamma_w$, where $\{e_1, e_2\}$ is an orthonormal basis for $\mathbb{C}^2$, and N_z is module multiplication on $\mathcal{R}_\Phi$,

(2) $\vee_{w \in \mathbb{D}} \gamma_w = \mathcal{R}_\Phi \subseteq \mathcal{R} \otimes \mathbb{C}^2$, and

(3) $\mathcal{R}_\Phi \in B_1(\mathbb{D})$.

Proof. That function γ_w is anti-holomorphic follows from the fact that $\overline{\varphi_1(w)}, \overline{\varphi_2(w)}$ and k_w are anti-holomorphic functions from $\mathbb{D}$ to $\mathbb{C}$, $\mathbb{C}$ and $\mathcal{R}$, respectively. Similarly, the same argument used in the proof of Proposition 3.2 shows that $N_z^* \gamma_w = \bar{w} \gamma_w$ for all $w \in \mathbb{D}$.

To show (2) we must prove for $G = g_1 \otimes e_1 + g_2 \otimes e_2 \in \mathcal{R} \otimes \mathbb{C}^2$, that $G \perp \bigvee_{w \in \mathbb{D}} \gamma_w$ implies that $G \in \text{ran } T_\Phi$. If $G \perp \gamma_w$ for every $w \in \mathbb{D}$, then

$$\begin{aligned} \langle G, \gamma_w \rangle &= \langle g_1, k_w \rangle \langle e_1, \overline{\varphi_2(w)} e_1 \rangle - \langle g_2, k_w \rangle \langle e_2, \overline{\varphi_1(w)} e_2 \rangle \\ &= g_1(w) \varphi_2(w) - g_2(w) \varphi_1(w) = 0, \end{aligned}$$

or

$$\det \begin{bmatrix} g_1(w) & \varphi_1(w) \\ g_2(w) & \varphi_2(w) \end{bmatrix} = 0, \tag{4.1}$$

for all $w \in \mathbb{D}$. Then there exists a unique nonzero function $\eta(w)$ satisfying $g_i(w) = \varphi_i(w)\eta(w)$ for $i = 1, 2$. Here we are using the fact that the rank of $\begin{bmatrix} \varphi_1(w) \\ \varphi_2(w) \end{bmatrix}$ is one for $w \in \mathbb{D}$.

The proof is completed once we show that $\eta \in \mathcal{R}$. Let ψ_i for $i = 1, 2$ be functions in $H^\infty(\mathbb{D})$ such that $\psi_1(w)\varphi_1(w) + \psi_2(w)\varphi_2(w) = 1$ for every $w \in \mathbb{D}$. Note that the existence of such functions follows from the corona theorem and the assumption that $\{\varphi_1, \varphi_2\}$ is a corona pair. Then the claim follows since we can express the function η as $\eta = (\psi_1\varphi_1 + \psi_2\varphi_2)\eta = \psi_1 g_1 + \psi_2 g_2$, where $\psi_i \in H^\infty(\mathbb{D})$ and $g_i \in \mathcal{R}$ for $i = 1, 2$. This completes the proof of (2).

Now, applying localization to the short exact sequence

$$0 \longrightarrow \mathcal{R} \otimes \mathbb{C} \xrightarrow{T_\Phi} \mathcal{R} \otimes \mathbb{C}^2 \longrightarrow \mathcal{R}_\Phi \longrightarrow 0,$$

we find that $\dim \ker (N_z - w)^* = 1$ for $w \in \mathbb{D}$. Hence, we have $\mathcal{R}_\Phi \in B_1^w(\mathbb{D})$. Finally, if we write

$$M_z \backsim \begin{bmatrix} * & * \\ 0 & N_z \end{bmatrix},$$

relative to the decomposition $\mathcal{R} = \text{ran} T_\Phi \oplus (\text{ran } T_\Phi)^\perp$, then we see that the assumption $\mathcal{R} \in B_1(\mathbb{D})$ implies M_z is onto, which implies N_z is onto for $w \in \mathbb{D}$ and hence $\mathcal{R}_\Phi \in B_1(\mathbb{D})$. ∎

If we assume that $H^\infty(\mathbb{D})$ is the multiplier algebra for a bounded quasi-free Hilbert module $\mathcal{R}$, then we do not need to assume that $\mathcal{R}$ is contractive.

We continue with some further consequences.

THEOREM 4.2. *Assume that the hypotheses of Theorem 4.1 hold. Consider the hermitian holomorphic line bundle L_Φ given by the exact sequence*

$$0 \longrightarrow \mathbb{C} \xrightarrow{\Phi} \mathbb{C}^2 \longrightarrow L_\Phi \longrightarrow 0;$$

that is, the fiber $L_\Phi(w) = \mathbb{C}^2/ran\,\Phi(w)$. Then

(1) $E^*_{\mathcal{R}_\Phi} \cong E^*_{\mathcal{R}} \otimes L^*_\Phi$, where $\otimes$ denotes the tensor product of vector bundles,

(2) $\mathcal{K}_{\mathcal{R}_\Phi} - \mathcal{K}_{\mathcal{R}} = \mathcal{K}_{L_\Phi}$, and

(3) $\mathcal{K}_{L_\Phi} = -\frac{1}{4} \bigtriangledown^2 \log(|\varphi_1(z)|^2 + |\varphi_2(z)|^2) dz \wedge d\bar{z}$.

Proof. The main observation is that we can use the anti-holomorphic section γ_w for $E^*_{\mathcal{R}_\Phi}$ to represent $E^*_{\mathcal{R}_\Phi}$ as $E^*_{\mathcal{R}} \otimes L^*_\Phi$ since k_w and and $\overline{\varphi_2(w)}e_1 - \overline{\varphi_1(w)}e_2$ are anti-holomorphic sections of $E^*_{\mathcal{R}}$ and L^*_Φ, respectively. To calculate the curvature of $E^*_{\mathcal{R}_\Phi}$, we note that

$$\log \|\gamma_w\|^2 = \log(\|\overline{\varphi_2(w)}e_1 - \overline{\varphi_1(w)}e_2\|^2 \|k_w\|^2) = \log(|\varphi_1(z)|^2 + |\varphi_2(z)|^2) + \log \|k_w\|^2,$$

and the result follows. ∎

Essentially this identification was used implicitly by Uchiyama [15] and Treil and the third author [9] when $\mathcal{R} = H^2(\mathbb{D})$.

We can use this result to show that the question of when two quotient Hilbert modules $\mathcal{R}_\Phi$ and $\mathcal{R}_\Psi$ are unitarily equivalent for some contractive quasi-free Hilbert module $\mathcal{R}$ is independent of $\mathcal{R}$.

COROLLARY 4.3. *Let $\mathcal{R}$ and $\tilde{\mathcal{R}}$ be two Hilbert modules satisfying the hypotheses of Theorem 4.1 and $\{\varphi_1, \varphi_2\}$, $\{\psi_1, \psi_2\}$ be two corona pairs. Then $\mathcal{R}_\Phi \cong \mathcal{R}_\Psi$ if and only if $\tilde{\mathcal{R}}_\Phi \cong \tilde{\mathcal{R}}_\Psi$.*

One can extend these results to the case of a quotient Hilbert module $\mathcal{R}_\Phi$, where Φ is a multiplier from $\mathbb{C}^m$ to $\mathbb{C}^{m+1}$. We can assume that $\mathcal{R}$ is contractive in which case its multiplier algebra is $H^\infty(\mathbb{D})$ or finesse that issue by assuming Φ has a left inverse in the multiplier algebra. Here we do the latter avoiding the use of the corona theorem.

THEOREM 4.4. *Let $\mathcal{R}$ be a reproducing kernel Hilbert module over $\mathbb{D}$ such that $\mathcal{R} \in B_1(\mathbb{D})$. Assume that $\Phi : \mathbb{D} \to \mathcal{L}(\mathbb{C}^m, \mathbb{C}^{m+1})$ is a multiplier which has a left inverse multiplier map $\Psi : \mathbb{D} \to \mathcal{L}(\mathbb{C}^{m+1}, \mathbb{C}^m)$. Then the multiplication operator T_Φ has closed range, rank $\Phi(w) = m$ for $w \in \mathbb{D}$ and $\Phi^*(z)\Phi(z) \geq \delta I_{\mathbb{C}^m}$ for some $\delta > 0$. If $\mathcal{R}_\Phi$ is the quotient Hilbert module $\mathcal{R}_\Phi = (\mathcal{R} \otimes \mathbb{C}^{m+1})/ran\ T_\Phi$, then*

(1) $\mathcal{R}_\Phi \in B_1(\mathbb{D})$,

(2) $L_\Phi(w) = \mathbb{C}^{m+1}/ran\,\Phi(w)$ defines a holomorphic line bundle such that $E^*_{\mathcal{R}_\Phi} \cong E^*_{\mathcal{R}} \otimes L^*_\Phi$, and

(3) $\mathcal{K}_{\mathcal{R}_\Phi} - \mathcal{K}_{\mathcal{R}} = \mathcal{K}_{L_\Phi}$.

Proof. Essentially the same proof as that used above will work once we exhibit a non-vanishing anti-holomorphic cross section of L^*_Φ and the resulting anti-holomorphic cross section of $\mathcal{R}_\Phi$. To that end express Φ as an $(m+1) \times m$ matrix of functions $\{\varphi_{ij}\} \subseteq \mathcal{M}(\mathcal{R})$, the multiplier algebra of $\mathcal{R}$, and let $\{e_1, \ldots, e_{m+1}\}$ be the standard orthonormal basis for $\mathbb{C}^{m+1}$. If we set

$$\Delta_\Phi(w) = \det \begin{bmatrix} e_1 & \varphi_{1,1} & \cdots & \varphi_{1,m} \\ \vdots & \vdots & \vdots & \vdots \\ e_{m+1} & \varphi_{m+1,1} & \cdots & \varphi_{m+1,m} \end{bmatrix}, \tag{4.2}$$

the formal determinant, for $w \in \mathbb{D}$, then standard properties of determinants show that $k_w \otimes \overline{\Delta_\Phi(w)}$ is orthogonal to ran T_Φ for $w \in \mathbb{D}$, where k_w is an anti-holomorphic cross section of $E^*_{\mathcal{R}}$ and $\overline{\Delta_\Phi(w)}$ is the complex conjugate of $\Delta_\Phi(w)$. Moreover, since the rank of $\Phi(w)$ is m, $\Delta_\Phi(w) \neq 0$ for $w \in \mathbb{D}$. Thus $\gamma_w = k_w \otimes \overline{\Delta_\Phi(w)} \neq 0$ for $w \in \mathbb{D}$ because the coefficients of $\Delta_\Phi(w)$ are the complex conjugate of the determinants of the principal minors of the matrix for $\Phi(w)$ and hence a cross section of $E^*_{\mathcal{R}_\Phi}$.

We claim that $k_w \otimes \overline{\Delta_\Phi(w)}$ is the desired anti-holomorphic cross-section of $E^*_{\mathcal{R}_\Phi}$; that is, $\bigvee_{w\in\mathbb{D}} k_w \otimes \overline{\Delta_\Phi(w)} = \mathcal{R}_\Phi$, which would complete the proof since the remainder of the argument is the same as that for Proposition 3.2. Let $G = \Sigma_{i=1}^{m+1} g_i \otimes e_i \in \mathcal{R} \otimes \mathbb{C}^{m+1}$ and assume $G \perp \gamma_w$ for every $w \in \mathbb{D}$. Then we have that $g_i(w) = \Sigma_{j=1}^{m} \eta_j(w)\varphi_{ij}(w)$ for $i = 1, ..., m+1$, for some unique functions $\{\eta_j(w)\}_{j=1}^m$ on $\mathbb{D}$. To show this one uses Cramer's rule to solve for the $\{\eta_j(w)\}_{j=1}^m$ as follows. For each $w_0 \in \mathbb{D}$ at least one of the principal minors of $\Phi(w_0)$ is non-zero. Since the determinant of the matrix

$$\begin{bmatrix} g_1(w_0) & \varphi_{1,1}(w_0) & \cdots & \varphi_{1,m}(w_0) \\ \vdots & \vdots & \vdots & \vdots \\ g_{m+1}(w_0) & \varphi_{m+1,1}(w_0) & \cdots & \varphi_{m+1,m}(w_0) \end{bmatrix}$$

is zero and the determinant of some principal minor is not, we can solve uniquely for the $\{\eta_j(w_0)\}_{j=1}^m$. To prove that the resulting functions on $\mathbb{D}$ are in $\mathcal{R}$, let $G = \begin{bmatrix} g_1 \\ \vdots \\ g_{m+1} \end{bmatrix}$ and $\Xi = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_m \end{bmatrix}$ so that $G(w) = \Phi(w)\Xi(w)$ for $w \in \mathbb{D}$. By hypothesis there exists a multiplier Ψ such that $\Psi\Phi = I$. Since $\Xi(w) = (\Psi(w)\Phi(w))\Xi(w) = \Psi(w)(\Phi(w)\Xi(w)) = \Psi(w)G(w)$ as functions on $\mathbb{D}$, we have that $\Xi(w)$ is indeed in the vector-valued $\mathcal{R} \otimes \mathbb{C}^m$, which completes the proof. ■

This result enables us to generalize Theorem 3.5 to a larger class of quasi-free Hilbert modules $\mathcal{R}$ and to multipliers from $\mathcal{R} \otimes \mathbb{C}^m$ to $\mathcal{R} \otimes \mathbb{C}^{m+1}$ for $m \in \mathbb{N}$.

Corollary 4.5. *Assuming the hypotheses of Theorem 4.4 hold, then the quotient Hilbert modules $\mathcal{R}_\Phi$ and $\mathcal{R}_\Psi$ are unitarily equivalent if and only if*

$$\bigtriangledown^2 log\ \|\Delta_\Phi\| = \bigtriangledown^2 log\ \|\Delta_\Psi\|,$$

where $\triangle_\Phi$ and $\triangle_\Psi$ are defined in the proof of Theorem 4.4.

Proof. The result follows since $k_w \otimes \overline{\Delta_\Phi(w)}$ and $k_w \otimes \overline{\Delta_\Psi(w)}$ are anti-holomorphic cross-sections of $E^*_{\mathcal{R}_\Phi}$ and $E^*_{\mathcal{R}_\Psi}$, respectively. ■

This representation of reproducing kernel Hilbert modules in $B_1(\mathbb{D})$ as a quotient module, does not include all of the quotient Hilbert modules in $B_1(\mathbb{D})$. For that, one would need to consider infinite dimensional coefficient Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, and a multiplier $\Phi : \mathbb{D} \to \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ such that $\dim \mathcal{E}_*/\operatorname{ran}\Phi(z) = 1$ for $z \in \mathbb{D}$. An explicit formula for an anti-holomorphic cross-section and the curvature of the quotient Hilbert module, however, would require some additional hypothesis such as the existence of the operator-valued determinant to define an analogue of $\triangle_\Phi(w)$. We will not pursue this matter in the current paper.

For $\mathcal{E}$ and $\mathcal{E}_*$ finite dimensional and a multiplier $\Phi : \mathbb{D} \to \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ with constant rank, one can define bundles with fibers $\ker \Phi(w)$ and $\operatorname{coker}\Phi(w) = \mathcal{E}_*/\operatorname{ran}\Phi(w)$ for $w \in \mathbb{D}$. Moreover, using the reproducing kernel, related Hilbert modules in $B_k(\mathbb{D})$ can be defined for $k \geq 1$. We consider here the most general case when Φ has no kernel and only some of the most direct results.

Theorem 4.6. *Suppose $\mathcal{R} \in B_1(\mathbb{D})$ is a reproducing kernel Hilbert module over $\mathbb{D}$. Assume that $\Phi : \mathbb{D} \to \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ is a multiplier and that $T_\Phi : \mathcal{R} \otimes \mathcal{E} \to \mathcal{R} \otimes \mathcal{E}_*$ has a left inverse T_Ψ for some multiplier $\Psi : \mathbb{D} \to \mathcal{L}(\mathcal{E}_*, \mathcal{E})$. Then the rank $\Phi(z) = \dim \mathcal{E} = m$ for $z \in \mathbb{D}$, the range of T_Φ is closed and there exists an n-dimensional hermitian holomorphic bundle V over $\mathbb{D}$ such that*

$$E^*_{\mathcal{R}_\Phi} \cong E^*_{\mathcal{R}} \otimes V^*,$$

or

$$E^*_{\mathcal{R}_\Phi}(w) \cong E^*_{\mathcal{R}}(w) \otimes V^*(w),$$

for $w \in \mathbb{D}$, where $\mathcal{R}_\Phi$ is the quotient Hilbert module defined by $(\mathcal{R} \otimes \mathcal{E}_*)/ran T_\Phi$, $V(w) = \mathcal{E}_*/ran\ \Phi(w)$ and $n = \dim\ \mathcal{E}_* - m$. Moreover, one has $\mathcal{K}_{\mathcal{R}_\Phi} - \mathcal{K}_{\mathcal{R}} \otimes I_V = I_{\mathcal{R}} \otimes \mathcal{K}_V$, or

$$\mathcal{K}_{\mathcal{R}_\Phi}(w) - \mathcal{K}_{\mathcal{R}}(w) \otimes I_V(w) = I_{\mathcal{R}}(w) \otimes \mathcal{K}_V(w),$$

for $w \in \mathbb{D}$. Finally, $\mathcal{R}_\Phi \in B_n(\mathbb{D})$.

Proof. First, the fact that Φ has a left inverse implies that rank $\Phi(w) = \dim\ \mathcal{E}$ for $w \in \mathbb{D}$. Now let V^* be the anti-holomorphic sub-bundle of $\mathbb{D} \times \mathcal{E}_*$ defined to be the orthogonal complement of ran $\Phi(w)$. Here we are using the fact that rank $\Phi(w)$ is constant. (The bundle V, the dual of V^*, is most naturally defined as the quotient bundle of $\mathbb{D} \times \mathcal{E}_*$ by ran $\Phi(w)$.) Since V^* is an anti-holomorphic vector bundle over $\mathbb{D}$, it is trivial and hence there exists an anti-holomorphic frame $\{h_i(w)\}_{i=1}^n$ for V^*. If k_w is a non-vanishing anti-holomorphic cross-section of the line bundle $E^*_{\mathcal{R}}$, then we show that $\{k_w \otimes h_i(w)\}_{i=1}^n$ is an anti-holomorphic frame for $E^*_{\mathcal{R}_\Phi}$. The rest of the proof is similar to that given for Theorem 4.4.

First, we observe that each $k_w \otimes h_i(w)$, $w \in \mathbb{D}$ and $i = 1, \ldots, n$, is orthogonal to the range of T_Φ. To prove $\mathcal{R}_\Phi \in B_n(\mathbb{D})$, we need to show that $\mathcal{R}_\Phi = \vee\{k_w \otimes h_i(w) : w \in \mathbb{D}, i = 1, \ldots, n\}$. Suppose $G = \sum_{i=1}^{\dim \mathcal{E}_*} g_i \otimes e_i \in \mathcal{R} \otimes \mathcal{E}_*$ is orthogonal to $\vee\{k_w \otimes h_i(w) : w \in \mathbb{D}, i = 1, \ldots, n\}$, where $\{e_i\}_{i=1}^{\dim\ \mathcal{E}_*}$ is an orthonormal basis for $\mathcal{E}_*$, or $\langle G, k_w \otimes h_i(w)\rangle = 0$ for $w \in \mathbb{D}$ and $i = 1, \ldots, n$. Since for $w_0 \in \mathbb{D}$, the rank of $\Phi(w_0)$ is m, we can identify an $m \times m$ sub-matrix of $\Phi(w_0)$ with non-zero determinant. Again, using Cramer's rule we can solve uniquely for an m-tuple of complex numbers, $\Xi(w_0) = \{\eta_i(w_0)\}_{i=1}^m \in \mathbb{C}^m$, such that $G(w_0) = \Phi(w_0)\Xi(w_0)$.

To show that the resulting functions $\{\eta_i\}_{i=1}^m$ on $\mathbb{D}$ are in $\mathcal{R}$, we use the left inverse $\Psi(w)$ of $\Phi(w)$ to conclude that

$$\Xi(w) = \Psi(w)\Phi(w)\Xi(w) = \Psi(w)G(w) \in \mathcal{R} \otimes \mathcal{E}_*.$$

Here we are using the fact that $G \in \mathcal{R} \otimes \mathcal{E}_*$ and Ψ is a multiplier in $\mathcal{L}(\mathcal{R} \otimes \mathcal{E}_*, \mathcal{R} \otimes \mathcal{E})$.

To establish the curvature formula, we first recall that the formula for the curvature of the Chern connection for a vector bundle is $\partial[H^{-1}\bar{\partial}H]$, where H is the gramian for an anti-holomorphic frame for the bundle (see [16]). If H_Φ is the gramian for the frame $\{k_w \otimes h_i(w)\}_{i=1}^m$, then $H_\Phi(w)$ is the $n \times n$ matrix

$$H_\Phi(w) = \big(\langle k_w \otimes h_i(w), k_w \otimes h_j(w)\rangle\big)_{i,j=1}^n = \|k_w\|^2\big(\langle h_i(w), h_j(w)\rangle\big)_{i,j=1}^n = \|k_w\|^2 H_h(w),$$

where H_h is the gramian for the anti-holomorphic frame $\{h_i(w)\}_{i=1}^n$ for V^*. Then

$$\begin{aligned}
\partial[H_\Phi^{-1}(\bar{\partial}H_\Phi)] &= \partial[\frac{1}{\|k_w\|^2}H_h^{-1}(\bar{\partial}(\|k_w\|^2 H_h))] \\
&= \partial[\frac{1}{\|k_w\|^2}H_h^{-1}((\bar{\partial}(\|k_w\|^2)H_h) + \|k_w\|^2\bar{\partial}H_h)] \\
&= \partial(\frac{1}{\|k_w\|^2}\bar{\partial}(\|k_w\|^2) + H_h^{-1}\bar{\partial}H_h)) \\
&= \partial(\frac{1}{\|k_w\|^2}\bar{\partial}(\|k_w\|^2)) + \partial[H_h^{-1}(\bar{\partial}H_h)].
\end{aligned}$$

Hence, expressing these matrices in terms of the respective frames and using the fact that the coordinates of a bundle and its dual can be identified, one has

$$\mathcal{K}_{\mathcal{R}_\Phi}(w) - \mathcal{K}_{\mathcal{R}}(w) \otimes I_V(w) = I_{\mathcal{R}}(w) \otimes \mathcal{K}_V(w),$$

for $w \in \mathbb{D}$, and its dual are adjoints of each other and hence can be identified since they are self-adjoint, which completes the proof. ■

As a consequence we can extend our corollary on the independence of unitary equivalence of the quotient Hilbert modules on the building block Hilbert module as follows.

Corollary 4.7. *Let $\mathcal{R}$ and $\tilde{\mathcal{R}}$ be Hilbert modules and $\Phi_i : \mathbb{D} \to \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ for $i = 1, 2$ be multiplier maps for $\mathcal{R}$ and $\tilde{\mathcal{R}}$, respectively, satisfying the hypotheses of Theorem 4.4. Then $\mathcal{R}_{\Phi_1} \cong \mathcal{R}_{\Phi_2}$ if and only if $\tilde{\mathcal{R}}_{\Phi_1} \cong \tilde{\mathcal{R}}_{\Phi_2}$.*

Proof. From the theorem we see that $E^*_{\mathcal{R}_{\Phi_i}} \cong E^*_{\mathcal{R}} \otimes V^*_{\Phi_i}$ and $E^*_{\tilde{\mathcal{R}}_{\Phi_i}} \cong E^*_{\tilde{\mathcal{R}}} \otimes V^*_{\Phi_i}$ for $i = 1, 2$. The result follows from the fact that $E^*_{\mathcal{R}} \otimes V^*_{\Phi_1}$ is isometrically isomorphic to $E^*_{\mathcal{R}} \otimes V^*_{\Phi_2}$ if and only if $V^*_{\Phi_1} \cong V^*_{\Phi_2}$, which follows from the fact that $E^*_{\mathcal{R}} \otimes V^*_{\Phi_1}$ is isomorphic to $V^*_{\Phi_1}$ and isometrically isomorphic to the latter in which the metric on each fiber of $V^*_{\Phi_1}(w)$ is multiplied by $\|k_w\|$. ■

5. $B_1(\mathbb{D})$ class and Similarity

In this section, we investigate a sufficient condition for certain quotient Hilbert modules, which are in $B_1(\mathbb{D})$, to be similar to the rank one quasi-free Hilbert module from which it is constructed.

THEOREM 5.1. *Let $\mathcal{R}$ be a multiplicity one quasi-free Hilbert module over $\mathbb{D}$. Let $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathcal{M}(\mathcal{R})$ such that $\varphi_1\psi_1 + \varphi_2\psi_2 = 1$. Then the range of T_Φ is closed, where $T_\Phi f = (\varphi_1 f, \varphi_2 f)$, and the quotient Hilbert module $\mathcal{R}_\Phi$ given by*

$$\cdots \longrightarrow \mathcal{R} \otimes \mathbb{C} \xrightarrow{T_\Phi} \mathcal{R} \otimes \mathbb{C}^2 \xrightarrow{\pi_\Phi} \mathcal{R}_\Phi \longrightarrow 0,$$

is similar to $\mathcal{R}$, where π_Φ is the quotient map.

Proof. Let $R_\Psi : \mathcal{R} \oplus \mathcal{R} \to \mathcal{R}$ be the bounded module map defined by $R_\Psi(f \oplus g) = \psi_1 f + \psi_2 g$ for $f, g \in \mathcal{R}$. By assumption,

$$R_\Psi T_\Phi = I.$$

Then for any $f \oplus g \in \mathcal{R} \oplus \mathcal{R}$, we have

$$f \oplus g = (T_\Phi R_\Psi(f \oplus g)) + (f \oplus g - T_\Phi R_\Psi(f \oplus g)),$$

with $T_\Phi R_\Psi(f \oplus g) \in \operatorname{ran} T_\Phi$ and $(f \oplus g - T_\Phi R_\Psi(f \oplus g)) \in \ker R_\Psi$. Also

$$\operatorname{ran} T_\Phi \cap \ker R_\Psi = \{0\}.$$

Consequently,

$$\mathcal{R} \oplus \mathcal{R} = \operatorname{ran} T_\Phi \dot{+} \ker R_\Psi.$$

Hence, there exists an idempotent $Q \in \mathcal{M}(\mathcal{R})$ such that $Q(\Phi f + g) = g$ for $f \in \mathcal{R}$ and $g \in \ker\, R_\Psi$. Moreover, $\operatorname{ran} T_\Phi = \ker Q$ and $\ker R_\Psi = \operatorname{ran} Q$. The invertible module map $Q \circ \pi_\Phi^{-1} : \mathcal{R}_\Phi \to \mathcal{R}$ is well defined and the required similarity. ■

COROLLARY 5.2. *Let $\{\varphi_1, \varphi_2\}$ be a corona pair in $H^\infty(\mathbb{D})$ and $\mathcal{R}$ be a multiplicity one, contractive quasi-free Hilbert module. Then the quotient Hilbert module $\mathcal{R}_\Phi$ given by*

$$\cdots \longrightarrow \mathcal{R} \otimes \mathbb{C} \xrightarrow{T_\Phi} \mathcal{R} \otimes \mathbb{C}^2 \xrightarrow{\pi_\Phi} \mathcal{R}_\Phi \longrightarrow 0,$$

is similar to $\mathcal{R}$.

Proof. We appeal to the corona theorem for $H^\infty(\mathbb{D})$ to get $\psi_1, \psi_2 \in H^\infty(\mathbb{D})$ such that $\varphi_1\psi_1 + \varphi_2\psi_2 = 1$. The corollary now follows from Theorem 5.1. ■

Another way to prove Corollary 5.2 is to use a Koszul complex type construction. In other words, if $\varphi_1\psi_1 + \varphi_2\psi_2 = 1$, then we let $T_{\tilde{\Psi}}$ be the module map in $\mathcal{L}(\mathcal{R}, \mathcal{R} \oplus \mathcal{R})$ defined by $T_{\tilde{\Psi}} f = (-\psi_2 f \oplus \psi_1 f)$. Therefore,

$$\mathcal{R} \otimes \mathbb{C}^2 = \operatorname{ran} T_\Phi \dot{+} \operatorname{ran} T_{\tilde{\Psi}},$$

and hence

$$\operatorname{ran} T_{\tilde{\Psi}} \simeq \mathcal{R}_\Phi.$$

Since $\ker T_{\tilde{\Psi}} = \{0\}$, the corollary follows.

We conclude the section with a result which essentially states that the similarity criterion for a certain class of quotient Hilbert modules is independent of the choice of the basic quasi-free Hilbert module "building blocks". We begin with the following theorem, which states that the splitting for a class of quotient Hilbert modules is an invariant property.

Theorem 5.3. *Suppose $\mathcal{R}, \hat{\mathcal{R}} \in B_1(\mathbb{D})$ are two reproducing kernel Hilbert module over $\mathbb{D}$ with multiplier spaces $\mathcal{M}(\mathcal{R})$ and $\mathcal{M}(\hat{\mathcal{R}})$, respectively, and let $\theta \in \mathcal{M}(\mathcal{R}) \cap \mathcal{M}(\hat{\mathcal{R}})$ such that both $T_\theta \in \mathcal{L}(\mathcal{R} \otimes \mathcal{E}, \mathcal{R} \otimes \mathcal{E}_*)$ and $\hat{T}_\theta \in \mathcal{L}(\hat{\mathcal{R}} \otimes \mathcal{E}, \hat{\mathcal{R}} \otimes \mathcal{E}_*)$ have multiplier left inverse with $\mathcal{E}$ and $\mathcal{E}_*$ finite dimensional Hilbert spaces. Then the quotient Hilbert modules $\mathcal{R}_\theta = (\mathcal{R} \otimes \mathcal{E}_*)/ran\ T_\theta$ and $\hat{\mathcal{R}}_\theta = (\hat{\mathcal{R}} \otimes \mathcal{E}_*)/ran\ \hat{T}_\theta$ are in $B_n(\mathbb{D})$. Moreover, the resolution*

$$\mathcal{R} \otimes \mathcal{E} \xrightarrow{T_\theta} \mathcal{R} \otimes \mathcal{E}_* \xrightarrow{\pi_\theta} \mathcal{R}_\theta \longrightarrow 0,$$

of $\mathcal{R}_\theta$ splits if and only if the analogous resolution of $\hat{\mathcal{R}}_\theta$ splits.

Proof. The first part follows from Theorem 4.6. To prove the second part, first, since $\mathcal{R}_\theta$ and $\hat{\mathcal{R}}_\theta$ are in $B_n(\mathbb{D})$, localizing at $z \in \mathbb{D}$, we have the following diagram

$$\begin{array}{ccccccc}
E_{\mathcal{R}} \otimes \mathcal{E} & \longrightarrow & E_{\mathcal{R}} \otimes \mathcal{E}_* & \longrightarrow & E_{\mathcal{R}} \otimes V_\theta & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{E}_z & \longrightarrow & \mathcal{E}_{*\,z} & \longrightarrow & V_{\theta,z} & \longrightarrow & 0
\end{array}$$

where $V_\theta = \sqcup_{w\in\Omega} \mathcal{E}_*/(\theta(w)\mathcal{E})$ and $\dim\left[\mathcal{E}_*/(\theta(w)\mathcal{E})\right] = n$ for all $w \in \mathbb{D}$. Now assume that $\mathcal{R}_\theta$ splits; that is, there exists a cross section $\sigma_\theta : \mathcal{R}_\theta \to \mathcal{R} \otimes \mathcal{E}_*$ such that $\pi_\theta \sigma_\theta = I_{\mathcal{R}_\theta}$. Again we localize this module map to obtain

$$V_{\theta\, z} \xrightarrow{\sigma_\theta(z)} \mathcal{E}_{*\,z}.$$

Moreover,

$$\pi_\theta(z)\sigma_\theta(z) = I_{V_{\theta z}}.$$

Then,

$$\mathcal{E}_{*z} = \operatorname{ran} \theta(z) \dot{+} \operatorname{ran} \sigma_\theta(z).$$

This decomposition is clearly independent of the choice of the Hilbert module $\mathcal{R}$. Moreover, given such a decomposition, one can obtain the cross section σ_θ. Hence, $\mathcal{R}_\theta$ splits if and only if $\hat{\mathcal{R}}_\theta$ also splits. ∎

By Theorem 3.2 of [2] and Theorem 5.3, we obtain

Theorem 5.4. *Let $\mathcal{R} \in B_1(\mathbb{D})$ be a reproducing kernel Hilbert module with $H^\infty(\mathbb{D})$ as the multiplier space and let $\theta \in H^\infty_{\mathcal{L}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$ with $\mathcal{E}$ and $\mathcal{E}_*$ finite dimensional Hilbert spaces such that $T_\theta \in \mathcal{L}(\mathcal{R} \otimes \mathcal{E}, \mathcal{R} \otimes \mathcal{E}_*)$ has a left inverse in $H^\infty_{\mathcal{L}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D})$. If the quotient Hilbert module $\mathcal{R}_\theta = (\mathcal{R} \otimes \mathcal{E}_*)/ran\ T_\theta$ is in $B_n(\mathbb{D})$, then the quotient Hilbert module $\mathcal{H}_\theta = (H^2(\mathbb{D}) \otimes \mathcal{E}_*)/ran\ T_\theta$ is similar to $\mathcal{R} \otimes \mathcal{F}$ if and only if $\mathcal{H}_\theta$ is similar to $H^2(\mathbb{D}) \otimes \mathcal{F}$ for the same Hilbert space $\mathcal{F}$.*

Proof. Restricting Theorem 3.2 in [2] to the one-variable context, we have that $\mathcal{H}_\theta$ is similar to $H^2(\mathbb{D})\otimes\mathcal{F}$ for some Hilbert space $\mathcal{F}$ if and only if the short exact module sequence defining $\mathcal{H}_\theta$ splits. The rest of the proof follows from the previous theorem. ■

Theorem 5.4 along with Theorem 0.1 in [9] provide a connection between the quotient Hilbert modules of the Hardy module and those of any other reasonable contractive reproducing kernel Hilbert module over $\mathbb{D}$ such as the Bergman module or the weighted Bergman modules.

Corollary 5.5. *With the assumptions in Theorem 5.4, if $\Pi(z)$ is the orthogonal projection onto the localization of π_θ at z, then the following statements are equivalent:*

(1) $\mathcal{R}_\theta$ is similar to $\mathcal{R}\otimes\mathcal{F}$ for some Hilbert space $\mathcal{F}$.

(2) $\mathcal{H}_\theta$ is similar to $H^2(\mathbb{D})\otimes\mathcal{F}$ for some Hilbert space $\mathcal{F}$.

(3) The eigenvector bundles of $\mathcal{R}_\theta$ and $\mathcal{R}\otimes\mathcal{F}$ are uniformly equivalent; that is, there exists an anti-holomorphic point-wise invertible bundle map $\Phi : E^*_{\mathcal{R}_\Phi} \to E^*_{\mathcal{R}}\otimes\mathcal{F}$ and a scalar $c>0$ such that $\|\Phi(w)\|\le c$ and $\|\Phi^{-1}(w)\|\le c$ for all $w\in\mathbb{D}$.

(4)There exists a bounded subharmonic function φ on $\mathbb{D}$ such that

$$\nabla^2\varphi(z) \ge \|\frac{\partial\Pi(z)}{\partial z}\|_2^2 - \frac{n}{(1-|z|^2)^2}, \quad (z\in\mathbb{D}),$$

for some $n=dim\ \mathcal{F}$ where $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm.

(5) The measure

$$(\|\frac{\partial\Pi(z)}{\partial z}\|_2^2 - \frac{n}{(1-|z|^2)^2})(1-|z|)dxdy$$

is a Carleson measure for some $n=dim\ \mathcal{F}$ and the estimate

$$(\|\frac{\partial\Pi(z)}{\partial z}\|_2^2 - \frac{n}{(1-|z|^2)^2})^{\frac{1}{2}} \le \frac{C}{1-|z|}$$

holds for some $C>0$.

6. Concluding remarks

In this paper we have confined our attention to multipliers with a left inverse and hence closed range and no zeros. Obviously, both assumptions are artificial restrictions. Given a pair $\{\varphi_1,\varphi_2\}$ in $\mathcal{M}(\mathcal{R})$, one can consider the quotient module defined by the submodule of $\mathcal{R}\otimes\mathbb{C}^2$ equal to the closure of the range of T_Φ. Many of the results obtained in Section 4 can be extended to this case with their proof based on the existence of an appropriate analogue of inner-outer factorization for functions in $\mathcal{R}$. Details will be provided in a subsequent paper.

Another possible direction for generalization concerns the case in which the pair of functions is allowed to have common zeros or to converge to zero at the boundary of $\mathbb{D}$. Consideration of the case in which $\|\Phi(z)\|$ is dominated by the absolute value of a singular inner function is instructive for what can happen. This phenomenon will also be considered in a subsequent paper.

In Sections 3 and 4 we took up the question of which quotient modules $\mathcal{R}_{\theta_1}$ and $\mathcal{R}_{\theta_2}$ are unitarily equivalent for $\theta_i(z):\mathbb{C}^m\to\mathbb{C}^{m+1}$ for $z\in\mathbb{D}$ and $i=1,2$ and obtained an explicit criterion. Of course, this question is of interest for the case of quotient modules defined by multipliers $\theta_i(z) : \mathbb{C}^m \to \mathbb{C}^l$ for $i = 1, 2$ and arbitrary non-negative integers m and l. The arguments for Theorem 4.6 and Corollary 4.7 indicate one approach to providing an answer using the characterization of finite dimensional hermitian holomorphic bundles and hence in terms of partial derivatives of the curvatures (see [1]). However, one might be able to obtain more concrete results paralleling the one in Theorem 4.4. Rather than attempt to suggest such general results here, we pose the problem of doing that for the case $\mathbb{C} \to \mathbb{C}^m$ for general m or, for example, for $\mathbb{C} \to \mathbb{C}^4$.

Another question concerns a possible generalization of Corollary 5.5 to the case in which one or both of $\mathcal{E}$ and $\mathcal{E}_*$ are infinite dimensional. Examples due to Treil [13] (see also [14]) show some of the pathology that can occur in a holomorphic sub-bundle S of the trivial bundle $\mathbb{D} \times \mathcal{H}$ for $\mathcal{H}$ an infinite dimensional Hilbert space even when S or $(\mathbb{D} \times \mathcal{H})/S$ is finite dimensional. Hence, any extension of Corollary 5.5 to such cases is likely to be very technical.

Another question concerns the extension of the results of this paper to a general bounded domain Ω in $\mathbb{C}$ or to bounded domain in $\mathbb{C}^m$ for $m > 1$. Some questions for the latter case will be considered in [7].

Finally, the natural framework for the kind of similarity question studied in the last section would be to ask when two quotient modules $\mathcal{R}_{\Phi_1}$ and $\mathcal{R}_{\Phi_2}$ are similar for corona pairs Φ_1 and Φ_2. Since Φ being a corona pair implies $\mathcal{R}_\Phi$ is similar to $\mathcal{R}$, this question only makes sense for more general pairs such as those covered by the following definition.

Definition 6.1. *A pair of functions* $\{\varphi_1, \varphi_2\}$ *in* $H^\infty(\mathbb{D})$ *is said to be a quasi-corona pair if*

$$|\varphi_1(z)|^2 + |\varphi_2(z)|^2 > 0,$$

for all $z \in \mathbb{D}$ *except for an at most countable set.*

The considerations of Theorems 3.5 and 3.7 hold if we replace the assumption that Φ is a corona pair by the assumption that Φ is a quasi-corona pair. Since the set of zeroes of a holomorphic function is countable, it is discrete. One would also need to consider the derivatives in the sense of distributions to handle the curvature of quotient modules defined by the closure of ranges of two quasi-corona pairs Φ_1 and Φ_2.

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