arXiv:1009.3242v3 [math.LO] 30 Sep 2010

REVERSE MATHEMATICS AND EQUIVALENTS OF THE AXIOM OF CHOICE

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Abstract. We study the reverse mathematics of countable analogues of several maximality principles that are equivalent to the axiom of choice in set theory. Among these are the principle asserting that every family of sets has a $\subseteq$-maximal subfamily with the finite intersection property and the principle asserting that if φ is a property of finite character then every set has a $\subseteq$-maximal subset of which φ holds. We show that these principles and their variations have a wide range of strengths in the context of second-order arithmetic, from being equivalent to Z_2 to being weaker than ACA_0 and incomparable with WKL_0. In particular, we identify a choice principle that, modulo Σ^0_2 induction, lies strictly below the atomic model theorem principle AMT and implies the omitting partial types principle OPT.

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Date: September 30, 2010.

The authors are grateful to Denis Hirschfeldt, Antonio Montalbán, and Robert Soare for valuable comments and suggestions. The first author was partially supported by an NSF Graduate Research Fellowship.

1. Introduction

A large number of statements in set theory are equivalent to the axiom of choice over Zermelo–Fraenkel set theory (ZF). In this paper, we examine what happens when some of these statements are interpreted in the setting of second-order arithmetic, where the only "sets" available are sets of natural numbers. This interpretation allows us to study computability-theoretic and proof-theoretic aspects of choice principles in the spirit of reverse mathematics. Our results show that the re-interpreted statements need not be trivial, as might be suspected. Instead, these principles demonstrate a wide range of reverse mathematical strengths.

The history of the axiom of choice is presented in detail by Moore [13]. The main facet of interest for our purposes is that, after Zermelo introduced the axiom of choice in 1904, set theorists began to obtain results proving other set-theoretic principles equivalent to it (relative to choice-free axiomatizations of set theory). These equivalence results, and their further development, now constitute a program in set theory, which has been documented in detail by Jech [9] and by Rubin and Rubin [15, 16].

This program provides us with a large collection of statements from which to choose. We begin in Section 2 with Zorn's lemma, which is perhaps the most well-known equivalent of the axiom of choice but which turns out to be of only limited interest in second-order arithmetic. In Sections 3 and 4, we turn to other maximality principles with more complex and interesting behavior. Our focus is on statements closely related to the following two equivalents of the axiom of choice:

- every family of sets has a $\subseteq$-maximal subfamily with the finite intersection property;
- if φ is a property of finite character and A is any set, there is a $\subseteq$-maximal subset B of A such that B has property φ.

We avoid studying principles that concern countable well-orderings. Such principles have been thoroughly explored in the context of reverse mathematics by Friedman and Hirst [5] and by Hirst [8]. We also do not study direct formalizations of choice principles in arithmetic. These have been studied by Simpson [18, Section VII.6].

The rest of this section is devoted to a brief overview of second-order arithmetic and reverse mathematics. We refer the reader to Simpson [18] for complete details on second-order arithmetic and to Soare [19] for background information on computability theory.

1.1. Second-order arithmetic.

Second-order arithmetic is, intuitively, a weak form of type theory in which there are only two kinds of primitive objects: natural numbers and sets of natural numbers. This system is sufficiently expressive that many theorems of classical mathematics can be formalized within it, provided that the theorems are put in an arithmetical context through appropriate coding conventions and countability assumptions.

We work in the language L_2 of second-order arithmetic, which has the signature $\langle 0, 1, +, \times, <, =_{\mathbb{N}}, \in \rangle$. Equality for sets of numbers is defined by extensionality: $X = Y$ is an abbreviation for $\forall n\,(n \in X \leftrightarrow n \in Y)$. The set of L_2 formulas is ramified into the arithmetical and analytical hierarchies, which are used to define induction and comprehension schemes.

The full second-order *induction scheme* consists of every instance of

$$(\varphi(0) \wedge (\forall n)[\varphi(n) \rightarrow \varphi(n+1)]) \rightarrow (\forall n)\,\varphi(n),$$

in which φ is an L_2-formula, possibly with set parameters. If Γ is Σ^i_n or Π^i_n for some $i \in \{0,1\}$ and $n \geq 0$, the scheme of Γ *induction* ($\mathsf{I}\Gamma$) consists of the restriction of the induction scheme to formulas in Γ.

The full second-order *comprehension scheme* consists of every instance of

$$(\exists X)(\forall n)[n \in X \leftrightarrow \varphi(n)]$$

in which φ is an L_2-formula that does not mention X but may have other set parameters. If Γ is Σ^i_n or Π^i_n, where $i \in \{0,1\}$ and $n \geq 0$, the scheme of Γ *comprehension* (Γ-CA) consists of the restriction of the induction scheme to formulas in Γ. We also have the scheme of Δ^i_n *comprehension* (Δ^i_n-CA), which contains every instance of

$$(\forall n)[\varphi(n) \leftrightarrow \psi(n)] \rightarrow (\exists X)(\forall n)[n \in X \leftrightarrow \phi(n)]$$

in which φ is Σ^i_n, ψ is Π^i_n, and neither of these formulas mentions X.

The theory Z_2 of (full) *second-order arithmetic* includes the axioms of a discrete ordered ring, the full comprehension scheme, and the full induction scheme.

Semantic interpretations of L_2-theories are given by L_2-*structures*. A general L_2-structure $\mathscr{M}$ includes a set $\mathbb{N}^{\mathscr{M}}$ of "numbers", a collection $\mathcal{S}^{\mathscr{M}}$ of "sets", and interpretations of the symbols of L_2 using $\mathbb{N}^{\mathscr{M}}$ and $\mathcal{S}^{\mathscr{M}}$. An L_2-structure $\mathscr{M}$ is an ω-*model* if $\mathbb{N}^{\mathscr{M}}$ is the set $\omega = \{0, 1, 2, \ldots\}$ of standard natural numbers, $\mathcal{S}^{\mathscr{M}} \subseteq \mathcal{P}(\omega)$, and all the symbols of L_2 are given their standard interpretations. We identify an ω-model with the collection of subsets of ω that it contains. As usual, the notation $\mathscr{M} \models \varphi$ indicates that the formula φ (which may have parameters from $\mathscr{M}$) is true in $\mathscr{M}$.

1.2. **Subsystems.** Fragments of Z_2 are called *subsystems of second-order arithmetic*. The program of reverse mathematics seeks to characterize statements in the language of second-order arithmetic according to the weakest subsystems that can prove them. These characterizations are obtained by proving a statement within a certain subsystem, and then proving in a weak base system that the statement implies all the axioms of that subsystem.

As is common in reverse mathematics, we will use the subsystem RCA_0 for this weak base system. RCA_0 includes the axioms of a discrete ordered semiring, Σ^0_1 induction, and Δ^0_1 comprehension. Intuitively, this subsystem corresponds to computable mathematics, and in fact it is satisfied by the

ω-model REC containing only computable sets. In this sense, RCA_0 is very weak. Nevertheless, it is able to establish many elementary properties of the natural numbers.

Two countable forms of equivalents of the axiom of choice are already provable in RCA_0. These are the principle that every set of natural numbers can be well ordered and the principle that every sequence of nonempty sets of natural numbers has a choice function. We will show that several other equivalents of the axioms of choice require stronger subsystems to prove.

These stronger systems are obtained by adding stronger set-existence axioms to RCA_0. The main ones we will be interested in are the following:

- ACA_0 is the subsystem obtained by adding the comprehension scheme for arithmetical formulas;

and for each $n \geq 1$,

- Π^1_n-CA_0 is the subsystem obtained by adding the scheme of Π^1_n comprehension;
- Δ^1_n-CA_0 is the subsystem obtained by adding the scheme of Δ^1_n comprehension.

There are two important subsystems that do not directly correspond to restrictions of the second-order comprehension scheme. The first of these, WKL_0, consists of RCA_0 along with a single axiom, known as *weak König's lemma*, which states any infinite subtree of $2^{<\mathbb{N}}$ contains an infinite path. The second, ATR_0, consists of RCA_0 along with an axiom scheme that states that any arithmetically-defined functional $F\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ may be iterated along any countable well-ordering, starting with any set. We will not make use of ATR_0 in this paper.

The following theorem summarizes the well-known relations between the subsystems we have mentioned, in terms of provability. For subsystems T and T', we write $T < T'$ if every axiom of T is provable in T' but some axiom of T' is not provable in T.

Theorem 1.1. *We have*

$$\mathsf{RCA}_0 < \mathsf{WKL}_0 < \mathsf{ACA}_0 < \Delta^1_1\text{-}\mathsf{CA}_0 < \mathsf{ATR}_0 < \Pi^1_1\text{-}\mathsf{CA}_0,$$

and for each $n \geq 1$,

$$\Pi^1_n\text{-}\mathsf{CA}_0 < \Delta^1_{n+1}\text{-}\mathsf{CA}_0 < \Pi^1_{n+1}\text{-}\mathsf{CA}_0.$$

2. Zorn's lemma

Zorn's lemma is one of the best known equivalents of the axiom of choice, so we begin by studying the strength of countable versions of this principle. The reverse mathematics results in this section are relatively elementary, providing a warm-up for the more technical results of the following sections.

Working in RCA_0, we define a *countable poset* to be a set $P \subseteq \mathbb{N}$ with a reflexive, antisymmetric, transitive relation $\leq_P$. As usual, we may freely convert $\leq_P$ into an irreflexive, transitive relation $<_P$.

Definition 2.1. The following principles are defined in RCA_0.

(ZL-1) If a nonempty countable poset has the property that every linearly ordered subset is bounded above, then every element of the poset is below some maximal element.

(ZL-2) If a nonempty countable poset has the property that every linearly ordered subset is bounded above, then there is a nonempty set consisting of the maximal elements of the poset.

(ZL-3) If a nonempty countable poset has the property that every linearly ordered subset is bounded above, then there is a function that assigns to each element of the poset a maximal element above it.

Of these three principles, ZL-1 is the most natural countable analogue of Zorn's lemma, but we will see that it is already provable in RCA_0. We will show that ZL-2 is equivalent to ACA_0 over RCA_0, as might be expected. Principle ZL-3 is of greater interest; it can be viewed as a uniform version of ZL-1. We will show it is also equivalent to ACA_0 over RCA_0.

Theorem 2.2. ZL-1 *is provable in* RCA_0.

Proof. Working in RCA_0, let $\langle P, \leq_P \rangle$ be a countable poset in which every linearly ordered subset of P is bounded above. Write $P = \langle p_i : i \in \mathbb{N} \rangle$. We will build a sequence $\langle q_i : i \in \mathbb{N} \rangle$ by induction. Let q_0 be an arbitrary element of P. At stage $i+1$, if $q_i <_P p_i$ then put $q_{i+1} = p_i$, and otherwise put $q_{i+1} = q_i$. This inductive construction can be carried out in RCA_0. Moreover, a Π^0_1 induction in RCA_0 shows that if $i < j$ then $q_i \leq_P q_j$.

Let $L = \{q_i : i \in \mathbb{N}\}$. To decide if a fixed $p_i \in P$ is in L, it is only necessary to simulate the construction up to stage $i+1$. Therefore L is a Δ^0_1 set, and so RCA_0 proves that L exists. Moreover, L is linearly ordered; if two elements of L are incomparable, then two elements of the original sequence $\langle q_i : i \in \mathbb{N} \rangle$ are incomparable, which is impossible.

By assumption, there is some $i \in \mathbb{N}$ such that p_i is an upper bound for L. In particular, it must be that $q_i <_P p_i$, which means by construction that $p_i = q_{i+1} \in L$. Moreover, because p_i is an upper bound for L, it must be that $q_{i+j} = p_i$ for all $j \geq 1$.

Now suppose there is some $p_j \in P$ with $p_i <_P p_j$. It cannot be that $j < i$, because this would imply $p_j \leq_P q_i \leq_P p_i$. However, if $i < j$ then, at stage j, the construction would select $q_{j+1} = p_j$, contradicting our result that $q_{j+1} = p_i$. Thus p_i is a maximal element above q_0. □

Theorem 2.3. *Each of* ZL-2 *and* ZL-3 *is equivalent to* ACA_0 *over* RCA_0.

Proof. For any countable poset P satisfying the hypothesis of ZL-2, the set of maximal elements of P is definable by an arithmetical formula and is nonempty by Theorem 2.2. Thus, ACA_0 implies ZL-2.

Next, we show that ZL-2 implies ZL-3 over RCA_0. Let $\langle P, \leq_P \rangle$ be any countable poset such that every element of P is below at least one maximal element, and by $\mathsf{ZL\text{-}2}$ let M be the set of maximal elements of P. Define a function $m\colon P \to P$ by the rule

$$m(p) = q \Leftrightarrow (q \in M) \wedge (p \leq_P q) \wedge (\forall r <_{\mathbb{N}} q)[p \leq_P r \rightarrow r \notin M].$$

Then m is a function with domain P such that for each p, $m(p)$ is a maximal element with $p \leq_P m(p)$. Moreover, the definition of m is Δ^0_0 relative to M and $\leq_P$, so we can form m in RCA_0.

Finally, we show that $\mathsf{ZL\text{-}3}$ implies ACA_0 over RCA_0. Fix any one-to-one function f. We will construct a poset $\langle P, \leq_P \rangle$ as follows. Let $P = \{p_{i,s} : i, s \in \mathbb{N}\}$. The order $\leq_P$ on P is defined by cases. If $i \neq j$ then $p_{i,s}$ and $p_{j,t}$ are incomparable for all $s, t \in \mathbb{N}$. Given $i, s, t \in \mathbb{N}$, with $s \neq t$, define $p_{i,t} <_P p_{i,s}$ to hold if either $f(s) = i$, or $f(t) \neq i$ and $t > s$. Thus, for a fixed i, if there is no s with $f(s) = i$ then we have a maximal chain

$$\cdots <_P p_{i,2} <_P p_{i,1} <_P p_{i,0},$$

while if $f(s) = i$ then, because f is one-to-one, we have a maximal chain

$$\cdots <_P p_{i,2} <_P p_{i,1} <_P p_{i,0} <_P p_{i,s}.$$

In particular, for each i, either $p_{i,0}$ is a maximal element of P or there is an s with $f(s) = i$ and $p_{i,s}$ is a maximal element of P. (This gives, as a corollary, a direct reversal of $\mathsf{ZL\text{-}2}$ to ACA_0 over RCA_0.)

Now, working in RCA_0, assume there is a function $m\colon P \to P$ taking each $p \in P$ to a $\leq_P$-maximal q with $p \leq_P q$. Fix $i \in \mathbb{N}$. Either $m(p_{i,0}) = p_{i,0}$, in which case i is in the range of f if and only if $f(0) = i$, or else $m(p_{i,0}) = p_{i,s}$ for some $s > 0$, in which case $f(s) = i$. Thus we have

$$i \in \text{range}(f) \Leftrightarrow (\exists s)[f(s) = i] \Leftrightarrow (\forall s)[m(p_{i,0}) = p_{i,s} \Rightarrow f(s) = i].$$

Therefore the range of f exists by Δ^0_1 comprehension. This completes the reversal. □

3. Intersection properties

We next study several principles asserting that every countable family of sets has a $\subseteq$-maximal subfamily with certain intersection properties (see Definition 3.2). We will show that, although these principles are all equivalent to the axiom of choice in set theory, they can have vastly different strengths when formalized in second-order arithmetic. In particular, we find new examples of principles weaker than ACA_0 and incomparable with WKL_0.

Definition 3.1. We define a *family of sets* to be a sequence $A = \langle A_i : i \in \omega \rangle$ of sets. A family A is *nontrivial* if $A_i \neq \emptyset$ for some $i \in \omega$.

Given a family of sets A and a set X, we say A *contains* X, and write $X \in A$, if $X = A_i$ for some $i \in \omega$. A family of sets B is a *subfamily* of A if every set in B is in A, that is, $(\forall i)(\exists j)[B_i = A_j]$. Two sets $A_i, A_j \in A$ are *distinct* if they differ extensionally as sets.

Our definition of a subfamily is intentionally weak; see Proposition 3.8 below and the remarks preceding it.

Definition 3.2. Let $A = \langle A_i : i \in \omega \rangle$ be a family of sets and fix $n \geq 2$. Then A has the

- D_n *intersection property* if the intersection of any n distinct sets in A is empty.
- $\overline{D}_n$ *intersection property* if the intersection of any n distinct sets in A is nonempty.
- F *intersection property* if for every $m \geq 2$, the intersection of any m distinct sets in A is nonempty.

Definition 3.3. Let $A = \langle A_i : i \in \omega \rangle$ and $B = \langle B_i : i \in \omega \rangle$ be families of sets, and let P be any of the properties in Definition 3.2. Then B is a *maximal* subfamily of A with the P intersection property if B has the P intersection property, and for every subfamily C of A that does also, if B is a subfamily of C then C is a subfamily of B.

It is straightforward to formalize Definitions 3.1–3.3 in RCA_0.

Given a family $A = \langle A_i : i \in \omega \rangle$ and some $J \in \omega^\omega$, we use the notation $\langle A_{J(i)} : i \in \omega \rangle$ for the subfamily $\langle B_i : i \in \omega \rangle$ where $B_i = A_{J(i)}$. We call this the subfamily *defined* by J. Given a finite set $\{j_0, \ldots, j_n\} \subset \omega$, we let $\langle A_{j_0}, \ldots, A_{j_n} \rangle$ denote the subfamily $\langle B_i : i \in \mathbb{N} \rangle$ where $B_i = A_{j_i}$ for $i \leq n$ and $B_i = A_{j_n}$ for $i > n$. Note that such a subfamily can still contain A_i for infinitely many i, because there could be a j such that $A_j = A_i$ for infinitely many i. We call a subfamily of A *finite* if it contains only finitely many distinct A_i.

We are interested in the following maximality principles.

Definition 3.4. Let P be any of the properties in Definition 3.2. The following principle is defined in RCA_0.

($P\mathsf{IP}$) Every nontrivial family of sets has a maximal subfamily with the P intersection property.

For $P = D_n$ and $P = \overline{D}_n$, the set-theoretic principle corresponding to $P\mathsf{IP}$ is, in the notation of Rubin and Rubin [16], $\mathsf{M}\,8\,(P)$. For $P = F$, it is $\mathsf{M}\,14$. For additional references concerning the set-theoretic forms, and for proofs of their equivalences with the axiom of choice, see Rubin and Rubin [16, pp. 54–56, 60].

Remark 3.5. Although we do not make it an explicit part of the definition, all of the families $\langle A_i : i \in \omega \rangle$ we construct in our results will have the property that for each i, A_i contains $2i$ and otherwise contains only odd numbers. This will have the advantage that if we are given an arbitrary subfamily $B = \langle B_i : i \in \omega \rangle$ of some such family, we can, for each i, uniformly B-computably find a j such that $B_i = A_j$. If A is computable, each subfamily B will then be of the form $\langle A_{J(i)} : i \in \omega \rangle$ for some $J \in \omega^\omega$ with $J \equiv_T B$.

3.1. **Implications over RCA_0, and equivalences to ACA_0.** The next sequence of propositions establishes the basic relations that hold among the principles we have defined. We begin with the following upper bound on their strength.

Proposition 3.6. *For any property P in Definition 3.2, $P\mathsf{IP}$ is provable in ACA_0.*

Proof. Suppose $A = \langle A_i : i \in \mathbb{N}\rangle$ is a nontrivial family of sets. If A has a finite maximal subfamily with the P intersection property, then we are done. Otherwise, we define a function $p\colon \mathbb{N} \to \mathbb{N}$ as follows. Let $p(0)$ be the least j such that $\langle A_j\rangle$ has the P intersection property, and given $i \in \mathbb{N}$, let $p(i+1)$ be the least $j > p(i)$ such that $\langle A_{p(0)}, \dots, A_{p(i)}, A_j\rangle$ has the P intersection property. Then p exists by arithmetical comprehension, and by assumption it is total. It is not difficult to see that $B = \langle A_{p(i)} : i \in \mathbb{N}\rangle$ is a maximal subfamily of A with the P intersection property. □

Proposition 3.7. *For each standard $n \geq 2$, the following are provable in RCA_0:*

1. *$F\mathsf{IP}$ implies $\overline{D}_n\mathsf{IP}$;*
2. *$\overline{D}_{n+1}\mathsf{IP}$ implies $\overline{D}_n\mathsf{IP}$.*

Proof. To prove (1), let $A = \langle A_i : i \in \mathbb{N}\rangle$ be a nontrivial family of sets. We may assume that A has no finite maximal subfamily with the $\overline{D}_n$ intersection property. Define a new family $\widetilde{A} = \langle \widetilde{A}_i : i \in \mathbb{N}\rangle$ by recursion as follows. For all $i \neq j$, let $2i \in \widetilde{A}_i$ and $2j \notin \widetilde{A}_i$. Now suppose s is such that the $\widetilde{A}_i$ have been defined precisely on the odd numbers less than $2s+1$. Consider all finite sets $F \subseteq \{0, \dots, s\}$ such that $|F| \geq n+1$ and for every $F' \subseteq F$ of size n there is an $x \leq s$ belonging to $\bigcap_{i \in F'} A_i$. If no such F exists, enumerate $2s+1$ into the complement of $\widetilde{A}_i$ for all i. Otherwise, list these sets as $F_0, \dots, F_k$. For each $j \leq k$, enumerate $2(s+j)+1$ into $\widetilde{A}_i$ if $i \in F_j$, and into the complement of $\widetilde{A}_i$ if $i \notin F_j$.

The family $\widetilde{A}$ exists by Δ^0_1 comprehension, and is nontrivial by construction. Let $\widetilde{B} = \langle \widetilde{B}_i : i \in \mathbb{N}\rangle$ be a maximal subfamily of $\widetilde{A}$ with the F intersection property. Now each $\widetilde{B}_i$ contains exactly one even number, and if $2j \in \widetilde{B}_i$ then $\widetilde{B}_i = \widetilde{A}_j$. We define a family $B = \langle B_i : i \in \mathbb{N}\rangle$, where $B_i = A_j$ for the unique j such that $2j \in \widetilde{B}_i$. We claim that this is a maximal subfamily of A with the $\overline{D}_n$ intersection property.

It is not difficult to see that B has the $\overline{D}_n$ intersection property. Indeed, let $A_{i_0}, \dots, A_{j_{n-1}}$ be any n distinct members of B, and assume the indices have been chosen so that $\widetilde{A}_{i_j} \in \widetilde{B}$ for all $j < n$. Then $\bigcap_{j<n} \widetilde{A}_{i_j} \neq \emptyset$, so by construction we can find a finite set F of size $\geq n+1$ such that $i_j \in F$ for all j and $\bigcap_{i \in F'} A_i \neq \emptyset$ for every n-element $F' \subset F$. In particular, $\bigcap_{j<n} A_{i_j} \neq \emptyset$.

To show that B is maximal, we first argue that it is not a finite subfamily. Assume otherwise. Say the distinct members of B are $A_{i_0}, \dots, A_{i_m}$, where the indices have been chosen so that $\widetilde{A}_{i_j} \in \widetilde{B}$ for all $j \leq m$. Now we can find a finite set F of size $\geq n+1$ such that $i_j \in F$ for all j and $\bigcap_{i \in F'} A_i \neq \emptyset$ for every n-element $F' \subset F$. If $m = 0$, this is because of our assumption on A, and if $m > 0$, this is because $\bigcap_{j \leq m} \widetilde{A}_{i_j} \neq \emptyset$. Our assumption on A also implies that the A_i for $i \in F$ cannot form a maximal subfamily with the $\overline{D}_n$ intersection property. We can therefore fix a k so that $A_k \neq A_i$ for all $i \in F$ and $\bigcap_{i \in F'} A_i \neq \emptyset$ for every n-element $F' \subset F \cup \{k\}$. Then by construction, $\widetilde{A}_k \cap \bigcap_{j \leq m} \widetilde{A}_{i_j} \neq \emptyset$. Of course, the same is true if we replace any i_j in the intersection by any i such that $A_i = A_{i_j}$. And since for every i such that $\widetilde{A}_i \in B$ we have $A_i = A_{i_j}$ for some $j \leq m$, it follows that the intersection of any finite number of members of $\widetilde{B}$ with $\widetilde{A}_k$ is nonempty. By maximality of $\widetilde{B}$, $\widetilde{A}_k \in \widetilde{B}$ and hence $A_k \in B$. This is the desired contradiction.

Now suppose $A_k \notin B$ for some k, so that necessarily $\widetilde{A}_k \notin \widetilde{B}$. Since $\widetilde{B}$ is maximal, and since B is not finite, we can consequently find a finite set F of size $\geq n+1$ such that

- for all $i \neq j$ in F, $A_i \neq A_j$;
- for all $i \in F$, $\widetilde{A}_i \in \widetilde{B}$;
- $\widetilde{A}_k \cap \bigcap_{i \in F} \widetilde{A}_i = \emptyset$.

By construction, this means there is an n-element subset F' of $F \cup \{k\}$ with $\bigcap_{i \in F'} A_i = \emptyset$, and clearly k must belong to F'. Since $A_i \in B$ for all $i \in F$, and in particular for all $i \in F' - \{k\}$, we conclude that B is maximal with respect to property $\overline{D}_n$. This completes the proof that $F\mathsf{IP}$ implies $\overline{D}_n\mathsf{IP}$.

A similar argument can be used to show (2). We have only to modify the construction of $\widetilde{A}$ by looking, instead of at finite sets $F \subseteq \{0, \dots, s\}$ with $|F| \geq n+1$, only at those with $|F| = n+1$. The details are left to the reader. □

An apparent weakness of our definition of subfamily is that we cannot, in general, effectively decide which members of a family are contained in a given subfamily. The next proposition demonstrates that if we strengthen the definition of subfamily to make this problem decidable, all the intersection principles we study become equivalent to arithmetical comprehension.

Proposition 3.8. *Let P be any of the properties in Definition 3.2. The following are equivalent over RCA_0:*

(1) ACA_0*;*
(2) *every nontrivial family of sets $\langle A_i : i \in \mathbb{N} \rangle$ has a maximal subfamily B with the P intersection property, and the set $I = \{i \in \mathbb{N} : A_i \in B\}$ exists.*

Proof. The argument that (1) implies (2) is a refinement of the proof of Proposition 3.6. In the case where A does not have a finite maximal subfamily with the P intersection property, we can take for I the range of the function p defined in that proof.

To show that (2) implies (1), we work in RCA_0 and let $f\colon \mathbb{N} \to \mathbb{N}$ be a one-to-one function. For each i, let

$$A_i = \{2i\} \cup \{2x+1 : (\exists y \le x)[f(y) = i]\}.$$

noting that $i \in \text{range}(f)$ if and only if A_i is not a singleton, in which case A_i contains cofinitely many odd numbers. Consequently, for every finite $F \subset \mathbb{N}$ of size ≥ 2, $\bigcap_{i \in F} A_i \ne \emptyset$ if and only if each $i \in F$ is in the range of f.

Apply (2) with $P = D_n$ to the family $A = \langle A_i : i \in \mathbb{N}\rangle$ to find the corresponding subfamily B and set I. Because B is a maximal subfamily with the D_n intersection property, there are at most $n-1$ many j such that $j \in \text{range}(f)$ and $A_j \in B$. And for each i not equal to any such j, we have

$$i \in \text{range}(f) \Leftrightarrow A_i \notin B \Leftrightarrow i \notin I.$$

Thus the range of f exists. We reach the same conclusion if we instead apply (2) with $P = F$ or $P = \overline{D}_n$ to A. In this case, B_i is not a singleton for all $i \in \mathbb{N}$, and we have

$$i \in \text{range}(f) \Leftrightarrow A_i \in B \Leftrightarrow i \in I. \qquad \square$$

We close this subsection by showing that the above reversal to ACA_0 goes through for $P = D_n$ even with our weak definition of subfamily.

Proposition 3.9. *For each standard $n \ge 2$, $D_n\mathsf{IP}$ is equivalent to ACA_0 over RCA_0.*

Proof. Fix a one-to-one function $f\colon \mathbb{N} \to \mathbb{N}$, and let A be the family defined in the preceding proposition. Let $B = \langle B_i : i \in \mathbb{N}\rangle$ be the family obtained from applying $D_n\mathsf{IP}$ to A. As above, there can be at most $n-1$ many j such that $j \in \text{range}(f)$ and $A_j \in B$. For i not equal to any such j, we have

$$i \in \text{range}(f) \Leftrightarrow A_i \notin B \Leftrightarrow (\forall k)[2i \notin B_k].$$

This gives us a Π^0_1 definition of the range of f. Since the range of f is also definable by a Σ^0_1 formula, it follows by Δ^0_1 comprehension that the range of f exists. $\square$

We do not know whether the implications from $F\mathsf{IP}$ to $\overline{D}_n\mathsf{IP}$ or from $\overline{D}_{n+1}\mathsf{IP}$ to $\overline{D}_n\mathsf{IP}$ are strict. However, all of our results in the sequel hold equally well for $F\mathsf{IP}$ as they do for $\overline{D}_2\mathsf{IP}$. Thus, we shall formulate all implications over RCA_0 involving these principles as being to $F\mathsf{IP}$ and from $\overline{D}_2\mathsf{IP}$.

3.2. **Non-implications and conservation results.** In contrast to Proposition 3.9, $F\mathsf{IP}$ and the principles $\overline{D}_n\mathsf{IP}$ for $n \ge 2$ are all strictly weaker than ACA_0. This section is dedicated to a proof of this nonimplication, as well as to results showing that $F\mathsf{IP}$ does not imply WKL_0 and $D_2\mathsf{IP}$ is not provable in WKL_0. These results will be further sharpened by Proposition 3.27 below.

Proposition 3.10. *There is an ω-model of $\mathsf{RCA}_0 + F\mathsf{IP}$ consisting entirely of low sets. Therefore $F\mathsf{IP}$ does not imply ACA_0 over RCA_0.*

Proof. Given a computable nontrivial family $A = \langle A_i : i \in \omega \rangle$ of sets, let $\mathbb{F}_A$ be the notion of forcing whose conditions are strings $\sigma \in \omega^{<\omega}$ such that some $x \leq \sigma(|\sigma| - 1)$ belongs to $A_{\sigma(i)}$ for all $i < |\sigma| - 1$, and $\sigma' \leq \sigma$ if $\sigma' \upharpoonright |\sigma'| - 1 \succeq \sigma \upharpoonright |\sigma| - 1$. Now fix any $A_i \neq \emptyset$, say with $x \in A_i$, and let $\sigma_0 = ix$. Given σ_{2e} for some $e \in \omega$, ask if there is a condition $\sigma \leq \sigma_{2e}$ such that $\Phi_e^{\sigma \upharpoonright |\sigma|-1}(e) \downarrow$. If so, let σ_{2e+1} be the least such σ of length greater than $|\sigma_{2e}|$, and if not, let $\sigma_{2e+1} = \sigma_{2e}$. Given σ_{2e+1}, ask if there is a condition $\sigma \leq \sigma_{2e+1}$ such that $\sigma(i) = e$ for some $i < |\sigma| - 1$. If so, let σ_{2e+2} be the least such σ, and if not, let $\sigma_{2e+2} = \sigma_{2e+1}$. A standard argument establishes that $J = \bigcup_{e \in \omega} (\sigma_e \upharpoonright |\sigma_e| - 1)$ is low, and hence so is $B = \langle A_{J(i)} : i \in \omega \rangle$. It is clear that B is a maximal subfamily of A with the F intersection property. Iterating and dovetailing this argument produces the desired ω-model.

The second part of the proposition follows from the fact that every ω-model of ACA_0 must contain a set of degree $\mathbf{0}'$, which is not low. $\square$

We will establish the result that $F\mathsf{IP}$ does not even imply WKL_0 by showing $F\mathsf{IP}$ is conservative for the following class of sentences.

Definition 3.11 (Hirschfeldt, Shore and Slaman [6, p. 5819])**.** A sentence in L_2 is *restricted* Π^1_2 if it is of the form

$$(\forall X)[\varphi(X) \to (\exists Y)\psi(X, Y)],$$

where φ is arithmetical and ψ is Σ^0_3.

Many familiar principles are equivalent to restricted Π^1_2 sentences over RCA_0, including the defining axiom of WKL_0. We discuss several others in the next subsection.

The study of restricted Π^1_2 conservativity was initiated by Hirschfeldt and Shore [6, Corollary 2.21] in the context of the principle COH. Subsequently, it was extended by Hirschfeldt, Shore, and Slaman [7, Corollary 3.15 and the penultimate paragraph of Section 4] to the principles AMT and $\Pi^0_1\mathsf{G}$ (see Definitions 3.21 and 3.26 below). The conservation proofs for the latter two principles differ from the original only in the choice of forcing notion (Mathias forcing for COH, Cohen forcing for AMT and $\Pi^0_1\mathsf{G}$). A similar proof goes through, *mutatis mutandis*, for the notion $\mathbb{F}_A$ from the proof of Proposition 3.10, giving the following conservation result. We refer the reader to either of the above-cited papers for details.

Theorem 3.12. *The principle $F\mathsf{IP}$ is conservative over RCA_0 for restricted Π^1_2 sentences. Therefore $F\mathsf{IP}$ does not imply WKL_0 over RCA_0.*

The preceding results lead to the question of whether $F\mathsf{IP}$, or any one of the principles $\overline{D}_n\mathsf{IP}$, is provable in RCA_0, or at least in WKL_0. We show in the following theorem that $F\mathsf{IP}$ fails in any ω-model of WKL_0 consisting entirely of sets of hyperimmune-free Turing degree. Recall that a Turing degree is *hyperimmune* if it bounds the degree of a function not dominated by any computable function, and a degree which is not hyperimmune is *hyperimmune-free*. A model of the kind we are interested in can be obtained by iterating and dovetailing the hyperimmune-free basis theorem of Jockusch and Soare [11, Theorem 2.4], which asserts that every infinite computable subtree of $2^{<\omega}$ has an infinite path of hyperimmune-free degree.

Theorem 3.13. *There exists a computable nontrivial family of sets for which any maximal subfamily with the $\overline{D}_2$ intersection property must have hyperimmune degree.*

To motivate the proof, which will occupy the rest of this subsection, we first discuss the simpler construction of a computable nontrivial family for which any maximal subfamiliy with the $\overline{D}_2$ intersection property must be noncomputable. This, in turn, is perhaps best motivated by thinking how a proof of the contrary could fail.

Suppose we are given a computable nontrivial family $A = \langle A_i : i \in \mathbb{N}\rangle$. The most direct method of building a maximal subfamily $B = \langle B_i : i \in \mathbb{N}\rangle$ with the $\overline{D}_2$ intersection property, assuming A has no finite such subfamily, is to let $B_0 = A_i$ for the least i so that $A_i \neq \emptyset$, then to let $B_1 = A_j$ for the least $j > i$ such that $A_i \cap A_j \neq \emptyset$, and so on. Of course, this subfamily will in general not be computable, but we could try to temper our strategy to make it computable. An obvious such attempt is the following. We first search through the members of A in some effective fashion until we find the first one that is nonempty, and we let this be B_0. Then, having defined $B_0, \ldots, B_n$ for some n, we search through A again until we find the first member not among the B_i but intersecting each of them, and let this be B_{n+1}. Now while this strategy yields a subfamily B which is indeed computable and has the $\overline{D}_2$ intersection property, B need not be maximal. For example, suppose the first nonempty set we discover is A_1, so that we set $B_0 = A_1$. It may be that A_0 intersects A_1, but that we discover this only after discovering that A_2 intersects A_1, so that we set $B_1 = A_2$. It may then be that A_0 also intersects A_2, but that we discover this only after discovering that A_3 intersects A_1 and A_2, so that we set $B_2 = A_3$. In this fashion, it is possible for us to never put A_0 into B, even though it ends up intersecting each B_i.

We can exploit precisely this difficulty to build a family $A = \langle A_i : i \in \omega\rangle$ for which neither the strategy above, nor any other computable strategy, succeeds. We proceed by stages, at each one enumerating at most finitely many numbers into at most finitely many A_i. By Remark 3.5, it suffices to ensure that for each e, either Φ_e is not total, or else $\langle A_{\Phi_e(i)} : i \in \omega\rangle$ is not a maximal subfamily with the $\overline{D}_2$ intersection property. We discuss how to satisfy a single such requirement. Of course, in the full construction there will be other requirements, but these will not interfere with one another.

At stage s, we look for the longest nonempty string $\sigma \in \omega^{<\omega}$ such that for all $i < |\sigma|$, $\Phi_e(i)[s] \downarrow= \sigma(i)$, and for all $i, j < |\sigma|$, $A_{\sigma(i)}$ and $A_{\sigma(j)}$ have been intersected by stage s. At the first stage that we find such a σ, we define t_e to be some number large enough that A_{t_e} does not yet intersect $A_{\Phi_e(i)}$ for any i. We then start defining numbers $p_{e,0}, p_{e,1}, \ldots$ as follows. At each stage, if we do not find a longer such σ, or if t_e is in the range of this σ, we do nothing. Otherwise, we choose the least n such that $p_{e,n}$ has not yet been defined, and define it be some number not yet in the range of Φ_e and large enough that $A_{p_{e,n}}$ does not intersect A_{t_e}. We call $p_{e,n}$ a *follower* for σ. Then for any $p_{e,m}$ that is already defined and is a follower for some $\tau \preceq \sigma$, we intersect $A_{p_{e,m}}$ with $A_{\sigma(i)}$ for all i. Also, if $\sigma(i) = p_{e,m}$ for some i and m, then for the largest such m and for all j with $\sigma(j) \neq p_{e,m}$, we intersect $A_{\sigma(j)}$ with A_{t_e}.

Now suppose that Φ_e is total and that the subfamily it defines is a maximal one with the $\overline{D}_2$ intersection property. The idea is that A_{t_e} should behave as A_0 did in the motivating example above, by never entering the subfamily but intersecting all of its members, thereby giving us a contradiction. For the first part, note that if $\Phi_e(i) = t_e$ for some i then $\sigma(i) = t_e$ for some string σ as above, and that necessarily $A_{\sigma(j)} \cap A_{t_e} = \emptyset$ for some j. But any string we find at a subsequent stage will extend σ and hence have t_e in its range, so we will never make $A_{\sigma(j)} = A_{\Phi_e(j)}$ intersect $A_{t_e} = A_{\Phi_e(i)}$. Thus, t_e cannot be the range of Φ_e. We conclude that $p_{e,n}$ is defined for every n. For the second part, note that since each $p_{e,n}$ is a follower for some initial segment of Φ_e, each $A_{\Phi_e(i)}$ is eventually intersected with $A_{p_{e,n}}$. By maximality, then, $p_{e,n}$ belongs to the range of Φ_e for all n, which means that each $A_{\Phi_e(i)}$ is eventually also intersected with A_{t_e}.

This basic idea is the same one that we now use in our proof of Theorem 3.13. But since here we are concerned with more than just computable subfamilies, it no longer suffices to just play against those of the form $\langle A_{\Phi_e(i)} : i \in \omega \rangle$. Instead, we must consider all possible subfamilies $\langle A_{J(i)} : i \in \omega \rangle$ for $J \in \omega^\omega$, and show that if J defines a maximal subfamily with the $\overline{D}_2$ intersection property then there exists a function $f \leq_T J$ such that for all e, either Φ_e is not total or it does not dominate f. Accordingly, we must now define followers $p_{e,n}$ not only for those $\sigma \in \omega^{<\omega}$ that are initial segments of Φ_e, but for all strings that look as though they can be extended to some such $J \in \omega^\omega$. We still enumerate the followers linearly as $p_{e,0}, p_{e,1}, \ldots$, even though the strings they are defined as followers for no longer have to be compatible.

Looking ahead to the verification, fix any J that defines a maximal subfamily with the $\overline{D}_2$ intersection property. We describe the intuition behind defining $f \leq_T J$ that escapes domination by a single computable function Φ_e. (Of course, there are much easier ways to define f to achieve this, but this definition is close to the one that will be used in the full construction.) Much as in the more basic argument above, the construction will ensure that there are infinitely many n such that $p_{e,n}$ is a follower for some initial segment of J and belongs to the range of J. Then, $f(x)$ can be thought of as telling us how far to go along J in order to find one more $p_{e,n}$ in its range. More precisely, f is defined along with a sequence $\sigma_0 \prec \sigma_1 \prec \cdots$ of initial segments of J. For each x, σ_{x+1} is an extension of σ_x whose range contains a follower $p_{e,n}$ for some τ with $\sigma_x \preceq \tau \prec \sigma_{x+1}$, and $f(x+1)$ is a number large enough to bound an element of $\bigcap_{i<|\sigma_{x+1}|} A_{\sigma_{x+1}(i)}$. The idea behind this definition is that if f actually is dominated by Φ_e, then we can modify our basic strategy so that in deciding which members of A to intersect with A_{t_e} in the construction, we consider not initial segments of Φ_e as before, but strings $\sigma \in \omega^{<\omega}$ that look like initial segments J. Then, just as before, we can show that no such string σ can have t_e in its range, and yet that $A_{\sigma(i)}$ is eventually intersected with A_{t_e} for all i. Thus we obtain the same contradiction we got above, namely that J does not have t_e in its range and hence cannot be maximal after all.

The main obstacle to this approach is that we do not know which computable function will dominate f, if f is in fact computably dominated, and so we cannot use its index in the definition of f. One way to remedy this is to make $f(x)$ large enough to find not only the next $p_{e,n}$ in the range of J for some fixed e, but the next $p_{e,n}$ for each $e < x$. This, in turn, demands that we define followers in such a way that $p_{e,n}$ is defined for every e and n, regardless of whether Φ_e is total. But then we must define followers $p_{e,n}$ even for strings that already contain t_e in their range, since we do not know ahead of time that this will not happen for all sufficiently long strings. In the construction, then, we distinguish between two types of followers, those defined as followers for strings that have t_e in their range, and those defined as followers for strings that do not. We will see in the verification that we can restrict ourselves to strings of the latter type, so this is not a serious complication.

We turn to the formal details. We adopt the convention that for all $e, x, y, s \in \omega$, if $\Phi_e(x)[s] \downarrow = y$, then $e, x, y \leq s$, and $\Phi_e(z)[s] \downarrow$ for all $z < x$. Let $s_{e,x}$ denote the least s such that $\Phi_e(x)[s] \downarrow$, which may of course be undefined if Φ_e is not total. Then to show that some function is not computably dominated it suffices to show it is not dominated by the map $x \mapsto s_{e,x}$ for any e.

Proof of Theorem 3.13. We build a computable $A = \langle A_i : i \in \omega \rangle$ by stages. Let $A_i[s]$ be the set of elements which have been enumerated into A_i by stage s, which will always be finite. Say a nonempty string $\sigma \in \omega^{<\omega}$ is *bounded* by s if

- $|\sigma| \leq s$;
- for all $i < |\sigma|$, $\sigma(i) \leq s$;
- for all $i, j < |\sigma|$, there is a $y \leq s$ with $y \in A_{\sigma(i)}[s] \cap A_{\sigma(j)}[s]$.

Construction. For all $i \neq j$, let $2i \in A_i$ and $2j \notin A_i$. At stage $s \in \omega$, assume inductively that for each e, we have defined finitely many numbers $p_{e,n}$, $n \in \omega$, each labeled as either a *type 1 follower* or a *type 2 follower* for some string $\sigma \in \omega^{<\omega}$. Call a number x *fresh* if x is larger than s and every number that has been mentioned during the construction so far.

We consider consecutive substages, at substage $e \leq s$ proceeding as follows.

Step 1. If t_e is undefined, define it to be a fresh number. If t_e is defined but $s_{e,0} = s$, redefine t_e to be a fresh large number. In the latter case, change any type 1 follower $p_{e,n}$ already defined to be a type 2 follower (for the same string).

Step 2. Consider any $\sigma \in \omega^{<\omega}$ bounded by s. Choose the least n such that $p_{e,n}$ has not been defined, and define it to be a fresh number. Then, for each $i < |\sigma|$, enumerate a fresh odd number into $A_{p_{e,n}} \cap A_{\sigma(i)}$. If there is an $i < |\sigma|$ such that $\sigma(i) = t_e$, call $p_{e,n}$ a type 1 follower for σ, and otherwise, call $p_{e,n}$ a type 2 follower for σ.

Step 3. Consider any $p_{e,n}$ defined at a stage before s, and any $\sigma \in \omega^{<\omega}$ bounded by s that extends the string that $p_{e,n}$ was defined as a follower for. If $p_{e,n}$ is a type 1 follower then, for each $i < |\sigma|$, enumerate a fresh odd number into $A_{p_{e,n}} \cap A_{\sigma(i)}$. If $p_{e,n}$ is a type 2 follower, then do this only for the σ such that $\sigma(i) \neq t_e$ for all i.

Step 4. Suppose there is an x such that $\Phi_e(x)[s]\downarrow$, and $s = s_{e,x}$ for the largest such x. Call a string $\sigma \in \omega^{<\omega}$ *viable* for e at stage s if there exist $\sigma_0 \prec \cdots \prec \sigma_x = \sigma$ satisfying

- $|\sigma_0| = 1$;
- for each $i \leq x$, σ_i is bounded by $s_{e,i}$;
- for each $i < x$ and $j \leq i$, there exists a k with $|\sigma_i| \leq k < |\sigma_{i+1}|$ and an n such that $p_{j,n}$ is defined and is a follower for some τ with $\sigma_i \preceq \tau \prec \sigma_{i+1}$, and $\sigma_{i+1}(k) = p_{j,n}$.

If $x > e$, let $k^{\sigma}_{x,e}$ be the least k that satisfies the last condition above for $i = x - 1$ and $j = e$.

Call s an *e-acceptable* stage if for every string σ viable for e at this stage,

- $k^{\sigma}_{e,x}$ is defined;
- $A_{\sigma(k^{\sigma}_{e,x})}[s] \cap A_{t_e}[s] = \emptyset$.
- there is an $i < k^{\sigma}_{e,x}$ such that
  - $\sigma(i) = p_{e,n}$ for some n;
  - $A_{\sigma(i)}[s] \cap A_{t_e}[s] = \emptyset$;
  - for all $j \leq i$ and all τ viable for e at stage s, $\sigma(j) \neq \tau(k^{\tau}_{e,x})$.

If s is e-acceptable, then for each viable σ, choose the largest such $i < k^{\sigma}_{x,e}$, and enumerate a fresh odd number into $A_{\sigma(j)} \cap A_{t_e}$ for each $j \leq i$.

Step 5. If $e < s$, go to the next substage. If $e < s$, then for each i and each x less than or equal to the largest number mentioned during the construction at stage s and and not enumerated into A_i, enumerate x into the complement of A_i. Then go to stage $s + 1$.

End construction.

Verification. It is clear that A is a computable nontrivial family. Suppose $B = \langle B_i : i \in \mathbb{N} \rangle$ is a maximal subfamily of A with the $\overline{D}_2$ intersection property. Choose the unique $J \in \omega^\omega$ such that $B_i = A_{J(i)}$ for all i.

Claim 3.14. *For each $e \in \omega$ and each $\sigma \prec J$, there is an $n \in \omega$ such that $p_{e,n}$ is a follower for some τ with $\sigma \preceq \tau \prec J$ and $A_{p_{e,n}} \in B$.*

Proof. First, notice that for each $\sigma \preceq J$, there are infinitely many s that bound σ. Hence, since at any such stage s of the construction (specifically, at step 2 of substage e), $p_{e,n}$ gets defined for a new $n \in \omega$, it follows that $p_{e,n}$ gets defined for all n. Second, note that t_e necessarily gets defined during the construction, and then gets redefined at most once. We use t_e henceforth to refer to its final value.

Fix $\sigma \prec J$ and $m \in \omega$, and let s be a stage by which $p_{e,n}$ has been defined for all $n \leq m$. Let τ be either σ if $A_{t_e} \notin B$ or $\sigma(i) = t_e$ for some $i < |\sigma|$, or an initial segment of J extending σ long enough that there exists a $i < |\tau|$ with $\tau(i) = t_e$. By our observation above, there exists a $t \geq \max\{s, e\}$ that bounds τ. Let $p_{e,n}$ be the follower for τ defined at stage t, substage e, step 2, of the construction, so that necessarily $n > m$. Note that $p_{e,n}$ is a type 2 follower if and only if $A_{t_e} \notin B$.

Choose any υ with $\tau \preceq \upsilon \prec J$, and let $u > t$ be large enough to bound υ. Then at stage u, substage e, step 3, of the construction, $A_{p_{e,n}}$ is made to intersect $A_{\upsilon(i)}$ for each $i < |\upsilon|$ (in case $p_{e,n}$ is a type 2 follower, this is because $\upsilon(i) \neq t_e$ for all i). Since υ was arbitrary, it follows that $A_{p_{e,n}} \cap A_{J(i)}$ for all $i \in \omega$. Hence, by maximality of B, it must be that $A_{p_{e,n}} \in B$. □

Now define a function $f : \mathbb{N} \to \mathbb{N}$, and a sequence $\sigma_0 \prec \sigma_1 \prec \cdots$ of initial segments of J, as follows. Let $\sigma_0 = J \upharpoonright 1$ and $f(0) = 2J(0)$, and assume that we have $f(x)$ and σ_x defined for some $x \geq 0$. Let $f(x+1)$ be the least s such that there exists a $\sigma \in \omega^{<\omega}$ satisfying

- $\sigma_x \prec \sigma \prec J$;
- σ is bounded by s;
- for each $j \leq x$, there exists a k with $|\sigma_x| \leq k < |\sigma|$ and an n such that $p_{j,n}$ is defined by stage s of the construction and is a follower for some τ with $\sigma_x \preceq \tau \prec \sigma$, and $\sigma(k) = p_{j,n}$.

Let σ_{x+1} be the least σ satisfying the above conditions. By the preceding claim, $f(x)$ and σ_x are defined for all x.

Clearly, $f \leq_T B$. Seeking a contradiction, suppose e is such that $f(x) \leq s_{e,x}$ for all x. A simple induction then shows that σ_x is viable for e at stage $s_{e,x}$. So in particular, for every x, there is a σ viable for e at stage $s_{e,x}$. We fix the present value of e for the remainder of the proof, including in the following claims.

Claim 3.15. *If σ is viable for e at stage $s_{e,0}$, then $A_{\sigma(0)}$ is not intersected with A_{t_e} before step 4 of substage e of the first e-acceptable stage.*

Proof. Note that necessarily $|\sigma| = 1$, and that $s_{e,0}$ is not e-acceptable. At step 1 of substage e of stage $s_{e,0}$, t_e gets redefined to be a fresh number. Viability at stage $s_{e,0}$ just means that σ is bounded by $s_{e,0}$, and hence $A_{\sigma(0)}$ cannot intersect A_{t_e} at the end of this step. Hence, if we let s be the stage at which $A_{\sigma(0)}$ is first intersected with A_{t_e}, then $s \geq s_{e,0}$. Suppose the intersection takes place at step k of substage i of stage s. Then in particular, this point in the construction comes strictly after step 4 of substage e of stage $s_{e,0}$.

It suffices to prove the claim under the following assumption: there is no σ' viable for e at stage $s_{e,0}$ such that $A_{\sigma(0)}$ is first intersected with A_{t_e} before step k of substage i of stage s. Note also that k must be 3 or 4, since the only other step at which different members of A are intersected is step 2, but one of the two sets intersected there is always indexed by a fresh number.

First suppose $k = 3$. Then it must be that for some n, and for some τ extending the string ρ for which $p_{i,n}$ is a follower, we are intersecting $A_{p_{i,n}}$ with $A_{\tau(i)}$ for all $i < |\tau|$. Since t_e cannot equal $p_{i,m}$ for any m, it must be that $\sigma(0) = p_{i,n}$, and hence that there is a a $j < |\tau|$ such that $\tau(j) = t_e$. Now $\sigma(0)$ is bounded by $s_{e,0}$ and hence is not fresh after step 4 of substage e of stage $s_{e,0}$, whereas $p_{i,n}$, when defined, is defined to be a fresh number. Thus, since $\sigma(0) = p_{i,n}$, $p_{i,n}$ must be defined as a follower for ρ before step 4 of substage e of stage $s_{e,0}$. At that point in the construction, by definition, ρ has to be bounded, so ρ must also be bounded by $s_{e,0}$. In particular, $\rho(j)$ must be viable for e at stage $s_{e,0}$, for every j. This means $\rho(j) \neq t_e$, since t_e is certainly not viable at stage $s_{e,0}$. But since τ has to be bounded by $s_{e,0}$ in order for us to be considering it, it must be that $A_{\rho(j)}$ and A_{t_e} are intersected at some earlier point in the construction. This contradicts our assumption above.

Now suppose $k = 4$ but $i \neq e$. Then it must be that s is i-acceptable. Since t_e cannot equal t_i, and since members of A are only intersected at step 4 with A_{t_i}, it must be that $\sigma(0) = t_i$. There must also be a $\tau \in \omega^{<\omega}$ such that τ is viable for i at stage s and $\tau(j) = t_e$ for some $j < |\tau|$. Since s is i-acceptable, $s_{i,0}$ is defined. Now $\sigma(0)$ is bounded by $s_{e,0}$ and hence is not fresh after step 4 of substage e of stage $s_{e,0}$, whereas at step 1 of substage i of stage $s_{i,0}$, t_i is redefined to be a fresh number. Thus, since $\sigma(0) = t_i$, step 1 of substage i of stage $s_{i,0}$ cannot happen after step 4 of substage e of stage $s_{e,0}$. So, since the one bit string $\tau(0)$ has to be viable for i at step $s_{i,0}$ by definition of viability, it follows that $\tau(0)$ is also viable at stage $s_{e,0}$. Hence, $\tau(0) \neq t_e$ since t_e is not viable at stage $s_{e,0}$. But since τ has to be bounded by $s_{e,0}$, it must be that $A_{\tau(0)}$ and A_{t_e} are intersected at some earlier point in the construction. This again gives us a contradiction.

We conclude that $k = 4$ and $i = e$, that is, that $A_{\sigma(0)}$ is first intersected with A_{t_e} at step 4 of substage e of stage s. This forces s to be e-acceptable, so the claim is proved. □

Claim 3.16. *Suppose $x > e$ and $\sigma \in \omega^{<\omega}$ is viable for e at stage $s_{e,x}$. Then for some $i < |\sigma|$, $\sigma(i) = p_{e,n}$ for some n and $A_{\sigma(i)}$ and A_{t_e} are disjoint through the end of stage $s_{e,x}$.*

Proof. We proceed by induction on x, beginning with $x = e + 1$. Fix σ. By construction, $s_{e,x}$ is the first stage that can be e-acceptable, so by the preceding claim, $A_{\sigma(0)}$ has empty intersection with A_{t_e} at the beginning of step 4 of substage e of this stage. Hence, $\sigma(i) \neq t_e$ for all $i < |\sigma|$ since σ must be bounded by $s_{e,x}$. Now by viability, there is an i and an n such that $\sigma(i) = p_{e,n}$ and is a follower for some τ with $\sigma(0) \preceq \tau \prec \sigma$. It follows that $p_{e,n}$ is a type 2 follower. Furthermore, it is easy to see that for any type 2 follower $p_{e,m}$, $A_{p_{e,m}}$ can only be made to intersect A_{t_e} at step 4 of substage e of an e-acceptable stage. Thus, $A_{\sigma(i)}$ must be disjoint from A_{t_e} at the beginning of step 4 of substage e of stage $s_{e,x}$. Additionally, if $s_{e,x}$ is not e-acceptable, then nothing is done at step 4 of substage e, and hence $A_{\sigma(i)}$ is not intersected with A_{t_e} during the course of the rest of the stage. If $s_{e,x}$ is e-acceptable, then in fact there must exist a i as above, namely $i = k^{\sigma}_{x,e}$, such that $A_{\sigma(i)}$ is not intersected with A_{t_e} at step 4 of substage e, and hence not during the course of the rest of the stage either. This proves the base case of the induction.

Now let $x > e$ be given and suppose the claim holds for x. Given $\sigma \in \omega^{<\omega}$ viable for e at stage $s_{e,x+1}$, there is some $\tau \prec \sigma$ viable for e at stage $s_{e,x}$. If $s_{e,x+1}$ is not e-acceptable, then the same i witnessing that the claim holds for x and τ witnesses also that it holds for $x+1$ and σ. This is because $\tau(i)$ is necessarily a type 2 follower, and $A_{\tau(i)}$ is consequently not intersected with A_{t_e} until step 4 of substage e of some e-acceptable stage after stage $s_{e,x}$. If $s_{e,x+1}$ is e-acceptable, then just as in the base case, viability of σ implies that for $i = k^{\sigma}_{x+1,e}$, $A_{\sigma(i)}$ does not intersect A_{t_e} at the beginning of step 4 of substage e of stage $s_{e,x+1}$, and is not made to do so by its end. □

Claim 3.17. *There exist infinitely many e-acceptable stages.*

Proof. Fix any stage $s = s_{e,x}$ for $x > e$, and assume there is not any e-acceptable stage greater than s. For each σ viable for e at stage s, let i_σ be the largest i satisfying the statement of the preceding claim. Then $\sigma(i_\sigma)$ is a type 2 follower, so by our assumption, $A_{\sigma(i_\sigma)}$ is never intersected with A_{t_e} during the course of the rest of the construction.

Now for each $y \geq x$ and each σ viable for e at stage $s_{e,y+1}$, $k^{\sigma}_{e,y+1}$ is defined and $\sigma(k^{\sigma}_{e,y+1})$ is a follower $p_{e,n}$ for some string extending a $\tau \prec \sigma$ viable for e at stage $s_{e,y}$. Since followers are always defined to be fresh numbers, if $k^{\tau}_{e,y}$ is defined then $\sigma(k^{\tau}_{e,y}) = p_{e,m}$ for some $p_{e,m}$ defined strictly before $p_{e,n}$ in the construction.

Thus, for any sufficiently large $y > x$, it must be that for each σ viable at stage $s_{e,y}$, $\sigma(k^{\sigma}_{e,y}) \neq \tau(k)$ for all τ viable at stage s and all $k \leq j_\tau$. Moreover, since $A_{\tau(i_\tau)} \cap A_{t_e} = \emptyset$ and $\sigma(k^{\sigma}_{e,y})$ is a follower for some extension of some such τ, it must be that $\sigma(k^{\sigma}_{e,y})$ is a type 2 follower. Hence, $A_{\sigma(k^{\sigma}_{e,y})}$ can only be intersected with A_{t_e} at step 4 of substage e of an e-acceptable stage, meaning at a stage at or before s. It follows that if y is additionally chosen large enough that, for each σ viable at stage $s_{e,y}$, the follower $\sigma(k^{\sigma}_{e,y})$ is not defined before stage s, then $A_{\sigma(k^{\sigma}_{e,y})}$ will be disjoint from A_{t_e}. But then in particular, $A_{\sigma(k^{\sigma}_{e,y})}[s_{e,y}] \cap A_{t_e}[s_{e,y}] = \emptyset$, so $s_{e,y}$ is an e-acceptable stage greater than s. This is a contradiction, so the claim is proved. □

We can now complete the proof. First note that $A_{t_e} \notin B$, for otherwise there would have to be an x and an $i < |\sigma_x|$ such that $\sigma_x(i) = t_e$. But then σ_x would be viable for e at stage $s = s_{e,x}$, and so is in particular it would be bounded by s, meaning $A_{\sigma_x(j)}[s]$ would have to intersect $A_{\sigma_x(i)}[s] = A_{t_e}[s]$ for all $j < |\sigma_x|$. This would contradict Claim 3.16. Now consider any e-acceptable stage $s = s_{e,x}$. By construction, there is an $i < |\sigma_x|$ such that $A_{\sigma_x(i)}$ is disjoint from A_{t_e} at the beginning of stage s, and each $A_{\sigma_x(j)}$ for $j \leq i$ is made to intersect A_{t_e} by the end of stage s. Since, by Claim 3.17, there are infinitely many e-acceptable stages, and since $J = \bigcup_x \sigma_x$, it follows that $A_{J(i)}$ intersects A_{t_e} for all i. In other words, B_i intersects A_{t_e} for all i, which contradicts the choice of B as a maximal subfamily of A with the $\overline{D}_2$ intersection property. □

Remark 3.18. Examination of the above proof shows that it can be formalized in RCA_0, because the construction is computable and the verification that the function f defined in it is total requires only Σ^0_1 induction. (See [18, Definition VII.1.4] for the formalizations of Turing reducibility and equivalence in RCA_0.)

As discussed above, this has as a consequence the following corollary.

Corollary 3.19. *The principle* $\overline{D}_2\mathsf{IP}$ *is not provable in* WKL_0.

Proof. Let $\mathscr{M}$ be an ω-model of WKL_0 such that every set in $\mathscr{M}$ is of hyperimmune-free degree. Let A be the family constructed by the formalized version of Theorem 3.13, noting that A belongs to REC and hence to $\mathscr{M}$. Suppose $B \in \mathscr{M}$ is a maximal subfamily of A with the $\overline{D}_2$ intersection property. Then by the preceding remark, $\mathscr{M} \models$ "B has hyperimmune degree". Now the property of having hyperimmune degree is defined by an arithmetical formula, and is thus absolute to ω-models. Therefore, B has hyperimmune degree, contradicting the construction of $\mathscr{M}$. □

3.3. **Relationships with other principles.** By the preceding results, $F\mathsf{IP}$ and the principles $D_n\mathsf{IP}$ are of the irregular variety that do not admit reversals to any of the main subsystems of Z_2 mentioned in the introduction. In particular, they lie strictly between RCA_0 and ACA_0, and are incomparable with WKL_0. Many principles of this kind have been studied in the literature, and collectively they form a rich and complicated structure. Partial summaries are given by Hirschfeldt and Shore [6, p. 199] and Dzhafarov and Hirst [4, p. 150]. Additional discussion of the princples is given by Montalbán [12, Section 1] and Shore [17]. In this subsection, we investigate where our intersection principles fit into the known collection of irregular principles.

We can already show that $F\mathsf{IP}$ does not imply Ramsey's theorem for pairs (RT^2_2) or any of of the main combinatorial principles studied by Hirschfeldt and Shore [6] (all of which follow from RT^2_2). See [4, Definition 3.2] for a concise list of definitions of the principles in the following corollary.

Corollary 3.20. *None of the following principles are implied by* $F\mathsf{IP}$ *over* RCA_0*:* RT^2_2*,* SRT^2_2*,* DNR*,* CAC*,* ADS*,* SADS*,* COH*.*

Proof. All but the last of these principles are equivalent to restricted Π^1_2 sentences, and so for them the corollary follows by the conservation result of Proposition 3.12. For COH, it follows by Proposition 3.10 and the fact that any ω-model of COH must contain a set of p-cohesive degree [1, p. 27], and such degrees are never low [10, Theorem 2.1]. □

Our next results require several basic model-theoretic concepts. We assume some suitable development of model theory in RCA_0 (compare [18, Section II.8]). Let T be a countable, complete, consistent theory.

- A *partial type* of T is a T-consistent set of formulas in a fixed number of free variables. A *complete type* is a $\subseteq$-maximal partial type.
- A model $\mathscr{M}$ of T *realizes* a partial type Γ if there is a tuple $\vec{a} \in |\mathscr{M}|$ such that $\mathscr{M} \models \varphi(\vec{a})$ for every $\varphi \in \Gamma$. Otherwise, $\mathscr{M}$ *omits* Γ.
- A partial type Γ is *principal* if there is a formula φ such that $T \vdash \varphi \to \psi$ for every formula $\psi \in \Gamma$. A model $\mathscr{M}$ of T is *atomic* if every partial type realized in $\mathscr{M}$ is principal.
- An *atom* of T is a formula φ such that for every formula ψ in the same free variables, exactly one of $T \vdash \varphi \to \psi$ or $T \vdash \varphi \to \neg\psi$ holds. T is *atomic* if for every T-consistent formula ψ, $T \vdash \varphi \to \psi$ for some atom φ.

A classical result states that a theory is atomic if and only if it has an atomic model. Hirschfeldt, Slaman, and Shore [7] studied the strength of this theorem in the following forms.

Definition 3.21 ([7, pp. 5808, 5831])**.** The following principles are defined in RCA_0.

(AMT) Every complete atomic theory has an atomic model.

(OPT) Let T be a complete theory and let S be a set of partial types of T. Then there is a model of T omitting all the nonprincipal partial types in S.

Over RCA_0, AMT is strictly implied by SADS ([7, Corollary 3.12 and Theorem 4.1]). The latter asserts that every linear order of type $\omega + \omega^*$ has a suborder of type ω or ω^*, and is one of the weakest principles studied in [6] that does not hold in the ω-model REC. Thus, AMT is especially weak even among principles lying below RT^2_2. It does, however, imply part (2) of the following theorem, and therefore also OPT ([7, Theorem 5.6 (2) and Corollary 5.8]).

Theorem 3.22 (Hirschfeldt, Shore, and Slaman [7, Theorem 5.7])**.** *The following are equivalent over* RCA_0*:*

1. OPT*;*
2. *for every set* X*, there exists a set of degree hyperimmune relative to* X*.*

This characterization was used by Hirschfeldt, Shore and Slaman [7, p. 5831] to conclude that WKL_0 does not imply OPT. It is of interest to us in light of Theorem 3.13 above, which links $F\mathsf{IP}$ with hyperimmune degrees. Specifically, by Remark 3.18, we have the following.

Corollary 3.23. *$\overline{D}_2\mathsf{IP}$ implies OPT over RCA_0.*

The next proposition and theorem provide a partial step towards the converse of this corollary.

Proposition 3.24. *Let $A = \langle A_i : i \in \mathbb{N}\rangle$ be a computable nontrivial family of sets. Every set D of degree hyperimmune relative to $\mathbf{0}'$ computes a maximal subfamily of A with the F intersection property.*

Proof. We may assume that A has no finite maximal subfamily with the F intersection property. And by deleting some of the members of A if necessary, we may further assume that $A_0 \neq \emptyset$. Define a $\emptyset'$-computable function $g\colon \mathbb{N} \to \mathbb{N}$ by letting $g(s)$ be the least y such that for all finite sets $F \subseteq \{0, \ldots, s\}$,

$$\bigcap_{j \in F} A_j \neq \emptyset \Rightarrow (\exists x \leq y)[x \in \bigcap_{j \in F} A_j].$$

Since D has hyperimmune degree relative to $\mathbf{0}'$, we may fix a function $f \leq_T D$ not dominated by any $\emptyset'$-computable function. In particular, f is not dominated by g.

Now define $J \in \omega^\omega$ as follows. Let $J(0) = 0$, and suppose inductively that we have defined $J(s)$ for some $s \geq 0$. Search for the least $i \leq s$ not yet in the range of J for which there exists an $x \leq f(s)$ with

$$x \in A_i \cap \bigcap_{j \leq s} A_{J(j)}.$$

If it exists, set $J(s+1) = i$, and otherwise, set $J(s+1) = 0$

Clearly, $J \leq_T f$. Moreover, $\bigcap_{i \leq s} A_{J(i)} \neq \emptyset$ for every s, so the subfamily defined by J has the F intersection property. We claim that for all i, if $A_i \cap \bigcap_{j \leq s} A_{J(j)} \neq \emptyset$ for every s then i is in the range of J. Suppose not, and let i be the least witness to this fact. Since f is not dominated by g, there exists an $s \geq i$ such that $f(s) \geq g(s)$ and for all $t \geq s$, $J(t) \neq j$ for any $j < i$. By construction, $J(j) \leq j$ for all j, so the set $F = \{i\} \cup \{J(j) : j \leq s\}$ is contained in $\{0, \ldots, s\}$. Consequently, there necessarily exists some $x \leq g(s)$ with $x \in A_i \cap \bigcap_{j \leq s} A_{J(j)}$. But then $x \leq f(s)$, so $J(s+1)$ is defined to be i, which is a contradiction. We conclude that $\langle A_{J(i)} : i \in \omega\rangle$ is maximal, as desired. □

Theorem 3.25. *Let $A = \langle A_i : i \in \mathbb{N}\rangle$ be a computable nontrivial family of sets. Every noncomputable computably enumerable set W computes a maximal subfamily of A with the F intersection property.*

Proof. As above, assume that A has no finite maximal subfamily with the F intersection property, and that $A_0 \neq \emptyset$. Fix a computable enumeration of W, denoting by $W[s]$ the set of elements enumerated into W by the end of stage s. We construct a limit computable set M by the method of permitting, denoting by $M[s]$ the approximation to it at stage s of the construction. For each i and each n, call $\langle i, n\rangle$ a *copy* of i.

Construction.

Stage 0. Enumerate $\langle 0, 0\rangle$ into $M[0]$.

Stage $s+1$. Assume that $M[s]$ has been defined, that it is finite and contains $\langle 0, 0\rangle$, and that each i has at most one copy in $M[s]$. For each i with no copy in $M[s]$, let $\ell(i, s)$ be the greatest k with a copy in $M[s]$, if it exists, such that there is an $x \leq s$ that belongs to A_i and to A_j for every $j \leq k$ with a copy in $M[s]$.

Now consider all $i \leq s$ such that

- $\ell(i, s)$ is defined;
- there is no j with a copy in $M[s]$ such that $\ell(i, s) < j < i$;
- for each $\langle j, n\rangle \in M[s]$, if $\ell(i, s) < j$ then $W[s]\restriction\langle j, n\rangle \neq W[s+1]\restriction\langle j, n\rangle$.

If there is no such i, let $M[s+1] = M[s]$. Otherwise, fix the least such i, and let $M[s+1]$ be the result of removing from $M[s]$ all $\langle j, n\rangle > \ell(i, s)$, and then enumerating into it the least copy of i greater than every element of $M[s]$ and $W[s+1] - W[s]$.

End construction.

For every m, if $M[s](m) \neq M[s+1](m)$ then $W[s] \restriction m \neq W[s+1] \restriction m$. Therefore, $M(m) = \lim_s M[s](m)$ exists for all m and is computable from W. Furthermore, note that $\bigcap_{\langle i,n\rangle \in M[s]} A_i \neq \emptyset$ for all s. Thus, if F is any finite subset of M, then $\bigcap_{\langle i,n\rangle \in F} A_i \neq \emptyset$ since F is necessarily a subset of $M[s]$ for some s. If we now let $J : \omega \to \omega$ be any W-computable function with range equal to $\{i : (\exists n)[\langle i, n\rangle \in M]\}$, it follows that $\langle A_{J(i)} : i \in \omega\rangle$ has the F intersection property.

We claim that this subfamily is also maximal. Seeking a contradiction, suppose not, and let i be the least witness to this fact. So $A_i \cap \bigcap_{\langle j,n\rangle \in F} A_j \neq \emptyset$ for every finite subset F of M, and no copy of i belongs to M. By construction, $\langle 0, 0\rangle \in M[s]$ for all s and hence also to M, so it must be that $i > 0$. Let $i_0, \ldots, i_r$ be the numbers less than i that have copies in M, and let these copies be $\langle i_0, n_0\rangle, \ldots, \langle i_r, n_r\rangle$, respectively. Let s be large enough so that

- there is an $x \leq s$ with $x \in A_i \cap \bigcap_{j \leq n} A_{i_j}$;
- for all $t \geq s$ and all $j \leq n$, $\langle i_j, n_j\rangle \in M[t]$.

Now for all $t \geq s$, $\ell(i, t)$ is defined, and its value must tend to infinity.

Note that no copy of i can be in $M[t]$ at any stage $t \geq s$. Otherwise, it would have to be removed at some later stage, which could only be done for the sake of enumerating a copy of some number $< i$. This, in turn, could not be a copy of any of $i_0, \dots, i_r$ by choice of s, and so it too would subsequently have to be removed. Continuing in this way would result in an infinite regress, which is impossible.

It follows that for each $t \geq s$ there is some $j > \ell(i,t)$ with a copy $\langle j, n\rangle$ in $M[t]$. Let $\langle j_t, n_t\rangle$ be the least such copy at stage t. Then $\langle j_t, n_t\rangle \leq \langle j_{t+1}, n_{t+1}\rangle$ for all t, since no $m < \langle j_t, n_t\rangle$ can be put into $M[t+1]$. Furthermore, for infinitely many t this inequality must be strict, since infinitely often $\ell(i, t+1) \geq j_t$.

Now fix any $t \geq s$ so that $\ell(i,u) \geq i$ for all $u \geq t$. Then for all $u \geq t$, $W[u] \restriction \langle j_t, n_t\rangle$ must be equal to $W[u+1] \restriction \langle j_t, n_t\rangle$. If not, we would necessarily have $W[u] \restriction \langle j_u, n_u\rangle \neq W[u+1] \restriction \langle j_u, n_u\rangle$, and hence $W[u] \restriction \langle j, n\rangle \neq W[u+1] \restriction \langle j, n\rangle$ for every $\langle j, n\rangle \in M[u]$ with $j > \ell(i,u)$. But then some copy of i would be enumerated into $M[u+1]$, which cannot happen. We conclude that for all $u \geq t$, $W[u] \restriction \langle j_u, n_u\rangle = W \restriction \langle j_u, n_u\rangle$. Thus, given any n, we can compute $W \restriction n$ simply by searching for a $u \geq t$ with $\langle j_u, n_u\rangle \geq x$. This contradicts the assumption that W is noncomputable. The proof is complete. □

The above is of special interest. Heuristically, one would expect to be able to adapt a permitting argument into one showing the same result but with "every noncomputable computably enumerable set" replaced by "every hyperimmune set". For example, the proof in [7] that OPT is implied over RCA_0 by the existence of a set of hyperimmune degree is an adaptation of a permitting argument of Csima [2, Theorem 1.2]. The basic idea is to translate receiving permissions into escaping domination by computable functions. We take a given function f not dominated by any computable one, and for each i define a a computable function g_i so that receiving permission for the ith requirement in the permitting argument (such as putting a copy of i into M) corresponds to having $f(s) \geq g_i(s)$ for some s. But if we try to do this in the case of Theorem 3.25, we run into the problem of seemingly needing to know f in order to define g. Intuitively, we are trying to put A_i into our subfamily at stage s, and are letting $g_i(s)$ be so large that it bounds a witness to the intersection of A_i and all the members of A put in so far. Thus, the definition of $g_i(s)$ depends on which A_j have been put in at a stage $t < s$, i.e., on which j had $f(t) > g_j(t)$ for some $t < s$. In the permitting argument this information is computable, but here it is not. We do not know of a way of get past this difficulty, and thus leave open the question of whether OPT reverses to $F\mathsf{IP}$ (or $\overline{D}_2\mathsf{IP}$) over RCA_0.

We also do not know whether the weaker implication from AMT to $F\mathsf{IP}$ is provable in RCA_0. However, the next proposition shows that it is provable in RCA_0 together with additional induction axioms. In particular, every ω-model of AMT is also a model of $F\mathsf{IP}$. Thus we have a firm connection between the model-theoretic principles AMT and OPT and the set-theoretic principles $F\mathsf{IP}$ and $\overline{D}_n\mathsf{IP}$.

Definition 3.26 (Hirschfeldt and Shore [6, p. 5823])**.** The following principle is defined in RCA_0.

($\Pi^0_1\mathsf{G}$) For any uniformly Π^0_1 collection of sets D_i, each of which is dense in $2^{<\mathbb{N}}$, there exists a set G such that for every i, $G \upharpoonright s \in D_i$ for some s.

Hirschfeldt, Shore and Slaman [7, Theorem 4.3, Corollary 4.5, and p. 5826] proved that $\Pi^0_1\mathsf{G}$ strictly implies AMT over RCA_0, and that AMT implies $\Pi^0_1\mathsf{G}$ over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$. As discussed in the previous subsection, $\mathsf{RCA}_0 + Pi^0_1 G$ is conservative over RCA_0 for restricted Π^1_2 sentences, and thus it does not imply WKL_0 over RCA_0.

Proposition 3.27. $\Pi^0_1\mathsf{G}$ *implies* $F\mathsf{IP}$ *over* RCA_0*.*

Proof. We argue in RCA_0. Let a nontrivial family $A = \langle A_i : i \in \mathbb{N}\rangle$ be given. We may assume A has no finite maximal subfamily with the F intersection property. Fix a bijection $c : \mathbb{N} \to \mathbb{N}^{<\mathbb{N}}$. Given $\sigma \in 2^{<\mathbb{N}}$, we say that a number $x < |\sigma|$ is *good for* σ if

- $\sigma(x) = 1$;
- $c(x) = \tau b$, which we call the *witness* of x, where
  - $\tau \in \mathbb{N}^{<\mathbb{N}}$,
  - $b \in \mathbb{N}$,
  - and there is a $y \leq b$ with $y \in \bigcap_{i<|\tau|} A_{\tau(i)}$.

We define the *good sequence* of σ to be either the empty string if there is no good number for σ, or else the longest sequence $x_0 \cdots x_n \in \mathbb{N}^{<\mathbb{N}}$, $n \geq 0$, where

- x_0 is the least good number for σ;
- each x_i is good, say with witness $\tau_i b_i$;
- for each $i < n$, x_{i+1} is the least good $x > x_i$ such that if τb is its witness then $\tau \succ \tau_i$.

Note that Σ^0_0 comprehension suffices to prove the existence of a function $2^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ which assigns to each $\sigma \in 2^{<\mathbb{N}}$ its good sequence.

Now for each $i \in \mathbb{N}$, let D_i be the set of all $\sigma \in 2^{<\mathbb{N}}$ that have a nonempty good sequence $x_0 \cdots x_n$, and if τb is the witness of x_n then

- either $\tau(j) = i$ for some $j < |\tau|$,
- or $A_i \cap \bigcap_{j<|\tau|} A_{\tau(j)} = \emptyset$.

The D_i are clearly uniformly Π^0_1, and it is not difficult to see that they are dense in $2^{<\mathbb{N}}$. Indeed, let $\sigma \in 2^{<\mathbb{N}}$ be given, and define b, j, and x as follows. If the good sequence of σ is empty, choose the least $j \geq i$ such that $A_j \neq \emptyset$ and let $b \geq \min A_j$ be large enough that $x = c^{-1}(jb) \geq |\sigma|$. If the good sequence of σ is some nonempty string $x_0 \cdots x_n$ and τb_n is the witness of x_n, choose the least $j \geq i$ such that $A_j \cap \bigcap_{k<|\tau|} A_{\tau(k)} \neq \emptyset$ and let $b \geq \min A_j \cap \bigcap_{k<|\tau|} A_{\tau(k)}$ be large enough that $x = c^{-1}(\tau j b) \geq |\sigma|$. In either case, j exists because of our assumption that A is nontrivial and has no finite maximal subfamily with the F intersection property. Now define

$\widetilde{\sigma} \in 2^{<\mathbb{N}}$ of length $x + 1$ by

$$\widetilde{\sigma}(y) = \begin{cases} \sigma(y) & \text{if } y < |\sigma|, \\ 0 & \text{if } |\sigma| \leq y < x, \\ 1 & \text{if } y = x \end{cases}$$

to get an extension of σ that belongs to D_i.

Apply $\Pi^0_1\mathsf{G}$ to the D_i to obtain a set G such that for all i, there is an s with $G \restriction s \in D_i$. Note, that by definition, each such s must be nonzero, and $G \restriction s$ must have a nonempty good sequence. Notice that if $s \leq t$ then the good sequence of $G \restriction t$ extends (not necessarily properly) the good sequence of $G \restriction s$. Furthermore, our assumption that A has no finite maximal subfamily with the F intersection property implies that the good sequences of the initial segments of G are arbitrarily long.

Now find the least s such that $G \restriction s$ has a nonempty good sequence, and for each $t \geq s$, if $x_0 \cdots x_n$ is the good sequence of $G \restriction t$, let $\tau_t b_t$ be the witness of x_n. By the preceding paragraph, we have $\tau_t \preceq \tau_{t+1}$ for all t, and $\lim_t |\tau_t| = \infty$. Let $J = \bigcup_{t \geq s} \tau_t$, which exists by Σ^0_0 comprehension. It is straightforward to check that $B = \langle A_{J(i)} : i \in \mathbb{N} \rangle$ is a maximal subfamily of A with the F intersection property. □

We end this section with the result that $F\mathsf{IP}$ does not imply $\Pi^0_1\mathsf{G}$ or even AMT. Csima, Hirschfeldt, Knight, and Soare [3, Theorem 1.5] showed that for every set $D \leq_T \emptyset'$, if every complete atomic decidable theory has an atomic model computable in D, then D is non-low$_2$. Thus AMT cannot hold in any ω-model all of whose sets have low$_2$ degree. In conjunction with Theorem 3.25 (2), this fact allows us to separate $F\mathsf{IP}$ and AMT.

Corollary 3.28. *For every noncomputable computably enumerable set W, there exists an ω-model $\mathscr{M}$ of $\mathsf{RCA}_0 + F\mathsf{IP}$ with $X \leq_T W$ for all $X \in \mathscr{M}$. Therefore $F\mathsf{IP}$ does not imply AMT over RCA_0.*

Proof. By Sacks's density theorem, there exist computably enumerable sets $\emptyset <_T W_0 <_T W_1 <_T \cdots < W$. Let $\mathscr{M}$ be the ω-model whose second-order part consists of all sets X such that $X \leq_T W_i$ for some i. For each i, Theorem 3.25 (2) relativized to W_i implies that every W_i-computable nontrivial family of sets has a W_{i+1}-computable maximal subfamily with the F intersection property. Thus, $\mathscr{M} \models F\mathsf{IP}$. The second part follows by building $\mathscr{M}$ with W low$_2$. □

4. Properties of finite character

The last family of choice principles we study makes use of properties of finite character, sometimes in conjunction with finitary closure operators (see Definitions 4.9 and 4.16). We will show that these principles are equivalent to well known subsystems of arithmetic, unlike the intersection principles of the last section.

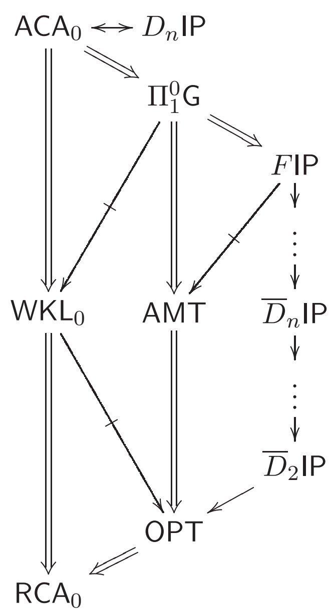

FIGURE 1. A summary of the results of Section 3, with $n \geq 2$ being arbitrary. Arrows denote implications provable in RCA$_0$, double arrows denote implications which are known to be strict, and negated arrows indicate nonimplications.

Definition 4.1. A formula φ with one free set variable X is said to be of *finite character* (or have the *finite character property*) if $\varphi(\emptyset)$ holds and, for every set A, $\varphi(A)$ holds if and only if $\varphi(F)$ holds for every finite $F \subseteq A$.

The following basic facts are provable in ZF.

Proposition 4.2. *Let $\varphi(X)$ be a formula of finite character.*

(1) *If $A \subseteq B$ and $\varphi(B)$ holds then $\varphi(A)$ holds.*
(2) *If $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ is a sequence of sets such that $\varphi(A_i)$ holds for each $i \in \omega$, then $\varphi(\bigcup_{i \in \omega} A_i)$ holds.*

We restrict our attention to formulas of second-order arithmetic, and consider countable analogues of several variants of the principle asserting that for every formula of finite character, every set has a maximal subset (under inclusion) satisfying that formula. Since the empty set satisfies any formula of finite character by definition, the validity of this principle can be seen by a simple application of Zorn's lemma.

The formalism here will be simpler than that in the previous section because we are dealing only with sets and their subsets, rather than with families of sets and their subfamilies. All the intersection properties studied in Section 3 can, in principle, be thought of as being defined by formulas of finite character. For example, given a family $A = \langle A_i : i \in \mathbb{N} \rangle$, the formula

$(\forall i)(\forall j)[A_i \cap A_j \neq \emptyset]$ has the finite character property, and if $J = \{j_0 < j_1 < \cdots\}$ is a maximal subset of $\mathbb{N}$ satisfying it, then $\langle A_{j_i} : i \in \mathbb{N}\rangle$ is a maximal subfamily of A with the $\overline{D}_2$ intersection property. However, such an analysis of $\overline{D}_2\mathsf{IP}$ would be too crude in light of Proposition 3.8. Therefore, our focus in this section will instead be on formulas of finite character in general, and on the strengths of principles based on formulas of finite character from restricted syntactic classes.

4.1. **The scheme FCP.**

We begin with various forms of the following principle.

Definition 4.3. The following scheme is defined in RCA_0.

(FCP) For each formula φ of finite character, which may have arbitrary parameters, every set A has a $\subseteq$-maximal subset B such that $\varphi(B)$ holds.

In set theory, FCP corresponds to the principle M 7 in the catalog of Rubin and Rubin [16], and is equivalent to the axiom of choice [16, p. 34 and Theorem 4.3].

In order to better gauge the reverse mathematical strength of FCP, we consider restrictions of the formulas to which it applies. As with other such ramifications, we will primarily be interested in restrictions to the classes in the arithmetical and analytical hierarchies. In particular, for each $i \in \{0, 1\}$ and $n \geq 0$, we make the following definitions:

- Σ^i_n-FCP is the restriction of FCP to Σ^i_n formulas;
- Π^i_n-FCP is the restriction of FCP to Π^i_n formulas;
- Δ^i_n-FCP is the scheme which says that for every Σ^i_n formula $\varphi(X)$ and every Π^i_n formula $\psi(X)$, if $\varphi(X)$ is of finite character and
$$(\forall X)[\varphi(X) \leftrightarrow \psi(X)],$$
then every set A has a $\subseteq$-maximal set B such that $\varphi(B)$ holds.

We also define QF-FCP to be the restriction of FCP to the class of quantifer-free formulas without parameters.

Our first result in this section is the following theorem, which will allow us to neatly characterize most of the above restrictions of FCP (see Corollary 4.6). We draw attention to part (2) of the theorem, where Σ^0_1 does not appear in the list of classes of formulas. The reason behind this will be made apparent by Proposition 4.7.

Theorem 4.4. *For $i \in \{0, 1\}$ and $n \geq 1$, let Γ be any of Π^i_n, Σ^i_n, or Δ^i_n.*

(1) *Γ-FCP is provable in Γ-CA_0;*
(2) *If Γ is Π^0_n, Π^1_n, Σ^1_n, or Δ^1_n, then Γ-FCP implies Γ-CA_0 over RCA_0.*

We will make use of the following technical lemma in the proof (as well as in the proof of Theorem 4.12 below). It is needed only because there are no term-forming operations for sets in L_2. For example, there is no term in L_2 that takes a set X and a canonical index n and returns $X \cup D_n$. (Recall that each finite (possibly empty) set of natural numbers is coded by a unique natural number known as its *canonical index*, and that D_n denotes the finite set with canonical index n.) The moral of the lemma is that such terms can be interpreted into L_2 in a natural way.

The coding of finite sets by their canonical indices can be formalized in RCA_0 in such a way that the predicate $i \in D_n$ is defined by a formula $\rho(i,n)$ with only bounded quantifiers, and such that the set of canonical indices is also definable by a bounded-quantifier formula [18, Theorem II.2.5]. Moreover, RCA_0 proves that every finite set has a canonical index. We use the notation $Y = D_n$ to abbreviate the formula $(\forall i)[i \in Y \leftrightarrow \rho(i,n)]$, along with similar notation for subsets of finite sets.

Lemma 4.5. *Let $\varphi(X)$ be a formula with one free set variable. There is a formula $\widehat{\varphi}(x)$ with one free number variable such that RCA_0 proves*

$$(\forall A)(\forall n)[A = D_n \to (\varphi(A) \leftrightarrow \widehat{\varphi}(n))]. \tag{4.5.1}$$

Moreover, we may take $\widehat{\varphi}$ to have the same complexities in the arithmetical and analytic hierarchies as φ.

Proof. Let $\rho(i,n)$ be the formula defining the relation $i \in D_n$, as discussed above. We may assume φ is written in prenex normal form. Form $\widehat{\varphi}(n)$ by replacing each occurrence $t \in X$ of φ, t a term, with the formula $\rho(t,n)$.

Let $\psi(X,\bar{Y},\bar{m})$ be the quantifier-free matrix of φ, where $\bar{Y}$ and $\bar{m}$ are sequences of variables that are quantified in φ. Similarly, let $\widehat{\psi}(n,\bar{Y},\bar{m})$ be the matrix of $\widehat{\varphi}$. Fix any model $\mathscr{M}$ of RCA_0 and fix $n, A \in \mathscr{M}$ such that $\mathscr{M} \models A = D_n$. A straightforward metainduction on the structure of ψ proves that

$$\mathscr{M} \models (\forall \bar{Y})(\forall \bar{m})[\psi(A,\bar{Y},\bar{m}) \leftrightarrow \widehat{\psi}(n,\bar{Y},\bar{m}).$$

The key point is that the atomic formulas in $\psi(A,\bar{Y},\bar{m})$ are the same as those in $\widehat{\psi}(n,\bar{Y},\bar{m})$, with the exception of formulas of the form $t \in A$, which have been replaced with the equivalent formulas of the form $\rho(t,n)$.

A second metainduction on the quantifier structure of φ shows that we may adjoin quantifiers to ψ and $\widehat{\psi}$ until we have obtained φ and $\widehat{\varphi}$, while maintaining logical equivalence. Thus every model of RCA_0 satisfies (4.5.1).

Because ρ has only bounded quantifiers, the substitution required to pass from φ to $\widehat{\varphi}$ does not change the complexity of the formula. □

If F is any finite set and n is its canonical index, we sometimes write $\widehat{\varphi}(F)$ for $\widehat{\varphi}(n)$.

Proof of Theorem 4.4. For (1), let $\varphi(X)$ and $A = \{a_i : i \in \mathbb{N}\}$ be an instance of Γ-FCP. Define $g \colon 2^{<\mathbb{N}} \times \mathbb{N} \to 2^{<\mathbb{N}}$ by

$$g(\tau,i) = \begin{cases} 1 & \text{if } \widehat{\varphi}(\{a_j : \tau(j)\downarrow= 1\} \cup \{a_i\}) \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

where $\widehat{\varphi}$ is as in the lemma, and for a finite set F, $\widehat{\varphi}(F)$ refers to $\widehat{\varphi}(n)$ where n is the canonical index of F. The function g exists by Γ comprehension.

By primitive recursion, there exists a function $h\colon \mathbb{N} \to 2^{<\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $h(i) = 1$ if and only if $g(h \restriction i, i) = 1$. For each $i \in \mathbb{N}$, let $B_i = \{a_j : j < i \wedge h(j) = 1\}$. An induction on φ shows that $\varphi(B_i)$ holds for every $i \in \mathbb{N}$.

Let $B = \{a_i : h(i) = 1\} = \bigcup_{i \in \mathbb{N}} B_i$. Because Proposition 4.2 is provable in ACA_0 and hence in Γ-CA_0, it follows that $\varphi(B)$ holds. By the same token, if $\varphi(B \cup \{a_k\})$ holds for some k then so must $\varphi(B_k \cup \{a_k\})$, and therefore $a_k \in B_{k+1}$, which means that $a_k \in B$. Therefore B is $\subseteq$-maximal, and we have shown that Γ-CA_0 proves Γ-FCP.

For (2), we assume Γ is one of Π^0_n, Π^1_n, or Σ^1_n; the proof for Δ^1_n is similar. We work in $\mathsf{RCA}_0 + \Gamma$-FCP. Let $\varphi(n)$ be a formula in Γ and let $\psi(X)$ be the formula $(\forall n)[n \in X \to \varphi(n)]$. It is easily seen that ψ is of finite character and belongs to Γ. By Γ-FCP, $\mathbb{N}$ contains a $\subseteq$-maximal subset B such that $\psi(B)$ holds. For any y, if $y \in B$ then $\varphi(y)$ holds. On the other hand, if $\varphi(y)$ holds then so does $\psi(B \cup \{y\})$, so y must belong to B by maximality. Therefore $B = \{y \in \mathbb{N} : \varphi(y)\}$, and we have shown that Γ-FCP implies Γ-CA_0. □

The corollary below summarizes the theorem as it applies to the various classes of formulas we are interested in. Of special note is part (5), which says that FCP itself (that is, FCP for arbitrary L_2-formulas) is as strong as any theorem of second-order arithmetic can be.

Corollary 4.6. *The following are provable in* RCA_0*:*

1. Δ^0_1-FCP, Σ^0_0-FCP, *and* QF-FCP*;*
2. *for each* $n \geq 1$, ACA_0 *is equivalent to* Π^0_n-FCP*;*
3. *for each* $n \geq 1$, Δ^1_n-CA_0 *is equivalent to* Δ^1_n-FCP*;*
4. *for each* $n \geq 1$, Π^1_n-CA_0 *is equivalent to* Π^1_n-FCP *and to* Σ^1_n-FCP*;*
5. Z_2 *is equivalent to* FCP*.*

The case of FCP for Σ^0_1 formulas is anomalous. The proof of part (2) of the theorem does not go through for Σ^0_1 because this class is not closed under universal quantification. As the proof of the next proposition shows, this limitation is quite significant. Intuitively, it means that a Σ^0_1 formula $\varphi(X)$ of finite character can only control a fixed finite piece of a set X. Hence, for the purposes of finding a maximal subset of which φ holds, we can replace φ by a formula with only bounded quantifiers.

Proposition 4.7. Σ^0_1-FCP *is provable in* RCA_0*.*

Proof. Let $\varphi(X)$ be a Σ^0_1 formula of finite character. We claim that there exists a finite subset F of $\mathbb{N}$ such that for every set A, if $F \cap A = \emptyset$ then $\varphi(A)$ holds. Let $\psi(X, x)$ be a bounded quantifier formula such that $\varphi(X) \equiv (\exists x)\, \psi(X, x)$, and fix n such that $\psi(\emptyset, n)$ holds. Note that $\psi(X, n)$ is a bounded quantifier formula with no free number variables. Any such formula is equivalent to a quantifier-free formula, because each quantifier will be bounded by a standard natural number. In turn, each quantifier-free formula can be written as a disjunction of conjunctions of atomic formulas and their negations. So we may assume $\psi(X, n)$ is in this form. Since $\psi(\emptyset, n)$ holds, there must be a disjunct $\theta(X)$ of $\psi(X, n)$ that holds of $\emptyset$. Clearly, $\theta(X)$ cannot have a conjunct of the form $t \in X$, t a term. Therefore, if we let F be the set of all terms t such $t \notin X$ is a conjunct of $\theta(X)$, we see that $\theta(A)$ holds whenever $F \cap A = \emptyset$. This completes the proof of the claim.

Now fix any set A. By the claim, there is a finite set F such that $\varphi(A - F)$ holds. By Σ^0_1 induction, there is such an F of smallest size. Then if $\varphi((A - F) \cup \{a\})$ holds for some $a \in A$, it cannot be that $a \in F$, as otherwise $F' = F - \{a\}$ would be a strictly smaller finite set than F such that $\varphi(A - F')$ holds. Thus it must be that $a \in A - F$, and we conclude that $A - F$ is a $\subseteq$-maximal subset of A of which φ holds. □

The above proof contains an implicit non-uniformity in the choice of F of smallest size. The following proposition shows that this non-uniformity is essential, by showing that a sequential form of Σ^0_1-FCP is a strictly stronger principle.

Proposition 4.8. *The following are equivalent over* RCA_0*:*

(1) ACA_0*;*
(2) *for every family* $A = \langle A_i : i \in \mathbb{N} \rangle$ *of sets, and every* Σ^0_1 *formula* $\varphi(X, x)$ *with one free set variable and one free number variable such that for all* $i \in \mathbb{N}$*, the formula* $\varphi(X, i)$ *is of finite character, there exists a family* $B = \langle B_i : i \in \mathbb{N} \rangle$ *of sets such that for all* i*,* B_i *is a* $\subseteq$*-maximal subset of* A_i *satisfying* $\varphi(X, i)$*.*

Proof. The forward implication follows by a straightforward modification of the proof of Theorem 4.4. For the reversal, let a one-to-one function $f \colon \mathbb{N} \to \mathbb{N}$ be given. For each $i \in \mathbb{N}$, let $A_i = \{i\}$, and let $\varphi(X, x)$ be the formula

$$(\exists y)[x \in X \to f(y) = x].$$

Then, for each i, $\varphi(X, i)$ has the finite character property, and for every set S that contains i, $\varphi(S, i)$ holds if and only if $i \in \text{range}(f)$. Thus, if $B = \langle B_i : i \in \mathbb{N} \rangle$ is the subfamily obtained by applying part (2) to the family $A = \langle A_i : i \in \mathbb{N} \rangle$ and the formula $\varphi(X, x)$, then

$$i \in \text{range}(f) \Leftrightarrow B_i = \{i\} \Leftrightarrow i \in B_i.$$

It follows that the range of f exists. □

Note that the proposition fails for the class of bounded-quantifier formulas of finite character in place of the class of Σ^0_1 such formulas, since part (2) is then clearly provable in RCA_0. Thus, in spite of the similarity between the two classes suggested by the proof of Proposition 4.7, the two do not coincide.

4.2. **Finitary closure operators.** We can strengthen FCP by imposing additional requirements on the maximal set being constructed. In particular, we now consider requiring the maximal set to satisfy a finitary closure property as well as to satisfy a property of finite character.

Definition 4.9. A *finitary closure operator* is a set of pairs $\langle F, n\rangle$ in which F is (the canonical index for) a finite (possibly empty) subset of $\mathbb{N}$ and $n \in \mathbb{N}$. A set $A \subseteq \mathbb{N}$ is *closed* under a finitary closure operator D, or D*-closed*, if for every $\langle F, n\rangle \in D$, if $F \subseteq A$ then $n \in A$.

Our definition of a closure operator is not the standard set-theoretic definition presented by Rubin and Rubin [16, Definition 6.3]. However, it is easy to see that for each operator of the one kind there is an operator of the other such that the same sets are closed under both. The above definition has the advantage of being readily formalizable in RCA_0.

The following fact expresses the monotonicity of finitary closure operators.

Proposition 4.10. *If D is a finitary closure operator and $A_0 \subseteq A_1 \subseteq A_2 \cdots$ is a sequence of sets such that each A_i is D-closed, then $\bigcup_{i\in\mathbb{N}} A_i$ is D-closed.*

The principle in the next definition is analogous to principle $\mathsf{AL}'3$ of Rubin and Rubin [16], which is equivalent to the axiom of choice by [16, p. 96, and Theorems 6.4 and 6.5].

Definition 4.11. The following scheme is defined in RCA_0.

(CE) If D is a finitary closure operator, φ is a formula of finite character, and A is any set, then every D-closed subset of A satisfying φ is contained in a maximal such subset.

In the terminology of Rubin and Rubin [16], this is a "primed" statement, meaning that it asserts the existence not merely of a maximal subset of a given set, but the existence of a maximal *extension* of any given subset. Primed versions of all of the principles considered above can be formed, and can easily be seen to be equivalent to the unprimed ones. By contrast, CE has only a primed form. This is because if A is a set, φ is a formula of finite character, and D is a finitary closure operator, A need not have any D-closed subset of which φ holds. For example, suppose φ holds only of $\emptyset$, and D contains a pair of the form $\langle \emptyset, a\rangle$ for some $a \in A$.

This leads to the observation that the requirements in the CE scheme that the maximal set must both be D-closed and satisfy a property of finite character are, intuitively, in opposition to each other. Satisfying a finitary closure property is a positive requirement, in the sense that forming the closure of a set usually requires adding elements to the set. Satisfying a property of finite character can be seen as a negative requirement in light of part (1) of Proposition 4.2.

We consider restrictions of CE as we did restrictions of FCP above. By analogy, if Γ is a class of formulas, we use the notation Γ-CE to denote the restriction of CE to the formulas in Γ. We begin with the following analogue of Theorem 4.4 (1) from the previous subsection.

Theorem 4.12. *For $i \in \{0,1\}$ and $n \geq 1$, let Γ be Π^i_n, Σ^i_n, or Δ^1_n. Then Γ-CE is provable in Γ-CA_0.*

Proof. We work in Γ-CA_0. Let φ be a formula of finite character in Γ, which may have parameters, and let D be a finitary closure operator. Let A be any set and let C be a D-closed subset of A such that $\varphi(C)$ holds.

For any $X \subseteq A$, let $\mathrm{cl}_D(X)$ denote the D-*closure* of X. That is, $\mathrm{cl}_D(X) = \bigcup_{i\in\mathbb{N}} X_i$, where $X_0 = X$ and for each $i \in \mathbb{N}$, X_{i+1} is the set of all $n \in \mathbb{N}$ such that either $n \in X_i$ or there is a finite set $F \subseteq X_i$ such that $\langle F, n\rangle \in D$. Because we take D to be a set, $\mathrm{cl}_D(X)$ can be defined using a Σ^0_1 formula with parameter D. Define a formula $\psi(\sigma, X)$ by

$$\psi(\sigma, X) \Leftrightarrow (\forall n)[(D_n \subseteq \mathrm{cl}_D(X \cup \{i : \sigma(i) = 1\})) \rightarrow \widehat{\varphi}(n)] \\ \wedge \mathrm{cl}_D(X \cup \{i : \sigma(i) = 1\}) \subseteq A,$$

where $\widehat{\varphi}$ is as in Lemma 4.5. Note that ψ is arithmetical if Γ is Π^0_n or Σ^0_n, and is in Γ otherwise.

Define the function $f\colon \mathbb{N} \rightarrow \{0,1\}$ inductively such that $f(i) = 1$ if and only if $\psi(\{j < i : f(j) = 1\} \cup \{i\}, C)$ holds. The characterization of the complexity of ψ ensures that f can be constructed using Γ comprehension. Now let

$$B_i = \mathrm{cl}_D(C \cup \{j < i : f(j) = 1\})$$

for each $i \in \mathbb{N}$, and let $B = \bigcup_{i\in\mathbb{N}} B_i$. The construction of f ensures that $\varphi(B_i)$ implies $\varphi(B_{i+1})$ for all i, and we have assumed that φ holds of $B_0 = \mathrm{cl}_D(C) = C$. Therefore, an instance of induction shows that φ holds of B_i for all $i \in \mathbb{N}$, and thus also of B by Proposition 4.2. This also shows that $B \subseteq A$. Similarly, because each B_i is D-closed, the formalized version of Proposition 4.10 implies B is D-closed.

Finally, we check that B is a maximal D-closed extension of C in A of which φ holds. Suppose that for some $i \in A$, $B \cup \{i\}$ is D-closed and $\varphi(B \cup \{i\})$ holds. Then since $B_i \subseteq B$, we have $\mathrm{cl}_D(B_i \cup \{i\}) \subseteq B \cup \{i\}$. Thus $\varphi(F)$ holds for every finite subset F of $\mathrm{cl}_D(B_i \cup \{i\})$, so by definition $f(i) = 1$ and $B_{i+1} = \mathrm{cl}_D(B_i \cup \{i\})$. Here we are using the fact that for all sets X and all $a, b \in \mathbb{N}$, $\mathrm{cl}_D(X \cup \{a, b\}) = \mathrm{cl}_D(\mathrm{cl}_D(X \cup \{a\}) \cup \{b\})$. Since $B_{i+1} \subseteq B$, we conclude that $i \in B$, as desired. □

It follows that for most standard classes Γ, Γ-CE is equivalent to Γ-FCP. Indeed, for any class Γ we have that Γ-CE implies Γ-FCP, because any instance of the latter can be regarded as an instance of the former by adding an empty finitary closure operator. And if Γ is Π^0_n, Π^1_n, Σ^1_n, or Δ^1_n, then Γ-FCP is equivalent to Γ-CA_0 by Theorem 4.4 (2), and hence reverses to Γ-CE. Thus, in particular, parts (2)–(5) of Corollary 4.6 hold for CE in place of FCP, and the full scheme CE itself is equivalent to Z_2.

The proof of the preceding theorem does not work for $\Gamma = \Delta^0_1$, because then Γ-CA_0 is just RCA_0, and we need at least ACA_0 to prove the existence of the function f defined there (the formula $\psi(\sigma, X)$ being arithmetical at best). The next proposition shows that this cannot be avoided, even for a class of considerably weaker formulas.

Proposition 4.13. QF-CE *implies* ACA_0 *over* RCA_0.

Proof. Assume a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ is given. Let $\varphi(X)$ be the quantifier-free formula $0 \notin X$, which trivially has finite character, and let $\langle p_i : i \in \mathbb{N}\rangle$ be an enumeration of all primes. Let D be the finitary closure operator consisting, for all $i, n \in \mathbb{N}$, of all pairs of the form

- $\langle \{p_i^{n+1}\}, p_i^{n+2}\rangle$;
- $\langle \{p_i^{n+2}\}, p_i^{n+1}\rangle$;
- $\langle \{p_i^{n+1}\}, 0\rangle$, if $f(n) = i$.

Notice that D exists by Δ^0_1 comprehension relative to f and our enumeration of primes.

Note that $\emptyset$ is a D-closed subset of $\mathbb{N}$ and $\varphi(\emptyset)$ holds. Thus, we may apply CE for quantifier-free formulas to obtain a maximal D-closed subset B of $\mathbb{N}$ such that $\varphi(B)$ holds. Then by definition of D, for every $i \in \mathbb{N}$, B either contains every positive power of p_i or no positive power. Now if $f(n) = i$ for some n, then no positive power of p can be in B, since otherwise p^{n+1} would necessarily be in B and hence so would 0. On the other hand, if $f(n) \neq i$ for all n then $B \cup \{p_i^{n+1} : n \in \mathbb{N}\}$ is D-closed and satisfies φ, so by maximality p_i^{n+1} must belong to B for every n. It follows that $i \in \text{range}(f)$ if and only if $p_i \in B$, so the range of f exists. □

Thus we are able to separate CE from FCP at least in terms of some of their strictest restrictions. In contrast to Corollary 4.6 (1) and Proposition 4.7, we consequently have:

Corollary 4.14. *The following are equivalent over* RCA_0*:*

(1) ACA_0;
(2) Σ^0_1-CE;
(3) Σ^0_0-CE;
(4) QF-CE.

We conclude this subsection with one additional illustration of how formulas of finite character can be used in conjunction with finitary closure operators. Recall the following concepts from order theory:

- A *countable join-semilattice* is a countable poset $\langle L, \leq_L\rangle$ with a maximal element 1_L and an operation $\vee_L \colon L \times L \to L$ such that for all $a, b \in L$, $a \vee_L b$, called the *join* of a and b, is the least upper bound of a and b.
- An *ideal* on a countable join-semilattice L is a subset I of L that is downward closed under $\leq_L$ and closed under $\vee_L$.

The principle in the following proposition is the countable analogue of a variant of $\mathsf{AL}'1$ in Rubin and Rubin [16]; compare with Proposition 4.19 below. For more on the computability theory of ideals on lattices, see Turlington [20].

Proposition 4.15. *Over* RCA_0*,* $\mathsf{QF\text{-}CE}$ *implies that every proper ideal on a countable join-semilattice extends to a maximal proper ideal.*

Proof. Let L be a countable join-semilattice. Let φ be the formula $1 \notin X$, and let D be the finitary closure operator consisting of all pairs of the form

- $\langle \{a, b\}, c\rangle$ where $a, b \in L$ and $c = a \vee b$;
- $\langle \{a\}, b\rangle$, where $b \leq_L a$.

Because we define a join-semilattice to come with both the order relation and the join operation, the set D is Δ^0_0 with parameters, so RCA_0 proves D exists. It is immediate that a set X is closed under D if and only if X is an ideal in L. □

4.3. **Nondeterministic finitary closure operators.** It appears that the underlying reason that the restriction of CE to arithmetical formulas is provable in ACA_0 (and more generally, why Γ-CE is provable in Γ-CA_0 if Γ is as in Theorem 4.12) is that our definition of finitary closure operator is very constraining. Intuitively, if D is such an operator and φ is an arithmetical formula, and we seek to extend some D-closed subset B satisfying φ to a maximal such subset, we can focus largely on ensuring that φ holds. Achieving closure under D is relatively straightforward, because at each stage we only need to search through all finite subsets F of our current extension, and then adjoin all n such that $\langle F, n\rangle \in D$. This closure process becomes far less trivial if we are given a choice of which elements to add. We now consider the case when each finite subset F can be associated with a possibly infinite set of numbers from which we must choose at least one to adjoin. We will show that this weaker notion of closure operator leads to a stronger analogue of CE.

Definition 4.16. A *nondeterministic finitary closure operator* is a sequence of sets of the form $\langle F, S\rangle$ where F is (the canonical index for) a finite (possibly empty) subset of $\mathbb{N}$ and S is a nonempty subset of $\mathbb{N}$. A set $A \subseteq \mathbb{N}$ is *closed* under a nondeterministic finitary closure operator N, or N-closed, if for each $\langle F, S\rangle$ in N, if $F \subseteq A$ then $A \cap S \neq \emptyset$.

Note that if D is a *deterministic* finitary closure operator, that is, a finitary closure operator in the stronger sense of the previous subsection, then for any set A there is a unique $\subseteq$-minimal D-closed set extending A. This is not true for nondeterministic finitary closure operators. Let N be the operator such that $\langle \emptyset, \mathbb{N}\rangle \in N$ and, for each $i \in \mathbb{N}$ and each $j > i$, $\langle \{i\}, \{j\}\rangle \in N$. Then any N-closed set extending $\emptyset$ will be of the form $\{i \in \mathbb{N} : i \geq k\}$ for some k, and any set of this form is N-closed. Thus there is no $\subseteq$-minimal N-closed set.

In this subsection we study the following nondeterministic version of CE.

Definition 4.17. The following scheme is defined in RCA_0.

(NCE) If N is a nondeterministic closure operator, φ is a formula of finite character, and A is any set, then every N-closed subset of A satisfying φ is contained in a maximal such subset.

Restrictions of NCE to various syntactical classes of formulas are defined as for CE and FCP. Note that, because the union of a chain of N-closed sets is again N-closed, NCE can be proved in set theory using Zorn's lemma.

Remark 4.18. We might expect to be able to prove NCE from CE by suitably transforming a given nondeterministic finitary closure operator N into a deterministic one. For instance, we could go through the members of N one by one, and for each such member $\langle F, S\rangle$ add $\langle F, n\rangle$ to D for some $n \in S$ (e.g., the least n). All D-closed sets would then indeed be N-closed. The converse, however, would not necessarily be true, because a set could have F as a subset for some $\langle F, S\rangle \in N$, yet it could contain a different $n \in S$ than the one chosen in defining D. In particular, a maximal D-closed subset (of some given set) would not need to be maximal among N-closed subsets.

The following result provides a simple but concrete example of this point. Recall that an *ideal* on a countable poset $\langle P, \leq_P\rangle$ is a subset I of P downward closed under $\leq_P$ and such that for all $p, q \in I$ there is an $r \in I$ with $p \leq_P r$ and $q \leq_P r$. The next proposition is similar to Proposition 4.15 above, which dealt with ideals on countable join-semilattices. In the proof of that proposition, we defined a deterministic finitary closure operator D in such a way that D-closed sets were closed under the join operation. For this we relied on the fact that for every two elements in the semilattice there is a unique element that is their join. The reason we need nondeterministic finitary closure operators below is that, for ideals on countable posets, there are no longer unique elements witnessing closure under the relevant operations.

Proposition 4.19. *Over* RCA_0, Π^0_2-NCE *implies that every ideal on a countable poset can be extended to a maximal ideal.*

Proof. We work in RCA_0. Let $\langle P, \leq_P\rangle$ be a countable poset. Without loss of generality we may assume $P = \{p_i : i \in \mathbb{N}\}$ is infinite. We form a nondeterministic closure operator $N = \langle N_i : i \in \mathbb{N}\rangle$ by considering the following two cases. For each $i \in \mathbb{N}$,

- if $i = 2\langle j, k\rangle$ and $p_j \leq_P p_k$, let $N_i = \langle \{p_k\}, \{p_j\}\rangle$;
- if $i = 2\langle j, k, l\rangle + 1$ and $p_j \leq_P p_l$ and $p_k \leq_P p_l$, let
$$N_i = \langle \{p_j, p_k\}, \{p_n : (p_j \leq_P p_n) \wedge (p_k \leq_P p_n)\}\rangle;$$
- otherwise, let $N_i = \langle \{p_i\}, \{p_i\}\rangle$.

This construction gives a quantifier-free definition of each N_i uniformly in i, so the sequence N exists.

Let $\varphi(X)$ be the Π^0_2 formula which says that every pair of elements in X has a common upper bound in P. A straightforward proof shows that φ is of finite character and that a set $I \subseteq P$ is an ideal on P if and only if I is N-closed and $\varphi(I)$ holds. □

Mummert [14, Theorem 2.4] showed that the proposition that every ideal on a countable poset extends to a maximal ideal is equivalent to Π^1_1-CA_0 over RCA_0. Hence, Π^0_2-NCE implies Π^1_1-CA_0. By Theorem 4.12, Π^0_2-CE is provable in ACA_0, so we see that the idea of Remark 4.18 fundamentally cannot work.

We will obtain the reversal of Π^0_2-NCE to Π^1_1-CA_0 in a sharper form in Theorem 4.21 below. First, we prove the following upper bound. The proof uses a technique involving countable coded β-models, parallel to Lemma 2.4 of Mummert [14]. In RCA_0, a *countable coded β-model* is defined as a sequence $\mathscr{M} = \langle M_i : i \in \mathbb{N}\rangle$ of subsets of $\mathbb{N}$ such that for every Σ^1_1 formula φ with parameters from $\mathscr{M}$, φ holds if and only if $\mathscr{M} \models \varphi$ [18, Definitions VII.2.1 and VII.2.3]. A general treatment of countable coded β-models is given by Simpson [18, Section VII.2].

Proposition 4.20. *Σ^1_1-NCE is provable in Π^1_1-CA_0.*

Proof. We work in Π^1_1-CA_0. Let φ be a Σ^1_1 formula of finite character (possibly with parameters) and let N be a nondeterministic closure operator. Let A be any set and let C be an N-closed subset of A such that $\varphi(C)$ holds.

Let $\mathscr{M} = \langle M_i : i \in \mathbb{N}\rangle$ be a countable coded β-model containing A, B, N, and any parameters of φ, which exists by [18, Theorem VII.2.10]. Using Π^1_1 comprehension, we may form the set $\{i : \mathscr{M} \models \varphi(M_i)\}$.

Working outside $\mathscr{M}$, we build an increasing sequence $\langle B_i : i \in \mathbb{N}\rangle$ of N-closed extensions of C. Let $B_0 = C$. Given i, ask whether there is a j such that

- M_j is an N-closed subset of A;
- $B_i \subseteq M_j$;
- $i \in M_j$;
- and $\varphi(M_j)$ holds.

If there is, choose the least such j and let $B_{i+1} = M_j$. Otherwise, let $B_{i+1} = B_i$. Finally, let $B = \bigcup_{i\in\mathbb{N}} B_i$.

Because the inductive construction only asks arithmetical questions about $\mathscr{M}$, it can be carried out in Π^1_1-CA_0, and so Π^1_1-CA_0 proves that B exists. Clearly $C \subseteq B \subseteq A$. An arithmetical induction shows that for all $i \in \mathbb{N}$, $\varphi(B_i)$ holds and B_i is N-closed. Therefore, the formalized version of Proposition 4.2 shows that $\varphi(B)$ holds, and the analogue of Proposition 4.10 to nondeterministic finitary closure operators shows that B is N-closed.

Now suppose that for some $i \in A$, $B \cup \{i\}$ is an N-closed subset of A extending C and satisfying φ. Because φ is Σ^1_1, and because N is a sequence, the property

$$(\exists X)[X \text{ is } N\text{-closed} \wedge B_i \subseteq X \subseteq A \wedge i \in X \wedge \varphi(X)] \tag{4.20.1}$$

is expressible by a Σ^1_1 sentence, and $B \cup \{i\}$ witnesses that it is true. Thus, because $\mathscr{M}$ is a β-model, this sentence must be satisfied by $\mathscr{M}$, which means that some M_j must also witness it. The inductive construction must therefore have selected such an M_j to be B_{i+1}, which means $i \in B_{i+1}$ and hence $i \in B$. It follows that B is maximal. □

The next theorem shows that NCE for quantifier-free formulas without parameters is already as strong as Σ^1_1-FCP and Σ^1_1-CE. In particular, in view of Corollary 4.14, it is considerably stronger than CE for quantifier-free formulas.

Theorem 4.21. *For each $n \geq 1$, the following are equivalent over* RCA_0*:*

(1) Π^1_1-CA_0;
(2) Σ^1_1-NCE;
(3) Σ^0_n-NCE;
(4) QF-NCE.

Proof. We have already proved (1) implies (2), and it is obvious that (2) implies (3) and (3) implies (4). The reversal of (4) to (1) splits into two steps.

For the first step, note that RCA_0 can convert any finitary closure operator D into a corresponding nondeterministic closure operator N such that the notions of D-closed and N-closed coincide (note that this is the opposite of what was discussed in Remark 4.18). Therefore NCE for quantifier-free formulas implies ACA_0 over RCA_0 by Proposition 4.13.

Next, for the second step, we work in ACA_0. Let $\langle T_i : i \in \mathbb{N}\rangle$ be a sequence of subtrees of $\mathbb{N}^{<\mathbb{N}}$. To prove Π^1_1-CA_0, it is sufficient to form the set of $i \in \mathbb{N}$ such that T_i has an infinite path [18, Lemma VI.1.1]. Let A be the set of all pairs $\langle i, \sigma\rangle$ such that $\sigma \in T_i$, along with one distinguished element z that is not a pair. Let $\varphi(X)$ be the formula $z \notin X$, which has no parameters provided that z is coded by a standard natural number. Clearly, φ has the finite character property.

Write $A - \{z\} = \{a_i : i \in \mathbb{N}\}$, and define a nondeterministic finitary closure operator $N = \langle N_i : i \in \mathbb{N}\rangle$ as follows. For each $j \in \mathbb{N}$, if $a_j = \langle i, \sigma\rangle$, then

- if σ is a dead end in T_i, let $N_j = \langle \{\langle i, \sigma\rangle\}, \{z\}\rangle$;
- if σ is not a dead end in T_i, let
$$N_j = \langle \{\langle i, \sigma\rangle\}, \{\langle i, \tau\rangle : \tau \in T_i \wedge \tau \succ \sigma \wedge |\tau| = |\sigma| + 1\}\rangle.$$

Notice that N can be formed by arithmetical comprehension.

Suppose B is an N-closed subset of A that satisfies φ (i.e., does not contain z). Then, for any i, whenever $\langle i, \sigma\rangle$ is in B there is some immediate extension τ of σ in T_i such that $\langle i, \tau\rangle$ is in B. Thus if $\langle i, \sigma\rangle$ is in B then there is an infinite path through T_i extending σ. So in particular, if $\langle i, \emptyset\rangle$ is in B then T_i has an infinite path. Conversely, if f is an infinite path through T_i, then $B \cup \{\langle i, f \upharpoonright n\rangle : n \in \mathbb{N}\}$ is N-closed and satisfies φ.

Because $\emptyset$ is N-closed and satisfies φ, we may apply NCE for quantifier-free formulas to get a maximal extension of it within A. By the previous paragraph and the maximality of B, T_i has a path if and only if $\langle i, \emptyset \rangle \in B$. Thus, the set of i such that T_i has a path exists, as desired. □

Our final results characterize the strength of NCE for formulas higher in the analytical hierarchy.

Proposition 4.22. *For each $n \geq 1$,*

1. *Σ^1_n-NCE and Π^1_n-NCE are provable in Π^1_n-CA_0;*
2. *Δ^1_n-NCE is provable in Δ^1_n-CA_0.*

Proof. We prove part (1), the proof of part (2) being similar. Let $\varphi(X)$ be a Σ^1_n formula of finite character, respectively a Π^1_n such formula. Let N be a nondeterministic closure operator, let A be any set, and let C be an N-closed subset of A such that $\varphi(C)$ holds.

By Lemma 4.5, let $\widehat{\varphi}$ be a Σ^1_n formula, respectively a Π^1_n formula, such that

$$(\forall X)(\forall n)[X = D_n \to (\varphi(X) \leftrightarrow \widehat{\varphi}(n))].$$

We may use Π^1_n comprehension to form the set $W = \{n : \widehat{\varphi}(n)\}$. Define $\psi(X)$ to be the arithmetical formula $(\forall n)[D_n \subseteq X \to n \in W]$.

We claim that for every set X, $\psi(X)$ holds if and only if $\varphi(X)$ holds. The definitions of W and ψ ensure that $\psi(X)$ holds if and only if $\varphi(D_n)$ holds for every finite $D_n \subseteq X$, which is true if and only if $\varphi(X)$ holds because φ has finite character. This establishes the claim.

By the claim, ψ is a property of finite character and $\psi(C)$ holds. Using Σ^1_1-NCE, which is provable in Π^1_1-CA_0 by Proposition 4.20 and thus in Π^1_n-CA_0, there is a maximal N-closed subset B of A extending C with property ψ. Again by the claim, B is a maximal N-closed subset of A extending B with property φ. □

Corollary 4.23. *The following are provable in RCA_0:*

1. *for each $n \geq 1$, Δ^1_n-CA_0 is equivalent to Δ^1_n-NCE;*
2. *for each $n \geq 1$, Π^1_n-CA_0 is equivalent to Π^1_n-NCE and to Σ^1_n-NCE;*
3. *Z_2 is equivelent to NCE.*

Proof. The implications from Δ^1_n-CA_0, Π^1_n-CA_0, and Z_2 follow by Proposition 4.22. On the other hand, each restriction of NCE trivially implies the corresponding restriction of FCP, so the reversals follow by Corollary 4.6. □

Remark 4.24. The characterizations in this section shed light on the role of the closure operator in the principles CE and NCE. For $n \geq 1$, we have shown that Σ^1_n-FCP, Σ^1_n-CE, and Σ^1_n-NCE are all equivalent over RCA_0. However, QF-FCP is provable in RCA_0, QF-CE is equvalent to ACA_0 over RCA_0, and QF-NCE is equivalent to Π^1_1-CA_0 over RCA_0. Thus the closure operators in the stronger principles serve as a sort of replacement for arithmetical quantification in the case of CE, and for Σ^1_1 quantification in the case of NCE. This allows these principles to have greater strength than might be suggested by the property of finite character alone. At higher levels of the analytical hierarchy, the principles become equivalent because the complexity of the property of finite character overtakes the complexity of the closure notions.

5. Questions

In this section we summarize the principal questions left over from our investigation. These concern the precise strength of $F\mathsf{IP}$ and the principles $\overline{D}_n\mathsf{IP}$. While we have closely located these principles' position in the structure of statements lying between RCA_0 and ACA_0, we do not know the answers to the following questions.

Question 5.1. Does $\overline{D}_2\mathsf{IP}$ imply $F\mathsf{IP}$ over RCA_0? Does $\overline{D}_n\mathsf{IP}$ imply $\overline{D}_{n+1}\mathsf{IP}$?

Question 5.2. Does AMT imply $\overline{D}_2\mathsf{IP}$ over RCA_0? Does OPT imply $\overline{D}_2\mathsf{IP}$?

By Proposition 3.27, the first part of the Question 5.2 has an affirmative answer over $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$. For the second part, it may be easier to ask whether the implication can at least be shown to hold in ω-models. An affirmative answer would likely follow from an affirmative answer to the following question.

Question 5.3. Given a computable nontrivial family A, does every set of hyperimmune degree compute a maximal subfamily of A with the F intersection property (or at least with the $\overline{D}_2$ intersection property)?

We conjecture the answer to be no.

Our final question is less directly related to our investigation. We mention it in view of Proposition 4.15 above.

Question 5.4. What is the strength of the principle asserting that every proper ideal on a countable join-semilattice extends to a maximal proper ideal?

This question is further motivated by work of Turlington [20, Theorem 2.4.11] on the similar problem of constructing prime ideals on computble lattices. However, because a maximal ideal on a countable lattice need not be a prime ideal, Turlington's results do not directly resolve our question.

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