PERMUTATION WEIGHTS AND MODULAR POINCARE POLYNOMIALS FOR AFFINE LIE ALGEBRAS

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Abstract

Poincare Polynomial of a Kac-Moody Lie algebra can be obtained by classifying the Weyl orbit $W(\rho)$ of its Weyl vector ρ. A remarkable fact for Affine Lie algebras is that the number of elements of $W(\rho)$ is finite at each and every depth level though totally it has infinite number of elements. This allows us to look at $W(\rho)$ as a manifold graded by depths of its elements and hence a new kind of Poincare Polynomial is defined. We give these polynomials for all Affine Kac-Moody Lie algebras, non-twisted or twisted. The remarkable fact is however that, on the contrary to the ones which are classically defined,these new kind of Poincare polynomials have modular properties, namely they all are expressed in the form of eta-quotients. When one recalls Weyl-Kac character formula for irreducible characters, it is natural to think that this modularity properties could be directly related with Kac-Peterson theorem which says affine characters have modular properties.

Another point to emphasize is the relation between these modular Poincare Polynomials and the Permutation Weights which we previously introduced for Finite and also Affine Lie algebras. By the aid of permutation weights, we have shown that Weyl orbits of an Affine Lie algebra are decomposed in the form of direct sum of Weyl orbits of its horizontal Lie algebra and this new kind of Poincare Polynomials count exactly these permutation weights at each and every level of weight depths.

I. INTRODUCTION

We know that any affine Lie algebra $\widehat{G}_N$ is related with a finite Lie algebra G_N which is called its horizontal Lie algebra. Let λ_i's and α_i's be respectively the fundamental weigths and simple roots of horizontal Lie algebra G_N where $i = 1, 2 \ldots, N$. They are determined by

$$\frac{2\ \kappa(\lambda_i, \alpha_j)}{\kappa(\alpha_i, \alpha_j)} \equiv \delta_{i,j}$$

where $\kappa(,)$ is symmetric scalar product which is known always to be exist via the relation

$$\frac{2\ \kappa(\alpha_i, \alpha_j)}{\kappa(\alpha_i, \alpha_j)} \equiv (A_N)_{i,j}$$

where A_N is the Cartan matrix of G_N. We follow the book of Humphreys [**1**] for finite and Kac [**2**] for Kac-Moody Lie algebras.

Let $\widehat{A}_N$ be the Cartan matrix and α_0 the extra simple root of $\widehat{A}_N$. Its dual λ_0 is to be introduced by hand via the relations

$$\kappa(\lambda_0, \lambda_0) = 0$$
$$\kappa(\lambda_0, \alpha_0) = 1$$

due to the fact that $\widehat{A}_N$ is singular. Note also that

$$\kappa(\lambda_0, \alpha_i) = 0 \quad , \quad i = 1, 2, \ldots, N.$$

and hence the name **horizontal** for G_N. Affine Lie algebras are also characterized by the existence of a unique isotropic root δ defined by

$$\delta = \sum_{\mu=0}^{N} k_\mu\ \alpha_\mu$$

where k_μ's are known to be **Kac labels** of $\widehat{G}_N$. Let $W_{\widehat{G}_N}$ be the weight lattice of $\widehat{G}_N$. For any element $\widehat{\lambda} \in \widehat{G}_N$, we know the following decomposition is always valid:

$$\widehat{\lambda} = \lambda + k\ \lambda_0 - M\ \delta \qquad (I.1)$$

where the **level** k is constant for the Weyl orbit $W(\widehat{\lambda})$ and the **depth** M is always defined to take values

$$M = 0, 1, \ldots, \infty.$$

For any fixed value of M, let us now define $W_M(\widehat{\lambda})$ to be the set of weights with the form (I.1). It is known that the orders of these sets are always finite, that is

$$| W_M(\widehat{\lambda}) | < \infty \qquad (I.2)$$

though their completion and hence the order of $W(\widehat{\lambda})$ is infinite. In view of (I.2), we suggest that $W(\widehat{\lambda})$ can be considered as a manifold graded by weight depths M and hence a Poincare polynomial $Q(\widehat{G}_N)$ is attributed by the following definition:

$$Q(\widehat{G}_N) \equiv \sum_{M=0}^{\infty} | W_M(\widehat{\lambda}) | \ t^M \qquad (I.3)$$

where t is taken to be an indeterminate here and also in the following. It is a priori clear that these polynomials are quite different from Affine Poincare polynomials $P(\widehat{G}_N)$ which are known to be defined by

$$P(\widehat{G}_N) = P(G_N) \prod_{i=1}^{N} \frac{1}{1 - t^{d_i - 1}} \ . \qquad (I.4)$$

by Bott theorem [**3**]. In (I.4), $P(G_N)$ is the Poincare polynomial and d_i's are exponents of G_N.

As we have shown in another work[**4**], an explicit calculation of Poincare polynomials of Hyperbolic Lie algebras can be carried out by classifying the Weyl orbit $W(\rho)$ in terms of lengths[**5**] of Weyl group elements. Such a calculation is extended in a direct way to a classification in terms of weight depths M. For simply-laced affine Lie algebras,

$$\widehat{G}_N = A_N^{(1)}, D_N^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$$

depicted in p.54 of [2], these calculations give the result

$$Q(\widehat{G}_N) = \frac{| W_{G_N} |}{P_N \ R_N} \qquad (I.5)$$

where

$$\begin{aligned} P_N &= \prod_{i=1}^{N} \prod_{k=0}^{\infty} (1 - q^{hk + d_i}) \\ R_N &= \prod_{k=0}^{\infty} \prod_{s=0}^{\infty} (1 + q^{2^s (2k+1) h^\vee})^{(s+1)N} \end{aligned} \qquad (I.6)$$

In above expressions, h and $h^\vee$ are coxeter and co-coxeter numbers of G_N and $\mid W_{G_N} \mid$ is the order of Weyl group W_{G_N} of finite Lie algebra G_N.

Although similar expressions could be obtained for a complete list of affine Lie algebras, this will be presented in the next section in which we expose modular properties of Q-Poincare polynomials defined in (I.3).

II. POINCARE POLYNOMIALS AS ETA-QUOTIENTS

There is quite vast litterature **[6]** on eta-quotients which are rational products of Dedekind eta functions with several arguments. Their relation with finite groups is also studied **[7]**. Let

$$\varphi(q) = \prod_{i=1}^{\infty}(1-q^i)$$

be Euler product and

$$\eta(\tau) \equiv q^{1/24} \ \varphi(q)$$

Dedekind eta function where $q = e^{2\pi i \tau}$. An eta-quotient is defined **[8]** to be a function $f(\tau)$ of the form

$$f(\tau) \equiv \prod_{i=1}^{d} \eta(s_i \tau)^{r_i} \qquad (II.1)$$

where $\{s_1, s_2, \ldots, s_d\}$ is a finite set of positive integers and $r_1, r_2, \ldots, r_d$ are arbitrary integers. Let us denote the collection of integers $r_1, s_1, r_2, s_2, \ldots, r_d, s_d$ defining $f(\tau)$ by the formal product

$$g = s_1^{r_1} s_2^{r_2} \ldots s_d^{r_d} \qquad (II.2)$$

and write $\eta_g(\tau)$ for the corresponding eta-quotient (II.1).

What's important here is to emphasize eta-products are in general meromorphic modular forms of weight $k \equiv \frac{1}{2}\sum_{i=1}^{d} r_i$ and multiplier system for some congruence subgroup of $SL_2(Z)$. This study is however outside the scope of this paper so we will only give here the complete list of Poincare series defined above in the notation of (II.2). To this end, we define

$$Q(\widehat{G}_N) = \mid W_{G_N} \mid \ q^{\frac{1}{24} \ \phi(\widehat{G}_N)} \ \eta_{g(\widehat{G}_N)}. \qquad (II.3)$$

The phase factors $q^{\frac{1}{24} \ \phi(\widehat{G}_N)}$ which stem from the difference between definitions of Euler product and η-function will also be given. Our results are given in the following Table-1 for non-twisted types in Kac's table Aff 1 (**p.54 of [2]**) and in Table-2 for twisted types of Table Aff 2 (**p.55 of [2]**):

Table-1

$$
\begin{aligned}
&g(A_N^{(1)}) = (h^\vee)^{(N+1)}\ 1^{-1} &&,\quad \phi(A_N^{(1)}) = -(N+1)h^\vee + 1\\
&g(B_N^{(1)}) = (2h^\vee)^{1}\ (h^\vee)^{(N-1)}\ 2^1\ 1^{-1} &&,\quad \phi(B_N^{(1)}) = -(N+1)h^\vee - 1\\
&g(C_N^{(1)}) = (2h^\vee)^{(N-1)}\ (h^\vee)^{1}\ 2^1\ 1^{-1} &&,\quad \phi(C_N^{(1)}) = -(N+1)h^\vee - 1\\
&g(D_N^{(1)}) = (h^\vee)^{(N+1)}\ (\frac{1}{2}h^\vee)^{-1}\ 2^1\ 1^{-1} &&,\quad \phi(D_N^{(1)}) = -(N+\frac{1}{2})h^\vee - 1\\
&g(G_2^{(1)}) = 12^1\ 6^{-1}\ 4^1\ 3^1\ 2^1\ 1^{-1} &&,\quad \phi(G_2^{(1)}) = -(2+1)\ 6\ \ +(6-2)\\
&g(F_4^{(1)}) = (18)^2\ 9^2\ 6^{-1}\ 3^1\ 2^1\ 1^{-1} &&,\quad \phi(F_4^{(1)}) = -(4+1)12\ \ +(12-4)\\
&g(E_6^{(1)}) = 12^7\ 6^{-1}\ 4^{-1}\ 3^1\ 2^1\ 1^{-1} &&,\quad \phi(E_6^{(1)}) = -(6+1)12\ \ +(12-6)\\
&g(E_7^{(1)}) = 18^8\ 9^{-1}\ 6^{-1}\ 3^1\ 2^1\ 1^{-1} &&,\quad \phi(E_7^{(1)}) = -(7+1)18\ \ +(18-7)\\
&g(E_8^{(1)}) = 30^9\ 15^{-1}\ 10^{-1}\ 6^{-1}\ 5^1\ 3^1\ 2^1\ 1^{-1} &&,\quad \phi(E_8^{(1)}) = -(8+1)30\ \ +(30-8)
\end{aligned}
$$

Table-2

$$
\begin{aligned}
&g(A_2^{(2)}) = 12^1\ \ 6^{-1}\ 4^{-1}\ 3^1\ 2^2\ 1^{-1} &&,\quad \phi(A_N^{(2)}) = -8\\
&g(A_{2N}^{(2)}) = (4h^\vee)^1\ (2h^\vee)^{(N-2)}\ (h^\vee)^1\ 4^{-1}\ 2^2\ 1^{-1} &&,\quad \phi(A_{2N}^{(2)}) = -(2N+1)h^\vee + 1\\
&g(A_{2N-1}^{(2)}) = (2h^\vee)^2\ (h^\vee)^{(N-3)}\ N^1\ 2^1\ 1^{-1} &&,\quad \phi(A_{2N-1}^{(2)}) = -(N+1)h^\vee - (N+1)\\
&g(D_{N+1}^{(2)}) = (2h^\vee)^N\ 4^{-1}\ 2^2\ 1^{-1} &&,\quad \phi(D_{N+1}^{(2)}) = -2Nh^\vee + 1\\
&g(E_6^{(2)}) = 12^1\ 8^{-1}\ 6^{-1}\ 4^1\ 3^1\ 2^1\ 1^{-1} &&,\quad \phi(E_6^{(2)}) = -6\\
&g(D_4^{(3)}) = 18^2\ 9^{-1}\ 6^{-1}\ 3^2\ 2^1\ 1^{-1} &&,\quad \phi(D_4^{(3)}) = -28
\end{aligned}
$$

At first glance, the examples $G_2^{(1)}, F_4^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ and D_4^3 are interesting due to a theorem **[6]** concerning modular forms for some congruence subgroups of $SL_2(Z)$. We leave however such a study in a subsequent paper.

III. POINCARE POLYNOMIALS AND PERMUTATION WEIGHTS FOR AFFINE LIE ALGEBRAS

We have defined Permutation Weights previously for finite Lie algebras **[9]** and also Affine Lie algebras **[10]**. In these works, it is shown that permutation weights can be calculated explicitly by the aid of a contructive corollary (**p.7 of [10]**).

Here, it is shown that the polynomials

$$\frac{Q(\widehat{G}_N)}{\mid W_{G_N} \mid} \qquad (III.1)$$

count permutation weights at each and every depth level, as will be exemplified in what follows. The present method provides a direct way to find permutation weights by explicit calculation of affine Weyl group elements to which the permutation weights are obtained. One could say that this is not generally so practical since it needs explicit calculations of Weyl group elements. The present method is however presented here as an independent investigation of the previous one.

All our Lie algebraic definitions are as in the sec.I. Let us first briefly remember our previous definition and determination **[10]** of permutation weights. Let $\widehat{\Lambda}^+$ and λ^+ be dominant weights of $\widehat{G}_N$ and G_N respectively, $W(\widehat{\Lambda}^+)$ and $W(\lambda^+)$ be corresponding Weyl orbits. We know that all the elements of $W(\widehat{\Lambda}^+)$ has the form (I.1) and among them the permutation weights are defined by the following specific form:

$$\lambda^+ + k \ \lambda_0 - M \ \delta \ \ , \ \ M = 1, 2, \ldots . \qquad (III.2)$$

In (III.2), for each and every value of M, we define $\mathcal{P}_M(\widehat{\Lambda}^+)$ to be the set of permutation weights of $\widehat{\Lambda}^+$ and $\mid \mathcal{P}_M(\widehat{\Lambda}^+) \mid$ be its order. Let also note that (III.1) can always be expressed in the following form:

$$\frac{Q(\widehat{G}_N)}{\mid W_{G_N} \mid} = \sum_{M=0}^{\infty} \ c_M \ q^M \qquad (III.3)$$

where c_M's are positive integers, $c_0 = 1$ and q is an indeterminate. One can show that

$$\mid \mathcal{P}_M(\widehat{\Lambda}^+) \mid = c_M \ \ , \ \ M = 1, 2, \ldots \qquad (III.4)$$

And hence $Q(\widehat{G}_N)$ states that all the elements $\lambda + k \ \lambda_0 - M \ \delta$ are belong to $W(\widehat{\Lambda}^+)$ where $\lambda \in W(\lambda^+)$. In other words, if

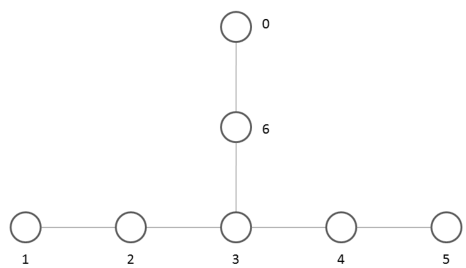
$$\lambda^+ + k\ \lambda_0 - M\ \delta \in \mathcal{P}_M(\widehat{\Lambda}^+)$$

is exist for any M, then one finds that

$$W(\lambda^+) + k\ \lambda_0 - M\ \delta \in W(\widehat{\Lambda}^+)$$

due to existence of Poincare series given in Table-1 and also Table-2. The existence of these new kind of Poincare series is the existence of permutation weights. One can formally say that this gives us an explicit way to decompose any Weyl orbit of an Affine Lie algebra as a direct sum of Weyl orbits of its horizontal Lie algebra. This reflects our main point of view to introduce permutation weights.

It is now useful to proceed in an example for which all our general framework is to be reflected. Let us consider the simply laced, affine Kac-Moody Lie algebra $E_6^{(1)}$ with the following Dynkin diagram:



From Table-1 of Sec.II, one finds that the similar of (III.3) is

$$\frac{Q(E_6^{(1)})}{\mid W_{E_6} \mid} = 1 + q + q^2 + q^3 + 2\ q^4 + 3\ q^5 + 3\ q^6 + 4\ q^7 + 6\ q^8 + 7\ q^9 + \ldots \qquad (III.1)$$

In view of (III.3) and (III.4), the following Table-3 is trivial:

Table-3

M	1	2	3	4	5	6	7	8	9
c_M	1	1	1	2	3	3	4	6	7

Let $W(E_6^{(1)})$ be the Weyl group of $E_6^{(1)}$. Then, all the elements which give us the permutation weights numbered in above Table-3 are given explicitly as in the following:

$$\Sigma_{1,1} = \sigma_0$$

$$\Sigma_{2,1} = \sigma_{0,6}$$

$$\Sigma_{3,1} = \sigma_{0,6,3}$$

$$\Sigma_{4,1} = \sigma_{0,6,3,2} \quad , \quad \Sigma_{4,2} = \sigma_{0,6,3,4}$$

$$\Sigma_{5,1} = \sigma_{0,6,3,2,1} \quad , \quad \Sigma_{5,2} = \sigma_{0,6,3,2,4} \quad , \quad \Sigma_{5,3} = \sigma_{0,6,3,4,5}$$

$$\Sigma_{6,1} = \sigma_{0,6,3,2,1,4} \quad , \quad \Sigma_{6,2} = \sigma_{0,6,3,2,4,3} \quad , \quad \Sigma_{6,3} = \sigma_{0,6,3,2,4,5}$$

$$\Sigma_{7,1} = \sigma_{0,6,3,2,1,4,3} \quad , \quad \Sigma_{7,2} = \sigma_{0,6,3,2,1,4,5}$$
$$\Sigma_{7,3} = \sigma_{0,6,3,2,4,3,5} \quad , \quad \Sigma_{7,4} = \sigma_{0,6,3,2,4,3,6}$$

$$\Sigma_{8,1} = \sigma_{0,6,3,2,1,4,3,2} \quad , \quad \Sigma_{8,2} = \sigma_{0,6,3,2,1,4,3,5}$$
$$\Sigma_{8,3} = \sigma_{0,6,3,2,1,4,3,6} \quad , \quad \Sigma_{8,4} = \sigma_{0,6,3,2,4,3,5,4}$$
$$\Sigma_{8,5} = \sigma_{0,6,3,2,4,3,5,6} \quad , \quad \Sigma_{8,6} = \sigma_{0,6,3,2,4,3,6,0}$$

$$\Sigma_{9,1} = \sigma_{0,6,3,2,1,4,3,2,5} \quad , \quad \Sigma_{9,2} = \sigma_{0,6,3,2,1,4,3,2,6},$$
$$\Sigma_{9,3} = \sigma_{0,6,3,2,1,4,3,5,4} \quad , \quad \Sigma_{9,4} = \sigma_{0,6,3,2,1,4,3,5,6},$$
$$\Sigma_{9,5} = \sigma_{0,6,3,2,1,4,3,6,0} \quad , \quad \Sigma_{9,6} = \sigma_{0,6,3,2,4,3,5,4,6},$$
$$\Sigma_{9,7} = \sigma_{0,6,3,2,4,3,5,6,0}$$

We assume here that Weyl group elements are expressed in the form of $\sigma_{\mu_1,\mu_2,\ldots\mu_k} \equiv \sigma_{\mu_1}\ \sigma_{\mu_2} \ldots \sigma_{\mu_k}$ where σ_μ's are simple Weyl reflections which are defined to be the Weyl group elements with respect to simple roots α_μ of $E_6^{(1)}$ with $\mu = 0, 1, 2, \ldots 6$. On the left-hand side, actions of Weyl group elements are defined on $E_6^{(1)}$ weight lattice by

$$\Sigma_{M,c_M}(\widehat{\Lambda}^+) \equiv \widehat{\Lambda}^+ + \kappa(\widehat{\Lambda}^+, \delta)\ \lambda_0 - M\ \delta.$$

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