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A NEW FORMULA FOR SOME LINEAR STOCHASTIC EQUATIONS WITH APPLICATIONS

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We give a representation of the solution for a stochastic linear equation of the form $X_t = Y_t + \int_{(0,t]} X_{s-}\,\mathrm{d}Z_s$ where Z is a càdlàg semimartingale and Y is a càdlàg adapted process with bounded variation on finite intervals. As an application we study the case where Y and $-Z$ are nondecreasing, jointly have stationary increments and the jumps of $-Z$ are bounded by 1. Special cases of this process are shot-noise processes, growth collapse (additive increase, multiplicative decrease) processes and clearing processes. When Y and Z are, in addition, independent Lévy processes, the resulting X is called a generalized Ornstein–Uhlenbeck process.

1. Introduction. In this paper we show that when Z is a càdlàg adapted semimartingale and Y is càdlàg adapted and with bounded variation on compact intervals, then the unique càdlàg adapted solution of $X_t = Y_t + \int_{(0,t]} X_{s-}\,\mathrm{d}Z_s$ is given via the representation $X_t = \int_{[0,t]} U_{u,t}\,\mathrm{d}Y_u$ where $U_{u,t}$ is defined by formula (2) below. This form seems to be new and we note that the integral with respect to Y is defined path-wise while the integral in the integral equation can be a stochastic integral. Of course when Y is a semimartingale, one cannot expect such a representation of the solution since $\{U_{u,t}|0 \le u \le t\}$ is not adapted as a process indexed by u.

We discuss an application to the case where Y and $-Z$ are nondecreasing processes jointly having stationary increments and subsequently specialize to cases where one or both also have independent increments (Lévy processes). This model is a generalization of both the shot-noise process as well as a growth–collapse process (e.g., see, [7, 11, 16] and references therein) or

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more generally an additive increase and multiplicative decrease process. The later have been used as models for the TCP window size in communication networks.

We note that Jacod ([8], Theorem 6.8, page 194) and Yoeurp and Yor [21] give a complete solution for the case where the integrator is a semimartingale and the driving process is càdlàg, Jaschke [9] gives a derivation for the case where the integrator does not have jumps of size -1, and Protter ([20], Theorems 52 and 53, pages 322–323) treats the case with a continuous integrator.

The literature related to generalized Ornstein–Uhlenbeck processes and their applications which are directly related to some of the special cases of the applications that we consider is huge and growing exponentially fast. We refer the reader to [1–6, 14, 15, 17–19, 22] and further references therein.

2. Main result. With respect to some standard (right continuous augmented) filtration, let $Y=\{Y_t|t\ge 0\}$ and $Z=\{Z_t|t\ge 0\}$ be two adapted càdlàg processes. Denote $Z_{0-}=0$, and for $t>0$, $Z_{t-}=\lim_{s\uparrow t} Z_s$. Set $\Delta Z_t = Z_t - Z_{t-}$ when Z is of bounded variation on compact intervals (BV); set $Z_t^c = Z_t - \sum_{0\le s\le t}\Delta Z_s$ and similarly for any other càdlàg process considered in this paper.

THEOREM 1. *Assume Y and Z are càdlàg and adapted, Y is BV and Z is a semimartingale. Then the unique càdlàg adapted solution to the equation $X_t = Y_t + \int_{(0,t]} X_{s-}\,\mathrm{d}Z_s$ is*

$$X_t = \int_{[0,t]} U_{u,t}\,\mathrm{d}Y_u, \tag{1}$$

where

$$U_{u,t} = \begin{cases} e^{Z_t - Z_u - (1/2)([Z,Z]_t^c - [Z,Z]_u^c)} \\ \qquad \times \prod_{u<s\le t} (1+\Delta Z_s) e^{-\Delta Z_s}, & 0\le u<t, \\ 1, & 0\le u=t \end{cases} \tag{2}$$

and $[Z,Z]$ is the quadratic variation process associated with Z. When Z is BV then (2) reduces to

$$U_{u,t} = \begin{cases} e^{Z_t^c - Z_u^c} \prod_{u<s\le t} (1+\Delta Z_s), & 0\le u<t, \\ 1, & 0\le u=t, \end{cases} \tag{3}$$

where Z^c is the continuous part of Z as defined earlier (rather than the continuous martingale part of Z as is customary in stochastic calculus).

PROOF. Note that with $T_0 = 0$ and for $n \geq 1$, $T_n = \inf\{t > T_{n-1} | \Delta Z_t = -1\}$, then for $T_n < u \leq t < T_{n+1}$

$$\frac{U_{T_n,t}}{U_{T_n,u-}} = U_{u,t}(1 + \Delta Z_u), \qquad \frac{U_{T_n,t}}{U_{T_n,u}} = U_{u,t}. \tag{4}$$

Also, since Y is a BV process, the covariation process $[Y, Z]$ is given via $[Y, Z]_t = \sum_{0 \leq s \leq t} \Delta Y_s \Delta Z_s$. If one follows the solution in equation (6.9) in Theorem (6.8) on page 194 of [8], then for $T_n \leq t < T_{n+1}$ we have that

$$\begin{aligned} X_t &= U_{T_n,t}\left(\Delta Y_{T_n} + \int_{(T_n,t]} U^{-1}_{T_n,u-}\,\mathrm{d}Y_u - \int_{(T_n,t]} U^{-1}_{T_n,u}\,\mathrm{d}[Y,Z]_u\right) \\ &= U_{T_n,t}\Delta Y_{T_n} + \int_{(T_n,t]} U_{u,t}(1 + \Delta Z_u)\,\mathrm{d}Y_u - \sum_{T_n < u \leq t} U_{u,t}\Delta Y_u \Delta Z_u \qquad (5) \\ &= \int_{[T_n,t]} U_{u,t}\,\mathrm{d}Y_u, \end{aligned}$$

where the second equality is justified since the first integral on the right-hand side of the first equality is a path-wise Stieltjes integral, and the second is a sum which is also defined path-wise. If Y was a general semimartingale, then interchanging $U_{T_n,t}$ with the integral sign like this would not be justified as the resulting integrand would no longer be adapted. Clearly if $n \geq 1$, then $U_{u,t} = 0$ for $u < T_n$, and thus

$$X_t = \int_{[0,t]} U_{u,t}\,\mathrm{d}Y_u. \tag{6}$$

Since this holds for all n, the proof for the more general case is complete. For the case where Z is BV, it is evident that $[Z, Z]^c = 0$, and it is easy to check that $\sum_{u<s\leq t} \Delta Z_s$ is convergent (actually, absolutely convergent), and hence the result follows. □

Of course one may also define the counting process,

$$N_t = \sum_{0<s\leq t} 1_{\{\Delta Z_s = -1\}}, \tag{7}$$

which is a.s. finite for all $t \geq 0$ and right-continuous (possibly a.s. identically zero or terminating), and write

$$X_t = \int_{[T_{N_t},t]} U_{u,t}\,\mathrm{d}Y_u. \tag{8}$$

It is worth while to note that for the case where Z is also a BV process, there is a more direct proof involving (path-wise) Stieltjes integration which can be taught in a classroom as follows. Write $Z = A - B$, where A and B are right-continuous and nondecreasing and have no jump points in common. Write $A_t^d = A_t - A_t^c = \sum_{0<s\le t} \max(\Delta Z_s, 0)$ and similarly for B. Observe that by right continuity ΔA_t, ΔB_t, $A_t^d - A_0$ and $B_t^d - B_0$ all converge to zero as $t \downarrow 0$. In particular, for every t for which $-1 \le \Delta B_s \,(\le 0)$ for $0 < s \le t$, we have that

$$1 + A_t^d - A_0 \le \prod_{0<s\le t} (1 + \Delta A_s) \le e^{A_t^d - A_0} \tag{9}$$

and

$$1 + B_t^d - B_0 \le \prod_{0<s\le t} (1 + \Delta B_s) \le e^{B_t^d - B_0} \tag{10}$$

which implies that

$$\prod_{0<s\le t} (1 + \Delta Z_s) = \left(\prod_{0<s\le t} (1 + \Delta A_s) \right) \left(\prod_{0<s\le t} (1 + \Delta B_s) \right) \to 1 \tag{11}$$

as $t \downarrow 0$.

Now note that with $C_t = e^{Z_t^c}$ and $D_t = \prod_{0<s\le t}(1 + \Delta Z_s)$, ordinary (Stieltjes) integration by parts yields

$$U_t \equiv C_t D_t = C_{0+} D_{0+} + \int_{(0,t]} D_{s-} \,\mathrm{d}C_s + \int_{(0,t]} C_{s-} \,\mathrm{d}D_s + \sum_{0<s\le t} \Delta C_s \Delta D_s, \tag{12}$$

and it is easy to check that the continuity of C and the fact that $\mathrm{d}C_t = C_t \,\mathrm{d}Z_t^c$ imply that

$$U_t = 1 + \int_{(0,t]} U_{s-} \,\mathrm{d}Z_s. \tag{13}$$

With this formula established, it is clear that if we denote $U_{u,t}$ as in (3), then in an identical way to which (13) was obtained we have (path-wise) that

$$U_{u,t} = 1 + \int_{(u,t]} U_{u,s-} \,\mathrm{d}Z_s \tag{14}$$

for all $0 \le u \le t$.

Now, if $X_t = \int_{[0,t]} U_{s,t} \,\mathrm{d}Y_s$, then $X_{t-} = \int_{[0,t)} U_{s,t-} \,\mathrm{d}Y_s$ and thus $\int_{(0,t]} X_{s-} \,\mathrm{d}Z_s$ is given by

$$\begin{aligned} \int_{(0,t]} \int_{[0,s)} U_{u,s-} \,\mathrm{d}Y_u \,\mathrm{d}Z_s &= \int_{[0,t)} \int_{(u,t]} U_{u,s-} \,\mathrm{d}Z_s \,\mathrm{d}Y_u \\ &= \int_{[0,t)} (U_{u,t} - 1) \,\mathrm{d}Y_u, \end{aligned} \tag{15}$$

but since $U_{t,t}=1$ we can include t in the domain of integration without changing the value which gives

$$\int_{(0,t]} X_{s-}\,\mathrm{d}Z_s = \int_{[0,t]} (U_{u,t}-1)\,\mathrm{d}Y_u = X_t - Y_t \tag{16}$$

as required.

3. Applications. Assume that Y and Z are right-continuous and nondecreasing jointly having stationary increments in the strong sense that the law of $\theta_s(Y,Z)$ is independent of s where

$$\theta_s(Y(t),Z(t)) = (Y(t+s)-Y(s), Z(t+s)-Z(s)). \tag{17}$$

It is standard to (uniquely) extend (Y,Z) to be a double sided process having stationary increments, that is, that $t\in\mathbb{R}$ rather than $t\ge 0$, thus we assume it at the outset. Finally we assume that Z has jumps bounded by 1. Without loss of generality let us assume that $Y_0=Z_0=0$, otherwise we perform what follows for $Y-Y_0$ and $Z-Z_0$ which also have stationary increments. We consider the unique process X defined via $X_t = X_0 + Y_t - \int_{(0,t]} X_{s-}\,\mathrm{d}Z_s$ for $t\ge 0$ where X_0 is almost surely finite; the unique solution of which is

$$X_t = X_0 e^{-Z_t^c}\prod_{0<s\le t}(1-\Delta Z_s) + \int_{(0,t]} e^{-(Z_t^c - Z_u^c)}\prod_{u<s\le t}(1-\Delta Z_s)\,\mathrm{d}Y_u, \tag{18}$$

where an empty product (when $u=t$ or when $t=0$ on the right) is defined to be 1.

Special cases of such processes are the shot-noise processes in which $Z_t = rt$ and Y are compound Poisson, growth collapse or additive increase multiplicative decrease (AIMD) processes in which $Y_t = rt$ and usually $Z = qN_\lambda$ where N_λ is a Poisson process with rate λ, and $0<q<1$, as well as clearing processes where Z is a Poisson process or, more generally, a renewal counting process (see, e.g., [10, 12]).

Consider the nondecreasing processes

$$J_t = Z_t^c - \sum_{0<s\le t}\log(1-\Delta Z_s)1_{\{\Delta Z_s<1\}}, \tag{19}$$

and $N_t = \sum_{0<s\le t} 1_{\{\Delta Z_s = 1\}}$. Then it is clear that Y, J, N jointly have stationary increments (in the strong sense), and from (18) we have

$$X_t = X_0 e^{-J_t} 1_{\{N_t=0\}} + \int_{(0,t]} e^{-(J_t-J_s)} 1_{\{N_t-N_s=0\}}\,\mathrm{d}Y_s. \tag{20}$$

If $\int_{(-\infty,0]} e^{J_s}\,\mathrm{d}Y_s$ is a.s. finite (recalling that for $s\le 0$, $J_s\le J_0=0$), then setting $X_t^* = \int_{(-\infty,t]} e^{-(J_t-J_s)}1_{\{N_t-N_s=0\}}\,\mathrm{d}Y_s$ it is clear that X^* is a stationary process. Moreover, if, in addition, either $\lim_{t\to\infty} N_t\ge 1$ a.s. (equivalently, $T_1=\inf\{t|\Delta Z_t = 1\}$ is a.s. finite) or $J_t\to\infty$ a.s. as $t\to\infty$, then

$|X_t^* - X_t| \to 0$ a.s. as $t \to \infty$, and thus for any a.s. finite initial X_0, a limiting distribution exists which is distributed like X_0^*.

In fact, when X_0 is independent of (Y, Z), then shifting by $-t$, noting that $\theta_{-t} J_s = J_{s-t} - J_{-t}$ (so that $\theta_{-t} J_t = 0$) and similarly for N and Y, it is clear that X_t has the same distribution as

$$\begin{aligned} & X_0 e^{J_{-t}} 1_{\{N_{-t}=0\}} + \int_{(0,t]} e^{J_{s-t}} 1_{\{N_{s-t}=0\}} \, \mathrm{d}Y_{s-t} \\ & \quad = X_0 e^{J_{-t}} 1_{\{N_{-t}=0\}} + \int_{(-t,0]} e^{J_s} 1_{\{N_s=0\}} \, \mathrm{d}Y_s. \end{aligned} \tag{21}$$

In particular, this implies that when $X_0 = 0$, then X_t is stochastically increasing in $t \ge 0$.

Let us summarize our findings as follows.

THEOREM 2. *If $\int_{(-\infty,0]} e^{J_s} \, \mathrm{d}Y_s < \infty$ a.s., and either $T_1 < \infty$ a.s. or $J_t \to \infty$ a.s. as $t \to \infty$, then X has the unique stationary version*

$$X_t^* = \int_{(-\infty,t]} e^{-(J_t - J_s)} 1_{\{N_t - N_s = 0\}} \, \mathrm{d}Y_s, \tag{22}$$

and for every initial a.s. finite X_0, X_t converges in distribution to X_0^*. Moreover, when $X_0 = 0$ a.s., then X_t is stochastically increasing in $t \ge 0$.

We note that when (Y, Z) also have independent increments so that they form a Lévy process, then the negative of the time reversed process is a left-continuous version of the forward process, and thus in this case [when X_0 is independent of (Y, Z)], X_t is also distributed like

$$X_0 e^{-J_t} 1_{\{N_t=0\}} + \int_{(0,t]} e^{-J_s} 1_{\{N_s=0\}} \, \mathrm{d}Y_s \tag{23}$$

which is also the consequence of the usual time reversal argument for Lévy processes. In what follows we will consider special cases of this structure.

We observe that in the general case N is a simple (i.e., a.s. $\Delta N_t \in \{0, 1\}$ for all t) counting process associated with a time stationary point process. Special cases of such processes are Poisson processes and delayed renewal processes where the delay has the stationary excess lifetime distribution associated with the subsequent i.i.d. inter-renewal times. We will consider this special case a bit later.

3.1. *EX_t for independent X_0, Y, Z.* Since Y has stationary increments, it follows that $EY_t = EY_1 t$. From (21) we have that when EY_1 and EX_0 are finite, then for $t \ge 0$,

$$EX_t = EX_0 E e^{J_{-t}} 1_{\{N_{-t}=0\}} + EY_1 \int_{-t}^{0} E e^{J_s} 1_{\{N_s=0\}} \, \mathrm{d}s, \tag{24}$$

and since for $s \le 0$, we have that $J_s = -(J_0 - J_s)$ is distributed like $-J_{-s} = -(J_{-s} - J_0)$, and similarly for N, we have that

$$EX_t = EX_0 E e^{-J_t} 1_{\{N_t=0\}} + EY_1 \int_0^t E e^{-J_s} 1_{\{N_s=0\}}\,\mathrm{d}s. \tag{25}$$

3.2. *EX_t for independent X_0, Y, Z with Lévy Z.* Here Z is a subordinator with Laplace–Stieltjes exponent $-\eta_z(\alpha) = \log E e^{-\alpha Z_1}$ where, for $\alpha \ge 0$,

$$\eta_z(\alpha) = c_z\alpha + \int_{(0,1]} (1 - e^{-\alpha x})\nu_z(\mathrm{d}x) \tag{26}$$

with $c_z \ge 0$ and $\int_{(0,1]} x\nu_z(\mathrm{d}x) < \infty$. Since the jumps of Z are bounded above by 1, then $\nu_z((1,\infty)) = 0$.

In this case $Z_t^c = c_z t$, N is a Poisson process with rate $\lambda = \nu_z\{1\}$ which is independent of the subordinator,

$$J_t = c_z t - \sum_{0<s\le t} \log(1 - \Delta Z_s) 1_{\{\Delta Z_s < 1\}}; \tag{27}$$

the Lévy measure of which, call it ν_j, is defined via $\nu_j((a,b]) = \nu_z((1 - e^{-a}, 1 - e^{-b}])$ for $0 < a < b < \infty$ and with exponent

$$\begin{aligned} \eta_j(\alpha) &= c_z\alpha + \int_{(0,\infty)} (1 - e^{-\alpha x})\nu_j(\mathrm{d}x) \\ &= c_z\alpha + \int_{(0,1)} (1 - (1-x)^\alpha)\nu_z(\mathrm{d}x), \end{aligned} \tag{28}$$

so that for $\alpha > 0$,

$$\eta_j(\alpha) + \lambda = c_z\alpha + \int_{(0,1]} (1 - (1-x)^\alpha)\nu_z(\mathrm{d}x). \tag{29}$$

We note that

$$\int_{(0,\infty)} \min(x,1)\nu_j(\mathrm{d}x) = \int_{(0,1)} \min(-\log(1-x),1)\nu_z(\mathrm{d}x), \tag{30}$$

and since $-\log(1-x) \le \frac{x}{1-x} \le xe$ for $0 < x \le 1 - e^{-1}$, the right-hand side is dominated above by $e\int_{(0,1)} x\nu_z(\mathrm{d}x) < \infty$, so that ν_j is indeed the proper Lévy measure of a subordinator. Now, for this case, $Ee^{-J_s} = e^{-\eta_j(1)s}$ where

$$\begin{aligned} \eta_j(1) &= c_z + \int_{(0,1)} (1 - (1-x)^1)\nu_z(\mathrm{d}x) \\ &= c_z + \int_{(0,1)} x\nu_z(\mathrm{d}x) = \eta_z'(0) - \lambda \end{aligned} \tag{31}$$

recalling $\lambda = \nu_z\{1\}$. Therefore, $Ee^{-J_s}1_{\{N_s=0\}} = e^{-(\eta_z'(0)-\lambda)s}e^{-\lambda s} = e^{-\eta_z'(0)s}$ so that in this case, since $\eta_z'(0) = c_z + \int_{(0,1]} x\nu_z(\mathrm{d}x) = EZ_1$, (25) becomes

$$EX_t = EX_0 e^{-EZ_1 t} + \frac{EY_1}{EZ_1}(1 - e^{-EZ_1 t}). \tag{32}$$

Recall that here Y need not have independent increments.

3.3. *Independent X_0, Y, Z with Lévy Y.* Since for every $0 = t_0 < t_1 < \cdots < t_n = t$ the independence between Y and Z and hence the independence of Y and J, yield

$$\begin{aligned} &E\left[\exp\left(-\alpha \sum_{i=1}^{n} e^{-J_{t_{i-1}}} 1_{\{N_{t_{i-1}}=0\}} (Y_{t_i} - Y_{t_{i-1}})\right)\Big| Z\right] \\ &\qquad = \prod_{i=1}^{n} \exp(-\eta_y(\alpha e^{-J_{t_{i-1}}} 1_{\{N_{t_{i-1}}=0\}})(t_i - t_{i-1})). \end{aligned} \tag{33}$$

It thus follows, as in equation (5.9) of [13] for the more general multivariate case and in Proposition 1 of [19] for the case where Y and Z are compound Poisson, that

$$\begin{aligned} &E\left[\exp\left(-\alpha \int_{(0,t]} e^{-J_s} 1_{\{N_s=0\}} \,\mathrm{d}Y_s\right)\Big| Z\right] \\ &\qquad = \exp\left(-\int_0^t \eta_y(\alpha e^{-J_s} 1_{\{N_s=0\}}) \,\mathrm{d}s\right). \end{aligned} \tag{34}$$

This implies, as in Theorem 5.1 of [13], that the conditional distribution of $\int_{(0,t]} e^{-J_s} 1_{\{N_s=0\}} \,\mathrm{d}Y_s$ given Z is infinitely divisible, as on the right-hand side, $-\eta_y/n$ is also a Laplace–Stieltjes exponent of a subordinator.

Equation (34), with $\xi_0(\alpha) = Ee^{-\alpha X_0}$, $a \wedge b = \min(a,b)$, and recalling

$$T_1 = \inf\{t | \Delta Z_t = 1\} = \inf\{t | N_t > 0\} \tag{35}$$

yields

$$\begin{aligned} Ee^{-\alpha X_t} &= E\xi_0(\alpha e^{-J_t} 1_{\{N_t=0\}}) \exp\left(-\int_0^t \eta_y(\alpha e^{-J_s}) 1_{\{N_s=0\}} \,\mathrm{d}s\right) \\ &= E\xi_0(\alpha e^{-J_t}) \exp\left(-\int_0^t \eta_y(\alpha e^{-J_s}) \,\mathrm{d}s\right) 1_{\{T_1 > t\}} \\ &\quad + E\exp\left(-\int_0^{T_1} \eta_y(\alpha e^{-J_s}) \,\mathrm{d}s\right) 1_{\{T_1 \le t\}}. \end{aligned} \tag{36}$$

Clearly, when either $T_1 < \infty$ a.s. or $J_t \to \infty$ a.s. as $t \to \infty$, then

$$\lim_{t\to\infty} Ee^{-\alpha X_t} = E\exp\biggl(-\int_0^{T_1} \eta_y(\alpha e^{-J_s})\,\mathrm{d}s\biggr). \tag{37}$$

We now observe that if N and J are independent, as for instance in the case where Z is a subordinator, and N is the counting process associated with a time stationary version of a renewal process the latter having inter-renewal time distribution F having a finite mean μ, then it is well known that N is a delayed renewal process in which the times between the $(i-1)$th and ith jumps are distributed F for $i \ge 2$ and the time until the first jump (i.e., the delay) has a distribution with density $f_e(t) = (1 - F(t))/\mu$. Therefore, in this case,

$$E\exp\biggl(-\int_0^{T_1} \eta_y(\alpha e^{-J_s})\,\mathrm{d}s\biggr) = \int_0^\infty E\exp\biggl(-\int_0^t \eta_y(\alpha e^{-J_s})\,\mathrm{d}s\biggr) f_e(t)\,\mathrm{d}t. \tag{38}$$

Differentiating the right-hand side of the first equality in (36) once and setting $\alpha = 0$ gives (25) as expected, while for the case where $X_0 = 0$ a.s., differentiating twice and setting $\alpha = 0$ yields

$$EX_t^2 = (\eta_y'(0))^2 E\biggl(\int_0^t e^{-J_s} 1_{\{N_s=0\}}\,\mathrm{d}s\biggr)^2 - \eta_y''(0) E\int_0^t e^{-2J_s} 1_{\{N_s=0\}}\,\mathrm{d}s. \tag{39}$$

3.4. *EX_t^2 for independent Y, Z with Lévy Y, Z and $X_0 = 0$.* We note that for every $\beta > 0$, $E\int_0^t e^{-\beta J_s} 1_{\{N_s=0\}}\,\mathrm{d}s = \frac{1-e^{-(\eta_j(\beta)+\lambda)t}}{\eta_j(\beta)+\lambda}$, where $\lambda = \nu_z\{1\}$. Also, note that since $N_u \le N_s$ for $u \le s$,

$$\begin{aligned}\biggl(\int_0^t e^{-J_s} 1_{\{N_s=0\}}\,\mathrm{d}s\biggr)^2 &= 2\int_0^t \int_0^s e^{-J_s - J_u} 1_{\{N_s=0\}}\,\mathrm{d}u\,\mathrm{d}s \\ &= 2\int_0^t \int_0^s e^{-(J_s - J_u)} e^{-2J_u} 1_{\{N_s=0\}}\,\mathrm{d}u\,\mathrm{d}s,\end{aligned} \tag{40}$$

and therefore (using Fubini and the stationary independent increments property of J), the expected value of the left-hand side is

$$\begin{aligned}&2\int_0^t \int_0^s e^{-(\eta_j(1)+\lambda)(s-u)} e^{-(\eta_j(2)+\lambda)u}\,\mathrm{d}u\,\mathrm{d}s \\ &\quad = 2\frac{(1 - e^{-(\eta_j(1)+\lambda)t})/(\eta_j(1)+\lambda) - (1 - e^{-(\eta_j(2)+\lambda)t})/(\eta_j(2)+\lambda)}{\eta_j(2) - \eta_j(1)}.\end{aligned} \tag{41}$$

Finally, we observe that for every positive integer n, we obtain [recall (29)]

$$\eta_j(n) + \lambda = c_z n + \int_{(0,1]} (1 - (1-x)^n)\nu_z(\mathrm{d}x)$$

$$= c_z n + \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \int_{(0,1]} x^k \nu_z(\mathrm{d}x), \tag{42}$$

and since, $\eta_z^{(0)}(0) = \eta_z(0) = 0$, $\eta_z'(0) = c_z + \int_{(0,1)} x\nu_z(\mathrm{d}x)$ and $\eta_z^{(k)}(0) = (-1)^{k-1}\int_{(0,1]} x^k \nu_z(\mathrm{d}x)$, for $k \geq 2$, it holds that

$$\eta_j(n) + \lambda = \sum_{k=0}^{n} \binom{n}{k} \eta_z^{(k)}(0). \tag{43}$$

In particular $\eta_j(1) + \lambda = \eta_z'(0) = c_z + \int_{(0,1]} x\nu(\mathrm{d}x)$ and $\eta_j(2) + \lambda = 2\eta_z'(0) + \eta_z''(0)$, so that $\eta_j(2) - \eta_j(1) = \eta_z'(0) + \eta_z''(0)$.

To summarize, when $EX_0 = 0$, we have

$$\begin{aligned} EX_t^2 &= 2(\eta_y'(0))^2 \\ &\quad \times ((1 - e^{-\eta_z'(0)t})/\eta_z'(0) - (1 - e^{-(2\eta_z'(0)+\eta_z''(0))t})/(2\eta_z'(0) + \eta_z''(0))) \\ &\qquad /(\eta_z'(0) + \eta_z''(0)) \\ &\quad - \eta_y''(0)\frac{1 - e^{-(2\eta_z'(0)+\eta_z''(0))t}}{2\eta_z'(0) + \eta_z''(0)} \end{aligned} \tag{44}$$

which converges to

$$\frac{2(\eta_y'(0))^2 - \eta_z'(0)\eta_y''(0)}{\eta_z'(0)(2\eta_z'(0) + \eta_z''(0))} = \frac{(\eta_y'(0)/\eta_z'(0))^2 - \eta_y''(0)/(2\eta_z'(0))}{1 + \eta_z''(0)/(2\eta_z'(0))} \tag{45}$$

as $t \to \infty$. We note that as $\nu_z(1,\infty) = 0$, then clearly whenever either $c_z > 0$ or $\nu_z(0,1) \neq 0$ (i.e., $Z - N$ is not identically zero), it holds that

$$\eta_z'(0) = c_z + \int_{(0,1]} x\nu_z(\mathrm{d}x) > \int_{(0,1]} x^2\nu_z(\mathrm{d}x) = -\eta_z''(0). \tag{46}$$

3.5. *Lévy Z, linear Y and $X_0 = x$.* It is of interest to consider the special case where $Y_t = rt$ for some $r > 0$ and $X_0 = x$ for some $x \geq 0$. For the case where Z is compound Poisson this model becomes the growth–collapse process from [16] where the computation of transient moments turns out to be especially tractable. Since

$$\frac{X_t}{r} = \frac{x}{r} + t - \int_{(0,t]} \frac{X_{s-}}{r}\,\mathrm{d}Z_s \tag{47}$$

we may without loss of generality assume that $r = 1$. Recall (23). Following the ideas in the proof of Proposition 3.1 of [4], we first write for $a \geq 0$ and

$b \geq 1$,

$$
\begin{aligned}
Ee^{-aJ_t}\left(\int_0^t e^{-J_s}\,\mathrm{d}s\right)^b &= bEe^{-aJ_t}\int_0^t\left(\int_u^t e^{-J_s}\,\mathrm{d}s\right)^{b-1} e^{-J_u}\,\mathrm{d}u\\
&= b\int_0^t Ee^{-a(J_t-J_u)}\left(\int_u^t e^{-(J_s-J_u)}\,\mathrm{d}s\right)^{b-1} e^{-(a+b)J_u}\,\mathrm{d}u \qquad (48)\\
&= b\int_0^t e^{-\eta_j(a+b)u} Ee^{-aJ_{t-u}}\left(\int_0^{t-u} e^{-J_s}\,\mathrm{d}s\right)^{b-1}.
\end{aligned}
$$

Thus, if $T \sim \exp(\theta)$ for some $\theta > 0$ and is independent of Z, then since the conditional distribution of $T-u$ given $T>u$ is the same as that of T (memoryless property), it readily follows that

$$
Ee^{-aJ_T}\left(\int_0^T e^{-J_s}\,\mathrm{d}s\right)^b = \frac{b}{\eta_j(a+b)+\theta} Ee^{-aJ_T}\left(\int_0^T e^{-J_s}\,\mathrm{d}s\right)^{b-1}. \qquad (49)
$$

For $a=0$ we have that, since $T_1 \wedge T \sim \exp(\lambda+\theta)$ and $\int_0^T e^{-J_s} 1_{\{N_s=0\}}\,\mathrm{d}s = \int_0^{T_1\wedge T} e^{-J_s}\,\mathrm{d}s$,

$$
E\left(\int_0^T e^{-J_s} 1_{\{N_s=0\}}\,\mathrm{d}s\right)^b = \frac{b}{\eta_j(b)+\lambda+\theta} E\left(\int_0^T e^{-J_s} 1_{\{N_s=0\}}\,\mathrm{d}s\right)^{b-1}. \qquad (50)
$$

For $a>0$ we have, from the fact that $T_1 \wedge T$ is independent of $1_{\{T_1>T\}}$, that

$$
\begin{aligned}
&Ee^{-aJ_T} 1_{\{N_T=0\}}\left(\int_0^T e^{-J_s} 1_{\{N_s=0\}}\,\mathrm{d}s\right)^b\\
&\qquad = Ee^{-aJ_{T_1\wedge T}} 1_{\{T_1>T\}}\left(\int_0^{T_1\wedge T} e^{-J_s}\,\mathrm{d}s\right)^b \qquad (51)\\
&\qquad = \frac{\theta}{\lambda+\theta} Ee^{-aJ_{T_1\wedge T}}\left(\int_0^{T_1\wedge T} e^{-J_s}\,\mathrm{d}s\right)^b
\end{aligned}
$$

and thus

$$
\begin{aligned}
&Ee^{-aJ_T} 1_{\{N_T=0\}}\left(\int_0^T e^{-J_s} 1_{\{N_s=0\}}\,\mathrm{d}s\right)^b\\
&\qquad = Ee^{-aJ_T} 1_{\{N_T=0\}}\left(\int_0^T e^{-J_s}\,\mathrm{d}s\right)^b \qquad (52)\\
&\qquad = \frac{b}{\eta_j(a+b)+\lambda+\theta} Ee^{-aJ_T} 1_{\{N_T=0\}}\left(\int_0^T e^{-J_s}\,\mathrm{d}s\right)^{b-1}.
\end{aligned}
$$

Clearly, when $b=0$ and $a>0$ we have that

$$
Ee^{-aJ_T} 1_{\{N_T=0\}} = e^{-(\eta_j(a)+\lambda)T} = \frac{\theta}{\eta_j(a)+\lambda+\theta}. \qquad (53)
$$

Now

$$
\begin{aligned}
EX_T^n &= E\left(xe^{-J_T}1_{\{N_T=0\}} + \int_0^T e^{-J_s}1_{\{N_s=0\}}\,\mathrm{d}s\right)^n \\
&= \sum_{k=1}^n \binom{n}{k} x^k E e^{-kJ_T} 1_{\{N_T=0\}} \left(\int_0^T e^{-J_s}\,\mathrm{d}s\right)^{n-k} \\
&\quad + E\left(\int_0^T e^{-J_s}1_{\{N_s=0\}}\,\mathrm{d}s\right)^n,
\end{aligned} \tag{54}
$$

and denoting [recall (43)]

$$
\mu_i = \eta_j(i) + \lambda = c_z i + \int_{(0,1]} (1-(1-x)^i)\nu_z(\mathrm{d}x) = \sum_{k=0}^i \binom{i}{k} \eta_z^{(k)}(0), \tag{55}
$$

it follows from (50), (52), (53) and (54), with some manipulations, that

$$
\begin{aligned}
EX_T^n = \frac{n!}{\prod_{i=1}^n \mu_i} \Biggl(\sum_{k=1}^n \frac{x^k \prod_{i=1}^k \mu_i}{k!} \left(\prod_{i=k+1}^n \frac{\mu_i}{\mu_i+\theta} - \prod_{i=k}^n \frac{\mu_i}{\mu_i+\theta} \right) \\
+ \prod_{i=1}^n \frac{\mu_i}{\mu_i+\theta} \Biggr),
\end{aligned} \tag{56}
$$

where an empty product is defined to be 1. Finally, noting that $EX_T^n = \int_0^\infty e^{-\theta t}\,\mathrm{d}EX_t^n$ it follows that if $\{E_i|i\ge 1\}$ are i.i.d. random variables with distribution $\exp(1)$, then $E_i/\mu_i \sim \exp(\mu_i)$. It is well known and easy to check that

$$
\prod_{i=k}^n \frac{\mu_i}{\mu_i+\theta} = \int_0^\infty e^{-\theta t}\,\mathrm{d}P\left[\sum_{i=k}^n \frac{E_i}{\mu_i} \le t\right]; \tag{57}
$$

hence, for $1 \le k \le n$,

$$
\prod_{i=k+1}^n \frac{\mu_i}{\mu_i+\theta} - \prod_{i=k}^n \frac{\mu_i}{\mu_i+\theta} = \int_0^\infty e^{-\theta t}\,\mathrm{d}P\left[\sum_{i=k+1}^n \frac{E_i}{\mu_i} \le t < \sum_{i=k}^n \frac{E_i}{\mu_i}\right], \tag{58}
$$

and thus we have the following somewhat curious result.

THEOREM 3. *Let $p_{ij}(t)$ be the transition matrix function of a pure death process $D = \{D_t|t\ge 0\}$ with death rates μ_i, $i \ge 1$ (0 is absorbing). Then*

$$
\begin{aligned}
EX_t^n &= \frac{n!}{\prod_{i=1}^n \mu_i}\left(p_{n0}(t) + \sum_{k=1}^n \frac{x^k \prod_{i=1}^k \mu_i}{k!} p_{nk}(t)\right) \\
&= \frac{n!}{\prod_{i=1}^n \mu_i} E\left[\prod_{i=1}^{D_t} \frac{x\mu_i}{i} \,\middle|\, D_0 = n\right],
\end{aligned} \tag{59}
$$

where an empty product is 1.

In particular, when $x=0$, then

$$EX_t^n = \frac{n!}{\prod_{i=1}^n \mu_i} p_{n0}(t)$$

$$= n! \int_{\substack{\sum_{i=1}^n x_i \le t \\ x_1,\dots,x_n \ge 0}} \cdots \int \exp\left(-\sum_{i=1}^n \mu_i x_i\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n \tag{60}$$

$$= n! t^n \int_{\substack{\sum_{i=1}^n x_i \le 1 \\ x_1,\dots,x_n \ge 0}} \cdots \int \exp\left(-t\sum_{i=1}^n \mu_i x_i\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

In fact, one may also give a finite simple algorithm with which to compute EX_t^n. For the sake of brevity we do it only for the case $x=0$. This can be done similarly to the Brownian motion in the proof of Theorem 1 on page 31 of [22] or, equivalently, directly from (60) as follows. Set $f_0 = 0$ and for $n \ge 1$ and $0 < a_1 < a_2 < \cdots < a_n$, let

$$f_n(a_1,\dots,a_n) = \int_{\substack{\sum_{i=1}^n x_i \le 1 \\ x_1,\dots,x_n \ge 0}} \cdots \int \exp\left(-\sum_{i=1}^n a_i x_i\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

$$= \int_{\substack{\sum_{i=2}^n x_i \le 1 \\ x_2,\dots,x_n \ge 0}} \cdots \int \left(\int_0^{1-\sum_{i=2}^n x_i} e^{-a_1 x_1} \mathrm{d}x_1\right) \times \exp\left(-\sum_{i=2}^n a_i x_i\right) \mathrm{d}x_2 \cdots \mathrm{d}x_n \tag{61}$$

$$= \frac{f_{n-1}(a_2,\dots,a_n) - e^{-a_1} f_{n-1}(a_2-a_1,\dots,a_n-a_1)}{a_1}.$$

Alternatively, if we denote $g_0 = 1$, and for $n \ge 1$ and $b_1,\dots,b_n > 0$,

$$g_n(b_1,\dots,b_n) = f_n(b_1, b_1+b_2,\dots,b_1+\cdots+b_n). \tag{62}$$

Then

$$g_n(b_1,\dots,b_n) = \frac{g_{n-1}(b_1+b_2, b_3,\dots,b_n) - e^{-b_1} g_{n-1}(b_2,b_3,\dots,b_n)}{b_1}. \tag{63}$$

From the above, it is also clear (see also [22], Theorem 1, page 31 for the case of a Brownian motion) that, in fact,

$$\begin{aligned} EX_t^n &= t^n n! f_n(\mu_1 t, \dots, \mu_n t) \\ &= t^n n! g_n(\mu_1 t, (\mu_2 - \mu_1)t, \dots, (\mu_n - \mu_{n-1})t) \end{aligned} \tag{64}$$

is a linear combination of exponentials. An algorithm for computing the coefficients of this linear combination is equivalent to the above simple algorithm which involves only a finite number of additions and multiplications.

We emphasize that the fact that Theorem 3 holds for all $n \geq 1$, and the algorithm for the computation of moments, also valid for all $n \geq 1$, is special for the case where Z is a nonzero subordinator. This is true since this is the only case where $\eta_j(n)$ is finite, strictly positive for all $n \geq 1$ and strictly increasing.

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