LATTICES AND COHOMOLOGY.

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Abstract

We give an interpretation of the cohomology of an arithmetically defined group as a set of equivalence classes of lattices. We use this interpretation to give a simpler proof of the connection established in [11] between genus and cohomology.

1 Introduction.

Galois cohomology is a fundamental tool for the classification of certain algebraic structures.

To be precise, let k be a field, G a linear algebraic group acting on a space V, both defined over k. It is known (see [4]), that if G is defined as the set of automorphisms of a tensor τ on V, e.g., a quadratic form or an algebra structure, the cohomology set $H^1(K/k, G_K)$ classifies the K/k-forms of τ, i.e., those tensors of the same type also defined over k that become isomorphic to τ over the larger field K (see section 4).

It would be reasonable to expect, therefore, that a similar theory were available for structures for which the corresponding automorphism group is not an algebraic group but an arithmetically defined subgroup of an algebraic group.

Such a theory is already hinted at in [11]. In this reference, two finiteness results are proven. The first one deals with the finiteness of the local cohomology set $H^1(\mathcal{G}_w, \Gamma_w)$, for an arithmetically defined group Γ. Notations are as in [11]. The second one deals with the finiteness of the kernel of the map

$$H^1(\mathcal{G}, \Gamma) \longrightarrow \prod_{v \text{ place of } k} H(\mathcal{G}_{w(v)}, \Gamma_{w(v)}),$$

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where we have fixed a place $w(v)$ of K dividing each place v of k. It is the proof of the second result which requires expressing the given kernel in terms of the set of double cosets

$$G_k \backslash G_{\mathbb{A}_k} / \prod_w \Gamma_w$$

(see corollary 3.3 in [11]). These double cosets are the same ones that classify the classes of lattices in a genus. This is the relation we want to pursue.

In this paper, we show that the results in [11] are part of a much more general theory that relates cohomology sets of arithmetically defined groups with certain equivalence classes of lattices in V.

Let k be a local or number fields, K/k a Galois extension with Galois group $\mathcal{G} = \mathcal{G}_{K/k}$. Let V_K, G_K denote the sets of K-points of V and G (see section 2). We establish the following result:

Proposition 1.1. *There exists a correspondence between G_k-orbits of $\mathcal{G}$-stable lattices in V_K, that are in the same G_K-orbit, and elements of*

$$\ker\left(H^1(\mathcal{G}, G_K^\Lambda) \xrightarrow{i_*} H^1(\mathcal{G}, G_K)\right),$$

where G_K^Λ is the stabilizer of one particular lattice, and i_* the map induced by the inclusion (see prop. 5.1).

The cocycles described above can be thought as equivalence classes of *lattices in the same space*. We develop a concept of *lattices in other spaces*, at least for the case that G is the stabilizer of a family of tensors (see 4). In this context, the following result is obtained:

Proposition 1.2. *Assume G is the stabilizer of a family of tensors $\mathfrak{T}$ on V. Then, the set $H^1(\mathcal{G}, G_K^\Lambda)$ is in one-to-one correspondence with the set of G_k-orbits of $\mathcal{G}$-invariant lattices in other spaces that are isomorphic over K. The set*

$$\ker\left(H^1(\mathcal{G}, G_K^\Lambda) \xrightarrow{i_*} H^1(\mathcal{G}, G_K)\right),$$

where i is the inclusion, corresponds to those lattices that are in the same space as Λ_k (see prop. 5.4).

The relation between the set of cohomology classes and the set of lattice classes is established by taking advantage of the long exact sequence in cohomology that arises from a short exact sequence over K.

We analyze the cohomology of the general linear group and its relation to classification of lattices. In particular, one obtains that the set of G_k-orbits of free lattices, that are isomorphic over K to a given lattice, corresponds to the set

$$\ker\left(H^1(\mathcal{G}_{K/k}, G_K^\Lambda) \longrightarrow H^1\left(\mathcal{G}_{K/k}, GL_K^\Lambda(V)\right)\right)$$

(see prop. 5.9).

A lattice is defined over k, if it is generated by its k-points. This is a stronger condition that $\mathcal{G}_{K/k}$-invariance. It is necessary to look at localizations to obtain a cohomological characterization of the set of lattice classes defined over k. We show that this set corresponds to the kernel

$$\ker\left(H^1(\mathcal{G}_{K/k}, G_K^{\Lambda}) \longrightarrow \prod_v H^1\Big(\mathcal{G}_w, GL_{K_w}^{\Lambda}(V)\Big)\right),$$

where $\mathcal{G}_w$ is the local Galois group (see prop. 5.14).

Section 5.1 is devoted to the study of the relation between integral cohomology and genera, ie., how cohomology can be used to study sets of lattices that become isometric over some extension. This is expressed in terms of the notion of *cohomological genus*. In particular, we recover the results in [11].

2 Notations.

In all of this article, k, K, E denote number or local fields of characteristic 0, or algebraic extensions of them. If k is a number field, $\Pi(k)$ denotes the set of places of k.

Remark 2.1. By an algebraic group, we mean a linear algebraic group. All algebraic groups are assumed to be subgroups of the general linear group of a vector space V, of finite dimension, over a sufficiently large algebraically closed field Ω of characteristic 0. We assume that all localizations of number fields inject into Ω. G denotes an algebraic group over Ω. $GL(V), SL(V)$ denote the general and special linear groups over Ω. When we work over a fixed local or number field k, we say that G is defined over k if the equations defining G have coefficients in k (see section 2.1.1 in [10]). *This is the case for all groups considered here.* For any field E, $k \subseteq E \subseteq \Omega$, we write G_E for the set of E-points of G, e.g., if $G = GL(V)$, the set of E points is denoted $GL_E(V)$. The same conventions apply to spaces and algebras. All spaces and algebras are assumed to be finite dimensional.

Exceptions to this rule are the multiplicative and additive groups. We denote the multiplicative group Ω^* by $\mathbb{G}_m$, and the additive group Ω by $\mathbb{G}_a$, when considered as algebraic groups. For the set of k-points we write k^*,k. Instead of $(\mathbb{G}_m)_k, (\mathbb{G}_a)_k$.

The orthogonal group of a quadratic form Q on V is written $\mathcal{O}_n(Q)$ or $\mathcal{O}_n(Q, V)$, where $n = \dim_\Omega(V)$. The set of E-points is denoted $\mathcal{O}_{n,E}(Q)$.

The field on which a particular lattice is defined is always written as a subindex. If K/k an extension of local or number fields and Λ_k is a lattice in V_k, Λ_K denotes the $\mathcal{O}_K$-lattice in V_K generated by Λ_k.

If G is an algebraic group acting on a space V, both defined over k, and Λ_k is a $\mathcal{O}_k$-lattice on V_k, the stabilizer of Λ_k in G_k is denoted G_k^{Λ}. If $G = GL(V)$, this set is denoted $GL_k^{\Lambda}(V)$. Similar conventions apply to special linear or orthogonal groups.

Remark 2.2. Whenever K/k is a Galois extension of a number field k, and v a place of k, w denotes a place of K dividing v. We assume that one fixed such w has been chosen for every v. This convention is also applied for infinite extension, e.g., $K = \bar{k}$.

Remark 2.3. $\mathcal{G}_{K/k}$ denotes the Galois group of the extension K/k. If there is no risk of confusion, we write simply $\mathcal{G}$. If K is not specified, we assume $K = \bar{k}$. If k is a number field and $v \in \Pi(k)$, we also use the notation $\mathcal{G}_w = \mathcal{G}_{K_w/k_v}$.

If Γ is a group acting on a set S, S/Γ denotes the set of orbits and S^Γ the set of invariant points. The action of $\gamma \in \Gamma$ is denoted $s \mapsto s^\gamma$, for $s \in S$.

3 Review of Galois cohomology.

In this section we recall some of the cohomology results that we need. The results in this section are found in chapter 1 in [5], and p. 13-26 in [10].

Definition 3.1. Let $\mathcal{G}$ be a finite group, A a group provided with a $\mathcal{G}$-action. $H^1(\mathcal{G}, A)$ is defined as the quotient

$$H^1(\mathcal{G}, A) = \Big\{\alpha : \mathcal{G} \mapsto A \Big| \alpha(hg) = \alpha(h)\alpha(g)^h\Big\} / \equiv,$$

where $\alpha \equiv \beta$ if and only if there exists $a \in A$ such that $\alpha(g) = a^{-1}\beta(g)a^g$ for all $g \in \mathcal{G}$. If $\mathcal{G}$ acts trivially on A, then $H^1(\mathcal{G}, A) \cong \mathrm{Hom}(\mathcal{G}, A)/A$, where A acts by conjugation. In what follows we write α_g instead of $\alpha(g)$.

In case that $A \subseteq B$ is a subgroup, there is a long exact sequence

$$0 \longrightarrow A^\mathcal{G} \longrightarrow B^\mathcal{G} \longrightarrow (B/A)^\mathcal{G} \longrightarrow H^1(\mathcal{G}, A) \longrightarrow H^1(\mathcal{G}, B),$$

and furthermore, under the natural action [1] of $B^\mathcal{G}$ on $(B/A)^\mathcal{G}$,

$$(B/A)^\mathcal{G}/B^\mathcal{G} \cong \ker\Big(H^1(\mathcal{G}, A) \longrightarrow H^1(\mathcal{G}, B)\Big). \tag{1}$$

To simplify notations, in all that follows we assume that whenever a sequence of pointed sets

$$\ldots \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow X \longrightarrow Y \longrightarrow Z$$

is written, X, Y, Z denote pointed sets, $W, V, U, \ldots$ denote groups, and W acts on X with

$$X/W \cong \ker(Y \longrightarrow Z).$$

If A is normal in B, we have in the sense just described a long exact sequence

$$0 \longrightarrow A^\mathcal{G} \longrightarrow B^\mathcal{G} \longrightarrow (B/A)^\mathcal{G} \longrightarrow H^1(\mathcal{G}, A) \longrightarrow H^1(\mathcal{G}, B) \longrightarrow H^1(\mathcal{G}, B/A). \tag{2}$$

[1] This result is not found in [5], but can be found in [10] p.22.

In case A is central in B, the higher order cohomology groups for A are also defined, and we have a long exact sequence

$$0 \longrightarrow A^{\mathcal{G}} \longrightarrow B^{\mathcal{G}} \longrightarrow (B/A)^{\mathcal{G}} \longrightarrow H^1(\mathcal{G}, A) \longrightarrow H^1(\mathcal{G}, B) \longrightarrow H^1(\mathcal{G}, B/A) \longrightarrow H^2(\mathcal{G}, A).$$

Finally, if A and B are both Abelian this sequence extends to cohomology of all orders (see [13] or [6]). All results of this section can be extended via direct limits to profinite groups acting continuously on discrete groups (see [13], p. 9 and p. 42).

We are specially interested in the case in which $\mathcal{G}$ is the Galois group $\mathcal{G}_{K/k}$ of a possibly infinite Galois extension K/k, where k is a local or number field. In what follows, the subgroups $A, B, \ldots$ are groups of algebraic or arithmetical nature.

The following is a known fact, (see [5], ex. 1, p. 16).

Proposition 3.2. *For any finite dimensional algebra A, defined over k, and any algebraic extension K/k, it holds that $H^1(\mathcal{G}_{K/k}, A_K^*) = \{1\}$.*

Example 3.3 (Hilbert's theorem 90)**.** $GL_K(V) \cong (\mathrm{End}_K(V))^*$. Therefore, $H^1\Big(\mathcal{G}, GL_K(V)\Big) = \{1\}$.

4 Tensors and K/k-forms.

By a tensor of type (l, m) on V, we mean an Ω-linear map $\tau : V^{\otimes l} \longrightarrow V^{\otimes m}$, where

$$V^{\otimes r} = \bigotimes_{i=1}^{r} V \text{ for } r \geq 1,\ V^{\otimes 0} = \Omega.$$

τ is said to be defined over k, if $\tau(V_k^{\otimes l}) \subseteq V_k^{\otimes m}$. *All tensors mentioned in this work are assumed to be defined over k.* $GL(V)$ acts on the set of tensors of type (l, m) by $g(\tau) = g^{\otimes m} \circ \tau \circ (g^{\otimes l})^{-1}$. It makes sense, therefore, to speak about the stabilizer of a tensor.

Let I be any set. By an I-family of tensors, we mean a map that associates, to each element $i \in I$, a tensor t_i of type (n_i, m_i). $GL(V)$ acts on the set of all I-families by acting in each coordinate. In all that follows, we say a family instead of an I-family unless the set of indices needs to be made explicit. Let $\mathfrak{T}$ be a family of tensors and $H = Stab_{GL(V)}(\mathfrak{T})$. Then, H is a linear algebraic group.

If K/k is a Galois extension with Galois group $\mathcal{G}$, we get an exact sequence

$$\{1\} \longrightarrow H_K \longrightarrow GL_K(V) \longrightarrow X_K \longrightarrow \{1\},$$

where X_K is the $GL_K(V)$-orbit of $\mathfrak{T}$. It follows from (1), and example 3.3, that $X_K^{\mathcal{G}}/GL_k(V) \cong H^1(\mathcal{G}, H_K)$. The elements of $X_K^{\mathcal{G}}/GL_k(V)$ can be thought of as isomorphism classes of pairs $(V'_k, \mathfrak{T}')$ that become isomorphic to $(V_k, \mathfrak{T})$ when extended to K. These classes are usually called K/k-forms of $(V, \mathfrak{T})$, or just k-forms if $K = \bar{k}$. Observe that two vector spaces of the same dimension are isomorphic, so we can always assume that the vector space V, in which all tensors are defined, is fixed.

Definition 4.1. We call a pair $(V, \mathfrak{T})$, where $\mathfrak{T}$ is a family of tensors on V, a *space with tensors*, or simply a *space*. By abuse of language, we identify $(V, \mathfrak{T})$ and $(V, \mathfrak{T}')$ whenever $\mathfrak{T}, \mathfrak{T}'$ are in the same $GL_k(V)$-orbit, i.e., if they correspond to the same K/k-form. We say that $(V, \mathfrak{T}')$ is a K/k-form of $(V, \mathfrak{T})$, if $\mathfrak{T}$ and $\mathfrak{T}'$ are in the same $GL_K(V)$ orbit.

Example 4.2. Let Q be a non-singular quadratic form on the space V. Then, $\mathcal{O}_n(Q) = Stab_{GL(V)}(Q)$. Equivalence classes of non-singular quadratic forms on V_k are classified by $H^1\Big(\mathcal{G}, \mathcal{O}_{n,\bar{k}}(Q)\Big)$. A space, in this case, is what is usually called a quadratic space.

5 Lattices.

Let k be a local or number field, K/k a Galois extension, $G \subseteq GL(V)$ an algebraic group defined over k, Λ_k a lattice on V_k, L_K a $\mathcal{G}$-invariant lattice on V_K. Let $\mathcal{G} = \mathcal{G}_{K/k}$.

Proposition 5.1. *If there is an element $\varphi \in G_K$ such that $\varphi(L_K) = \Lambda_K$, then $a_\sigma = \varphi^\sigma \varphi^{-1}$ is a cocycle, and its class in $H^1(\mathcal{G}, G_K^\Lambda)$ is independent of the choice of φ, depending only on the orbit of L_K under G_k. The correspondence assigning, to every such G_k-orbit of $\mathcal{O}_K$-lattices, an equivalence class of cocycles, is an injection. The image of this map is*

$$\ker \Big(H^1(\mathcal{G}, G_K^\Lambda) \xrightarrow{i_*} H^1(\mathcal{G}, G_K) \Big),$$

where i is the inclusion.

Proof. G_K acts on the set of $\mathcal{O}_K$-lattices in V_K. Let X be the orbit of Λ_K. We have an exact sequence

$$\{1\} \longrightarrow G_K^\Lambda \longrightarrow G_K \longrightarrow X \longrightarrow \{1\}.$$

Hence, by (1), we get $X^{\mathcal{G}}/G_k \cong \ker \Big(H^1(\mathcal{G}, G_K^\Lambda) \longrightarrow H^1(\mathcal{G}, G_K) \Big)$. $\square$

Example 5.2. Using the fact that $H^1\Big(\mathcal{G}, GL_K(V)\Big) = \{1\}$, we obtain that the set of $GL_k(V)$-orbits of $\mathcal{G}$-invariant $\mathcal{O}_K$-lattices isomorphic to Λ_K is in correspondence with $H^1\Big(\mathcal{G}, GL_K^\Lambda(V)\Big)$.

If G is defined as the stabilizer of a family of tensors, e.g., the unitary group of a hermitian form or the automorphism group of an algebra, we get a more precise result.

Recall that in section 4 we identified K/k-forms of $(V, \mathfrak{T})$ with the corresponding $GL_k(V)$-orbits of families of tensors.

Definition 5.3. Let $(V, \mathfrak{T})$ be a space. A lattice in $(V_K, \mathfrak{T})$ is a pair $(\Lambda_K, \mathfrak{T})$, where Λ_K is a lattice in V_K. $GL_K(V)$ acts on the set of pairs $(\Lambda_K, \mathfrak{T}')$, for all families of tensors $\mathfrak{T}'$, by acting on each component. Two lattices $(\Lambda_K, \mathfrak{T})$, $(L_K, \mathfrak{T}')$ are said to be *in the same space* if $\mathfrak{T}, \mathfrak{T}'$ are in the same $GL_k(V)$-orbit.

Proposition 5.4. *Assume that G is the stabilizer of a family of tensors $\mathfrak{T}$ on V. The set $H^1(\mathcal{G}, G_K^\Lambda)$ is in one-to-one correspondence with the set of G_k-orbits of $\mathcal{G}$-invariant $\mathcal{O}_K$-lattices in the same G_K-orbit, in all spaces that are K/k-forms of $(V, \mathfrak{T})$. The kernel of the map*

$$H^1(\mathcal{G}, G_K^\Lambda) \xrightarrow{i_*} H^1(\mathcal{G}, G_K),$$

where i is the inclusion, corresponds to the subset of orbits of lattices that are in the same space as Λ_K.

Proof. We have an action of $GL_K(V)$ on the set of all pairs $(L_K, \mathfrak{T}')$, where L_K is a lattice and $\mathfrak{T}'$ an I-family of tensors with a fixed index set I. If T is the orbit of $(\Lambda_K, \mathfrak{T})$, we have a sequence

$$\{1\} \longrightarrow G_K^\Lambda \longrightarrow GL_K(V) \longrightarrow T \longrightarrow \{1\},$$

and the same argument as before applies. Last statement follows from the fact that spaces $(V_K, \mathfrak{T}')$ are classified by $H^1(\mathcal{G}_{K/k}, G_K)$, (see section 4 or [5], p. 15). □

Remark 5.5. Recall that $\Lambda_K = \Lambda_k \otimes_{\mathcal{O}_k} \mathcal{O}_K$. If L_K is in the same G_k-orbit as Λ_K, $L_K = L_k \otimes_{\mathcal{O}_k} \mathcal{O}_K$, since G_k also acts on V_k. Recall that we defined the cocycle corresponding to L by the formula $a_\sigma = \varphi^\sigma \varphi^{-1}$ (see prop. 5.1). This definition does not depend on G, as long as $\varphi \in G$. It follows that the set of G_k-orbits of lattices in V_k that are isomorphic as $\mathcal{O}_k$-modules, and whose extensions to K are in the same G_K orbit, corresponds to

$$\ker\left(H^1(\mathcal{G}, G_K^\Lambda) \longrightarrow H^1(\mathcal{G}, G_K) \times H^1\Big(\mathcal{G}, GL_K^\Lambda(V)\Big)\right). \qquad (3)$$

In the case that G is the stabilizer of a family of tensors,

$$\ker\left(H^1(\mathcal{G}, G_K^\Lambda) \longrightarrow H^1\Big(\mathcal{G}, GL_K^\Lambda(V)\Big)\right)$$

corresponds to the set of G_k-orbits of such lattices in all spaces that are K/k-forms of $(V, \mathfrak{T})$.

Example 5.6. If Λ_k is free, (3) corresponds to the set of G_k-orbits of free lattices on V_k, whose extensions to K are in the same G_K-orbit.

Definition 5.7. We say that an $\mathcal{O}_K$-lattice Λ_K is *defined over* k, if $\Lambda_K \cong \mathcal{O}_K \otimes_{\mathcal{O}_k} \Lambda_k$ for some Λ_k. We say that Λ_K is a *k-free lattice*, if Λ_k is free.

Assume first that G is the stabilizer of a family of tensors.

Definition 5.8. Let $a \in H^1(\mathcal{G}, G_K^\Lambda)$. We say that a is defined over k, k-free or in $(V, \mathfrak{T})$ if some (hence any), lattice in the class corresponding to a has this property. Define

$$\begin{array}{rcl} \mathcal{L}_{\mathrm{def}}(G, K/k, \Lambda) & = & \{a \in H^1(\mathcal{G}, G_K^\Lambda) | a \text{ is defined over } k\}, \\ \mathcal{L}_{\mathrm{fr}}(G, K/k, \Lambda) & = & \{a \in \mathcal{L}_{\mathrm{def}}(G, K/k, \Lambda) | a \text{ is } k\text{-free}\}, \\ \mathcal{L}^V(G, K/k, \Lambda) & = & \{a \in H^1(\mathcal{G}, G_K^\Lambda) | a \text{ is in } (V_K, \mathfrak{T})\}, \\ \mathcal{L}^V_{\mathrm{def}}(G, K/k, \Lambda) & = & \mathcal{L}^V(G, K/k, \Lambda) \cap \mathcal{L}_{\mathrm{def}}(G, K/k, \Lambda), \\ \mathcal{L}^V_{\mathrm{fr}}(G, K/k, \Lambda) & = & \mathcal{L}^V(G, K/k, \Lambda) \cap \mathcal{L}_{\mathrm{fr}}(G, K/k, \Lambda). \end{array}$$

Let

$$F_1 : H^1(\mathcal{G}, G_K^\Lambda) \longrightarrow H^1(\mathcal{G}, G_K), \tag{4}$$

$$F_2 : H^1(\mathcal{G}, G_K^\Lambda) \longrightarrow H^1(\mathcal{G}, GL_K^\Lambda(V)), \tag{5}$$

be the maps defined by the inclusions. Then, we have the following proposition:

Proposition 5.9. *Assume that Λ_k is free. The following identities hold:*

$$\begin{array}{rcl} \mathcal{L}^V(G, K/k, \Lambda) & = & \ker F_1, \\ \mathcal{L}_{\mathrm{fr}}(G, K/k, \Lambda) & = & \ker F_2, \\ \mathcal{L}^V_{\mathrm{fr}}(G, K/k, \Lambda) & = & \ker F_1 \cap \ker F_2. \square \end{array}$$

Later, we give a similar interpretation to $\mathcal{L}_{\mathrm{def}}$.

Example 5.10.

$$\mathcal{L}_{\mathrm{fr}}(\mathcal{O}_n(Q), \bar{k}/k, \Lambda) = \ker \left(H^1(\mathcal{G}, \mathcal{O}^\Lambda_{n,\bar{k}}(Q)) \longrightarrow H^1\Big(\mathcal{G}, GL^\Lambda_{\bar{k}}(V)\Big) \right)$$

is in correspondence with the set of isometry classes of free quadratic lattices that become isometric to Λ_k over some extension.

Remark 5.11. Notice that $\mathcal{L}^V, \mathcal{L}^V_{\mathrm{def}}, \mathcal{L}^V_{\mathrm{fr}}$ can be defined, even if G is not the stabilizer of a family of tensors, as follows:

$$\begin{array}{rcl} \mathcal{L}^V(G, K/k, \Lambda) & = & \ker \Big(H^1(\mathcal{G}, G_K^\Lambda) \longrightarrow H^1(\mathcal{G}, G_K) \Big), \\ \mathcal{L}^V_{\mathrm{def}}(G, K/k, \Lambda) & = & \Big\{ a \in \mathcal{L}^V(G, K/k, \Lambda) \Big| a \text{ is defined over } k \Big\}, \\ \mathcal{L}^V_{\mathrm{fr}}(G, K/k, \Lambda) & = & \Big\{ a \in \mathcal{L}^V(G, K/k, \Lambda) \Big| a \text{ is free} \Big\}. \end{array}$$

In this case, the first and last identities of proposition 5.9 still hold. Notice that we can still interpret $\mathcal{L}^V$ as a set of equivalence classes of lattices, because of proposition 5.1.

The set $H^1(\mathcal{G}, U_K)$ and the ideal group. Let k be a local or number field, K/k a finite Galois extension. Let $\mathcal{G} = \mathcal{G}_{K/k}$, and let $U_K = \mathcal{O}_K^*$ denote the group of units of $\mathcal{O}_K$.

For any local or number field E, let I_E be its group of fractional ideals, P_E the subgroup of principal fractional ideals. There is a natural map $\alpha : I_k \to I_K$ defined by $\alpha(\mathcal{A}) = \mathcal{A} \bigotimes_{\mathcal{O}_k} \mathcal{O}_K$. Clearly $\alpha(P_k) \subseteq P_K$, so we get a map $\alpha' : I_k/P_k \to I_K/P_K$.

We apply the general theory to $\Lambda_k = \mathcal{O}_k$, $G = \mathbb{G}_m$. Any $\lambda \in K^*$ acts by $\mathcal{A} \mapsto \lambda\mathcal{A}$, for $\mathcal{A} \in I_K$, whence $G_K^\Lambda = U_K$. It follows that,

$$H^1(\mathcal{G}, U_K) \cong (P_K)^{\mathcal{G}}/\alpha(P_k).$$

Non-zero prime ideals of $\mathcal{O}_K$ form a set of free generators for I_K (see [7], p. 18). Let $\mathcal{A} \in I_K$. We can write

$$\mathcal{A} = \prod_{\wp\in\Pi(k)} (\prod_{\mathcal{P}|\wp} \mathcal{P}^{\beta(\mathcal{P})}).$$

If $\mathcal{A}$ is $\mathcal{G}$-invariant, all the powers $\beta(\mathcal{P})$ corresponding to prime divisors of the same prime of k must be equal. In other words:

$$\mathcal{A} = \prod_{\wp\in\Pi(k)} (\prod_{\mathcal{P}|\wp} \mathcal{P})^{\beta(\wp)}, \tag{6}$$

where $\beta(\wp)$ is the common value of $\beta(\mathcal{P})$ for all $\mathcal{P}$ dividing $\wp$. This ideal is in $\alpha(I_k)$ if and only if the ramification degree $e_\wp$ divides $\beta(\wp)$ for all $\wp$. Hence, we have an exact sequence

$$0 \longrightarrow \ker\alpha' \longrightarrow (P_K)^{\mathcal{G}}/\alpha(P_k) \longrightarrow \prod_{\wp\in\Pi(k)} (\mathbb{Z}/e_\wp),$$

where the image of the last map corresponds to those ideals of the form (6) that are principal in K. The image of $\ker\alpha'$ is what we call $\mathcal{L}_{\text{def}}(G, K/k, \Lambda)$. In particular, since all ideals become principal in some extension, we can take a direct limit, to obtain the long exact sequence:

$$0 \longrightarrow I_k/P_k \longrightarrow H^1(\mathcal{G}_{\bar{k}/k}, U_{\bar{k}}) \longrightarrow \prod_{\wp\in\Pi(k)} (\mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

A refinement of this argument gives

$$H^1(\mathcal{G}_{\bar{k}/k}, U_{\bar{k}}) \cong (I_k \otimes_{\mathbb{Z}} \mathbb{Q}) \,/\, (P_k \otimes_{\mathbb{Z}} \mathbb{Z}), \quad \mathcal{L}_{\text{def}}(G, K/k, \Lambda) = I_k/P_k.$$

Recall remarks 2.2 and 2.3. Assume k is a number field. There exist natural localization maps

$$F_v : H^1(\mathcal{G}, G_K^\Lambda) \to H^1(\mathcal{G}_w, G_{K_w}^\Lambda),$$

defined by inclusion and restriction. We define $G_{K_w}^\Lambda = G_{K_w}$ if w is Archimedean.

Lemma 5.12. *Let* $F_1 : H^1(\mathcal{G}, G_K^\Lambda) \to H^1(\mathcal{G}, G_K)$ *be the map induced by the inclusion. If the natural map*

$$\tau : H^1(\mathcal{G}, G_K) \to \prod_{v \in \Pi(k)} H^1(\mathcal{G}_w, G_{K_w})$$

is injective, then $\ker F_1 \supseteq \bigcap_v \ker F_v$.

Proof of lemma. Immediate from the following commutative diagram:

$$\begin{array}{ccc} H^1(\mathcal{G}, G_K^\Lambda) & \xrightarrow{F_1} & H^1(\mathcal{G}, G_K) \\ \big\downarrow {\scriptstyle \prod_v F_v} & & \big\downarrow {\scriptstyle \tau} \\ \prod_v H^1(\mathcal{G}_w, G_{K_w}^\Lambda) & \longrightarrow & \prod_v H^1(\mathcal{G}_w, G_{K_w}). \square \end{array}$$

Remark 5.13. If the hypothesis of this lemma is satisfied, one says that G satisfies the Hasse principle over k.

Characterisation of $\mathcal{L}_{\text{def}}$. $\mathcal{L}_{\text{def}}(G, K/k, \Lambda)$ is the set of equivalence classes of lattices defined over k that become isomorphic over K. A lattice L_K is defined over k if and only if it is generated by its k-points, i.e.,

$$L_K = \mathcal{O}_K(L_K \cap V_k).$$

This is a local property, being an equality of lattices. On the other hand, for any local place v, all lattices defined over k_v are k_v-free, i.e.,

$$\mathcal{L}_{\text{def}}(GL(V), K_w/k_v, \Lambda) = \mathcal{L}_{\text{fr}}(GL(V), K_w/k_v, \Lambda).$$

The following result is immediate from this observation.

Proposition 5.14.

$$\mathcal{L}_{\text{def}}(G, K/k, \Lambda) = \ker\left(H^1(\mathcal{G}, G_K^\Lambda) \longrightarrow \prod_v H^1\Big(\mathcal{G}_w, GL_{K_w}^\Lambda(V)\Big)\right). \square$$

5.1 Genus and cohomology.

Assume that In all of section 5.1, k is a number field.

Definition 5.15. Let F_v be the localization map. Define

$$C_{\text{gen}}(G, K/k, \Lambda) = \ker\left(\prod_v F_v\right).$$

We call this set the *cohomological genus* of Λ with respect to G.

Proposition 5.16. *For any linear algebraic group G, it holds that*

$$C_{\text{gen}}(G, K/k, \Lambda) \subseteq \mathcal{L}_{\text{def}}(G, K/k, \Lambda).$$

Proof. This follows from proposition 5.14 and the commutative diagram

$$\begin{array}{ccc} H^1(\mathcal{G}, G_K^\Lambda) & & \\ \downarrow & \searrow & \\ \prod_{v\in\Pi(k)} H^1(\mathcal{G}_w, G_{K_w}^\Lambda) & \longrightarrow & \prod_{v\in\Pi(k)} H^1\Big(\mathcal{G}_w, GL_{K_w}^\Lambda(V)\Big).\square \end{array}$$

Remark 5.17. Assume G is the stabilizer of a family of tensors. This result tells us that the cohomological genus corresponds to a set of equivalence classes ol lattices defined over k. In fact, $a \in C_{\text{gen}}(G, K/k, \Lambda)$ if and only if a corresponds to a lattice, in some K/k-form of $(V, \mathfrak{T})$, that is in the same G_{k_v}-orbit, at every place v.

Definition 5.18. We define the VC-genus of Λ_k by the formula

$$VC_{\text{gen}}(G, K/k, \Lambda) = C_{\text{gen}}(G, K/k, \Lambda) \cap \mathcal{L}^V(G, K/k, \Lambda).$$

In other words, it is the kernel of the map

$$H^1(\mathcal{G}, G_K^\Lambda) \longrightarrow H^1(\mathcal{G}.G_K) \times \prod_{v\in\Pi(k)} H^1(\mathcal{G}_w, G_{K_w}^\Lambda). \tag{7}$$

Let G be an arbitrary linear algebraic group. The VC-genus corresponds to a set of G_k-orbits of lattices in V_k. In fact, it corresponds to a subset of the set of double cosets $G_k\backslash G_{\mathbb{A}_k}/G_{\mathbb{A}_k}^\Lambda$, i.e., the *genus* of G (see [10], p. 440). In particular, the following proposition holds.

Proposition 5.19. *If G has class number 1 with respect to a lattice Λ_k, then (7) has trivial kernel for every Galois extension K/k (compare with corollary 4 on p. 491 of [10]).*$\square$

This, in particular, applies to a group having absolute strong approximation (see [10]). However, we have a stronger result.

Proposition 5.20. *If G has absolute strong approximation over k, the map (7) is injective.*

Proof. Recall remark 2.2. Let M_K,L_K be two $\mathcal{G}$-invariant $\mathcal{O}_K$-lattices in V_K, that are locally in the same G_{k_v}-orbit for all v. Then, we can choose elements $\sigma_v \in G_{k_v}$, such that $\sigma_v M_{K_w} = L_{K_w}$ for every place v, and $\sigma_v = 1$ at all but a finite number of places. Now, any global element σ, close enough to σ_v at all finite places where $\sigma_v \neq 1$, and stabilizing $M_{K_w} = L_{K_w}$ at the remaining finite places, satisfies $\sigma M_K = L_K$, as claimed. $\square$

The following result is just a restatement of lemma 5.12.

Proposition 5.21. *If G satisfies the Hasse principle over k, then*

$$VC_{\text{gen}}(G, K/k, \Lambda) = C_{\text{gen}}(G, K/k, \Lambda).\square$$

This result tells us that, in the presence of Hasse principle, the cohomological genus corresponds to a subset of the genus (compare with [11], thm 3.3, p. 198).

Let $G \subseteq GL(V)$ be a semi-simple group with universal cover $\widetilde{G}$ and fundamental group μ_n. Let $K = \bar{k}$. The short exact sequence

$$\{1\} \longrightarrow \mu_n \longrightarrow \widetilde{G}_K \longrightarrow G_K \longrightarrow \{1\},$$

defines a map $\theta : G_k \longrightarrow H^1(\mathcal{G}, F) = k^*/(k^*)^n$.

Let Λ_k be any lattice in V_k. The following proposition holds.

Proposition 5.22. *With the above notations, $VC_{\text{gen}}(G, K/k, \Lambda)$ is in one-to-one correspondence with the genus of G* (compare with theorem 8.13 in [10], p. 490)*.*

Proof. It suffices to show that any two G_k-orbits in the same genus are identified over some extension. Without loss of generality, we assume k is non-real. It suffices to check that they are in the same spinor genus (see [2]). Spinor genera are classified by

$$J_k/J_k^n k^* \Theta_{\mathbb{A}}(G^{\Lambda}_{\mathbb{A}_k}),$$

where $\Theta_{\mathbb{A}}(G^{\Lambda}_{\mathbb{A}_k})$ is the image of the local spinor norm (see[2] [1] or [2]). This is a finite set, and the representing adeles can be chosen to have trivial coordinates at almost all places. Therefore, it suffices to take an extension that contains the n-roots of unity, and n roots of a finite set of local elements.□

This result allows us to use cohomology to study the genus of any Semisimple group.

6 Determinant class of a lattice.

Let $[\mathcal{A}]$ be the k^*-orbit of the $\mathcal{O}_K$-ideal $\mathcal{A}$. Assume that

$$\Lambda_k = \underbrace{\mathcal{O}_k \oplus \ldots \oplus \mathcal{O}_k}_{n \text{ times}}.$$

The map $\det_* : H^1\Big(\mathcal{G}, GL_K^{\Lambda}(V)\Big) \longrightarrow H^1(\mathcal{G}, U_K)$ is the map induced in cohomology by the determinant. It is surjective, since det has a right inverse. However, in general it is not injective, as the example below shows.

Definition 6.1. Let L_K be a $\mathcal{G}$-invariant lattice in V_K, and let a be the cocycle class corresponding to the $GL_k(V)$-orbit of L_K. We define the determinant class of L_K, which we denote $\det_*(L_K)$, by:

$$\det_*(L_K) = \det_*(a) \in H^1(\mathcal{G}, U_K) \cong P_K^{\mathcal{G}}/\alpha(P_k),$$

and we identify it with the corresponding ideal class.

[2] The case of an orthogonal group is already considered in [3].

Example 6.2. Using the standard embedding $GL(V) \times GL(W) \longrightarrow GL(V \oplus W)$, it is easy to prove that $\det_*(\Lambda_K \oplus L_K) = \det_*(\Lambda_K)\det_*(L_K)$. In particular, we obtain that $\det_*(\mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_n) = [\mathcal{A}_1 \ldots \mathcal{A}_n]$.

Assume $k \subseteq K$ are local fields with maximal ideals $\wp$, $\mathcal{P}$. Assume that $\wp\mathcal{O}_K = \mathcal{P}^e$. Then,

$$\det_*(\underbrace{\mathcal{P} \oplus \ldots \oplus \mathcal{P}}_{e}) = [\mathcal{P}^e] = 1 = \det_*(\underbrace{\mathcal{O}_K \oplus \ldots \oplus \mathcal{O}_K}_{e}),$$

but the latter lattice is defined over k and the first one is not.

Let $\mathcal{L}_{\mathrm{def}} = \mathcal{L}_{\mathrm{def}}(GL(V), K/k, \Lambda)$. We have the following result:

Lemma 6.3. $\mathcal{L}_{\mathrm{def}} \cap \ker(\det_*) = \{1\}$.

Proof of lemma. This follows from the fact that all k-defined lattices are of the form $\mathcal{A}_k \oplus \mathcal{O}_k \oplus \ldots \oplus \mathcal{O}_k$ (see [8], (81:5)). It can also be proved by a diagram chasing argument. □

Now observe that, for any algebraic group $G \subseteq GL(V)$, we have

$$\mathcal{L}_{\mathrm{def}}(G, K/k, \Lambda) = i_*^{-1}(\mathcal{L}_{\mathrm{def}}),$$

where i_* is the cohomology map induced by the inclusion.

Proposition 6.4. *If $G \subseteq SL(V)$, then $i_*^{-1}(\mathcal{L}_{\mathrm{def}}) = \ker(i_*)$.*

Proof of proposition. It is immediate from the commutative diagram

$$\begin{array}{ccc}
H^1(\mathcal{G}, G_K^\Lambda) & \longrightarrow & H^1\Big(\mathcal{G}, SL_K^\Lambda(V)\Big) \\
{\scriptstyle i_*}\searrow & \circlearrowleft & \swarrow \\
 & H^1\Big(\mathcal{G}, GL_K^\Lambda(V)\Big) & \\
 & \Big\downarrow {\scriptstyle \det_*} & \\
 & H^1(\mathcal{G}, U_K) &
\end{array}$$

that $\mathrm{im}(i_*) \subseteq \ker(\det_*)$. Now recall lemma 6.3.□

In particular, such a group cannot identify a free lattice to a non-free k-defined lattice over any extension, although it can identify a free lattice to a non-k-defined lattice.

In this case, a description of $\mathcal{L}_{\mathrm{fr}}$ is equivalent to a description of $\mathcal{L}_{\mathrm{def}}$, hence $\mathcal{L}_{\mathrm{def}}(G, K/k, \Lambda)$ can be described without resorting to localization.

7 Example:Commutative algebras

Let $A_k = k^n$ as a k-algebra. This is a space with a tensor of type $(2,1)$. Then, the group of automorphisms of A_K is the symmetric group S_n for any algebraic extension K/k. Asume henceforth that K is an algebraic closure of k.

The k-forms of A are all semisimple commutative k-algebras. This set is classified by $H^1(\mathcal{G}, S_n) = Hom(\mathcal{G}, S_n)/\equiv$, where $\equiv$ denotes the conjugation (as an equivalence relation). Recall that any semisimple commutative algebra is a sum of fields. If ψ is the map that corresponds to an algebra $L_k = \bigoplus_i L_{i,k}$, a simple computation shows that there is correspondence between the fields $L_{i,k}$ and the orbits of $\mathrm{im}(\psi)$.

Let R_k denote an $\mathcal{O}_k$-subalgebra of A_k (not necessarily with 1) of maximal rank as a lattice. Let $\Gamma = S_n^R$. Then, the image of the map $H^1(\mathcal{G}, \Gamma) \longrightarrow H^1(\mathcal{G}, S_n)$ corresponds to the set of isomorphisms classes of algebras whose extensions to K contain a $\mathcal{G}$-invariant algebra isomorphic to R_K. In particular, this includes all lattices defined over k, i.e., all $\mathcal{O}_k$-algebras whose extensions to K are isomorphic to R_K.

Example 7.1. If Γ is not transitive, then no field can contain an $\mathcal{O}_k$-algebra whose extension to K is isomorphic to R_K. This is the case for example of the algebra $R_k = \mathcal{O}_k^{n-1} \oplus \mathcal{A}$ or $R_k = \{a \in \mathcal{O}_k^n | a_{n-1} - a_n \in \mathcal{A}\}$ if $\mathcal{A}$ is an ideal different from (1), and $n \geq 3$.

Remark 7.2. All result in this paper apply also to lattices over rings of S-integers. Absolute strong approximation must be replaced by strong approximation with respect to S.

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