The algebraic structure of the universal complicial sets

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Abstract. The nerve of a strict omega-category is a simplicial set with additional structure, making it into a so-called complicial set, and strict omega-categories are in fact equivalent to complicial sets. The nerve functor is represented by a sequence of strict omega-categories, called orientals, which are associated to simplexes. In this paper we give a detailed algebraic description of the morphisms between orientals. The aim is to describe complicial sets algebraically, by operators and equational axioms.

1. Introduction

The orientals or oriented simplexes are a sequence of strict ω-categories

$$\mathcal{O}_0, \mathcal{O}_1, \ldots$$

associated to simplexes. They were discovered by Street, who described them as fundamental objects in nature [**1**]. A strict ω-category X has a *nerve*, consisting of the sequence of morphism sets

$$\operatorname{Hom}(\mathcal{O}_0, X), \operatorname{Hom}(\mathcal{O}_1, X), \ldots;$$

Verity [**2**] has shown that the nerve functor makes the category of strict ω-categories equivalent to a category of simplicial sets with additional structure, called *complicial sets*.

By definition, a complicial set is a simplicial set with a distinguished class of elements, called *thin elements*, subject to certain axioms; Verity's theorem therefore amounts to a description of strict ω-categories in combinatorial terms. There is an analogous cubical theory [**3**] which gives a more algebraic description, in terms of cubical sets with additional operations and equational axioms. This paper is part of a programme aimed at producing a similar algebraic description for complicial sets, using operations and equational axioms rather than distinguished subsets.

In this paper we consider the universal examples; in other words, we consider the nerves of the orientals themselves. This is in fact a purely algebraic problem. The category of orientals can be embedded in the category of chain complexes and chain maps [**4**]: the objects are the chain complexes of the standard simplexes; the morphisms are the augmentation-preserving chain maps taking standard basis elements to sums of standard basis elements. This gives a simple algebraic description of the nerves of orientals, as subsets of graded abelian groups, but we really need

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an internal description independent of any supersets. We will therefore solve the following problem: find an algebraic structure on graded sets, consisting of internal operations and equational axioms, such that the nerve of $\mathcal{O}_n$ is freely generated by its identity endomorphism ι_n. The equational axioms which solve this problem will be called *complicial identities;* they are listed in Definition 6.1.

The structure of the paper is as follows. In Section 2 we recall the description of orientals in terms of chain maps. In Section 3 we describe the additional operations (they were introduced with different notation and terminology in [**4**]). In Section 4 we show that the nerve of $\mathcal{O}_n$ is generated by ι_n; this was also done in [**4**], but here we give more precise details and in effect obtain canonical forms for the elements of the nerve. In Section 5 we give some additional properties of the nerves, for later use. In Section 6 we describe the complicial identities, and we show that they are satisfied in the nerves of orientals. In Section 7 we describe certain consequences of the complicial identities, and in Section 8 we prove the main theorem (Theorem 8.7), showing that the nerve of $\mathcal{O}_n$ is the set with complicial identities freely generated by ι_n.

2. Orientals and chain complexes

From [**4**] we recall the description of the category of orientals in terms of chain maps. For $n = 0, 1, \ldots$ let $\mathbf{Z}\Delta(n)$ be the cellular chain complex of the standard n-simplex. We regard $\mathbf{Z}\Delta(n)$ as a free graded abelian group with a prescribed basis. The basis elements, written in the form $[a_0, \ldots, a_q]$, correspond to the $(q+1)$-tuples of integers $a_0, \ldots, a_q$ such that

$$0 \leq a_0 < a_1 < \ldots < a_q \leq n;$$

a basis element $[a_0, \ldots, a_q]$ is homogeneous of degree q. The boundary homomorphism $\partial\colon \mathbf{Z}\Delta(n) \to \mathbf{Z}\Delta(n)$, which lowers degrees by 1, is given on basis elements of positive degree by

$$\partial[a_0, \ldots, a_q] = \sum_{i=0}^{q} (-1)^i [a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_q].$$

There is also an augmentation homomorphism $\epsilon\colon \mathbf{Z}\Delta(n) \to \mathbf{Z}$, which is given on basis elements of degree 0 by

$$\epsilon[a_0] = 1,$$

and which vanishes on basis elements of positive degree.

We will write $\mathbf{Z}\Delta(m, n)$ for the abelian group consisting of the chain maps from $\mathbf{Z}\Delta(m)$ to $\mathbf{Z}\Delta(n)$; thus a member of $\mathbf{Z}\Delta(m, n)$ is a degree-preserving abelian group homomorphism $f\colon \mathbf{Z}\Delta(m) \to \mathbf{Z}\Delta(n)$ such that $\partial f = f\partial$. We will also write $\mathcal{O}(m, n)$ for the subset of $\mathbf{Z}\Delta(m, n)$ consisting of the chain maps f which are augmentation-preserving ($\epsilon f = \epsilon$) and which take basis elements to sums of basis elements (if a is a basis element then $f(a) = b_1 + \ldots + b_k$ for some $k \geq 0$ and for some basis elements $b_1, \ldots, b_k$). It is clear that there is a category $\mathbf{Z}\Delta$ with objects $0, 1, 2, \ldots$ and with morphism sets $\mathbf{Z}\Delta(m, n)$, using ordinary function composition. It is also clear that $\mathbf{Z}\Delta$ has a subcategory $\mathcal{O}$ with objects $0, 1, 2, \ldots$ and with morphism sets $\mathcal{O}(m, n)$. This category $\mathcal{O}$ is the *category of orientals.*

For $n = 0, 1, \ldots$ let $\mathbf{Z}\Delta(-, n)$ be the graded abelian group consisting of the groups

$$\mathbf{Z}\Delta(0, n),\ \mathbf{Z}\Delta(1, n),\ \ldots$$

and let $\mathcal{O}(-,n)$ be the graded set consisting of the subsets

$$\mathcal{O}(0,n),\ \mathcal{O}(1,n),\ \ldots,$$

so that $\mathcal{O}(-,n)$ is the nerve of the oriental $\mathcal{O}_n$; we are interested in the structure of $\mathcal{O}(-,n)$.

3. Operations in the nerves of orientals

In this section we construct three families of operations in $\mathcal{O}(-,n)$: *face operations*, *degeneracy operations* and *wedge operations*. They will be restrictions of operations in $\mathbf{Z}\Delta(-,n)$. The face and degeneracy operations come from the obvious simplicial set structure on $\mathbf{Z}\Delta(-,n)$, and we begin by recalling the definition of a simplicial set.

Definition 3.1. A *simplicial set* X is a sequence of sets $X_0, X_1, \ldots$ together with face operations

$$\partial_i\colon X_m \to X_{m-1} \quad (m > 0,\ 0 \le i \le m)$$

and degeneracy operations

$$\epsilon_i\colon X_m \to X_{m+1} \quad (0 \le i \le m)$$

such that

$$\begin{aligned}
&\partial_i\partial_j = \partial_{j-1}\partial_i \quad (i < j),\\
&\partial_i\epsilon_j = \epsilon_{j-1}\partial_i \quad (i < j),\\
&\partial_i\epsilon_i = \partial_{i+1}\epsilon_i = \mathrm{id},\\
&\partial_i\epsilon_j = \epsilon_j\partial_{i-1} \quad (i > j+1),\\
&\epsilon_i\epsilon_j = \epsilon_{j+1}\epsilon_i \quad (i \le j).
\end{aligned}$$

We will now describe the face and degeneracy operations in $\mathbf{Z}\Delta(-,n)$ in terms of basis elements. For these basis elements we use notations such as $[\mathbf{b},\mathbf{c}]$ or $[\mathbf{b},i,\mathbf{c}]$, where $\mathbf{b}$ and $\mathbf{c}$ are suitable sequences of integers.

Definition 3.2. Let x be a chain map in $\mathbf{Z}\Delta(m,n)$.

For $m > 0$ and $0 \le i \le m$, the *face* $\partial_i x$ is the chain map in $\mathbf{Z}\Delta(m-1,n)$ given on basis elements by

$$(\partial_i x)[\mathbf{b},\mathbf{c}] = x[\mathbf{b},\mathbf{c}'],$$

where the terms of $\mathbf{b}$ are less than i, the terms of $\mathbf{c}$ are greater than or equal to i, and the terms of $\mathbf{c}'$ are got from those of $\mathbf{c}$ by adding 1.

For $0 \le i \le m$ the *degeneracy* $\epsilon_i x$ is the chain map in $\mathbf{Z}\Delta(m+1,n)$ given on basis elements by

$$\begin{aligned}
&(\epsilon_i x)[\mathbf{b},\mathbf{c}] = x[\mathbf{b},\mathbf{c}''],\\
&(\epsilon_i x)[\mathbf{b},i,\mathbf{c}] = (\epsilon_i x)[\mathbf{b},i+1,\mathbf{c}] = x[\mathbf{b},i,\mathbf{c}''],\\
&(\epsilon_i x)[\mathbf{b},i,i+1,\mathbf{c}] = 0,
\end{aligned}$$

where the terms of $\mathbf{b}$ are less than i, the terms of $\mathbf{c}$ are greater than $i+1$, and the terms of $\mathbf{c}''$ are got from those of $\mathbf{c}$ by subtracting 1.

We get the following result.

Proposition 3.3. *The face and degeneracy operations*

$$\partial_i\colon \mathbf{Z}\Delta(m,n)\to\mathbf{Z}\Delta(m-1,n),\quad \epsilon_i\colon \mathbf{Z}\Delta(m,n)\to\mathbf{Z}\Delta(m+1,n)$$

are group homomorphisms making $\mathbf{Z}\Delta(-,n)$ *into a simplicial set. They restrict to operations*

$$\partial_i\colon \mathcal{O}(m,n)\to\mathcal{O}(m-1,n),\quad \epsilon_i\colon \mathcal{O}(m,n)\to\mathcal{O}(m+1,n)$$

making $\mathcal{O}(-,n)$ *into a simplicial set.*

Proof. It is clear that the operations in $\mathbf{Z}\Delta(-,n)$ are homomorphisms satisfying the simplicial identities. If $x\in\mathcal{O}(-,n)$, so that x is augmentation-preserving and takes basis elements to sums of basis elements, then $\partial_i x$ and $\epsilon_i x$ clearly belong to $\mathcal{O}(-,n)$ as well. □

Degeneracies can be characterised as follows.

Proposition 3.4. *Let x be a morphism in $\mathcal{O}(-,n)$. Then x is in the image of ϵ_i if and only if $xa=0$ for every basis element a including i and $i+1$.*

Proof. By definition, if x is in the image of ϵ_i then x vanishes on every basis element including i and $i+1$.

Conversely, suppose that x vanishes on every basis element including i and $i+1$. Since x is a chain map,

$$x[\mathbf{b},i,\mathbf{c}]=x[\mathbf{b},i+1,\mathbf{c}]$$

for all basis elements of the form $[\mathbf{b},i,i+1,\mathbf{c}]$, and it follows that $x=\epsilon_i\partial_i x$. □

Next we define the wedge operations.

Definition 3.5. Let m and i be integers with $0\le i\le m-1$. If x and y are chain maps in $\mathbf{Z}\Delta(m,n)$ such that $\partial_i x=\partial_{i+1}y$, then the *wedge* $x\wedge_i y$ is the chain map in $\mathbf{Z}\Delta(m+1,n)$ given by

$$x\wedge_i y=\epsilon_{i+1}x-\epsilon_i^2\partial_{i+1}y+\epsilon_i y.$$

In terms of basis elements, if a does not include i then $(x\wedge_i y)a=(\epsilon_i y)a$, if a does not include $i+2$ then $(x\wedge_i y)a=(\epsilon_{i+1}x)a$, and

$$\begin{aligned}&(x\wedge_i y)[\mathbf{b},i,i+2,\mathbf{c}]=(\epsilon_{i+1}x+\epsilon_i y)[\mathbf{b},i,i+2,\mathbf{c}],\\&(x\wedge_i y)[\mathbf{b},i,i+1,i+2,\mathbf{c}]=0.\end{aligned}$$

Geometrically, if x and y are regarded as functions on the m-simplex then $x\wedge_i y$ acts on a point of the $(m+1)$-simplex in the following way: project the point onto the union of the faces opposite vertex i and vertex $i+2$; apply y if the projection is in the face opposite i; apply x if the projection is in the face opposite $i+2$.

We get the following results.

Proposition 3.6. *If $x\wedge_i y$ is defined in $\mathbf{Z}\Delta(-,n)$ then*

$$\partial_i(x\wedge_i y)=y,\quad \partial_{i+2}(x\wedge_i y)=x.$$

Proof. This is clear from the definition. □

Proposition 3.7. *Let x and y be morphisms in $\mathcal{O}(m,n)$ such that $\partial_i x=\partial_{i+1}y$. Then $x\wedge_i y$ is a morphism in $\mathcal{O}(m+1,n)$.*

Proof. Given that x and y are augmentation-preserving and that they take basis elements to sums of basis elements, we must show that $x \wedge_i y$ has the same properties. This is straightforward. □

Proposition 3.8. *Let z be a morphism in $\mathcal{O}(m+1,n)$, and let i be an integer with $0 \le i \le m-1$. Then the following are equivalent.*

1. *There are morphisms x, y in $\mathcal{O}(m,n)$ with $\partial_i x = \partial_{i+1} y$ such that $z = x \wedge_i y$.*
2. *There are chain maps u, v in $\mathbf{Z}\Delta(m,n)$ such that $z = \epsilon_i u + \epsilon_{i+1} v$.*
3. *One has $za = 0$ for every basis element a including $i, i+1, i+2$.*

Proof. We show that (1) ⇒ (2) ⇒ (3) ⇒ (1).

If $z = x \wedge_i y$ then z has the form $\epsilon_i u + \epsilon_{i+1} v$ by definition.

If $z = \epsilon_i u + \epsilon_{i+1} v$ then clearly $za = 0$ for every basis element a including $i, i+1, i+2$.

Suppose that $za = 0$ for every basis element a including $i, i+1, i+2$. Let

$$x = \partial_{i+2} z, \quad y = \partial_i z.$$

It follows from Proposition 3.3 that x and y are morphisms in $\mathcal{O}(m,n)$ and that $\partial_i x = \partial_{i+1} y$; the wedge $x \wedge_i y$ therefore exists. The morphisms z and $x \wedge_i y$ then agree on basis elements not including i, on basis elements not including $i+2$, and on basis elements including $i, i+1, i+2$. Since z and $x \wedge_i y$ are chain maps, they must also agree on basis elements including i and $i+2$ but not $i+1$. Therefore $z = x \wedge_i y$.

This completes the proof. □

4. Canonical forms

Let ι_n be the identity morphism in $\mathcal{O}(n,n)$. In this section we show that the elements of $\mathcal{O}(-,n)$ can be expressed in terms of ι_n by using the face, degeneracy and wedge operations. In effect we find canonical forms for the morphisms in $\mathcal{O}(-,n)$ (see Theorem 4.11). The argument is based on the following result.

Theorem 4.1. *Let x be a morphism in $\mathcal{O}(m,n)$. Then there are integers $x_0, \ldots, x_m$ with $0 \le x_0 \le x_1 \le \ldots \le x_m \le n$ such that*

$$x[0] = [x_0],\ \ldots,\ x[m] = [x_m].$$

If a is a basis element in $\mathbf{Z}\Delta(m)$ of the form $[s, a_1, \ldots, a_{q-1}, t]$ then xa is a sum of basis elements $[b_0, \ldots, b_q]$ with

$$x_s \le b_0 < b_1 < \ldots < b_q \le x_t.$$

Proof. Let $[i]$ be a zero-dimensional basis element in $\mathbf{Z}\Delta(m)$. Then $x[i]$ is a sum of zero-dimensional basis elements in $\mathbf{Z}\Delta(n)$, and this sum has exactly one term because x is augmentation-preserving. Therefore $x[i] = [x_i]$ for some integer x_i with $0 \le x_i \le n$.

For $0 < i \le m$ we have

$$\partial x[i-1,i] = x\partial[i-1,i] = x([i] - [i-1]) = [x_i] - [x_{i-1}].$$

But $x[i-1,i]$ is a sum of basis elements $[j,k]$, so $\partial x[i-1,i]$ is a sum of expressions $[k] - [j]$ with $j < k$. Therefore $x_{i-1} \le x_i$.

Let a be a basis element of the form $[s, a_1, \ldots, a_q]$; we will show by induction on q that xa is a sum of basis elements $[b_0, \ldots, b_q]$ with $x_s \le b_0$. The result is clear when $q = 0$, and is trivial when $xa = 0$. From now on, suppose that

$q > 0$ and $xa \neq 0$, so that xa is a non-empty sum of basis elements of positive dimension. Let $[k_0, \dots, k_q]$ be a term in this sum such that k_0 is as small as possible and, subject to this condition, such that k_1 is as large as possible. Then the basis element $[k_0, k_2, \dots, k_q]$ has a negative coefficient in ∂xa because there is no possibility of cancellation. Since $\partial xa = x\partial a$, it follows that $[k_0, k_2, \dots, k_q]$ is a term in $x[s, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_q]$ for some odd value of i, and it then follows from the inductive hypothesis that $x_s \leq k_0$. Since k_0 is minimal, xa is a sum of basis elements $[b_0, \dots, b_q]$ with $x_s \leq b_0$.

Similarly, if a is a basis element of the form $[a_0, \dots, a_{q-1}, t]$, then xa is a sum of basis elements $[b_0, \dots, b_q]$ with $b_q \leq x_t$.

This completes the proof. $\square$

The canonical forms for morphisms in $\mathcal{O}(-, n)$ will depend on parameters called *terminus*, *rank* and *corank*, which are defined as follows.

Definition 4.2. Let x be a morphism in $\mathcal{O}(m, n)$. Then the *terminus* of x, denoted $\operatorname{terminus} x$, is the integer t such that $x[m] = [t]$; the *rank* of x, denoted $\operatorname{rank} x$, is the number of zero-dimensional basis elements $[a]$ in $\mathbf{Z}\Delta(m)$ such that $x[a] \neq x[m]$; the *corank* of x, denoted $\operatorname{corank} x$, is the number of zero-dimensional basis elements $[a]$ in $\mathbf{Z}\Delta(m)$ such that $a < m$ and $x[a] = x[m]$.

From Theorem 4.1 we immediately draw the following conclusion.

Proposition 4.3. *Let x be a morphism in $\mathcal{O}(m, n)$ with terminus t, rank r and corank s. Then t, r and s are nonnegative integers such that $t \leq n$ and $r + s = m$.*

We will find the canonical forms by an induction on terminus. The passage to morphisms of terminus t from morphisms of lower terminus is based on a cone construction, as follows.

Definition 4.4. Let x be a morphism in $\mathcal{O}(m, n)$ with terminus less than t. Then the *t-cone* on x is the morphism $\rho_t x$ in $\mathcal{O}(m + 1, n)$ given on basis elements as follows: $(\rho_t x)[m + 1] = [t]$; if

$$x[\mathbf{a}] = \sum_{\mathbf{b}} x_{\mathbf{ab}}[\mathbf{b}]$$

then

$$(\rho_t x)[\mathbf{a}] = \sum_{\mathbf{b}} x_{\mathbf{ab}}[\mathbf{b}], \quad (\rho_t x)[\mathbf{a}, m + 1] = \sum_{\mathbf{b}} x_{\mathbf{ab}}[\mathbf{b}, t].$$

It is easy to see that $\rho_t x$ as defined here really is a morphism in $\mathcal{O}(m + 1, n)$, with terminus t and corank zero. It is also easy to verify the following result.

Proposition 4.5. *Let S_{t-1} be the subset of $\mathcal{O}(-, n)$ consisting of the morphisms with terminus less than t. Then S_{t-1} is closed under the face, degeneracy and wedge operations. The t-cone construction*

$$\rho_t \colon S_{t-1} \to \mathcal{O}(-, n)$$

commutes with the operations in S_{t-1}, and

$$\rho_t \partial_t^{n-t+1} \iota_n = \partial_{t+1}^{n-t} \iota_n.$$

We will use Proposition 4.5 as part of an induction on terminus. To complete the argument, we will show that arbitrary morphisms with terminus at most t can be expressed in terms of $\partial_{t+1}^{n-t}\iota_n$ and of t-cones.

Let x be a morphism in $\mathcal{O}(m,n)$ with terminus t and rank r; we will construct an associated morphism γx in $\mathcal{O}(r,n)$ with terminus t, rank r and corank zero. Note that, by Theorem 4.1, if $a = [a_0,\dots,a_q]$ is a basis element in $\mathbf{Z}\Delta(m)$ with $a_q < r$, then xa is a sum of basis elements $[b_0,\dots,b_q]$ with $b_q < t$. We deduce that there is a chain map

$$\gamma x\colon \mathbf{Z}\Delta(r) \to \mathbf{Z}\Delta(n)$$

given on basis elements as follows: $(\gamma x)[r] = [t]$; for $r > 0$, if $[\mathbf{a}]$ is a basis element for $\mathbf{Z}\Delta(r-1)$ and if

$$x[\mathbf{a},r] = \sum_{\mathbf{b}}(x'_{\mathbf{ab}}[\mathbf{b}] + x''_{\mathbf{ab}}[\mathbf{b},t])$$

with the sum running over basis elements $[\mathbf{b}]$ for $\mathbf{Z}\Delta(t-1)$, then

$$(\gamma x)[\mathbf{a}] = \sum_{\mathbf{b}} x''_{\mathbf{ab}}[\mathbf{b}], \quad (\gamma x)[\mathbf{a},r] = \sum_{\mathbf{b}} x''_{\mathbf{ab}}[\mathbf{b},t].$$

It is easy to see that γx has the following properties.

Proposition 4.6. *Let x be a morphism in $\mathcal{O}(-,n)$ with terminus t and rank r. Then γx is a morphism in $\mathcal{O}(r,n)$ with terminus t, rank r and corank zero. If $r = 0$ then $\gamma x = \partial_0^t\partial_{t+1}^{n-t}\iota_n$; if $r > 0$ then γx is a t-cone.*

Given a morphism x in $\mathcal{O}(-,n)$ with terminus t, rank r and corank s, we now construct further chain maps as follows: let

$$\beta x\colon \mathbf{Z}\Delta(r+s) \to \mathbf{Z}\Delta(n)$$

be given by

$$\beta x = \epsilon_r^s\gamma x;$$

for $0 \le p < r$ let

$$\alpha_p x\colon \mathbf{Z}\Delta(p+s+1) \to \mathbf{Z}\Delta(n)$$

be given by

$$\alpha_p x = \partial_{p+1}^{r-p-1}(x - \beta x) + \epsilon_p\partial_{p+1}^{r-p}\beta x.$$

Example 4.7. Let x be a morphism in $\mathcal{O}(r,n)$ which is equal to $\partial_0^t\partial_{t+1}^{n-t}\iota_n$ or is a t-cone. Then $\operatorname{rank} x = r$, $\operatorname{corank} x = 0$, $\beta x = \gamma x = x$, and

$$\alpha_p x = \epsilon_p\partial_{p+1}^{r-p}x \quad (0 \le p < r).$$

In general we get the following result.

Proposition 4.8. *Let x be a morphism in $\mathcal{O}(-,n)$ with terminus t, rank r and corank s. If $s = 0$ and $0 \le p < r$ then $\alpha_p x$ is a morphism in $\mathcal{O}(p+s+1,n)$ with terminus less than t. If $s > 0$ and $0 \le p < r$ then $\alpha_p x$ is a morphism in $\mathcal{O}(p+s+1,n)$ with terminus t and corank $s-1$.*

Proof. It is clear that $\alpha_p x$ is an augmentation-preserving chain map from $\mathbf{Z}\Delta(p+s+1)$ to $\mathbf{Z}\Delta(n)$. To show that $\alpha_p x$ is a morphism in $\mathcal{O}$, we must show that $\alpha_p x$ takes each basis element a to a sum of basis elements.

Suppose that a does not have a term $p+1$. Then $(\alpha_p x)a = (\partial_{p+1}^{r-p-1}x)a$ because $(\partial_{p+1}^{r-p-1}\beta x)a = (\epsilon_p\partial_{p+1}^{r-p}\beta x)a$, and it follows that $(\alpha_p x)a$ is a sum of basis elements.

Suppose that a has a term $p+1$ and a term greater than $p+1$. Then $(\partial_{p+1}^{r-p-1}\beta x)a = 0$, because βx has the form $\epsilon_r^s\gamma x$, and it again follows that $(\alpha_p x)a$ is a sum of basis elements.

Suppose that $a = [\mathbf{a}, p+1]$. Then $(\partial_{p+1}^{r-p-1}\beta x)a$ is the sum of the terms of the form $[\mathbf{b}, t]$ in $(\partial_{p+1}^{r-p-1}x)a$, so $\{\partial_{p+1}^{r-p-1}(x-\beta x)\}a$ is the sum of the remaining terms in $(\partial_{p+1}^{r-p-1}x)a$, and it again follows that $(\alpha_p x)a$ is a sum of basis elements.

This shows that $(\alpha_p x)a$ is a sum of basis elements in all cases; therefore $\alpha_p x$ is a morphism in $\mathcal{O}(p+s+1, n)$.

We will now consider the terminus and corank of $\alpha_p x$. For $p+2 \leq i \leq p+s+1$ it follows from the calculations above and from Proposition 4.3 that

$$(\alpha_p)[i] = (\partial_{p+1}^{r-p-1}x)[i] = x[r+i-p-1] = [t];$$

on the other hand,

$$(\alpha_p x)[p+1] = (x-\beta x)[r] + (\beta x)[p] = [t] - [t] + (\beta x)[p] = (\gamma x)[p] \neq [t].$$

If $s = 0$ it now follows that $\alpha_p x$ has terminus less than t; if $s > 0$ it follows that $\alpha_p x$ has terminus t and corank $s-1$.

This completes the proof. □

We will express a morphism x in terms of the morphisms $\alpha_p x$ and γx by using the following notation.

Notation 4.9. In a set with operations ∂_i and $\wedge_i$ we write

$$u(\wedge_k v) = u \wedge_k v,$$

regarding $(\wedge_k v)$ as an operator which acts on the right, and for $l \geq 1$ we write

$$u \wedge_k^l v = u(\wedge_k \partial_{k+1}^{l-1}v)(\wedge_k \partial_{k+1}^{l-2}v)\ldots(\wedge_k \partial_{k+1}v)(\wedge_k v).$$

We also write

$$(u \wedge_k^l)v = u \wedge_k^l v,$$

regarding $(u \wedge_k^l)$ as an operator which acts on the left, and for $r \geq 0$ we write

$$\Lambda^r(u_{r-1}, \ldots, u_0, v) = \partial_r(u_{r-1} \wedge_{r-1}^1)\partial_{r-1}(u_{r-2} \wedge_{r-2}^2)\ldots\partial_1(u_0 \wedge_0^r)v.$$

Note in particular that $\Lambda^0(v) = v$.

In $\mathcal{O}(-, n)$ an induction based on Definition 3.5 gives the following result.

Proposition 4.10. *If u and v are morphisms in $\mathcal{O}(-, n)$ and if $\partial_k u = \partial_{k+1}^l v$, then $u \wedge_k^l v$ is defined and*

$$u \wedge_k^l v = \epsilon_{k+1}^l u - \epsilon_k^{l+1}\partial_{k+1}^l v + \epsilon_k v.$$

The canonical form is now as follows.

Theorem 4.11. *Let x be a morphism in $\mathcal{O}(-, n)$ with rank r and with corank s. Then*

$$x = \Lambda^r(\alpha_{r-1}x, \ldots, \alpha_0 x, \epsilon_r^s\gamma x).$$

Proof. Recall that

$$\alpha_p x = \partial_{p+1}^{r-p-1}(x - \beta x) + \epsilon_p\partial_{p+1}^{r-p}\beta x,$$

where $\beta x = \epsilon_r^s\gamma x$. For $0 \leq p \leq r$, let

$$v_p = \epsilon_p^{r-p}\partial_p^{r-p}(x - \beta x) + \beta x,$$

so that, in particular, $v_r = x$. It suffices to show that

$$\begin{aligned} &v_0 = \epsilon_r^s \gamma x, \\ &v_{p+1} = \partial_{p+1}(\alpha_p x \wedge_p^{r-p} v_p) \quad (0 \le p < r); \end{aligned}$$

we proceed as follows.

The morphisms $\partial_0^r x$ and $\partial_0^r \beta x$ are morphisms in $\mathcal{O}(s,n)$ such that

$$(\partial_0^r x)[0] = (\partial_0^r x)[s] = [t], \quad (\partial_0^r \beta x)[0] = (\partial_0^r \beta x)[s] = [t].$$

By Theorem 4.1, these two morphisms both annihilate all basis elements of positive dimension; therefore $\partial_0^r x = \partial_0^r \beta x$. It follows that $v_0 = \beta x = \epsilon_r^s \gamma x$.

For $0 \le p < r$ it is straightforward to verify that $\partial_p \alpha_p x = \partial_{p+1}^{r-p} v_p$ and that

$$v_{p+1} = \partial_{p+1}(\epsilon_{p+1}^{r-p} \alpha_p x - \epsilon_p^{r-p+1} \partial_{p+1}^{r-p} v_p + \epsilon_p v_p);$$

therefore $v_{p+1} = \partial_{p+1}(\alpha_p x \wedge_p^{r-p} v_p)$.

This completes the proof. □

We can now prove the main result.

Theorem 4.12. *Every morphism in $\mathcal{O}(-,n)$ can be expressed in terms of ι_n by using the face, degeneracy and wedge operations.*

Proof. Let S_t be the set of morphisms with terminus at most t. We will show inductively that S_t is generated by $\partial_{t+1}^{n-t} \iota_n$; the case $t = n$ will then give the result.

It follows from Theorem 4.11, Proposition 4.6 and Proposition 4.8 that S_t has a set of generators consisting of the morphism $\partial_0^t \partial_{t+1}^{n-t} \iota_n$, the t-cones, and the morphisms in S_{t-1}; it therefore suffices to show that these generators are expressible in terms of $\partial_{t+1}^{n-t} \iota_n$.

For $t = 0$ the result holds because the only generator is $\partial_1^n \iota_n$.

For $t > 0$, suppose as an inductive hypothesis that S_{t-1} is generated by $\partial_t^{n-t+1} \iota_n$. It is evident that $\partial_0^t \partial_{t+1}^{n-t} \iota_n$ can be expressed in terms of $\partial_{t+1}^{n-t} \iota_n$. It follows from the inductive hypothesis and Proposition 4.5 that t-cones can be expressed in terms of $\partial_{t+1}^{n-t} \iota_n$. It also follows from the inductive hypothesis that the morphisms in S_{t-1} can be expressed in terms of $\partial_{t+1}^{n-t} \iota_n$, because $\partial_t^{n-t+1} \iota_n = \partial_t \partial_{t+1}^{n-t} \iota_n$.

This completes the proof. □

5. Further properties of the nerves of orientals

We have shown in Theorem 4.12 that $\mathcal{O}(-,n)$ is generated by ι_n. In the later sections of this paper we will find identities determining the structure of $\mathcal{O}(-,n)$ completely. Here we obtain some additional results needed in those sections.

First we give some formulae for faces.

Proposition 5.1. *Let x be a morphism in $\mathcal{O}(-,n)$ with rank r, and let $\partial_i x$ be a face with $0 \le i < r$. Then*

$$\begin{aligned} &\gamma \partial_i x = \partial_i \gamma x, \\ &\alpha_p \partial_i x = \alpha_p x \quad (0 \le p < i), \\ &\alpha_p \partial_i x = \partial_i \alpha_{p+1} x \quad (i \le p < r-1). \end{aligned}$$

Proof. Let the terminus of x be t and the corank be s. It follows from Proposition 4.3 that $\partial_i x$ has terminus t, rank $r-1$ and corank s. It is clear from the construction that $\gamma\partial_i x = \partial_i\gamma x$, it follows that

$$\beta\partial_i x = \epsilon^s_{r-1}\gamma\partial_i x = \epsilon^s_{r-1}\partial_i\gamma x = \partial_i\epsilon^s_r\gamma x = \partial_i\beta x,$$

and for $0 \leq p < r-1$ it follows that

$$\alpha_p\partial_i x = \partial^{r-p-2}_{p+1}(\partial_i x - \partial_i\beta x) + \epsilon_p\partial^{r-p-1}_{p+1}\partial_i\beta x.$$

If now $0 \leq p < i$ then

$$\alpha_p\partial_i x = \partial^{r-p-1}_{p+1}(x - \beta x) + \epsilon_p\partial^{r-p}_{p+1}\beta x = \alpha_p x;$$

if $i \leq p < r-1$ then

$$\alpha_p\partial_i x = \partial_i\partial^{r-p-2}_{p+2}(x - \beta x) + \partial_i\epsilon_{p+1}\partial^{r-p-1}_{p+2}\beta x = \partial_i\alpha_{p+1}x.$$

□

Next we consider the image set of the wedge operation $\wedge_i$, which is denoted $\operatorname{im}\wedge_i$.

Proposition 5.2. *Let x be a morphism of rank r in $\mathcal{O}(-,n)$.*

If $x \in \operatorname{im}\wedge_i$ with $i \leq r-3$ then $\gamma x \in \operatorname{im}\wedge_i$, and $\alpha_p x \in \operatorname{im}\wedge_i$ for $i+2 \leq p < r$.

If $x \in \operatorname{im}\wedge_{r-2}$ then $\gamma x \in \operatorname{im}\epsilon_{r-2}$ and $\alpha_{r-1}x \in \operatorname{im}\wedge_{r-2}$.

If $x \in \operatorname{im}\wedge_{r-1}$ then $\alpha_{r-1}x \in \operatorname{im}\wedge_{r-1}$.

If $x \in \operatorname{im}\wedge_i$ with $r \leq i$ then $\alpha_p x \in \operatorname{im}\wedge_{i-r+p+1}$ for $0 \leq p < r$.

Proof. Let the corank of x be s, so that

$$\alpha_p x = \partial^{r-p-1}_{p+1}x - \partial^{r-p-1}_{p+1}\epsilon^s_r\gamma x + \epsilon_p\partial^{r-p}_{p+1}\epsilon^s_r\gamma x,$$

and recall from Proposition 3.8 that a morphism is in $\operatorname{im}\wedge_i$ if and only if it annihilates all basis elements including $i, i+1, i+2$.

Suppose that $x \in \operatorname{im}\wedge_i$ with $i \leq r-3$. Then γx annihilates basis elements including $i, i+1, i+2$ because x annihilates basis elements including $i, i+1, i+2, r$; therefore $\gamma x \in \operatorname{im}\wedge_i$. For $i+2 \leq p < r$ it follows that $\partial^{r-p-1}_{p+1}x$, $\partial^{r-p-1}_{p+1}\epsilon^s_r\gamma x$ and $\epsilon_p\partial^{r-p}_{p+1}\epsilon^s_r\gamma x$ annihilate all basis elements including $i, i+1, i+2$, and it then follows that $\alpha_p x \in \operatorname{im}\wedge_i$.

Suppose that $x \in \operatorname{im}\wedge_{r-2}$. In this case γx annihilates all basis elements including $r-2, r-1$ because x annihilates all basis elements including $r-2, r-1, r$; hence, by Proposition 3.4, $\gamma x \in \operatorname{im}\epsilon_{r-2}$. As in the previous case, $\alpha_{r-1}x \in \operatorname{im}\wedge_{r-2}$.

Suppose that $x \in \operatorname{im}\wedge_{r-1}$. Then $s \geq 1$, so that $\epsilon^s_r\gamma x$ and $\epsilon_{r-1}\partial_r\epsilon^s_r\gamma x$ are in $\operatorname{im}\wedge_{r-1}$. As before, it follows that $\alpha_{r-1}x \in \operatorname{im}\wedge_{r-1}$.

Finally, suppose that $x \in \operatorname{im}\wedge_i$ with $r \leq i$, and suppose that $0 \leq p < r$. Then $i \leq r+s-2$, so $\partial^{r-p-1}_{p+1}x$, $\partial^{r-p-1}_{p+1}\epsilon^s_r\gamma x$ and $\epsilon_p\partial^{r-p}_{p+1}\epsilon^s_r\gamma x$ are all in $\operatorname{im}\wedge_{i-r+p+1}$; therefore $\alpha_p x \in \operatorname{im}\wedge_{i-r+p+1}$. □

We finish this section by giving bounds for the termini and coranks of the morphisms in expressions of the form $\Lambda^r(u_{r-1}, \ldots, u_0, v)$.

Proposition 5.3. *Let* $\Lambda^r(u_{r-1},\dots,u_0,v)$ *be an expression defined in* $\mathcal{O}(-,n)$*, and let*

$$\begin{aligned} &v_0 = v,\\ &v_{p+1} = \partial_{p+1}(u_p \wedge_p^{r-p} v_p) \quad (0 \le p < r),\\ &w_p = u_p \wedge_p^{r-p} v_p \quad (0 \le p < r). \end{aligned}$$

Then the morphisms v_p *and* w_p *all have the same terminus as* v*. If* $\operatorname{rank} v \ge r$ *then the morphisms* v_p *and* w_p *all have the same corank as* v*.*

Proof. By definition, $v_{p+1} = \partial_{p+1} w_p$. From Proposition 3.6 we see that $\partial_p w_p = v_p$. For $i \ge r$ it follows that

$$v_r[i-1] = w_{r-1}[i] = v_{r-1}[i-1] = \ldots = w_0[i] = v_0[i-1] = v[i-1];$$

in particular it follows that the morphisms v_p and w_p all have the same terminus as v. Let this common terminus be t. If $\operatorname{rank} v \ge r$, then $v[r-1] \ne [t]$, from which it follows that v_p and w_p have the same corank as v. □

6. Complicial identities

In this section we define sets with complicial identities; they will be simplicial sets with wedge operations subject to certain axioms. We will then show that the simplicial sets $\mathcal{O}(-,n)$ satisfy these axioms.

Definition 6.1. A *set with complicial identities* is a simplicial set X, together with *wedges*

$$x \wedge_i y \in X_{m+1},$$

defined when $x, y \in X_m$ and $\partial_i x = \partial_{i+1} y$, such that the following axioms hold.

(1) If $x \wedge_i y$ is defined with $x, y \in X_m$, then

$$\begin{aligned} &\partial_j(x \wedge_i y) = \partial_j x \wedge_{i-1} \partial_j y \quad (0 \le j < i),\\ &\partial_i(x \wedge_i y) = y,\\ &\partial_{i+2}(x \wedge_i y) = x,\\ &\partial_j(x \wedge_i y) = \partial_{j-1} x \wedge_i \partial_{j-1} y \quad (i+3 \le j \le m+1). \end{aligned}$$

(2) If $x \in X_m$ and $0 \le i < m$ then

$$\epsilon_i x = \epsilon_i \partial_{i+1} x \wedge_i x, \quad \epsilon_{i+1} x = x \wedge_i \epsilon_i \partial_i x.$$

(3) If A is of the form $b \wedge_i (y \wedge_i z)$ then

$$A = (\partial_{i+2} b \wedge_i y) \wedge_{i+1} \partial_{i+1} A.$$

(4) If A is of the form $(x \wedge_i y) \wedge_{i+1} c$ then

$$A = \partial_{i+2} A \wedge_i (y \wedge_i \partial_i c).$$

(5) The equality

$$[x \wedge_i \partial_{i+1}(y \wedge_i z)] \wedge_i (y \wedge_i z) = (x \wedge_i y) \wedge_{i+1} [\partial_{i+1}(x \wedge_i y) \wedge_i z]$$

holds whenever either side is defined.

(6) Let A be an element of the form $\partial_{i+2}[(x \wedge_{i+1} y) \wedge_{i+1} (y \wedge_i z)]$; then the equality

$$A \wedge_i (w \wedge_{i+1} \partial_i A) = (\partial_{i+3} A \wedge_i w) \wedge_{i+2} A$$

holds whenever either side is defined.

(7) If $i \leq j-3$ then the equality

$$(x \wedge_i y) \wedge_j (z \wedge_i w) = (x \wedge_{j-1} z) \wedge_i (y \wedge_{j-1} w)$$

holds whenever either side is defined.

A *morphism of sets with complicial identities* $f\colon X \to Y$ is a sequence of functions $f\colon X_m \to Y_m$ commuting with the face, degeneracy and wedge operations.

Remark 6.2. There is some redundancy in these axioms. Indeed, the degeneracy operations on elements of positive dimension are redundant because of axiom (2). For similar reasons, one can omit some of the simplicial identities, retaining only $\partial_i\partial_j = \partial_{j-1}\partial_i$ (for $i < j$) and $\partial_i\epsilon_i = \partial_{i+1}\epsilon_i = \mathrm{id}$.

Remark 6.3. Axiom (3) could be written simply as an equality

$$b \wedge_i (y \wedge_i z) = (\partial_{i+2} b \wedge_i y) \wedge_{i+1} \partial_{i+1}[b \wedge_i (y \wedge_i z)],$$

required when either side is defined. A similar remark applies to axioms (4) and (6).

Remark 6.4. Note that axiom (1) does not give a formula for $\partial_{i+1}(x \wedge_i y)$. In a sense, the operation $\wedge_i$ exists in order to construct the operation

$$(x, y) \mapsto \partial_{i+1}(x \wedge_i y).$$

By using axioms (1), (2) and (5), one can show that this binary operation is associative and that it makes the m-dimensional elements into the morphisms of a category. The objects are the $(m-1)$-dimensional elements, the source and target of a morphism x are $\partial_i x$ and $\partial_{i+1} x$, and the identity of an object a is $\epsilon_i a$.

Remark 6.5. I intend to show in a future paper that sets with complicial identities are equivalent to the complicial sets used by Verity in [**2**].

We will now show that the axioms of Definition 6.1 apply to orientals.

Proposition 6.6. *The graded sets $\mathcal{O}(-,n)$ are sets with complicial identities.*

Proof. From Section 3 we know that $\mathcal{O}(-,n)$ is a simplicial set and that it has wedge operations with the correct domains and codomains.

It is straightforward to verify axiom (1).

Next we verify axiom (5). The existence of the expression on one side is equivalent to the existence of the expression on the other, because the existence of either expression is equivalent to the truth of the equalities

$$\partial_i x = \partial_{i+1} y, \quad \partial_i y = \partial_{i+1} z.$$

Also, if the two expression do exist, they have the same value, namely

$$\epsilon_{i+3}\epsilon_{i+2} x - \epsilon_i^3 \partial_{i+1} y + \epsilon_{i+3}\epsilon_i y - \epsilon_i^3 \partial_{i+1} z + \epsilon_{i+1}\epsilon_i z.$$

The remaining axioms all say that expressions of certain forms are equal to wedges. They can be proved by using Proposition 3.8, which says that an element of $\mathcal{O}(-,n)$ is in the image of $\wedge_i$ if and only if it is in $\operatorname{im}\epsilon_i + \operatorname{im}\epsilon_{i+1}$. For example, suppose that the expression on the left of axiom (6) is defined, and let

$$B = A \wedge_i (w \wedge_{i+1} \partial_i A).$$

We find that $B \in \operatorname{im}\epsilon_{i+2} + \operatorname{im}\epsilon_{i+3}$; therefore B has the form $B' \wedge_{i+2} B''$. Using axiom (1) we then find that

$$\begin{aligned} B' &= \partial_{i+4} B = \partial_{i+3} A \wedge_i \partial_{i+3}(w \wedge_{i+1} \partial_i A) = \partial_{i+3} A \wedge_i w, \\ B'' &= \partial_{i+2} B = A; \end{aligned}$$

therefore $B = (\partial_{i+3}A \wedge_i w) \wedge_{i+2} A$.

This completes the proof. $\square$

7. Consequences of the complicial identities

We have shown in Theorem 4.12 that $\mathcal{O}(-,n)$ is generated by the identity morphism ι_n. In Section 8 we will show that $\mathcal{O}(-,n)$ is freely generated by ι_n subject to the complicial identities of Definition 6.1; in other words, we will show that the identities determine the entire structure. In order to do this, we now consider consequences of the identities in general.

First we consider the expressions $u \wedge_k^l v$ (see Notation 4.9). The domains of definition and most of the faces are as follows.

Proposition 7.1. *An expression $u \wedge_k^l v$ in a set with complicial identities exists if and only if $\partial_k u = \partial_{k+1}^l v$. If the expression does exist, then*

$$\begin{aligned}
&\partial_i(u \wedge_k^l v) = \partial_i u \wedge_{k-1}^l \partial_i v \quad (i < k),\\
&\partial_k(u \wedge_k^l v) = v,\\
&\partial_{k+2}(u \wedge_k^1 v) = u,\\
&\partial_{i+1}(u \wedge_k^l v) = u \wedge_k^{l-1} \partial_i v \quad (l > 1,\ k < i \leq k+l),\\
&\partial_{i+1}(u \wedge_k^l v) = \partial_{i-l+1} u \wedge_k^l \partial_i v \quad (k+l < i).
\end{aligned}$$

Proof. We have $x \wedge_k y$ defined if and only if $\partial_k x = \partial_{k+1} y$, and we have $\partial_k(x \wedge_k y) = y$ when $x \wedge_k y$ is defined; therefore $u \wedge_k^l v$ is defined if and only if $\partial_k u = \partial_{k+1}^l v$. The formulae for the faces are proved by induction on l, using Definition 6.1(1) when $l = 1$ and using the formula

$$u \wedge_k^l v = (u \wedge_k^{l-1} \partial_{k+1} v) \wedge_k v$$

when $l > 1$. $\square$

For the expressions $\Lambda^r(u_{r-1}, \ldots, u_0, v)$ the domains of definition are as follows.

Proposition 7.2. *An expression $\Lambda^r(u_{r-1}, \ldots, u_0, v)$ exists in a set with complicial identities if and only*

$$\partial_p u_p = \Lambda^p(u_{p-1}, \ldots, u_0, \partial_{p+1}^{r-p} v)$$

for $0 \leq p < r$.

Proof. For $0 \leq p \leq r$ let

$$v_p = \partial_p(u_{p-1} \wedge_{p-1}^{r-p+1}\ )\partial_{p-1}(u_{p-2} \wedge_{p-2}^{r-p+2}\ ) \ldots \partial_1(u_0 \wedge_0^r\ )v,$$

so that $\Lambda^r(u_{r-1}, \ldots, u_0, v) = v_r$. For $0 \leq p < r$, by Proposition 7.1, v_{p+1} exists if and only if $\partial_p u_p = \partial_{p+1}^{r-p} v_p$; it therefore suffices to show that

$$\partial_{p+1}^{r-p} v_p = \Lambda^p(u_{p-1}, \ldots, u_0, \partial_{p+1}^{r-p} v).$$

But for $0 < q \leq p$ it follows from Proposition 7.1 that

$$\partial_{p+1}^{r-p}\partial_q(u_{q-1} \wedge_{q-1}^{r-q+1}\ ) = \partial_q \partial_{p+2}^{r-p}(u_{q-1} \wedge_{q-1}^{r-q+1}\ ) = \partial_q(u_{q-1} \wedge_{q-1}^{p-q+1}\ )\partial_{p+1}^{r-p},$$

and this gives the result. $\square$

Nest we compute the faces of these expressions.

Proposition 7.3. *In a set with complicial identities the following equalities are valid whenever their left sides are defined:*

$$\begin{aligned}
&\partial_i\Lambda^r(u_{r-1},\ldots,u_0,v) = \Lambda^{r-1}(\partial_i u_{r-1},\ldots,\partial_i u_{i+1},u_{i-1},\ldots,u_0,\partial_i v) \quad (i<r),\\
&\partial_0\Lambda^0(v) = \partial_0 v,\\
&\partial_r\Lambda^r(u_{r-1},\ldots,u_0,v) = \partial_r u_{r-1} \quad (r>0),\\
&\partial_i\Lambda^r(u_{r-1},\ldots,u_0,v) = \Lambda^r(\partial_i u_{r-1},\partial_{i-1}u_{r-2},\ldots,\partial_{i-r+1}u_0,\partial_i v) \quad (i>r).
\end{aligned}$$

Proof. Suppose first that $i<r$. Using Proposition 7.1, for $r\geq p>i+1$ we get

$$\partial_i\partial_p(u_{p-1}\wedge_{p-1}^{r-p+1}\ ) = \partial_{p-1}\partial_i(u_{p-1}\wedge_{p-1}^{r-p+1}\ ) = \partial_{p-1}(\partial_i u_{p-1}\wedge_{p-2}^{r-p+1}\ )\partial_i,$$

we then get

$$\partial_i\partial_{i+1}(u_i\wedge_i^{r-i}\ ) = \partial_i\partial_i(u_i\wedge_i^{r-i}\ ) = \partial_i,$$

and for $i\geq p>0$ we get

$$\partial_i\partial_p(u_{p-1}\wedge_{p-1}^{r-p+1}\ ) = \partial_p\partial_{i+1}(u_{p-1}\wedge_{p-1}^{r-p+1}\ ) = \partial_p(u_{p-1}\wedge_{p-1}^{r-p}\ )\partial_i;$$

therefore

$$\partial_i\Lambda^r(u_{r-1},\ldots,u_0,v) = \Lambda^{r-1}(\partial_i u_{r-1},\ldots,\partial_i u_{i+1},u_{i-1},\ldots,u_0,\partial_i v).$$

It is obvious that $\partial_0\Lambda^0(v)=\partial_0 v$, because $\Lambda^0(v)=v$.

For $r>0$ the equality $\partial_r\Lambda^r(u_{r-1},\ldots,u_0,v)=\partial_r u_{r-1}$ is immediate from Proposition 7.1.

Finally, suppose that $i>r$. For $r\geq p>0$ it follows from Proposition 7.1 that

$$\partial_i\partial_p(u_{p-1}\wedge_{p-1}^{r-p+1}\ ) = \partial_p\partial_{i+1}(u_{p-1}\wedge_{p-1}^{r-p+1}\ ) = \partial_p(\partial_{i-r+p}u_{p-1}\wedge_{p-1}^{r-p+1}\ )\partial_i;$$

therefore

$$\partial_i\Lambda^r(u_{r-1},\ldots,u_0,v) = \Lambda^r(\partial_i u_{r-1},\partial_{i-1}u_{r-2},\ldots,\partial_{i-r+1}u_0,\partial_i v).$$

□

Next we give a collapsing result.

Proposition 7.4. *If x is an element of dimension at least r in a set with complicial identities, then*

$$x = \Lambda^r(\epsilon_{r-1}\partial_r x,\ \epsilon_{r-2}\partial_{r-1}^2 x,\ \ldots,\ \epsilon_0\partial_1^r x,\ x).$$

Proof. Repeated applications of Definition 6.1(2) show that for $0\leq p<r$ we have

$$\partial_{p+1}(\epsilon_p\partial_{p+1}^{r-p}x\wedge_p^{r-p}x) = \partial_{p+1}\epsilon_p x = x.$$

The result follows. □

In the rest of this section, we aim to find conditions implying that elements of the form $\Lambda^r(u_{r-1},\ldots,u_0,v)$ are wedges.

Proposition 7.5. *Let $u\wedge_k v$ be a wedge in a set with complicial identities. If $i=k-1$ or $i=k$ and if $u,v\in\operatorname{im}\wedge_i$ then $\partial_{k+1}(u\wedge_k v)\in\operatorname{im}\wedge_i$.*

Proof. Suppose that $u = u' \wedge_{k-1} u''$ and $v = v' \wedge_{k-1} v''$. The existence of the wedge $u \wedge_k v$ implies that $v' = \partial_{k+1} v = \partial_k u = \partial_k (u' \wedge_{k-1} u'')$; therefore

$$u \wedge_k v = (u' \wedge_{k-1} u'') \wedge_k [\partial_k (u' \wedge_{k-1} u'') \wedge_{k-1} v''].$$

It now follows from Definition 6.1(5) that

$$\partial_{k+1}(u \wedge_k v) = u' \wedge_{k-1} \partial_k (u'' \wedge_{k-1} v'').$$

If $u = u' \wedge_k u''$ and $v = v' \wedge_k v''$, then it similarly follows that $u'' = \partial_{k+1}(v' \wedge_k v'')$ and that

$$\partial_{k+1}(u \wedge_k v) = \partial_{k+1}(u' \wedge_k v') \wedge_k v''.$$

□

Proposition 7.6. *In a set with complicial identities, let A be an element of the form $x \wedge_{i+1} y$ or $y \wedge_i z$ or $\partial_{i+2}[(x \wedge_{i+1} y) \wedge_{i+1} (y' \wedge_i z)]$. Then elements of the form $A \wedge_i (w \wedge_{i+1} v)$ are in $\operatorname{im} \wedge_{i+2}$, and elements of the form $(u \wedge_i w) \wedge_{i+2} A$ are in $\operatorname{im} \wedge_i$.*

Proof. Note first that A must be m-dimensional with $0 \le i \le m-3$, because of the existence of $A \wedge_i (w \wedge_{i+1} v)$ or of $(u \wedge_i w) \wedge_{i+2} A$.

Suppose that $B = A \wedge_i (w \wedge_{i+1} v)$; then $\partial_i A = \partial_{i+1}(w \wedge_{i+1} v) = v$, so

$$B = A \wedge_i (w \wedge_{i+1} \partial_i A).$$

Because of Definition 6.1(6), it suffices to show that A is of the form

$$\partial_{i+2}[(x \wedge_{i+1} y) \wedge_{i+1} (y \wedge_i z)],$$

and we do this as follows: if $A = x \wedge_{i+1} y$ then

$$\begin{aligned} A &= \partial_{i+2} \epsilon_{i+2} A \\ &= \partial_{i+2}(A \wedge_{i+1} \epsilon_{i+1} \partial_{i+1} A) \\ &= \partial_{i+2}(A \wedge_{i+1} \epsilon_{i+1} y) \\ &= \partial_{i+2}[(x \wedge_{i+1} y) \wedge_{i+1} (y \wedge_i \epsilon_i \partial_i y)]; \end{aligned}$$

if $A = y \wedge_i z$ then similarly

$$A = \partial_{i+2}[(\epsilon_{i+1} \partial_{i+2} y \wedge_{i+1} y) \wedge_{i+1} (y \wedge_i z)];$$

if $A = \partial_{i+2}[(x \wedge_{i+1} y) \wedge_{i+1} (y' \wedge_i z)]$ then $y = \partial_{i+1}(x \wedge_{i+1} y) = \partial_{i+2}(y' \wedge_i z) = y'$, so

$$A = \partial_{i+2}[(x \wedge_{i+1} y) \wedge_{i+1} (y \wedge_i z)].$$

The argument for elements of the form $(u \wedge_i w) \wedge_{i+2} A$ is similar. □

Proposition 7.7. *In a set with complicial identities, if $r \ge i+3$ then*

$$\epsilon_r (x \wedge_i y) = \epsilon_{r-1} x \wedge_i \epsilon_{r-1} y$$

whenever $x \wedge_i y$ is defined.

Proof. The proof is by induction on r. We use Definition 6.1(1) and (2) repeatedly.

In the case $r = i+3$ we have

$$\begin{aligned} \epsilon_{i+3}(x \wedge_i y) &= (x \wedge_i y) \wedge_{i+2} \epsilon_{i+2} \partial_{i+2} (x \wedge_i y) \\ &= (x \wedge_i y) \wedge_{i+2} \epsilon_{i+2} x \\ &= (x \wedge_i y) \wedge_{i+2} (x \wedge_{i+1} \epsilon_{i+1} \partial_{i+1} x). \end{aligned}$$

It follows from Proposition 7.6 that $\epsilon_{i+3}(x \wedge_i y)$ is in $\operatorname{im} \wedge_i$, and it then follows that

$$\begin{aligned}\epsilon_{i+3}(x \wedge_i y) &= \partial_{i+2}\epsilon_{i+3}(x \wedge_i y) \wedge_i \partial_i\epsilon_{i+3}(x \wedge_i y)\\ &= \epsilon_{i+2}\partial_{i+2}(x \wedge_i y) \wedge_i \epsilon_{i+2}\partial_i(x \wedge_i y)\\ &= \epsilon_{i+2}x \wedge_i \epsilon_{i+2}y.\end{aligned}$$

For $r > i+3$ it follows from the inductive hypothesis and Definition 6.1(7) that

$$\begin{aligned}\epsilon_r(x \wedge_i y) &= (x \wedge_i y) \wedge_{r-1} \epsilon_{r-1}\partial_{r-1}(x \wedge_i y)\\ &= (x \wedge_i y) \wedge_{r-1} \epsilon_{r-1}(\partial_{r-2}x \wedge_i \partial_{r-2}y)\\ &= (x \wedge_i y) \wedge_{r-1} (\epsilon_{r-2}\partial_{r-2}x \wedge_i \epsilon_{r-2}\partial_{r-2}y)\\ &= (x \wedge_{r-2} \epsilon_{r-2}\partial_{r-2}x) \wedge_i (y \wedge_{r-2} \epsilon_{r-2}\partial_{r-2}y)\\ &= \epsilon_{r-1}x \wedge_i \epsilon_{r-1}y.\end{aligned}$$

This completes the proof. □

Proposition 7.8. *In a set with complicial identities, an element of the form*

$$(u \wedge_i u') \wedge_k (v \wedge_i v')$$

is in $\operatorname{im} \wedge_i$ *for* $i \leq k$ *and is in* $\operatorname{im} \wedge_{i+1}$ *for* $i \geq k-1$.

Proof. For $i = k-1$ and for $i = k$ the results are trivial or are contained in Definition 6.1(3) and (4); for $i = k-2$ and for $i = k+1$ the results follow from Proposition 7.6; for $i \leq k-3$ and for $i \geq k+2$ the results are contained in Definition 6.1(7). □

Proposition 7.9. *In a set with complicial identities, an element of the form*

$$\partial_{i+3}[u \wedge^l_{i+2} \partial_{i+2}(u' \wedge^{l+1}_{i+1} v)]$$

is in $\operatorname{im} \wedge_i$ *if* u *and* v *are in* $\operatorname{im} \wedge_i$.

Proof. Let

$$z = u \wedge^l_{i+2} \partial_{i+2}(u' \wedge^{l+1}_{i+1} v).$$

Because of Definition 6.1(1), it suffices to show that $z \in \operatorname{im} \wedge_i$. We will do this by induction on l.

Let

$$A = \partial_{i+2}(u' \wedge^{l+1}_{i+1} v)$$

and let

$$u'' = \begin{cases} u & (l = 1),\\ u \wedge^{l-1}_{i+2} \partial_{i+3}A & (l > 1),\end{cases}$$

so that

$$z = u'' \wedge_{i+2} A.$$

Note that

$$A = \partial_{i+2}[(u' \wedge^l_{i+1} \partial_{i+2}v) \wedge_{i+1} v]$$

with $u' \wedge^l_{i+1} \partial_{i+2}v \in \operatorname{im} \wedge_{i+1}$ by the definition of $\wedge^l_{i+1}$ and with $v \in \operatorname{im} \wedge_i$ by hypothesis. By Proposition 7.6, to show that $z \in \operatorname{im} \wedge_i$ it suffices to show that $u'' \in \operatorname{im} \wedge_i$.

Suppose that $l = 1$. Then $u'' = u$, so $u'' \in \operatorname{im} \wedge_i$ by hypothesis.

Suppose that $l > 1$. Using Proposition 7.1 we get

$$\begin{aligned} u'' &= u \wedge_{i+2}^{l-1} \partial_{i+3} A \\ &= u \wedge_{i+2}^{l-1} \partial_{i+2}\partial_{i+4}(u' \wedge_{i+1}^{l+1} v) \\ &= u \wedge_{i+2}^{l-1} \partial_{i+2}(u' \wedge_{i+1}^{l} \partial_{i+3} v). \end{aligned}$$

Since $v \in \operatorname{im} \wedge_i$, it follows from Definition 6.1(1) that $\partial_{i+3} v \in \operatorname{im} \wedge_i$, and it then follows from the inductive hypothesis that $u'' \in \operatorname{im} \wedge_i$.

This completes the proof. □

Proposition 7.10. *In a set with complicial identities, let v' be an element of the form $\partial_{k+1}(u \wedge_k^l v)$ with $v \in \operatorname{im} \wedge_i$. Then $v' \in \operatorname{im} \wedge_i$ whenever one of the following sets of conditions is satisfied:*

(1) *$i \leq k-2$ and $u \in \operatorname{im} \wedge_i$;*
(2) *$k \leq i < k+l$ and $l \geq 2$;*
(3) *$k+l \leq i$ and $u \in \operatorname{im} \wedge_{i-l+1}$.*

Proof. Let $u_0 = u$, and for $0 < j \leq l$ let

$$u_j = u_{j-1} \wedge_k \partial_{k+1}^{l-j} v;$$

we must show in each case that $\partial_{k+1} u_l \in \operatorname{im} \wedge_i$.

(1) Since $v \in \operatorname{im} \wedge_i$ with $i \leq k-2$, it follows from Definition 6.1(1) that $\partial_{k+1}^{l-j} v \in \operatorname{im} \wedge_i$ for $0 \leq j < l$. Since $u_0 \in \operatorname{im} \wedge_i$, it follows from Proposition 7.8 that $u_1, \ldots, u_l \in \operatorname{im} \wedge_i$, and it then follows from Definition 6.1(1) that $\partial_{k+1} u_l \in \operatorname{im} \wedge_i$.

(2) Suppose first that $i = k$; we must show that

$$\partial_{k+1}(u_{l-1} \wedge_k v) \in \operatorname{im} \wedge_k.$$

But $u_{l-1} \in \operatorname{im} \wedge_k$ because $l \geq 2$, and $v \in \operatorname{im} \wedge_k$ because $i = k$, so the result follows from Proposition 7.5.

Now suppose that $k < i < k+l$. We have $u_{k+l-i} \in \operatorname{im} \wedge_k$ because $i < k+l$, and we have $\partial_{k+1}^{i-k-1} v \in \operatorname{im} \wedge_{k+1}$ by Definition 6.1(1); therefore $u_{k+l-i+1} \in \operatorname{im} \wedge_{k+2}$ by Proposition 7.6. We also have $\partial_{k+1}^{i-k-2} v \in \operatorname{im} \wedge_{k+2}$ by Definition 6.1(1), so $u_{k+l-i+2} \in \operatorname{im} \wedge_{k+3}$ by Proposition 7.8. By repeating the use of Proposition 7.8, we eventually get $u_l \in \operatorname{im} \wedge_{i+1}$. Definition 6.1(1) now gives us $\partial_{k+1} u_l \in \operatorname{im} \wedge_i$.

(3) In this case $u_0 \in \operatorname{im} \wedge_{i-l+1}$ by hypothesis and $\partial_{k+1}^{l-1} v \in \operatorname{im} \wedge_{i-l+1}$ by Definition 6.1(1); therefore $u_1 \in \operatorname{im} \wedge_{i-l+2}$ by Proposition 7.8. In the same way $u_2 \in \operatorname{im} \wedge_{i-l+3}$, and so on. Eventually we get $u_l \in \operatorname{im} \wedge_{i+1}$. As before, Definition 6.1(1) now gives us $\partial_{k+1} u_l \in \operatorname{im} \wedge_i$. □

We conclude with a result converse to Proposition 5.2.

Proposition 7.11. *Let x be an element of the form*

$$x = \Lambda^r(u_{r-1}, \ldots, u_0, \epsilon_r^s w)$$

in a set with complicial identities.

If $i \leq r-3$, if $w \in \operatorname{im} \wedge_i$, and if $u_p \in \operatorname{im} \wedge_i$ for $i+2 \leq p < r$, then $x \in \operatorname{im} \wedge_i$.

If $w \in \operatorname{im} \epsilon_{r-2}$ and $u_{r-1} \in \operatorname{im} \wedge_{r-2}$ then $x \in \operatorname{im} \wedge_{r-2}$.

If $s \geq 1$ and $u_{r-1} \in \operatorname{im} \wedge_{r-1}$ then $x \in \operatorname{im} \wedge_{r-1}$.

If $r \leq i \leq r+s-2$, and if $u_p \in \operatorname{im} \wedge_{i-r+p+1}$ for $0 \leq p < r$, then $x \in \operatorname{im} \wedge_i$.

Proof. Let $v_0 = \epsilon_r^s w$, and for $0 \le p < r$ let

$$v_{p+1} = \partial_{p+1}(u_p \wedge_p^{r-p} v_p);$$

thus $x = v_r$. We claim that $v_p \in \operatorname{im} \wedge_i$ for $0 \le p \le r$, except possibly for cases with $i \le r-3$ and $p = i+2$.

First we suppose that $i \le r-3$, that $w \in \operatorname{im} \wedge_i$, and that $u_p \in \operatorname{im} \wedge_i$ for $i+2 \le p < r$. We have $v_0 \in \operatorname{im} \wedge_i$ by Proposition 7.7, and we get

$$v_1 \in \operatorname{im} \wedge_i, \ \dots, \ v_{i+1} \in \operatorname{im} \wedge_i$$

by repeated applications of Proposition 7.10(2). We now have

$$v_{i+3} = \partial_{i+3}[u_{i+2} \wedge_{i+2}^{r-i-2} \partial_{i+2}(u_{i+1} \wedge_{i+1}^{r-i-1} v_{i+1})]$$

with u_{i+2} and v_{i+1} in $\operatorname{im} \wedge_i$, so $v_{i+3} \in \operatorname{im} \wedge_i$ by Proposition 7.9. Since $u_p \in \operatorname{im} \wedge_i$ for $i+3 \le p < r$, repeated applications of Proposition 7.10(1) now show that the elements

$$v_{i+4}, v_{i+5}, \dots, v_r$$

are in $\operatorname{im} \wedge_i$.

Next we suppose that $w \in \operatorname{im} \epsilon_{r-2}$ and $u_{r-1} \in \operatorname{im} \wedge_{r-2}$. We have $v_0 \in \operatorname{im} \epsilon_{r-2}$ because $\epsilon_r^s \epsilon_{r-2} = \epsilon_{r-2} \epsilon_{r-1}^s$; hence, by Definition 6.1(2), $v_0 \in \operatorname{im} \wedge_{r-2}$. As in the previous case, we get

$$v_1 \in \operatorname{im} \wedge_{r-2}, \ \dots, \ v_{r-1} \in \operatorname{im} \wedge_{r-2}.$$

Since $v_r = \partial_r(u_{r-1} \wedge_{r-1} v_{r-1})$ and since $u_{r-1} \in \operatorname{im} \wedge_{r-2}$, it follows from Proposition 7.5 that $v_r \in \operatorname{im} \wedge_{r-2}$.

Next we suppose that $s \ge 1$ and $u_{r-1} \in \operatorname{im} \wedge_{r-1}$. Then $v_0 \in \operatorname{im} \epsilon_r$, and it follows from Definition 6.1(2) that $v_0 \in \operatorname{im} \wedge_{r-1}$. The rest of the argument is as in the previous case.

Finally we suppose that $r \le i \le r+s-2$ and that $u_p \in \operatorname{im} \wedge_{i-r+p+1}$ for $0 \le p < r$. We get $v_0 = \epsilon_{i+1} \epsilon_r^{s-1} w$; therefore $v_0 \in \operatorname{im} \wedge_i$ by Definition 6.1(2). We then get

$$v_1 \in \operatorname{im} \wedge_i, \ \dots, \ v_r \in \operatorname{im} \wedge_i$$

by repeated applications of Proposition 7.10(3).

This completes the proof. □

8. Freeness

We have shown that $\mathcal{O}(-,n)$ is a set with complicial identities generated by the identity morphism ι_n (see Proposition 6.6 and Theorem 4.12). We will now show that $\mathcal{O}(-,n)$ is freely generated by ι_n; that is to say, given an n-dimensional element u in a set with complicial identities U, we will show that there is a unique morphism $f\colon \mathcal{O}(-,n) \to U$ with $f\iota_n = u$.

We will construct morphisms on $\mathcal{O}(-,n)$ by combining suitable functions on subsets of $\mathcal{O}(-,n)$. The functions concerned are called *partial morphisms*, and are defined as follows.

Definition 8.1. Let S be a subset of a set with complicial identities, let U be a set with complicial identities, and let k be a nonnegative integer. Then a *partial morphism of degree k* from S to U is a function $f\colon S \to U$ which increases degrees by k and which satisfies the following conditions.

(1) If x is an m-dimensional member of S with $m > 0$ and if $0 \le i \le m$, then $\partial_i x \in S$ and $f\partial_i x = \partial_i f x$.

(2) If x is a 1-dimensional member of S and $x \in \operatorname{im} \epsilon_0$ then $fx \in \operatorname{im} \epsilon_0$.
(3) If $x \in S$ and $x \in \operatorname{im} \wedge_i$ then $fx \in \operatorname{im} \wedge_i$.

Partial morphisms have all the good properties that one might expect.

Proposition 8.2. *Let $f \colon S \to U$ be a partial morphism and let x be a member of S.*
(1) If $x = \epsilon_i y$ for some y, then $y \in S$ and $fx = \epsilon_i fy$.
(2) If $x = y \wedge_i z$ for some y and z, then $y, z \in S$ and $fx = fy \wedge_i fz$.

Proof. We prove these statements in reverse order.

(2) Suppose that $x = y \wedge_i z$. Then $y = \partial_{i+2} x$ and $z = \partial_i x$ by Definition 6.1(1); therefore $y, z \in S$. Since also $fx \in \operatorname{im} \wedge_i$, it then follows that

$$fx = \partial_{i+2} fx \wedge_i \partial_i fx = f\partial_{i+2} x \wedge_i f\partial_i x = fy \wedge_i fz.$$

(1) Suppose that $x = \epsilon_i y$. Then $y = \partial_i x$, so that $y \in S$. We will now prove that $fx = \epsilon_i fy$ by induction on the dimension of x.

Suppose that $x = \epsilon_0 y$ and x is 1-dimensional. Then $fx \in \operatorname{im} \epsilon_0$, and it follows that

$$fx = \epsilon_0 \partial_0 fx = \epsilon_0 f\partial_0 x = \epsilon_0 fy.$$

Suppose that $x = \epsilon_0 y$ and that the dimension of x is greater than 1. Then $x = \epsilon_0 \partial_1 y \wedge_0 y$ by Definition 6.1(2). From the inductive hypothesis and from what we have already proved, it now follows that

$$fx = f\epsilon_0 \partial_1 y \wedge_0 fy = \epsilon_0 f\partial_1 y \wedge_0 y = \epsilon_0 \partial_1 fy \wedge_0 fy = \epsilon_0 fy.$$

Finally suppose that $x = \epsilon_i y$ with $i > 0$. Then $x = y \wedge_{i-1} \epsilon_{i-1} \partial_{i-1} y$ by Definition 6.1(2), and it follows as in the previous case that $fx = \epsilon_i fy$.

This completes the proof. □

For partial morphisms on subsets of $\mathcal{O}(-, n)$ we deduce the following result.

Proposition 8.3. *Let f be a partial morphism on a subset S of $\mathcal{O}(-, n)$, let*

$$x = \Lambda^r(u_{r-1}, \ldots, u_0, v)$$

be defined in $\mathcal{O}(-, n)$, let the terminus of v be t, and let the corank of v be s. Suppose that S contains all morphisms with terminus t; alternatively, suppose that $\operatorname{rank} v \geq r$ and that S contains all morphisms with terminus t and corank s. Then the morphisms $x, u_{r-1}, \ldots, u_0, v$ are members of S, and

$$fx = \Lambda^r(fu_{r-1}, \ldots, fu_0, fv).$$

Proof. This follows from Propositions 5.3 and 8.2. □

We will construct partial morphisms on subsets of $\mathcal{O}(-, n)$ by recursion on terminus and corank. For $t \geq 0$ let S_t be the subset of $\mathcal{O}(-, n)$ consisting of the morphisms of terminus at most t. For $t \geq 0$ and $s \geq -1$ let S_t^s be the subset of $\mathcal{O}(-, n)$ consisting of the following morphisms: the morphisms with terminus less than t; the morphism $\partial_0^t \partial_{t+1}^{n-t} \iota_n$; the t-cones, in the case that $t > 0$; the morphisms with terminus t and with corank at most s. We now proceed as follows.

Lemma 8.4. *Let u be an element in a set with complicial identities U. Then there is a partial morphism $f \colon S_0^{-1} \to U$ such that $f\partial_1^n \iota_n = u$.*

Proof. The only morphism in S_0^{-1} is the zero-dimensional morphism $\partial_1^n\iota_n$. The condition $f\partial_1^n\iota_n = u$ therefore defines a function $f\colon S_0^{-1}\to U$, and this function is trivially a partial morphism. $\square$

Lemma 8.5. *For $t>0$, let $F\colon S_{t-1}\to U$ be a partial morphism of degree $k>0$. Then there is a partial morphism $f\colon S_t^{-1}\to U$ of degree $k-1$ such that $f\partial_{t+1}^{n-t}\iota_n = F\partial_t^{n-t+1}\iota_n$.*

Proof. We define a function $f\colon S_t^{-1}\to U$ as follows: if x is a member of $\mathcal{O}(m,n)$ with terminus less than t, then

$$fx = \partial_{m+1}Fx;$$

if $x = \partial_0^t\partial_{t+1}^{n-t}\iota_n$ then

$$fx = \partial_0^t F\partial_t^{n-t+1}\iota_n;$$

if x is a t-cone in $\mathcal{O}(m+1,n)$ then $\partial_{m+1}x$ has terminus less than t, and we make the definition

$$fx = F\partial_{m+1}x.$$

It is clear that f increases degrees by $k-1$ and that

$$f\partial_{t+1}^{n-t}\iota_n = F\partial_t\partial_{t+1}^{n-t}\iota_n = F\partial_t^{n-t+1}\iota_n;$$

it therefore remains to verify conditions (1)–(3) of Definition 8.1.

(1) Let $\partial_i x$ be a face of an m-dimensional member of S_{t-1}, so that $m>0$ and $0\le i\le m$. Then $\partial_i x$ is an $(m-1)$-dimensional member of S_{t-1}, so that $\partial_i x\in S_t^{-1}$, and

$$f\partial_i x = \partial_m F\partial_i x = \partial_m\partial_i Fx = \partial_i\partial_{m+1}Fx = \partial_i fx.$$

The zero-dimensional morphism $\partial_0^t\partial_{t+1}^{n-t}\iota_n$ does not have any faces.

Let x be a 1-dimensional t-cone, and consider the face $\partial_0 x$. This is the zero-dimensional morphism $\partial_0^t\partial_{t+1}^{n-1}\iota_n$; therefore $\partial_0 x\in S_t^{-1}$ and $f\partial_0 x = \partial_0^t F\partial_t^{n-t+1}\iota_n$. The zero-dimensional morphism $\partial_1 x$ must have the form $[0]\mapsto[j]$ with $0\le j\le t-1$, and it can be expressed as $\partial_0^j\partial_{j+1}^{t-j-1}\partial_t^{n-t+1}\iota_n$. Therefore

$$\partial_0 fx = \partial_0 F\partial_1 x = \partial_0\partial_0^j\partial_{j+1}^{t-j-1}F\partial_t^{n-t+1}\iota_n = \partial_0^t F\partial_t^{n-t+1}\iota_n,$$

and it follows that $f\partial_0 x = \partial_0 fx$.

Let $\partial_i x$ be a face of an $(m+1)$-dimensional t-cone such that $m>0$ and $0\le i\le m$. Then $\partial_i x$ is an m-dimensional t-cone, so that $\partial_i x\in S_t^{-1}$ and

$$f\partial_i x = F\partial_m\partial_i x = F\partial_i\partial_{m+1}x = \partial_i F\partial_{m+1}x = \partial_i fx.$$

Finally, let x be an $(m+1)$-dimensional t-cone with $m\ge 0$ and consider the face $\partial_{m+1}x$. This is an m-dimensional member of S_{t-1}; therefore $\partial_{m+1}x\in S_t^{-1}$ and

$$f\partial_{m+1}x = \partial_{m+1}F\partial_{m+1}x = \partial_{m+1}fx.$$

(2) Let x be a 1-dimensional member of S_t^{-1} lying in the image of ϵ_0. Then $x\in S_{t-1}$, so that $fx = \partial_2 Fx$, and we also have $x = \epsilon_0\partial_0 x$. It now follows from Proposition 8.2 that $fx\in\operatorname{im}\epsilon_0$, because

$$fx = \partial_2 F\epsilon_0\partial_0 x = \partial_2\epsilon_0 F\partial_0 x = \epsilon_0\partial_1 F\partial_0 x.$$

(3) Let x be an m-dimensional member of S_{t-1} lying in the image of $\wedge_i$. Then $fx = \partial_{m+1}Fx$. We also have $x = \partial_{i+2}x\wedge_i\partial_i x$ by Definition 6.1(1), and we must

have $0 \le i \le m-2$. It now follows from Proposition 8.2 and Definition 6.1(1) that $fx \in \operatorname{im} \wedge_i$, because

$$fx = \partial_{m+1} F(\partial_{i+2}x \wedge_i \partial_i x) = \partial_{m+1}(F\partial_{i+2}x \wedge_i F\partial_i x) = \partial_m F\partial_{i+2}x \wedge_i \partial_m F\partial_i x.$$

The zero-dimensional morphism $\partial_0^t \partial_{t+1}^{n-t}\iota_n$ cannot belong to the image of a wedge operation.

Let x be an $(m+1)$-dimensional t-cone, and suppose that $x \in \operatorname{im} \wedge_i$ with $0 \le i \le m-2$. Then $fx \in \operatorname{im} \wedge_i$ as before, because

$$fx = F\partial_{m+1}(\partial_{i+2}x \wedge_i \partial_i x) = F(\partial_m\partial_{i+2}x \wedge_i \partial_m\partial_i x) = F\partial_m\partial_{i+2}x \wedge_i F\partial_m\partial_i x.$$

Finally, suppose that x is an $(m+1)$-dimensional t-cone lying in the image of $\wedge_{m-1}$. We must have $x = \epsilon_{m-1}\partial_{m-1}x$ (see Propositions 3.8 and 3.4). It now follows from Proposition 8.2 and Definition 6.1(2) that $fx \in \operatorname{im} \wedge_{m-1}$, because

$$fx = F\partial_{m+1}\epsilon_{m-1}\partial_{m-1}x = F\epsilon_{m-1}\partial_m\partial_{m-1}x = \epsilon_{m-1}F\partial_m\partial_{m-1}x.$$

This completes the proof. □

Lemma 8.6. *For $t \ge 0$ and $s \ge 0$, if f is a partial morphism on S_t^{s-1}, then there is a partial morphism f' on S_t^s which is an extension of f.*

Proof. Let x be a morphism in S_t^s with terminus less than t or with terminus t and corank less than s; then x belongs to S_t^{s-1} and we make the definition

$$f'x = fx.$$

Now let x be a morphism with terminus t and corank s, and let the rank of x be r. According to Theorem 4.11, x has a canonical form given by

$$x = \Lambda^r(\alpha_{r-1}x, \dots, \alpha_0 x, \epsilon_r^s \gamma x).$$

According to Propositions 4.8 and 4.6, the morphisms $\alpha_p x$ and γx are members of S_t^{s-1}, and we will obtain $f'x$ by applying f to each of these morphisms.

In order to show that this is possible, we use Proposition 7.2. For $0 \le p < r$ it follows from this proposition and from the existence of the canonical form that

$$\partial_p \alpha_p x = \Lambda^p(\alpha_{p-1}x, \dots, \alpha_0 x, \partial_{p+1}^{r-p}\epsilon_r^s \gamma x).$$

If $s = 0$ then the argument $\partial_{p+1}^{r-p}\epsilon_r^s\gamma x$ has terminus less than t; if $s > 0$ then it has terminus t, rank $p+1$ and corank $s-1$; in both cases, it follows from Proposition 8.3 that

$$\partial_p f\alpha_p x = f\partial_p\alpha_p x = \Lambda^p(f\alpha_{p-1}x, \dots, f\alpha_0 x, f\partial_{p+1}^{r-p}\epsilon_r^s\gamma x).$$

If now $s = 0$ then

$$f\partial_{p+1}^{r-p}\epsilon_r^s\gamma x = f\partial_{p+1}^{r-p}\gamma x = \partial_{p+1}^{r-p} f\gamma x = \partial_{p+1}^{r-p}\epsilon_r^s\gamma x;$$

if $s > 0$ then $\epsilon_r^{s-1}\gamma x \in S_t^{s-1}$ and we get

$$f\partial_{p+1}^{r-p}\epsilon_r^s\gamma x = f\partial_{p+1}^{r-p-1}\epsilon_r^{s-1}\gamma x = \partial_{p+1}^{r-p-1}\epsilon_r^{s-1} f\gamma x = \partial_{p+1}^{r-p}\epsilon_r^s f\gamma x;$$

in any case we get

$$\partial_p f\alpha_p x = \Lambda^p(f\alpha_{p-1}x, \dots, f\alpha_0 x, \partial_{p+1}^{r-p}\epsilon_r^s f\gamma x).$$

Because of Proposition 7.2, we can now define $f'x$ by the formula

$$f'x = \Lambda^r(f\alpha_{r-1}x, \dots, f\alpha_0 x, \epsilon_r^s f\gamma x).$$

We will now show that f' is an extension of f. For $s > 0$ there is nothing to do, because there are no morphisms of terminus t and corank s in S_t^{s-1}. In the case $s = 1$ we must show that $f'x = fx$ when $x = \partial_0^t \partial_{t+1}^{n-t+1} \iota_n$ or x is a t-cone. To do this, let the rank of x be r. It then follows from Example 4.7 that

$$f'x = \Lambda^r(f\epsilon_{r-1}\partial_r x, \dots, f\epsilon_0 \partial_1^r x, fx),$$

it follows from Proposition 8.2 that

$$f'x = \Lambda^r(\epsilon_{r-1}\partial_r fx, \dots, \epsilon_0 \partial_1^r fx, fx),$$

and it follows from Proposition 7.4 that $f'x = fx$ as required.

It remains to show that f' is a partial morphism by verifying the conditions of Definition 8.1. It clearly suffices to consider morphisms with terminus t and corank s, and we argue as follows.

(1) Let x be a morphism with terminus t, with corank s and with rank r, and consider a face $\partial_i x$ with $0 \leq i < r$. In this case $\partial_i x$ has terminus t, corank s and rank $r - 1$, so that $\partial_i x \in S_t^s$ and it follows from Proposition 5.1 that

$$f'\partial_i x = \Lambda^{r-1}(f\partial_i \alpha_{r-1} x, \dots, f\partial_i \alpha_{i+1} x, f\alpha_{i-1} x, \dots, f\alpha_0 x, \epsilon_{r-1}^s f\partial_i \gamma x).$$

Since $f\partial_i \alpha_p x = \partial_i f\alpha_p x$ for $i < p < r$ and since

$$\epsilon_{r-1}^s f\partial_i \gamma x = \epsilon_{r-1}^s \partial_i f\gamma x = \partial_i \epsilon_r^s f\gamma x,$$

it now follows from Proposition 7.3 that

$$f'\partial_i x = \partial_i \Lambda^r(f\alpha_{r-1} x, \dots, f\alpha_0 x, \epsilon_r^s f\gamma x) = \partial_i f'x.$$

Now let x be a morphism with terminus t, with corank s and with rank zero, and consider the face $\partial_0 x$. This exists only in cases with $s > 0$. It has terminus t, corank $s - 1$ and rank zero, and it therefore belongs to S_t^{s-1}. It follows that $\partial_0 x \in S_t^s$. It also follows from Theorem 4.11 and Proposition 7.3 that

$$f'\partial_0 x = f\partial_0 x = f\partial_0 \Lambda^0(\epsilon_0^s \gamma x) = f\partial_0 \epsilon_0^s \gamma x = f\epsilon_0^{s-1} \gamma x = \epsilon_0^{s-1} f\gamma x$$

and that

$$\partial_0 f'x = \partial_0 \Lambda^0(\epsilon_0^s f\gamma x) = \partial_0 \epsilon_0^s f\gamma x = \epsilon_0^{s-1} f\gamma x;$$

therefore $f'\partial_0 x = \partial_0 f'x$.

Now let x be a morphism with terminus t, with corank s, and with rank $r > 0$, and consider the face $\partial_r x$. If $s = 0$ then $\partial_r x$ has terminus less than t; if $s > 0$ then $\partial_r x$ has terminus t and corank $s - 1$; in both cases, $\partial_r x \in S_t^{s-1}$. It follows that $\partial_r x \in S_t^s$. It also follows from Theorem 4.11 and Proposition 7.3 that

$$f'\partial_r x = f\partial_r x = f\partial_r \Lambda^r(\alpha_{r-1} x, \dots, \alpha_0 x, \epsilon_r^s \gamma x) = f\partial_r \alpha_{r-1} x = \partial_r f\alpha_{r-1} x$$

and

$$\partial_r f'x = \partial_r \Lambda^r(f\alpha_{r-1} x, \dots, f\alpha_0 x, \epsilon_r^s f\gamma x) = \partial_r f\alpha_{r-1} x;$$

therefore $f'\partial_r x = \partial_r f'x$.

Finally, let x be a morphism with terminus t, with corank s, and with rank r, and consider $\partial_i x$ for $r < i \leq r + s$. In this case $\partial_i x$ has terminus t and corank $s - 1$, so that $\partial_i x \in S_t^{s-1}$; therefore $\partial_i x \in S_t^s$. It now follows from Theorem 4.11 and Proposition 7.3 that

$$\begin{aligned}
f'\partial_i x &= f\partial_i x \\
&= f\partial_i \Lambda^r(\alpha_{r-1} x, \dots, \alpha_0 x, \epsilon_r^s \gamma x) \\
&= f\Lambda^r(\partial_i \alpha_{r-1} x, \dots, \partial_{i-r+1} \alpha_0 x, \partial_i \epsilon_r^s \gamma x) \\
&= f\Lambda^r(\partial_i \alpha_{r-1} x, \dots, \partial_{i-r+1} \alpha_0 x, \epsilon_r^{s-1} \gamma x).
\end{aligned}$$

Here $\epsilon_r^{s-1}\gamma x$ has terminus t, rank r and corank $s-1$, so it follows from Propositions 8.3 and 8.2 that

$$\begin{aligned} f'\partial_i x &= \Lambda^r(f\partial_i\alpha_{r-1}x,\dots,f\partial_{i-r+1}\alpha_0 x, f\epsilon_r^{s-1}\gamma x)\\ &= \Lambda^r(\partial_i f\alpha_{r-1}x,\dots,\partial_{i-r+1}f\alpha_0 x, \epsilon_r^{s-1}f\gamma x). \end{aligned}$$

On the other hand, it follows from Proposition 7.3 that

$$\begin{aligned} \partial_i f'x &= \Lambda^r(\partial_i f\alpha_{r-1}x,\dots,\partial_{i-r+1}f\alpha_0 x, \partial_i\epsilon_r^{s}f\gamma x)\\ &= \Lambda^r(\partial_i f\alpha_{r-1}x,\dots,\partial_{i-r+1}f\alpha_0 x, \epsilon_r^{s-1}f\gamma x); \end{aligned}$$

therefore $f'\partial_i x = \partial_i f'x$.

(2) Suppose that x is a 1-dimensional morphism with terminus t and positive corank s lying in the image of ϵ_0. This can occur only in the case that $s=1$, and the rank of x must be zero; therefore $f'x = \Lambda^0(\epsilon_0 f\gamma x) = \epsilon_0 f\gamma x$. It follows that $f'x$ is in the image of ϵ_0.

(3) Suppose that x is a morphism with terminus t and corank s lying in the image of $\wedge_i$. Let the rank of x be r, so that $0 \le i \le r+s-2$. It follows from Proposition 5.2 that certain morphisms $\alpha_p x$ are in certain sets $\operatorname{im}\wedge_j$, and it may also follow that γx is in $\operatorname{im}\wedge_i$ or $\operatorname{im}\epsilon_i$, The images $f\alpha_p x$ and $f\gamma x$ then satisfy the same conditions, and it follows from Proposition 7.11 that $f'x \in \operatorname{im}\wedge_i$.

This completes the proof. □

From Lemmas 8.4–8.6 we get the main result.

Theorem 8.7. *For $n \ge 0$, the graded set $\mathcal{O}(-,n)$ is the set with complicial identities freely generated by the identity morphism ι_n in $\mathcal{O}(n,n)$.*

Proof. First, by Proposition 6.6, $\mathcal{O}(-,n)$ is a set with complicial identities.

Now let u be an n-dimensional element in a set with complicial identities U. We must show that there is a unique morphism from $\mathcal{O}(-,n)$ to U sending ι_n to u.

We construct a suitable morphism as follows. By Lemma 8.4 there is a partial morphism on S_0^{-1} sending $\partial_1^n\iota_n$ to u; by repeated applications of Lemma 8.6, there is a partial morphism on the entire set S_0 sending $\partial_1^n\iota_n$ to u; by Lemma 8.5, there is a partial morphism on S_1^{-1} sending $\partial_2^{n-1}\iota_n$ to u; by Lemma 8.6 there is a partial morphism on the entire set S_1 sending $\partial_2^{n-1}\iota_n$ to u; etc. Eventually we obtain a partial morphism $f\colon \mathcal{O}(-,n)\to U$ such that $f\iota_n = u$. By Proposition 8.2, f is in fact a morphism of sets with complicial identities. Since, according to Theorem 4.12, $\mathcal{O}(-,n)$ is generated by ι_n, it follows that f is the only morphism from $\mathcal{O}(-,n)$ to U with $f\iota_n = u$.

This completes the proof. □

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