Optimal Control of Risk Process in a Regime Switching Environment

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Abstract

This paper is concerned with cost optimization of an insurance company. The surplus of the insurance company is modeled by a controlled regime switching diffusion, where the regime switching mechanism provides the fluctuations of the random environment. The goal is to find an optimal control that minimizes the total cost up to a stochastic exit time. A weaker sufficient condition than that of (Fleming and Soner, 2006, Section V.2) for the continuity of the value function is obtained. Further, the value function is shown to be a viscosity solution of a Hamilton-Jacobian-Bellman equation.

Keywords. Regime switching diffusion, continuity of the value function, exit time control, viscosity solution.

AMS subject classification. 93E20, 60J60.

1 Introduction

Recently the optimal risk control and dividend distribution problems have drawn growing attention from researchers. Some recent developments can be found in Cadenillas et al. (2006); Choulli et al. (2003); Irgens and Paulsen (2005); Paulsen and Gjessing (1997); Paulsen et al. (2005); Schmidli (2001, 2002); Taksar and Hunderup (2007); Taksar and Markussen (2003); Touzi (2000) and the references therein. In those works, the liquid assets of the insurance companies are modeled by some stochastic processes (usually diffusions or jump diffusions). At any time t, the insurance companies can choose different business activities such as reinsurance, investment, dividend payment, etc. The decisions are based upon the information

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available to them by time t as well as the pre-given economic or political criteria. Different business activities lead to different dynamics in the evolution of the surplus and hence different economic or political returns. This sets a scene for an optimal stochastic control model for the surplus of the insurance company. In the literature, the most commonly used criteria are: (i) maximizing expected utility at or up to the time of ruin (Irgens and Paulsen (2005); Touzi (2000)), (ii) minimizing the probability of ultimate ruin (Schmidli (2001, 2002); Taksar and Markussen (2003)), and (iii) maximizing the cumulative expected discounted dividends (Cadenillas et al. (2006); Choulli et al. (2003); Paulsen and Gjessing (1997)).

In contrast to the aforementioned references, in this work, we focus on the problem of cost optimization for the insurance company. In addition to the usual operating cost of an insurance company such as corporate debt, bond liability, loan amortization, etc, in practice, almost every insurance claim is accompanied by a certain amount of business cost resulting from claim appraisal, investigation, settlement negotiation, and so on. Minimizing the cost may increase the profit of the insurance company and lower premiums for its customers.

We should also note that the word cost can be used in a general sense: it may represent any monetary amount such as claims, penalties, dividends, utilities, and so on. As demonstrated in Cai et al. (2009) and Feng (2009), the total discounted cost actually covers a number of ruin-related quantities frequently analyzed in ruin literature such as the expected present value of penalty at ruin and the total dividends under various dividend strategies. Therefore this work can be applied to a broad range of aspects of risk management such as utility and cumulative dividends maximization.

Another feature of this work is the consideration of regime switching. Most of the existing literature on optimal control of risk processes are based on the framework of diffusion approximation model (Grandell (1991)). That is, the surplus of an insurance company is usually modeled or approximated by a (jump) diffusion process. Roughly speaking, if the surplus of the insurance company is much larger than the individual claims, then the classical homogeneous Poisson model can be approximated by a diffusion model. Along another line, Asmussen (1989) proposed a Markovian-modulated risk model. The model is a hybrid system, in which continuous dynamics are intertwined with discrete events. More specifically, the evolutions of the surplus (continuous dynamics) is subject to jumps or switches of the economic or political environment (discrete events), and the dynamics of the surplus in different environments are markedly different. As demonstrated in Asmussen (1989) (see also Yang and Yin (2004)), this model can capture the features of insurance policies that depend on the economic or political environment changes. The states of the discrete event process can model for example, certain types of epidemics in health insurance, weather types in automobile insurance, the El Niño/La Niña phenomenon in property insurance, or economic conditions in unemployment insurance. Also, in many practical situations, people are more concerned with the short term results of the business activities. For instance, the manager and/or shareholders of an insurance company want to determine the short-term benefits of a particular business activity. Inspired by these arguments, we consider a controlled surplus process modeled by a regime switching diffusion over a finite time horizon $[s, T]$, where $T > 0$ is a fixed constant and $s \in [0, T)$.

By the nature of the risk process, the insurance company may default at some finite time; at which point we say that the surplus is ruined and denote the ruin time by $\tilde{\tau}$. Consequently, we need only to consider the total cost up to the ruin time or the terminal time T, whichever comes first. That is, our control problem is over the interval $[s, T \wedge \tilde{\tau}]$, with $T \wedge \tilde{\tau} = \min\{T, \tilde{\tau}\}$ being a random time (stopping time). This is generally termed as exit time control or stochastic control with exit time in the literature (Fleming and Soner (2006)). As we shall see in Example 3.1, the stopping time $T \wedge \tilde{\tau}$ depends on the initial surplus $X(s) = x$. As a result, the value function defined in (2.6) is not necessarily continuous with respect to x. See Example 3.1 for detailed discussions. Then, a problem of great interest is: Under what condition(s), is the value function continuous? (Fleming and Soner, 2006, p. 202) proposed a condition on the drift of the underlying (1-dimensional) controlled diffusion under which the continuity of the value function is guaranteed. The condition in Fleming and Soner (2006) was recently generalized in Bayraktar et al. (2010) by considering both the drift and the diffusion coefficients. In this work, we propose a new condition in terms of the regularity of the boundary point to obtain the continuity of the value function. Our condition is another generalization of the one in Fleming and Soner (2006) and possibly applicable in more situations than that of Bayraktar et al. (2010) if the underlying controlled diffusion is multi-dimensional. See Theorem 3.4, Proposition 3.7, and Example 3.8 for more details.

We emphasize that the continuity of the value function is very important and useful for the following reasons. Firstly, with the continuity of the value function, one has the dynamic programming principle (Fleming and Soner (2006)), which in turn, helps to establish the viscosity solution property for the value function. Secondly, the continuity of the value function plays a vital role in the study of numerical approximation to the value function. Typically, one constructs a controlled locally consistent discrete sequence $\{X_n^h, n \in \mathbb{N}\}$ and find the associated value function V^h for the discrete problem, where $h > 0$ is the stepsize of the discretization. If the value function is not continuous, then as $h \downarrow 0$, V^h may not converge to the true value function V, even though X^h converges to X in some suitable sense, where X^h is the continuous parameter interpolated process of $\left\{X_n^h, n \in \mathbb{N}\right\}$. In other words, X^h approximates X, but V^h may still differ significantly from V. See Example 3.1 and also Kushner and Dupuis (2001) for more details. In fact, this work is largely motivated by this aspect of consideration.

The classical approach to stochastic control problem is the verification theorem, see for example Fleming and Rishel (1975). Typically, this approach requires the value function to be sufficiently smooth. As we mentioned earlier, there are many examples where the value function is not even continuous. In fact, due to the dependence of the terminal time $T \wedge \tilde{\tau}$ on the initial data $X(s) = x$, the value function in our setup is not necessarily continuous. Therefore the verification approach does not apply in our problem. In this paper, we use the viscosity solution approach. With the aid of the dynamic programming principle, we prove that the value function is a viscosity solution of the Hamilton-Jacobian-Bellman (HJB) equation. Since the germinal work Crandall et al. (1992); Lions (1983) and others, the viscosity solution characterization have been exploited in many control problems under various settings. But the related results in the content of regime switching is relatively scarce. Moreover, the proof for our case is not a trivial extension of the existing results. The presence of regime switching adds much difficulty in the proof. See Theorem 4.3 for more details.

The rest of the paper is arranged as follows. We formulate the problem in Section 2. Section 3 is devoted to the continuity of the value function. We obtain several sufficient conditions for continuity. In Section 4, we show that the value function is a viscosity solution of the HJB equation (2.12). We conclude the paper with a few remarks in Section 5. Appendix A provides a result on regularity of the boundary point.

A few words about notations are necessary at this point. We use I_A to denote the indicator function of a set A. If $a, b \in \mathbb{R}$, then $a \wedge b := \min\{a, b\}$. Throughout the paper, K is a generic positive constant whose exact value may differ in different appearances. For any $x_0 \in \mathbb{R}$ and $r > 0$, $B(x_0, r) = \{x \in \mathbb{R} : |x - x_0| < r\}$.

2 Problem Formulation

Let $X(t)$ denote the surplus of a large insurance company at time $t \in [s, T]$, where $T > 0$ is fixed and $s \in [0, T)$. As we indicated in Section 1, the surplus process X often displays abrupt changes according to the different economic, political, or natural environments facing the insurance company. Following Asmussen (1989), we use a continuous time Markov chain $\alpha(\cdot)$ to model the variations of the external environments of the insurance company. For simplicity, we assume the Markov chain has a finite state space $\mathcal{M} = \{1, \dots, m\}$ and is generated by $Q = (q_{ij})$, that is,

$$\mathbf{P}\{\alpha(t+\Delta t) = j | \alpha(t) = i, \alpha(s), s \le t\} = \begin{cases} q_{ij}\Delta t + o(\Delta t), & \text{if } j \neq i, \\ 1 + q_{ii}\Delta t + o(\Delta t), & \text{if } j = i, \end{cases} \tag{2.1}$$

where $q_{ij} \geq 0$ for $i, j = 1, \ldots, m$ with $j \neq i$ and $\sum_{j=1}^{m} q_{ij} = 0$ for each $i = 1, \ldots, m$. At any time t, we denote by $u(t)$ one of the possible business activities available for the insurance company. For instance, $u(t)$ may represent a reinsurance strategy, an investment plan, or a dividend payment scheme, etc.

Suppose X satisfies the following stochastic differential equation with regime switching:

$$dX(t) = b(t, X(t), \alpha(t), u(t))dt + \sigma(t, X(t), \alpha(t), u(t))dw(t), \tag{2.2}$$

with initial conditions

$$X(s) = x > 0 \quad \text{and} \quad \alpha(s) = \alpha \in \mathcal{M}, \tag{2.3}$$

where $w(\cdot)$ is a standard Brownian motion independent of the Markov chain α. Note that the independence between w and α is a standard assumption in the literature (Mao and Yuan (2006)). Denote

$$\mathcal{F}_t := \sigma \left\{ w(r), \alpha(r) : 0 \leq r \leq t \right\}.$$

Without loss of generality, we assume that $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets. Suppose throughout the paper that the control policy u is $\mathcal{F}_t$ adapted and that for any t, $u(t) \in U$, where U is a compact subset of $\mathbb{R}$. Any control u satisfying the above conditions is said to be an *admissible control*. Let $\mathcal{U}$ denote the collection of all admissible controls.

Set

$$\tau := \{t \geq s : (t, X(t)) \notin [0, T) \times (0, \infty)\} = T \wedge \tilde{\tau}, \tag{2.4}$$

where $\tilde{\tau} := \inf \{t \geq s : X(t) = 0\}$ denotes the ruin time. For a given control $u \in \mathcal{U}$, the expected total cost is

$$J(s, x, \alpha, u(\cdot)) = \mathbf{E}_{s,x,\alpha} \left[\int_s^\tau l(t, X(t), \alpha(t), u(t))dt + g(\tau, X(\tau), \alpha(\tau)) \right], \tag{2.5}$$

where $l : [0, T] \times \mathbb{R} \times \mathcal{M} \times U \mapsto \mathbb{R}_+$ represents the running cost, $g : [0, T] \times [0, \infty) \times \mathcal{M} \mapsto \mathbb{R}_+$ is the terminal cost, and $\mathbf{E}_{s,x,\alpha}$ denotes the expectation with respect to the probability law such that the regime switching diffusion $X(t)$ in (2.2) starts with initial condition specified in (2.3).

The goal is to find an optimal control $u^* \in \mathcal{U}$ that minimizes the total cost

$$V(s, x, \alpha) = J(s, x, \alpha, u^*) = \inf_{u \in \mathcal{U}} J(s, x, \alpha, u), \quad \forall (s, x, \alpha) \in [0, T) \times (0, \infty) \times \mathcal{M}. \tag{2.6}$$

Note that the terminal and boundary conditions are

$$V(T, x, \alpha) = g(T, x, \alpha), \quad \forall x \in (0, \infty), \alpha \in \mathcal{M}, \tag{2.7}$$

and

$$V(s, 0, \alpha) = g(s, 0, \alpha), \quad \forall s \in [0, T], \alpha \in \mathcal{M}. \tag{2.8}$$

Throughout the paper, we assume

Assumption A1. For each $\alpha \in \mathcal{M}$, the functions $b(\cdot,\cdot,\alpha,\cdot)$, $\sigma(\cdot,\cdot,\alpha,\cdot)$, $l(\cdot,\cdot,\alpha,\cdot)$, and $g(\cdot,\alpha)$ are uniformly continuous. Moreover, there exist positive constants κ_0 and p such that for any $t \in [0,T]$, $x, y \in \mathbb{R}$, $\alpha \in \mathcal{M}$, and $u \in U$, we have

$$\begin{array}{l} |\varphi(t,x,\alpha,u) - \varphi(t,y,\alpha,u)| \le \kappa_0 |x-y|, \text{ for } \varphi = b, \sigma, l, \text{ and } g, \\ |b(t,x,\alpha,u)| + |\sigma(t,x,\alpha,u)| \le \kappa_0(1+|x|), \\ |l(t,x,\alpha,u)| + |g(t,x,\alpha)| \le \kappa_0(1+|x|^p). \end{array} \tag{2.9}$$

It is well known (Mao and Yuan (2006)) that under Assumption A1, for any $u \in \mathcal{U}$ and any $(s,x,\alpha) \in [0,T) \times (0,\infty) \times \mathcal{M}$, there exists a unique solution X to (2.2) with initial condition (2.3) under the control u. Moreover, J in (2.5) is well-defined. In the sequel, we denote such a solution by $X^{s,x,\alpha}$ or $X^{s,x,\alpha;u}$ if the emphasis on the initial conditions and the control is needed. Similarly, $\alpha^{s,\alpha}$ denotes the Markov chain with initial condition $\alpha(s) = \alpha$.

For convenience of later presentations, we introduce the operator $\mathcal{L}_t^u$. For any $h(t,\cdot,\alpha) \in C^2, t \in [0,T], \alpha \in \mathcal{M}$ and $u \in U$, we define

$$\mathcal{L}_t^u h(t,x,\alpha) = b(t,x,\alpha,u)\frac{\partial}{\partial x}h(t,x,\alpha) + \frac{1}{2}\sigma^2(t,x,\alpha,u)\frac{\partial^2}{\partial x^2}h(t,x,\alpha) + \sum_{j=1}^{m} q_{\alpha j} h(t,x,j). \tag{2.10}$$

The following verification theorem can be established using the standard argument as in Fleming and Rishel (1975) and Fleming and Soner (2006), together with the generalized Itô formula (Mao and Yuan (2006)). We shall omit the proof for brevity.

Theorem 2.1. *Suppose there exists a function $\varphi : [s,T] \times \mathbb{R}_+ \times \mathcal{M} \mapsto \mathbb{R}_+$ such that*

(i) $\varphi(\cdot,\cdot,\alpha) \in C^{1,2}$ *for each* $\alpha \in \mathcal{M}$,

(ii) φ *satisfies the polynomial growth condition, that is, for some positive constants* p *and* K*, we have*

$$|\varphi(t,x,\alpha)| \le K(1+|x|^p), \text{ for any } t \in [s,T] \text{ and } \alpha \in \mathcal{M}, \tag{2.11}$$

(iii) φ *satisfies the Hamilton-Jacobian-Bellman* (HJB) *equation*

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi(t,x,\alpha) + \min\limits_{u \in U}[\mathcal{L}_t^u \varphi(t,x,\alpha) + l(t,x,\alpha,u)] = 0, \\ \hfill t \in (s,T), x > 0, \alpha \in \mathcal{M}, \\ \varphi(s,0,\alpha) = g(s,0,\alpha), \ \varphi(T,x,\alpha) = g(T,x,\alpha), \ s \in [0,T), x > 0, \alpha \in \mathcal{M}. \end{cases} \tag{2.12}$$

Then $\varphi(s,x,\alpha) \le J(s,x,\alpha,u(\cdot))$ for any initial condition $(s,x,\alpha) \in [0,T) \times (0,\infty) \times \mathcal{M}$ and any admissible feedback control $u(\cdot)$.

Moreover, if $u^*(\cdot)$ is an admissible feedback control such that

$$\begin{aligned}\mathcal{L}_t^{u^*}\varphi(t,x,\alpha)+l(t,x,\alpha,u^*(t,x,\alpha)) = \min_{u\in U}[\mathcal{L}_t^u\varphi(t,x,\alpha)+l(t,x,\alpha,u)] = 0,\\ \forall (s,x,\alpha)\in[0,T)\times(0,\infty)\times\mathcal{M},\end{aligned} \tag{2.13}$$

Then $\varphi(s,x,\alpha) = V(s,x,\alpha)$ for all $(s,x,\alpha)\in[0,T)\times(0,\infty)\times\mathcal{M}$ and $u^*(\cdot)$ is an optimal control.

3 Continuity

Theorem 2.1 provides conditions under which a sufficiently smooth function coincides with the value function. In particular, it indicates that if φ solves the HJB equation (2.12), then it provides a lower bound for the value function V. In addition, if we can find a feedback control $u^*(\cdot)$ satisfying (2.13), then $\varphi = V$ and $u^*(\cdot)$ is an optimal control. It is natural to ask whether the converse is true: "Does the value function V defined in (2.6) always satisfy (2.12)?" In general, the answer is no, since the value function V is not necessarily smooth enough (with respect to the variables s and x). More specifically, in our setup, both the stopping time τ and the control u may depend on the initial value $X(s) = x$. Consequently, the value function may not be even continuous. To illustrate, we consider the following uncontrolled deterministic system, where the value function is discontinuous. The example is inspired by the tangency problem presented in (Kushner and Dupuis, 2001, pp. 277–279).

Example 3.1. Consider

$$\begin{cases} dX(t) = 2(t-1)dt, \quad t\in[s,2]\\ X(s) = x > 0,\end{cases} \tag{3.1}$$

where $s\in[0,2]$. The solution of (3.1) is

$$X(t) = X^{s,x}(t) = (t-1)^2 + x - (s-1)^2, \quad t\in[s,2).$$

Let $\tau = \inf\{t > s : X(t) = 0\}\wedge 2$ and $V(s,x) = \tau$. Note that Assumption A1 is satisfied for this example. Consider the case when $s\in[0,1]$. Then it is obvious that $X(t)$ first decreases to its minimum $x-(s-1)^2$ then increases to ∞ as $t\to\infty$. As a result, we have

$$\tau\begin{cases} = 2, & \text{if } x-(s-1)^2 > 0,\\ \le 1, & \text{if } x-(s-1)^2\le 0.\end{cases}$$

Hence it follows that the value function $V(s,x) = \tau$ is not continuous on the parabola $\{(s,x)\in[0,1]\times(0,\infty) : x-(s-1)^2 = 0\}$. See the demonstration in Figure 3.1.

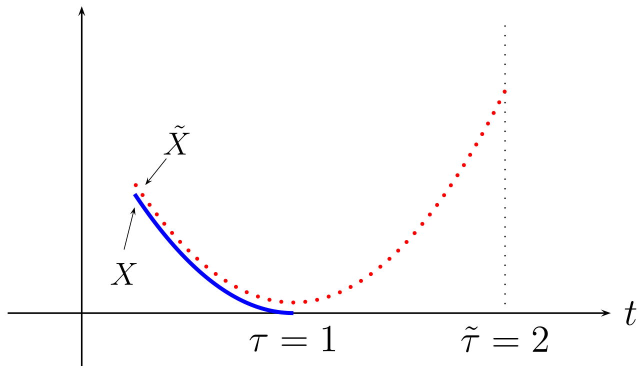

Figure 1: Discontinuous Value Function

Example 3.1 naturally motivates us to the following question: "Under what conditions is the value function continuous?" To answer this question, we follow the treatment in (Fleming and Soner, 2006, Chapter V). We first consider an auxiliary control problem, whose value function V^ε is continuous. Then we propose conditions under which V^ε converges uniformly to the original value function V as $\varepsilon \downarrow 0$, from which the continuity of V is established. As we shall see in Theorem 3.4, Proposition 3.7, and Example 3.8, our condition is more general than that in (Fleming and Soner, 2006, Section V.2).

Let $\psi : \mathbb{R} \times \mathcal{M} \mapsto \mathbb{R}$ be a function such that

$$\left|\psi^+(x,\alpha) - \psi^+(y,\alpha)\right| \le L\,|x-y|\,, \text{ for any } x, y \in \mathbb{R}, \alpha \in \mathcal{M}, \tag{3.2}$$

where $L > 0$ is a constant. For any $\varepsilon > 0$, we define

$$\Gamma^\varepsilon(t) := \exp\left\{-\frac{1}{\varepsilon}\int_s^t \psi^+(X(r),\alpha(r))dr\right\}. \tag{3.3}$$

Note that Γ depends on the processes X, α, and the underlying control u as well. But for notational simplicity, we have omitted those dependence. Consider the auxiliary control problem

$$J^\varepsilon(s,x,\alpha,u(\cdot)) = \mathbf{E}_{s,x,\alpha}\left[\int_s^T \Gamma^\varepsilon(t) l(t,X(t),\alpha(t),u(t))dt + \Gamma^\varepsilon(T) g(T,X(T),\alpha(T))\right], \tag{3.4}$$

$$V^\varepsilon(s,x,\alpha) = \inf_{u(\cdot)\in\mathcal{U}} J^\varepsilon(s,x,\alpha,u(\cdot)). \tag{3.5}$$

Lemma 3.2. *Assume Assumption* A1 *and* (3.2)*. For any $u \in \mathcal{U}$, denote $X_i(t) = X^{s,x_i,\alpha;u}(t)$ and*

$$\Gamma_i^\varepsilon(t) = \exp\left\{-\frac{1}{\varepsilon}\int_s^t \psi^+(X_i(r),\alpha(r))dr\right\}, \quad t \in [s,T],$$

where $s \in [0,T)$, $x_i > 0$, $\alpha \in \mathcal{M}$, *and* $i = 1,2$. *Then we have*

$$\mathbf{E} \sup_{t\in[0,T]} |X_1(t) - X_2(t)|^2 \le K\, |x_1 - x_2|^2 , \tag{3.6}$$

and

$$|\Gamma_1^\varepsilon(t) - \Gamma_2^\varepsilon(t)| \le \frac{L}{\varepsilon}(t-s) \sup_{r\in[s,t]} |X_1(r) - X_2(r)| . \tag{3.7}$$

Proof. Note that virtually the same argument as that of (Yin and Zhu, 2010, Lemma 2.14) yields (3.6). Therefore it remains to prove (3.7). It is easy to see that $\left|e^{-a} - e^{-b}\right| \le |a-b|$ for any $a, b \ge 0$. Thus it follows from (3.2) that

$$\begin{aligned}
|\Gamma_1^\varepsilon(t) - \Gamma_2^\varepsilon(t)| &= \left| \exp\left\{ -\frac{1}{\varepsilon}\int_s^t \psi^+(X_1(r),\alpha(r))dr \right\} - \exp\left\{ -\frac{1}{\varepsilon}\int_s^t \psi^+(X_2(r),\alpha(r))dr \right\} \right| \\
&\le \frac{1}{\varepsilon} \left| \int_s^t \left[\psi^+(X_1(r),\alpha(r)) - \psi^+(X_2(r),\alpha(r)) \right] dr \right| \\
&\le \frac{1}{\varepsilon} \int_s^t L\, |X_1(r) - X_2(r)|\, dr \\
&\le \frac{L}{\varepsilon}(t-s) \sup_{r\in[s,t]} |X_1(r) - X_2(r)| .
\end{aligned}$$

This completes the proof. □

Theorem 3.3. *Under Assumption* A1*, the function* $V^\varepsilon(s,x,\alpha)$ *is continuous with respect to the variables* s *and* x *for each* $\alpha \in \mathcal{M}$,.

Proof. We divide the proof into several steps.

Step 1. For $\phi = l, g$ and $t \in [0,T]$, with notations as in Lemma 3.2, it follows from the Cauchy-Schwartz inequality and Assumption A1 that

$$\begin{aligned}
&\mathbf{E}\, |\Gamma_1^\varepsilon(t)\phi(t, X_1(t), \alpha(t), u(t)) - \Gamma_2^\varepsilon(t)\phi(s, X_2(s), \alpha(s), u(t))| \\
&\le \mathbf{E}\, |(\Gamma_1^\varepsilon(t) - \Gamma_2^\varepsilon(t))\phi(t, X_1(t), \alpha(t), u(t))| \\
&\quad + \mathbf{E}\, |\Gamma_2^\varepsilon(t)[\phi(t, X_1(t), \alpha(t), u(t)) - \phi(t, X_2(t), \alpha(t), u(t))]| \\
&\le \mathbf{E}^{1/2} |\Gamma_1^\varepsilon(t) - \Gamma_2^\varepsilon(t)|^2\, \mathbf{E}^{1/2} |\phi(t, X_1(t), \alpha(t), u(t))|^2 \\
&\quad + \mathbf{E}^{1/2} |\Gamma_2^\varepsilon(t)|^2\, \mathbf{E}^{1/2} |\phi(t, X_1(t), \alpha(t), u(t)) - \phi(t, X_2(t), \alpha(t), u(t))|^2 \\
&\le K\mathbf{E}^{1/2} |\Gamma_1^\varepsilon(t) - \Gamma_2^\varepsilon(t)|^2\, \mathbf{E}^{1/2}(1 + |X_1(t)|^{2p}) + K\mathbf{E}^{1/2} |X_1(t) - X_2(t)|^2 .
\end{aligned}$$

Note that by virtue of (Mao and Yuan, 2006, Theorem 3.24), we have

$$\mathbf{E} \sup_{t\in[0,T]} |X_1(t)|^{2p} \le K = K(x_1, T, p). \tag{3.8}$$

This, together with Lemma 3.2, leads to

$$\begin{aligned}
&\mathbf{E}\left|\Gamma_1^\varepsilon(t)\phi(t,X_1(t),\alpha(t),u(t)) - \Gamma_2^\varepsilon(t)\phi(s,X_2(s),\alpha(s),u(t))\right| \\
&\le K\frac{L}{\varepsilon}(t-s)\mathbf{E}^{1/2}\left(\sup_{r\in[s,t]}|X_1(r)-X_2(r)|\right)^2 + K\mathbf{E}^{1/2}\left|X_1(t)-X_2(t)\right|^2 \\
&\le K\left|x_1-x_2\right|,
\end{aligned} \tag{3.9}$$

where $K = K(\varepsilon, x_1, T, L, p)$ does not depend on x_2, t, or u.

Step 2. Now it follows from (3.9) that

$$\begin{aligned}
&|V^\varepsilon(s,x_1,\alpha) - V^\varepsilon(s,x_2,\alpha)| \\
&\le \sup_{u\in\mathcal{U}}\mathbf{E}\bigg[\int_s^T |\Gamma_1^\varepsilon(t)l(t,X_1(t),\alpha(t),u(t))dt - \Gamma_2^\varepsilon(t)l(t,X_2(t),\alpha(t),u(t))|\,dt \\
&\qquad\qquad + |\Gamma_1^\varepsilon(t)g(T,X_1(T)) - \Gamma_2^\varepsilon(t)g(T,X_2(T))|\bigg] \\
&\le K\left|x_1-x_2\right|.
\end{aligned} \tag{3.10}$$

This shows that $V^\varepsilon(s,\cdot,\alpha)$ is continuous for any $s\in[0,T]$ and $\alpha\in\mathcal{M}$.

Step 3. Next we prove that V^ε is continuous with respect to s as well. To this purpose, we consider $s_1 < s_2 \le T$. By virtue of (Fleming and Soner, 2006, Section IV.7), V^ε satisfies the dynamic programming principle. Therefore for any $\delta > 0$, we can choose a $u_1 \in \mathcal{U}$ such that

$$\begin{aligned}
V^\varepsilon(s_1,x,\alpha) &\le \mathbf{E}\left[\int_{s_1}^{s_2}\Gamma^\varepsilon(t)l(t,X(t),\alpha(t),u_1(t))dt + V^\varepsilon(s_2,X(s_2),\alpha(s_2))\right] \\
&< V^\varepsilon(s_1,x,\alpha) + \delta/3,
\end{aligned}$$

where $\Gamma^\varepsilon(t) = \exp\{-\frac{1}{\varepsilon}\int_{s_1}^t \psi^+(X(r),\alpha(r))dr\}$ and $X = X^{s_1,x,\alpha,u_1}$. Then we have from Assumption A1 that

$$\begin{aligned}
&|V^\varepsilon(s_1,x,\alpha) - V^\varepsilon(s_2,x,\alpha)| - \delta/3 \\
&\le \left|\mathbf{E}\left[\int_{s_1}^{s_2}\Gamma^\varepsilon(t)l(t,X(t),\alpha(t),u_1(t))dt + V^\varepsilon(s_2,X(s_2),\alpha(s_2))\right] - V^\varepsilon(s_2,x,\alpha)\right| \\
&\le \int_{s_1}^{s_2} K(1+\mathbf{E}\left|X(t)\right|^p)dt + \mathbf{E}\left|V^\varepsilon(s_2,X(s_2),\alpha) - V^\varepsilon(s_2,x,\alpha)\right| \\
&\quad + \mathbf{E}\left|V^\varepsilon(s_2,X(s_2),\alpha(s_2)) - V^\varepsilon(s_2,X(s_2),\alpha)\right| \\
&:= I + II + III.
\end{aligned} \tag{3.11}$$

Using (3.8), we obtain

$$I = \int_{s_1}^{s_2} K(1+\mathbf{E}\left|X(t)\right|^p)dt \le K(s_2-s_1). \tag{3.12}$$

For the term last term, we first notice that the definition of V^ε in (3.5), Assumption A1, and (3.8) imply that that $V^\varepsilon(s_2,X(s_2),j) \le K$, where $K = K(x,T,p)$ is a constant. Then it follows that

$$\begin{aligned} III &= \mathbf{E}\left|V^{\varepsilon}(s_2, X(s_2), \alpha(s_2)) - V^{\varepsilon}(s_2, X(s_2), \alpha)\right| \\ &= \mathbf{E}\left[\left|V^{\varepsilon}(s_2, X(s_2), \alpha(s_2)) - V^{\varepsilon}(s_2, X(s_2), \alpha)\right| I_{\{\alpha(s_2)\neq\alpha\}}\right] \\ &\le K\mathbf{P}\left\{\alpha(s_2) \neq \alpha\right\} \\ &\le K(s_2 - s_1), \end{aligned} \tag{3.13}$$

where in the last inequality, we used (2.1). Further, since

$$X(s_2) = x + \int_{s_1}^{s_2} b(t, X(t), \alpha(t), u(t))dt + \int_{s_1}^{s_2} \sigma(t, X(t), \alpha(t), u(t))dw(t),$$

using Assumption A1 and (3.8), we can readily verify that

$$\mathbf{E}\left|X(s_2) - x\right| \le K\left|s_2 - s_1\right|^{1/2}.$$

Hence by virtue of (3.10), we have

$$II = \mathbf{E}\left|V^{\varepsilon}(s_2, X(s_2), \alpha) - V^{\varepsilon}(s_2, x, \alpha)\right| \le K\mathbf{E}\left|X(s_2) - x\right| \le K\left|s_2 - s_1\right|^{1/2}. \tag{3.14}$$

Combing the above estimates (3.12)–(3.14) into (3.11), we arrive at

$$\left|V^{\varepsilon}(s_1, x, \alpha) - V^{\varepsilon}(s_2, x, \alpha)\right| - \delta/3 \le K\left|s_2 - s_1\right|^{1/2} + K(s_2 - s_1).$$

Since $\delta > 0$ is arbitrary, the continuity of V^{ε} with respect to s is established, as desired. □

Assumption A2. Suppose there exists a function $\psi : \mathbb{R} \times \mathcal{M} \mapsto \mathbb{R}$ satisfying (3.2) and that

$$\psi(x, j) \le 0, \quad \forall (x, j) \in [0, \infty) \times \mathcal{M}. \tag{3.15}$$

Moreover, there exists some $u \in U$ such that

$$\int_s^t \psi^+(X(r), \alpha(r))dr > 0, \text{ a.s. for any } t \in (s, T], \tag{3.16}$$

where $X = X^{s,0,i;u}$ is the controlled process under the constant control $u(t) \equiv u, t \in [s, T]$, $\alpha = \alpha^{s,i}$, $s \in [0, T)$, and $i \in \mathcal{M}$.

Theorem 3.4. *In addition to Assumptions* A1 *and* A2*, suppose also that* $g(\cdot, \cdot, \alpha) \in C^{1,2}$ *for each* $\alpha \in \mathcal{M}$ *and that*

$$\frac{\partial}{\partial t} g(t, x, \alpha) + \mathcal{L}_t^u g(t, x, \alpha) + l(t, x, \alpha, u) \ge 0, \quad (t, x, \alpha, u) \in (0, T) \times (0, \infty) \times \mathcal{M} \times U, \tag{3.17}$$

Then $V(\cdot, \cdot, i)$ *is continuous for each* $i \in \mathcal{M}$.

Proof. We divide the proof into two steps. The first step is concerned with the special case when $g \equiv 0$ while the second step deals with the general case.

Step 1. First assume $g \equiv 0$. Fix $(s, i) \in [0, T) \times \mathcal{M}$. Let u, X, and α as in Assumption A2. Then (3.16) implies that for any $t > s$

$$\lim_{\varepsilon \downarrow 0} \Gamma^{\varepsilon}(t) = \lim_{\varepsilon \downarrow 0} \exp\left\{ -\frac{1}{\varepsilon} \int_s^t \psi^+(X(r), \alpha(r)) dr \right\} = 0, \quad \text{a.s.}$$

and hence by virtue of the dominated convergence theorem and the definition of J^{ε} in (3.4), we obtain

$$\lim_{\varepsilon \downarrow 0} J^{\varepsilon}(s, 0, \alpha, u) = 0.$$

Since $J^{\varepsilon}(s, 0, \alpha, u) \geq V^{\varepsilon}(s, 0, \alpha) \geq 0$, it follows that

$$\lim_{\varepsilon \downarrow 0} V^{\varepsilon}(s, 0, \alpha) = 0.$$

Let $0 < \varepsilon_1 < \varepsilon_2$. Then, noting the nonnegativity of the functions ψ^+ and l, we can readily verify that $J^{\varepsilon_1}(s, 0, \alpha, u) \leq J^{\varepsilon_2}(s, 0, \alpha, u)$ and hence $V^{\varepsilon_1}(s, 0, \alpha) \leq V^{\varepsilon_2}(s, 0, \alpha)$. We have shown in Theorem 3.3 that V^{ε} is continuous. Thus Dini's theorem implies that $\lim_{\varepsilon \downarrow 0} V^{\varepsilon}(s, 0, \alpha) = 0$ uniformly on $[0, T]$ for each $\alpha \in \mathcal{M}$. Now let

$$h(\varepsilon) := \sup \left\{ V^{\varepsilon}(s, 0, \alpha) : s \in [0, T], \alpha \in \mathcal{M} \right\}.$$

Then we have $\lim_{\varepsilon \to 0} h(\varepsilon) = 0$ thanks to the uniform convergence of V^{ε}.

For any $(s, x, \alpha) \in [0, T] \times (0, \infty) \times \mathcal{M}$, thanks to the dynamic programming principle for V^{ε} and the definition of h, we have[1]

$$\begin{aligned} V^{\varepsilon}(s, x, \alpha) &= \inf_{u \in \mathcal{U}} \mathbf{E} \left[\int_s^{\tau} l(t, X(t), \alpha(t), u(t)) dt + V^{\varepsilon}(\tau, X(\tau), \alpha(\tau)) \right] \\ &\leq \inf_{u \in \mathcal{U}} \mathbf{E} \int_s^{\tau} l(t, X(t), \alpha(t), u(t)) dt + h(\varepsilon) \\ &= V(s, x, \alpha) + h(\varepsilon). \end{aligned}$$

Thanks to the definition of J^{ε} in (3.4) and the assumption that $g \equiv 0$, we have

$$\begin{aligned} J^{\varepsilon}(s, x, i, u) &= \mathbf{E} \int_s^T \Gamma^{\varepsilon}(t) l(t, X(t), \alpha(t), u) dt \\ &= \mathbf{E} \left[\int_s^{\tau} \Gamma^{\varepsilon}(t) l(t, X(t), \alpha(t), u(t)) dt + I_{\{\tau < T\}} \int_{\tau}^T \Gamma^{\varepsilon}(t) l(t, X(t), \alpha(t), u(t)) dt \right]. \end{aligned}$$

[1]Note that if $\tau = T$, then $V^{\varepsilon}(T, X(T), \alpha(T)) = V(T, X(T), \alpha(T)) = g(T, X(T), \alpha(T)) = 0$ by our assumption on g.

Note that for all $t \in [s, \tau]$, $X(t) \in [0, \infty)$. Thus it follows from Assumption A2 and the definition of Γ^ε in (3.3) that $\Gamma^\varepsilon(t) = 1$ and hence

$$J^\varepsilon(s, x, \alpha, u) = J(s, x, \alpha, u) + \mathbf{E}\left[I_{\{\tau < T\}} \int_\tau^T \Gamma^\varepsilon(t) l(t, X(t), \alpha(t), u(t)) dt \right].$$

Furthermore, since $l \geq 0$, we have

$$V(s, x, \alpha) \leq V^\varepsilon(s, x, \alpha) \leq V(s, x, \alpha) + h(\varepsilon).$$

This implies that $V^\varepsilon \to V$ uniformly on $[0, T] \times (0, \infty) \times \mathcal{M}$. Since V^ε is continuous, V is also continuous.

Step 2. For general $g \geq 0$, let $\tilde{l}(t, x, \alpha, u) = l(t, x, \alpha, u) + \frac{\partial}{\partial t} g(t, x, \alpha) + \mathcal{L}_t^u g(t, x, \alpha)$ and $\tilde{g} \equiv 0$. Then $\tilde{l} \geq 0$ by virtue of (3.17) and hence Step 1 implies that the function

$$\widetilde{V}(s, x, \alpha) := \inf_{u \in \mathcal{U}} \widetilde{J}(s, x, \alpha, u) = \inf_{u \in \mathcal{U}} \mathbf{E} \int_s^\tau \tilde{l}(t, X(t), \alpha(t), u(t)) dt$$

is continuous. Apply Itô's formula to g,

$$\mathbf{E} g(\tau, X(\tau), \alpha(\tau)) - g(s, x, \alpha) = \mathbf{E} \int_s^\tau \left(\frac{\partial}{\partial t} + \mathcal{L}_t^u \right) g(t, X(t), \alpha(t)) dt.$$

Then it follows that

$$\begin{aligned}
\widetilde{J}(s, x, \alpha, u) &= \mathbf{E} \int_s^\tau \tilde{l}(t, X(t), \alpha(t), u(t)) dt \\
&= \mathbf{E} \int_s^\tau l(t, X(t), \alpha(t), u(t)) dt + \mathbf{E} \int_s^\tau \left(\frac{\partial}{\partial t} + \mathcal{L}_t^u \right) g(t, X(t), \alpha(t)) dt \\
&= \mathbf{E} \int_s^\tau l(t, X(t), \alpha(t), u(t)) dt + \mathbf{E} g(\tau, X(\tau), \alpha(\tau)) - g(s, x, \alpha) \\
&= J(s, x, \alpha, u) - g(s, x, \alpha).
\end{aligned}$$

Therefore we conclude that $V(s, x, \alpha) = \widetilde{V}(s, x, \alpha) + g(s, x, \alpha)$ is also continuous. □

Remark 3.5. With the continuity of the value function at our hands, we have the dynamic programming principle by virtue of Fleming and Soner (2006):

$$V(s, x, \alpha) = \inf_{u(\cdot) \in \mathcal{U}} \mathbf{E} \left\{ \int_s^{\theta \wedge \tau} l(t, X(t), \alpha(t), u(t)) dt + V(\theta \wedge \tau, X(\theta \wedge \tau), \alpha(\theta \wedge \tau)) \right\}, \quad (3.18)$$

where θ is a stopping time.

Remark 3.6. Note that Assumption A2, in particular (3.16), is the crucial condition in the proof of Theorem 3.4. One may wonder when Assumption A2 is true? Next we present several sufficient conditions.

Proposition 3.7. *Either one of the following conditions implies Assumption* A2*:*

(i) *There exists a function* $\psi : \mathbb{R} \times \mathcal{M} \mapsto \mathbb{R}$ *satisfying* (3.2) *and* (3.15)*, and that for some* $u \in U$*,* $\{\psi(X(t), \alpha(t)), t \in [s, T]\}$ *is a strict local submartingale, where* $X = X^{s,0,i,u}$*,* $\alpha = \alpha^{s,i}$*,* $s \in [0, T)$*, and* $i \in \mathcal{M}$*.*

(ii) *There exists a twice continuously differentiable function* $\psi : \mathbb{R} \times \mathcal{M} \mapsto \mathbb{R}$ *satisfying* (3.2) *and* (3.15)*, and that for some* $u \in U$*,*

$$\mathcal{L}_t^u \psi(0, i) = b(t, 0, i, u)\psi'(0, i) + \frac{1}{2}\sigma^2(t, 0, i, u)\psi''(0, i) + \sum_{j=1}^{m} q_{ij}\psi(0, j) > 0, \qquad (3.19)$$

where $t \in [0, T]$ *and* $i \in \mathcal{M}$*.*

(iii) *There exists a* $u \in U$ *such that the boundary point* 0 *is* regular *for the domain* $(0, \infty)$ *for the process* $(X, \alpha) = (X^{s,0,i,u}, \alpha^{s,u})$*, where* $s \in [0, T), i \in \mathcal{M}$*.*

Proof. (i) Since $\{\psi(X(t), \alpha(t)), t \in [s, T]\}$ is a strict local submartingale and $\psi^+ \geq 0$, we must have (3.16).

(ii) This is obvious since (3.19) implies that

$$\mathcal{L}_t^u \psi(x, i) = b(t, x, i, u)\psi'(x, i) + \frac{1}{2}\sigma^2(t, x, i, u)\psi''(x, i) + \sum_{j=1}^{m} q_{ij}\psi(x, j) > 0, \quad t \in [0, T]$$

for all $(x, i) \in N \times \mathcal{M}$, where N is a neighborhood of 0. As a result, $\{\psi(X(t), \alpha(t)), t \in [s, T]\}$ is a strict local submartingale.

(iii) If 0 is a regular boundary point, then the function $\psi(x, i) := -x$ satisfies the conditions in Assumption A2. We refer to the appendix and Theorem A.1 for more discussions on regular boundary point. □

Example 3.8. Consider a uncontrolled surplus process given by

$$dX(t) = 2(t - 1)dt + (t - X(t))^+ dw(t), \quad t \in [s, 2],$$

with initial surplus

$$X(s) = x > 0,$$

where $s \geq 0$ and w is a one-dimensional standard Brownian motion. Similar to Example 3.1, we define $\tau = \tau^{s,x} = \inf\{t > s : X(t) = 0\} \wedge 2$ and $V(s, x) = \tau$.

As in Fleming and Soner (2006), the signed distance to the boundary point 0 is $\hat{\rho}(x) = -x, x \in \mathbb{R}$. Then

$$\mathcal{L}_t \hat{\rho}(0) = -2(t - 1) \begin{cases} > 0 & \text{for } t \in (0, 1) \\ < 0 & \text{for } t \in (1, 2). \end{cases}$$

Hence the sufficient condition ($\mathcal{L}_t\hat{\rho}(0) > 0$, for all $t \in [0,2]$) for continuity of the value function given in Fleming and Soner (2006) fails. See (Fleming and Soner, 2006, equation (2.8), p. 202) for more details.

Nevertheless, we claim that 0 is a regular boundary point for the domain $(0,\infty)$ and hence Assumption A2 still holds true by virtue of Proposition 3.7. Consequently the value function is continuous. To this end, we consider the function $\varphi(x) = -x^2 + x, x \in (-\frac{1}{2}, \frac{1}{2})$. It is easy to see that φ satisfies conditions (i) and (ii) in Theorem A.1. Next we show that φ satisfies condition (iii) in Theorem A.1 as well. In fact,

$$\begin{aligned}\mathcal{L}_t\varphi(0) &= 2(t-1)\varphi'(0) + \frac{1}{2}\left((t-0)^+\right)^2\varphi''(0)\\ &= 2(t-1) - t^2\\ &= -(t-1)^2 - 1 \leq -1 < 0, \quad \forall t \in [0,2].\end{aligned}$$

Then it follows from continuity that $\mathcal{L}_t\varphi(x) < 0$ for all $t \in [0,2]$ and $x \in U$, where $U \subset (-\frac{1}{2}, \frac{1}{2})$ is a neighborhood of 0. Consequently, $\{\varphi(X(t)), t \in [s,2]\}$ is superharmonic in $(0,\infty) \cap U$. Therefore Theorem A.1 implies that 0 is a regular boundary point and hence Assumption A2 is verified. Further, we can readily that all other conditions in Theorem 3.4 are satisfied and hence the value function V is continuous.

Example 3.9. Suppose a controlled surplus process X satisfies

$$dX(t) = b(t, X(t), \alpha(t), u(t))dt + \sigma(t, X(t), \alpha(t), u(t))dw(t), \quad t \geq s \geq 0, \tag{3.20}$$

with initial conditions

$$X(s) = x > 0, \quad \alpha(s) = i \in \{1,2\},$$

where w is a one-dimensional standard Brownian motion, $\alpha \in \{1,2\}$ is a continuous time Markov chain generated by $Q = \begin{pmatrix} -3 & 3 \\ 4 & -4 \end{pmatrix}$, $u(t) \in [0,1]$ denotes the retention rate (so $1-u(t)$ is the proportion reinsured to a reinsurance company) at time t, and

$$\begin{aligned}b(t,x,1,u) &= \sin t + x + 0.4 - 0.8(1-u), \qquad \sigma(t,x,1,u) = \sin t + 0.5x + 0.5u,\\ b(t,x,2,u) &= \cos t + 3x + 1 - 2(1-u), \qquad \sigma(t,x,2,u) = \cos t + x + 2u.\end{aligned}$$

Note that (3.20) represents a surplus process subject to non-cheap reinsurance, investment in a Markovian-modulated Black-Scholes model, and seasonal fluctuations in premium collection. This is motivated by the model considered in Taksar and Markussen (2003). Denote

$$\tau := \inf\{t > s : X(t) = 0\} \wedge 100.$$

The payoff for a reinsurance strategy $u(\cdot)$ is

$$J(s,x,i,u) = \mathbf{E}_{s,x,i}\int_s^\tau e^{-rt}X(t)dt, \quad s \in [0,100), \ x > 0, \ i = 1,2,$$

where $r > 0$ is the discounting factor. The objective is to maximize the payoff and find a reinsurance strategy $u^*(\cdot)$ such that

$$V(s,x,i) = \sup_{u\in\mathcal{U}} J(s,x,i,u) = J(s,x,i,u^*), \quad s \in [0,100),\ x > 0,\ i = 1,2. \tag{3.21}$$

We claim that the value function V is continuous with respect to the variables s and x by virtue of Theorem 3.4. In fact, it is obvious that all conditions in Assumptions A1 are satisfied. Next we use Theorem A.1 and Proposition 3.7 to show that Assumption A2 is also true and hence the claim follows. To this end, we consider $\varphi(x,1) = -x^2 + 0.5x$ and $\varphi(x,2) = -x^2 + 2x$, where $x \in U := (-0.25, 0.25)$. Then we can easily verify that conditions (i) and (ii) in Theorem A.1 are satisfied. To verify condition (iii), we let $u = 0.5$ and compute

$$\begin{aligned}\mathcal{L}_t^{u=0.5}\varphi(0,1) &= b(t,0,1,0.5)\varphi'(0,1) + 0.5\sigma^2(t,0,1,0.5)\varphi''(0,1) - 3\varphi(0,1) + 3\varphi(0,2)\\ &= 0.5\sin t + 0.5(\sin t + 0.5\cdot 0.5)^2(-2)\\ &= -\sin^2 t - 0.0625 < 0, \quad \text{for any } t \in [0,100],\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_t^{u=0.5}\varphi(0,2) &= b(t,0,2,0.5)\varphi'(0,2) + 0.5\sigma^2(t,0,2,0.5)\varphi''(0,2) + 4\varphi(0,1) - 4\varphi(0,2)\\ &= 2\cos t + 0.5(\cos t + 1)^2(-2)\\ &= -1 - \cos^2 t < 0, \quad \text{for any } t \in [0,100].\end{aligned}$$

Hence it follows that φ is superharmonic in $((0,\infty)\cap U) \times \{1,2\}$ and condition (iii) in Theorem A.1 is verified. Thus by virtue of Theorem A.1 and Proposition 3.7, we conclude that Assumption A2 holds and hence V defined in (3.21) is continuous.

4 Viscosity Solution

With the continuity of the value function and the dynamic programming principle, we can now characterize the value function to be a viscosity solution of the HJB equation (2.12). First we recall the notion of viscosity solution from Fleming and Soner (2006).

Definition 4.1. A function v is called a *viscosity subsolution* (*viscosity supersolution*, resp.) of (2.12) if for any $\varphi(\cdot,\cdot,\alpha) \in C^{1,2}, \alpha \in \mathcal{M}$, whenever $v - \varphi$ attains a maximum (minimum, resp.) at (s,x,α) with $v(s,x,\alpha) = \varphi(s,x,\alpha)$, we have

$$\frac{\partial}{\partial t}\varphi(s,x,\alpha) + \inf_{u\in U}\{\mathcal{L}_s^u\varphi(s,x,\alpha) + l(s,x,\alpha,u)\} \geq 0 \ (\leq 0, \text{resp.}) \tag{4.1}$$

Further, a function v is called a *viscosity solution* of (2.12) if it is both a viscosity subsolution and supersolution of (2.12).

We first state a lemma, whose proof can be found in Bayraktar et al. (2010).

Lemma 4.2. *For any* $(s, x, \alpha) \in [0, T) \times (0, \infty) \times \mathcal{M}$ *and* $u \in \mathcal{U}$*, define*

$$\theta := \inf \{t > s : X(t) \notin B(x, h)\} \wedge [s, s + h^2),$$

where $h \in (0, 1)$ *and* $X = X^{s,x,\alpha,u}$*. Then there exists a positive constant* κ *such that*

$$\mathbf{E}[\theta - s] \geq \kappa h^2.$$

Moreover, $\kappa = \kappa(s, x)$ *is independent of the control* u*.*

Theorem 4.3. *Assume Assumptions* 1 *and* 2*. Then the value function* (2.6) *is a viscosity solution of* (2.12)*.*

Proof. The proof is inspired by Bayraktar et al. (2010); we use similar ideas. We first establish the viscosity subsolution property of the value function V in Step 1, followed by viscosity supersolution in Step 2.

Step 1. We first prove that V is a viscosity subsolution of (2.12). Suppose it was not the case, then there would exist some $\varphi \in C^{1,2}$, a $u \in U$, and a maximizer $(s_0, x_0, \alpha_0) \in [0, T) \times (0, \infty) \times \mathcal{M}$ of $V - \varphi$ with $V(s_0, x_0, \alpha_0) = \varphi(s_0, x_0, \alpha_0)$, but

$$\frac{\partial}{\partial t}\varphi(s_0, x_0, \alpha_0) + \mathcal{L}^u_{s_0}\varphi(s_0, x_0, \alpha_0) + l(s_0, x_0, \alpha_0, u) < -\delta < 0, \tag{4.2}$$

where $\delta > 0$. Then by the continuity of the function $l(\cdot, \cdot, \alpha_0, u) + (\frac{\partial}{\partial t} + \mathcal{L}^u_t)\varphi(\cdot, \cdot, \alpha_0)$, there exists an $h \in (0, 1)$ such that

$$\frac{\partial}{\partial t}\varphi(s, x, \alpha_0) + \mathcal{L}^u_s\varphi(s, x, \alpha_0) + l(s, x, \alpha_0, u) < -\delta/2 < 0, \tag{4.3}$$

for all $(s, x) \in [s_0, s_0 + h^2) \times B(x_0, h)$. Without loss of generality, we assume that $h < 1$ is sufficiently small so that $[s_0, s_0 + h^2) \times B(x_0, h) \subset [0, T) \times (0, \infty)$. Denote $X = X^{s_0,x_0,\alpha_0,u}$ and define

$$\theta := \inf \{t > s_0 : X(t) \notin B(x_0, h)\} \wedge (s_0 + h^2).$$

Note that $\theta < \tau$ a.s. By virtue of the dynamic programming principle, we have

$$V(s_0, x_0, \alpha_0) \leq \mathbf{E}\left[\int_{s_0}^{\theta} l(r, X(r), \alpha(r), u)dr + V(\theta, X(\theta), \alpha(\theta))\right]. \tag{4.4}$$

Using the assumptions on φ, we can derive from (4.4) that

$$0 \leq \mathbf{E}\left[\int_{s_0}^{\theta} l(r, X(r), \alpha(r), u)dr + \varphi(\theta, X(\theta), \alpha(\theta)) - \varphi(s_0, x_0, \alpha_0)\right].$$

Apply Itô's formula to φ,

$$\mathbf{E}\varphi(\theta, X(\theta), \alpha(\theta)) - \varphi(s_0, x_0, \alpha_0) = \mathbf{E}\int_{s_0}^{\theta}\left(\frac{\partial}{\partial t} + \mathcal{L}_r^u\right)\varphi(r, X(r), \alpha(r))dr.$$

Hence it follows from (4.3) and Lemma 4.2 that

$$\begin{aligned} 0 &\leq \mathbf{E}\int_{s_0}^{\theta}\left[l(r, X(r), \alpha(r), u) + \left(\frac{\partial}{\partial t} + \mathcal{L}_r^u\right)\varphi(r, X(r), \alpha(r))\right]dr \\ &= \mathbf{E}\int_{s_0}^{\theta}\left[l(r, X(r), \alpha_0, u) + \left(\frac{\partial}{\partial t} + \mathcal{L}_r^u\right)\varphi(r, X(r), \alpha_0)\right]dr + A \\ &\leq \mathbf{E}\int_{s_0}^{\theta}(-\frac{\delta}{2})dr + A \\ &\leq -\frac{\delta}{2}\kappa h^2 + A, \end{aligned} \tag{4.5}$$

where κ is the constant in Lemma 4.2, and

$$\begin{aligned} A = \mathbf{E}\int_{s_0}^{\theta}\Bigg[&l(r, X(r), \alpha(r), u) + \left(\frac{\partial}{\partial t} + \mathcal{L}_r^u\right)\varphi(r, X(r), \alpha(r)) \\ &- l(r, X(r), \alpha_0, u) - \left(\frac{\partial}{\partial t} + \mathcal{L}_r^u\right)\varphi(r, X(r), \alpha_0)\Bigg]dr. \end{aligned}$$

Next we show that A is negligible compared to the term $-\frac{\delta}{2}\kappa h^2$. To this end, we denote

$$H(s, x, \alpha, u) := l(s, x, \alpha, u) + \left(\frac{\partial}{\partial t} + \mathcal{L}_s^u\right)\varphi(s, x, \alpha).$$

Then for each $\alpha \in \mathcal{M}$ and $u \in U$, as a function of (s, x), H is continuous and hence bounded on the compact $[s_0, s_0 + 1] \times \bar{B}(x_0, 1)$. Therefore we compute from (2.1) that

$$\begin{aligned} A &= \mathbf{E}\int_{s_0}^{\theta}[H(r, X(r), \alpha(r), u) - H(r, X(r), \alpha_0, u)]dr \\ &\leq \mathbf{E}\int_{s_0}^{\theta}\sum_{j\neq\alpha_0}|H(r, X(r), j, u) - H(r, X(r), \alpha_0, u)|\, I_{\{\alpha(r)=j\}}dr \\ &\leq K\int_{s_0}^{s_0+h^2}\mathbf{P}\left\{\alpha(r) = j|\alpha(s_0) = \alpha_0\right\}dr \\ &\leq K\int_{s_0}^{s_0+h^2}(r - s_0)dr = Kh^4, \end{aligned} \tag{4.6}$$

here K is some constant independent of h and u. Then it follows from (4.5) and (4.6) that for $h > 0$ sufficiently small, we have

$$0 \leq -\frac{\delta}{2}\kappa h^2 + Kh^4 < 0.$$

This is a contradiction. Hence the value function V must be a viscosity subsolution of (2.12).

Step 2. Now we show that V is a viscosity supersolution of (2.12). Again, we use a contradiction argument. Suppose on the contrary that V was not a viscosity supersolution of (2.12). Then there would exist a $\phi \in C^{1,2}$ and a minimizer $(s_0, x_0, \alpha_0) \in [0,T) \times (0,\infty) \times \mathcal{M}$ of $V - \phi$ with $V(s_0, x_0, \alpha_0) = \phi(s_0, x_0, \alpha_0)$, but

$$\frac{\partial}{\partial t}\phi(s_0, x_0, \alpha_0) + \inf_{u\in U}\left\{\mathcal{L}^u_{s_0}\phi(s_0, x_0, \alpha_0) + l(s_0, x_0, \alpha_0, u)\right\} = \delta > 0, \tag{4.7}$$

where δ is a constant. By Assumption A1, the function

$$(s, x) \mapsto \inf_{u\in U}\{\mathcal{L}^u_s\phi(s, x, \alpha) + l(s, x, \alpha, u)\}$$

is continuous for each $\alpha \in \mathcal{M}$. Hence we can find an $h > 0$ such that

$$\inf_{u\in U}\{\mathcal{L}^u_s\phi(s, x, \alpha_0) + l(s, x, \alpha_0, u)\} > \frac{\delta}{2}, \text{ for all } (s, x) \in [s_0, s_0 + h^2) \times B(x_0, h). \tag{4.8}$$

Let $\varepsilon = \frac{\delta}{4}\kappa h^2$, where κ is the constant in Lemma 4.2. Let $u \in \mathcal{U}$ be an ε-optimal control and denote $X = X^{s_0,x_0,\alpha_0,u}$. Put $\theta := \inf\{t > s_0 : X(t) \notin B(x_0, h)\} \wedge (s_0 + h^2)$. Then it follows that

$$V(s_0, x_0, \alpha_0) \geq \mathbf{E}\int_{s_0}^{\theta} l(r, X(r), \alpha(r), u(r))dr + V(\theta, X(\theta), \alpha(\theta)) - \varepsilon. \tag{4.9}$$

As in Step 1, using the assumptions and Itô's formula on ϕ, we can rewrite (4.9) as

$$\begin{aligned} 0 &\geq \mathbf{E}\left[\int_{s_0}^{\theta} l(r, X(r), \alpha(r), u(r))dr + \phi(\theta, X(\theta), \alpha(\theta)) - \phi(s_0, x_0, \alpha_0)\right] - \varepsilon \\ &= \mathbf{E}\int_{s_0}^{\theta}\left[l(r, X(r), \alpha(r), u(r)) + \left(\frac{\partial}{\partial t} + \mathcal{L}^{u(r)}_r\right)\phi(r, X(r), \alpha(r))\right]dr - \varepsilon. \end{aligned}$$

But $\varepsilon = \frac{\delta}{4}\kappa h^2$. This, together with Lemma 4.2, leads to

$$\begin{aligned} 0 &\geq \mathbf{E}\int_{s_0}^{\theta}\left[l(r, X(r), \alpha(r), u(r)) + \left(\frac{\partial}{\partial t} + \mathcal{L}^{u(r)}_r\right)\phi(r, X(r), \alpha(r))\right]dr - \frac{\delta}{4}\mathbf{E}[\theta - s_0] \\ &= \mathbf{E}\int_{s_0}^{\theta}\left[l(r, X(r), \alpha(r), u(r)) + \left(\frac{\partial}{\partial t} + \mathcal{L}^{u(r)}_r\right)\phi(r, X(r), \alpha(r)) - \frac{\delta}{4}\right] \\ &= \mathbf{E}\int_{s_0}^{\theta}\left[l(r, X(r), \alpha_0, u(r)) + \left(\frac{\partial}{\partial t} + \mathcal{L}^{u(r)}_r\right)\phi(r, X(r), \alpha_0) - \frac{\delta}{4}\right] + B, \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} B = \mathbf{E}\int_{s_0}^{\theta}\Bigg[&l(r, X(r), \alpha(r), u(r)) + \left(\frac{\partial}{\partial t} + \mathcal{L}^{u(r)}_r\right)\phi(r, X(r), \alpha(r)) \\ &- l(r, X(r), \alpha_0, u(r)) - \left(\frac{\partial}{\partial t} + \mathcal{L}^{u(r)}_r\right)\phi(r, X(r), \alpha_0)\Bigg]dr. \end{aligned}$$

Using the same argument as that in Step 1, we deduce that for some constant K independent of h and u,

$$|B| \leq K h^4. \tag{4.11}$$

Therefore it follows from (4.8)–(4.11) that

$$0 > \mathbf{E} \int_{s_0}^{\theta} (\frac{\delta}{2} - \frac{\delta}{4}) dr - |B| \geq \frac{\delta}{4} \kappa h^2 - K h^4 > 0,$$

for $h > 0$ sufficiently small. This is a contradiction. Therefore V is a viscosity supersolution of (2.12). This completes the proof. □

5 Conclusions and Remarks

In this work, we considered cost optimization problem for an insurance company. The surplus of the insurance company was modeled by a controlled regime switching diffusion. We presented a sufficient condition for the continuity of the value function and further characterized it as a viscosity solution of the HJB equation (2.12). The consideration of regime switching mechanism provides a better approximation to the real-world dynamics. The novelty of this work also includes a new sufficient condition for continuity of the value function. The sufficient condition in this paper is a new generalization of the one in Fleming and Soner (2006).

A number of other questions deserve further investigations. In particular, we were not able to obtain the explicit form of the value function and an optimal control by solving (2.12). The reason for this is that (2.12) is a coupled system of nonlinear second order ordinary differential equations, rendering extreme difficulty in finding a closed form solution of (2.12). Therefore a viable alternative is to employ numerical approximations. The Markov chain approximation method developed in Kushner and Dupuis (2001) will be utilized in the near future. We may also consider more complicated stochastic models such as regime switching diffusions with jumps as well as other optimality criteria such as dividend maximization and ruin probability minimization problems.

A Regular Boundary Point

The appendix provides a result on regular boundary points. For notational simplicity, we shall present the result when the continuous component X is 1-dimensional. The multi-dimensional case can be handled in a similar fashion. Note that the notations in the appendix are not necessarily the same as those in the main part of the paper.

Let $(X,\alpha) \in \mathbb{R} \times \mathcal{M}$ be a switching diffusion process, where $\mathcal{M} = \{1,\dots,m\}$. The generator $\mathcal{G}$ of (X,α) is defined as follows. For any $h(\cdot,\cdot,i) \in C^{1,2}, i \in \mathcal{M}$, we define

$$\begin{aligned}\mathcal{G}h(t,x,i) =& \frac{\partial}{\partial t}h(t,x,i) + h'(t,x,i)b(t,x,i) \\ &+ \frac{1}{2}h''(t,x,i)\sigma^2(t,x,i) + \sum_{j=1}^{m} q_{ij}[h(t,x,j) - h(t,x,i)],\end{aligned} \tag{A.1}$$

where h' and h'' denote the first and second order derivatives of h with respect to the variable x, respectively, $b,\sigma : [0,\infty)\times\mathbb{R}\times\mathcal{M} \mapsto \mathbb{R}$ are given functions, and q_{ij} are constants satisfying $q_{ij} \geq 0$ for $i \neq j$ and $q_{ii} = -\sum_{j\neq i} q_{ij}$. Further, we assume that b and σ satisfy

$$\begin{aligned}&|b(t,x,i) - b(t,y,i)| + |\sigma(t,x,i) - \sigma(t,y,i)| \leq K\,|x-y|\,, \\ &|b(t,x,i)| + |\sigma(t,x,i)| \leq K(1+|x|), \quad \forall t \in [0,\infty), x,y \in \mathbb{R}, i \in \mathcal{M},\end{aligned} \tag{A.2}$$

where K is a positive constant. It is well-known (Mao and Yuan (2006)) that under these conditions, for any $s \geq 0$, $x \in \mathbb{R}$, and $i \in \mathcal{M}$, the generator (A.1) uniquely determines a switching process $(X(\cdot),\alpha(\cdot))$ with initial conditions $X(s) = x$ and $\alpha(s) = i$. Denote such a process by $(X^{s;x,i},\alpha^{s;i})$ if the emphasis on initial conditions are needed.

Let G be an open subset of $\mathbb{R}$ and $a \in \partial G$. The point a is said to be *regular* for the process (X,α) in G if for any $s \geq 0$ and $i \in \mathcal{M}$ we have

$$\mathbf{P}\{\tau = s\} = 1,$$

where

$$\tau = \tau^{s;a,i} := \inf\left\{t > s : X^{s;a,i}(t) \notin G\right\}$$

denotes the first exit time for the process (X,α) from G.

Theorem A.1. *The point $a \in \partial G$ is a regular point if there exist a neighborhood U of a and a function $\varphi : U \times \mathcal{M} \mapsto \mathbb{R}$ such that*

(i) *$\varphi(x,i) > 0$ for all $x \in G \cap U - \{a\}$ and each $i \in \mathcal{M}$;*

(ii) *$\lim_{x\to a, x\in G} \varphi(x,i) = 0$ for each $i \in \mathcal{M}$; and*

(iii) *φ is* superharmonic *in $(G\cap U)\times\mathcal{M}$, that is, φ is bounded below and continuous in $(G\cap U)\times\mathcal{M}$ and satisfies*

$$\varphi(x,i) \geq \mathbf{E}\varphi(X^{s;x,i}(\tau_V),\alpha^{s;i}(\tau_V)), \quad \forall (x,i) \in V, \tag{A.3}$$

where $s \geq 0$, $V \subset (G\cap U)\times M$, and $\tau_V = \inf\{t > s : (X^{s;x,i}(t),\alpha^{s;i}(t)) \notin V\}$.

Proof. The proof is motivated by (Dynkin, 1965, Chapter 13), we use similar ideas. We divide the proof into several steps.

Step 1. Let U and φ be as in the statement of the theorem. Without loss of generality, we assume

$$\sup_{(x,i)\in U\times\mathcal{M}} \varphi(x,i) \le 1. \tag{A.4}$$

In fact, it is not hard to see that if φ satisfies (i)–(iii), then so does the function $\varphi \wedge 1 = \min(\varphi, 1)$.

Suppose that a is not regular, then we have

$$\mathbf{P}\left\{\tau > s_0\right\} > 0 \text{ for some } s_0 \ge 0 \text{ and } \ell \in \mathcal{M}, \tag{A.5}$$

where $\tau = \tau^{s_0;a,\ell} = \inf\left\{t > s_0 : X^{s_0;a,\ell}(t) \notin G\right\}$. Then, by virtue of the Blunmenthal Zero-One Law Karatzas and Shreve (1991),

$$\mathbf{P}\left\{\tau > s_0\right\} = 1. \tag{A.6}$$

Obviously,

$$\{X(s_0) = a, \alpha(s_0) = \ell, \tau > s_0\} \subset \left\{\sup_{t\in[s_0,\tau)} \operatorname{dist}(a, X(t)) > 0\right\}.$$

Hence it follows that

$$\mathbf{P}\left\{\sup_{t\in[s_0,\tau)} \operatorname{dist}(a, X(t)) > 0\right\} > 0.$$

Therefore for some $\delta > 0$, we have

$$\mathbf{P}\left\{\sup_{t\in[s_0,\tau)} \operatorname{dist}(a, X(t)) > \delta\right\} > 0. \tag{A.7}$$

Step 2. Now set $G_0 = G \cap B(a,\delta)$ and $\tau_0 = \inf\left\{t > s_0, X(t) \notin G_0\right\}$. Note that by choosing δ sufficiently small, we may without loss of generality assume that $G_0 \subset U$. Note that for any $t > s_0$, we can write

$$\begin{aligned}
&\mathbf{E}\left[I_{\{\tau_0<\tau\}}\varphi(X(\tau_0), \alpha(\tau_0))\right] \\
&\quad = \mathbf{E}\left[I_{\{\tau_0<\tau,\tau_0\le t\}}\varphi(X(\tau_0), \alpha(\tau_0))\right] + \mathbf{E}\left[I_{\{\tau_0<\tau,\tau_0>t\}}\varphi(X(\tau_0), \alpha(\tau_0))\right] \\
&\quad := I_1(t) + I_2(t).
\end{aligned} \tag{A.8}$$

Using (A.4), (A.6), and the continuity of the sample paths of X, we have

$$\lim_{t\downarrow s_0} I_1(t) \le \lim_{t\downarrow s_0} \mathbf{P}\left\{\tau_0 \le t\right\} \le \mathbf{P}\left\{\tau_0 = s_0\right\} = 0. \tag{A.9}$$

On the other hand, it follows from the strong Markov property and (A.3) that

$$\begin{aligned} I_2(t) &= \mathbf{E}\left[I_{\{\tau_0<\tau,\tau_0>t\}}\varphi(X(\tau_0),\alpha(\tau_0))\right] \\ &= \mathbf{E}\left[I_{\{\tau_0>t\}}\mathbf{E}_{X(t),\alpha(t)}[I_{\{\tau_0<\tau\}}\varphi(X(\tau_0),\alpha(\tau_0))]\right] \\ &\le \mathbf{E}\left[I_{\{\tau_0>t\}}\varphi(X(t),\alpha(t))\right]. \end{aligned} \tag{A.10}$$

Thanks to condition (ii), for any $\varepsilon > 0$, we can choose a neighborhood N of a such that

$$\varphi(x,i) < \varepsilon, \text{ for any } (x,i) \in (N \cap G) \times \mathcal{M}. \tag{A.11}$$

Also, since the sample paths of X are continuous (see, for example, Mao and Yuan (2006) or Yin and Zhu (2010)), we can choose some $D \subset N$ such that

$$\mathbf{P}\{\beta > s_0\} = 1, \text{ where } \beta = \inf\{t > s_0 : X(t) \notin D\}. \tag{A.12}$$

Then it follows from (A.4), (A.10), and (A.11) that

$$\begin{aligned} I_2(t) &\le \mathbf{E}\left[I_{\{\tau_0>t\}}\varphi(X(t),\alpha(t))\right] \\ &= \mathbf{E}\left[I_{\{\tau_0>t,\beta>t\}}\varphi(X(t),\alpha(t))\right] + \mathbf{E}\left[I_{\{\tau_0>t,\beta\le t\}}\varphi(X(t),\alpha(t))\right] \\ &\le \varepsilon + \mathbf{P}\{\beta \le t\}. \end{aligned}$$

By virtue of (A.12), we further obtain $\limsup_{t\downarrow s_0} I_2(t) \le e$. But $\varepsilon > 0$ is arbitrary, it therefore follows that

$$\lim_{t\downarrow s_0} I_2(t) = 0. \tag{A.13}$$

A combination of (A.8), (A.9), and (A.13) leads to

$$\mathbf{E}\left[I_{\{\tau_0<\tau\}}\varphi(X(\tau_0),\alpha(\tau_0))\right] = 0. \tag{A.14}$$

Step 3. Now we set $A = \{\tau_0 < \tau, X(\tau_0) \in G\}$. Then it is obvious that

$$\left\{\sup_{t\in[s_0,\tau)} \operatorname{dist}(a, X(t)) > \delta\right\} \subset A.$$

This, together with (A.7), implies that $\mathbf{P}(A) > 0$. Therefore it follows from condition (i) that

$$\mathbf{E}\left[I_{\{\tau_0<\tau\}}\varphi(X(\tau_0),\alpha(\tau_0))\right] \ge \mathbf{E}\left[\varphi(X(\tau_0),\alpha(\tau_0))I_A\right] > 0. \tag{A.15}$$

Finally, the contradiction between (A.14) and (A.15) implies that a must be a regular boundary point. □

Remark A.2. If the process (X,α) is assumed to be strong Feller (Zhu and Yin (2009)), then we can show that the conditions in Theorem A.1 are also necessary. The argument is similar to that in (Dynkin, 1965, Chapter 13). We shall omit the details here.

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