Distributed STBCs with Partial Interference Cancellation Decoding

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I. Introduction

***Abstract*—Recently, Guo and Xia introduced low complexity decoders called Partial Interference Cancellation (PIC) and PIC with Successive Interference Cancellation (PIC-SIC), which include the Zero Forcing (ZF) and ZF-SIC receivers as special cases, for point-to-point MIMO channels. In this paper, we show that PIC and PIC-SIC decoders are capable of achieving the full cooperative diversity available in wireless relay networks. We give sufficient conditions for a Distributed Space-Time Block Code (DSTBC) to achieve full diversity with PIC and PIC-SIC decoders and construct a new class of DSTBCs with low complexity full-diversity PIC-SIC decoding. We also show that almost all known full-diversity PIC/PIC-SIC decodable codes constructed for point-to-point MIMO channels can be used as full-diversity PIC/PIC-SIC decodable DSTBCs in relay networks. The proposed DSTBCs with low complexity, full-diversity PIC/PIC-SIC decoding achieve higher rates (in complex symbols per channel use) than the multigroup ML decodable DSTBCs available in the literature. Simulation results show that the proposed codes have better bit error rate performance than the best known low complexity, full-diversity DSTBCs.**

Spatial diversity in wireless channels helps mitigate the negative effects of fading. In systems where the terminals can not have multiple transmit/receive antennas due to space considerations, such as wireless sensor networks or cellular networks for mobile phones, spatial diversity called *cooperative diversity* can be achieved by using the antennas of other users (relays) in the network to aid the communication of messages from a single source [1], [2]. The amplify-and-forward (AF) protocol is widely studied and is more practical since it involves simpler processing at the relays than other cooperative protocols. In [3], a two phase cooperative protocol based on AF was given. In this scheme, the source transmits information to the relays in the first phase. Each of the relays linearly processes the signal that it receives in the first phase and transmits the processed signal to the destination in the second phase. The destination effectively sees a Space-Time Block Code (STBC) being transmitted by the relays. Since this code has been generated by the relays in a distributed fashion, it is called a *Distributed STBC* (DSTBC). If the number of independent real information symbols in the DSTBC is K, then the rate of the DSTBC is $R = \frac{K}{2T}$ complex symbols per channel use (cspcu), where T is the combined duration of the first and second phases. For practical significance, it is desirable that the DSTBC be of high rate and have a low complexity, full-diversity decoding algorithm. In this paper, we consider the situation where the relay nodes have no channel state information (CSI), while the destination has full CSI. This assumption is more practical than assuming CSI at the relays, since it reduces the burden on the relays, which may be other users in the network.

A DSTBC is said to be *g-group maximum-likelihood (ML) decodable* if the K information symbols can be partitioned into g groups, $g > 1$, such that each group of symbols can be ML decoded independent of the symbols of the other groups. If the maximum number of symbols in any group is λ, then the code is also said to be *λ-real symbol* or *$\frac{\lambda}{2}$-complex symbol ML decodable*. Since the decoding complexity is determined by λ, DSTBCs with small λ and large g are desirable. In [4], single-real symbol ML decodable full-diversity DSTBCs called Distributed Orthogonal Space-Time Block codes were constructed with rate at the most $\frac{2}{2+N}$ for any number of relays $N \geq 2$. In [5], single-real ML decodable DSTBCs with rate $\frac{1}{4}$ were constructed for any number of relays N using real orthogonal designs [6]. In [7], single complex symbol ML decodable DSTBCs were constructed for any number of relays N. The rate these codes is upper bounded by $\frac{4}{4+N}$. In [8], 4-group ML decodable DSTBCs with rate $\frac{1}{2}$ were constructed for any number of relays N using matrix representations of Extended Clifford Algebras. All the codes discussed in this paragraph rely on optimal decoding in order to tap full cooperative diversity.

In [9], PIC and PIC-SIC decoders were introduced for decoding STBCs for point-to-point MIMO channels. A PIC decoder partitions the information symbols of the code into multiple groups. A PIC receiver decodes each group of symbols independently of other groups. In order to decode a particular group of symbols, a PIC decoder first implements a linear filter to eliminate the interference from symbols in all other groups and then decodes all the symbols of the current group jointly. A PIC-SIC receiver uses successive interference cancellation along with PIC decoding. If λ is the maximum number of symbols in any group, we say that the PIC or PIC-SIC decoder performs *λ-real symbol PIC* or *PIC-SIC decoding* respectively. When $\lambda = 1$, the PIC (PIC-SIC) decoder reduces to ZF (ZF-SIC) decoder. A criterion for an STBC, in point-to-point MIMO channel, to achieve full-diversity with PIC and PIC-SIC decoding were given in [10], [11]. Since the complexity of PIC/PIC-SIC implementation depends on λ, one would like the code to have a full-diversity PIC/PIC-SIC decoding algorithm with full-diversity. Code constructions for point-to-point MIMO channels with full-diversity and low complexity PIC/PIC-SIC decoding were given in [9], [11], [12], [13], [14] and with ZF/ZF-SIC decoding were given in [15], [16].

The contributions and organization of this paper are as follows.

- For a two phase AF based cooperative protocol, we give sufficient conditions for a DSTBC to achieve full cooperative diversity when PIC and PIC-SIC decoders are used at the destination. As a special case, we also obtain full-diversity criteria for ZF and ZF-SIC decoding at the destination. In particular, we show that for a N-relay network, where the relays and the source are equipped with 1 antenna and the destination is equipped with N_D antennas, a diversity of $N\left(1-\frac{log(logP)}{logP}\right)$ is achievable when $N_D=1$ and a diversity of N is achievable when $N_D>1$ with PIC/PIC-SIC decoding (Section III).
- It is shown that PIC/PIC-SIC decoding is capable of achieving the full cooperative diversity offered by the wireless relay network. Since the PIC and PIC-SIC decoders are of low complexity, the proposed full-diversity criteria enable us to construct DSTBCs with high rates and low decoding complexity, while achieving full cooperative diversity (Section III).
- We construct a new class of full-diversity λ-real symbol PIC-SIC decodable DSTBCs for all even number of relays N with $N\geq 2\lambda$, $\lambda\geq 1$, with rates arbitrarily close to $\frac{\lambda}{\lambda+1}$ cspcu. The new single real symbol decodable codes have rates close to $\frac{1}{2}$ while the single real symbol ML decodable codes in the literature have rate at the most $\frac{1}{4}$ for $N\geq 6$ and $\frac{2}{5}$ for $2<N<6$. The new single complex symbol decodable codes have rate close to $\frac{2}{3}$ while the single complex symbol ML decodable codes in the literature have rate at the most $\frac{1}{2}$ cspcu (Section IV-A).
- We show that a family of codes given in [11] in the context of point-to-point MIMO channels can be used as full-diversity PIC-SIC decodable DSTBCs. This family includes λ-real symbol decodable codes for all $N\geq\lambda\geq 1$ with rates close to $\frac{\lambda}{\lambda+1}$. However, for the same amount of delay and λ, these codes have lower rates than the new codes proposed in Section IV-A. We also show that almost all known full-diversity PIC/PIC-SIC decodable codes constructed for point-to-point MIMO channels can be used as full-diversity PIC/PIC-SIC decodable DSTBCs (Section IV-B).
- We show that the proposed full-diversity DSTBCs achieve higher rates when compared with the multigroup ML decodable DSTBCs of same decoding complexity available in the literature (Table I in Section V summarizes the comparison of the proposed codes in this paper with other low complexity DSTBCs). Moreover, simulation results show that the new PIC-SIC decodable codes have a better bit error rate performance than the best known multigroup ML decodable DSTBCs (Section V).

The system model is explained in Section II. Some of the open problems related to PIC/PIC-SIC decoding in wireless relay networks are discussed in Section VI.

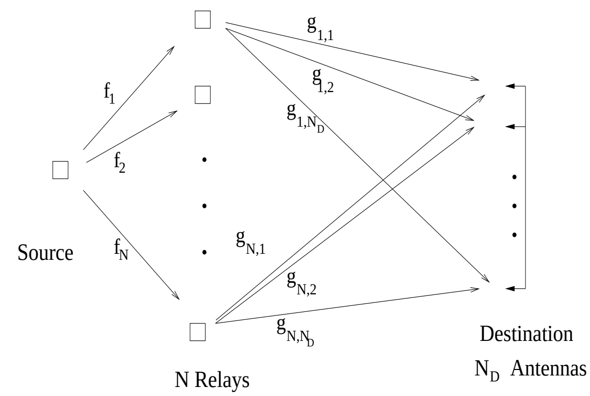

Fig. 1. Relay network model

Notation: For a complex matrix A the transpose, the conjugate and the conjugate-transpose are denoted by A^T,A^* and A^H respectively. $||A||_F$ is the Frobenius norm of the matrix A. I_n is the $n\times n$ identity matrix, $\mathbf{0}$ is the all zero matrix of appropriate dimension, $\mathbf{1}\{\cdot\}$ is the indicator function and $i=\sqrt{-1}$. The cardinality of a set Γ is denoted by $|\Gamma|$. The complement of a set Γ with respect to a universal set U is denoted by Γ^c, whenever U is clear from context. For a square matrix A, $det(A)$ is the determinant of A and $Tr(A)$ is the trace of A. For a complex matrix A, A_{Re} and A_{Im} denote its real and imaginary parts respectively. Vectorization of a matrix A is denoted by $vec(A)$ and the expectation operator is denoted by $\mathsf{E}(\cdot)$.

II. System Model

We consider a wireless relay network with $N+2$ nodes: a source node, N relay nodes and a destination node. The source and the relay nodes are equipped with single antennas and the destination has N_D antennas as shown in Fig. 1. This model captures the scenario corresponding to the uplink of a cellular system or a wireless local area network (WLAN). We consider an AF cooperative protocol in this paper in which each transmission cycle is composed of two phases: a broadcast phase of duration T_1 and a cooperation phase of duration T_2. During the broadcast phase, the source transmits to all the relays. In the cooperation phase, the relays process the signal received in the broadcast phase and transmit it to the destination. The channel gain from the source to the j^{th} relay is f_j, $j=1,\ldots,N$. The channel gain from the j^{th} relay to the l^{th} receive antenna at the destination is $g_{j,l}$, for $j=1,\ldots,N$ and $l=1,\ldots,N_D$. We make the following assumptions in our model: (i) All the nodes have half-duplex constraint, (ii) the channel gains f_j, $j=1,\ldots,N$, and $g_{j,l}$, $j=1,\ldots,N$, $l=1,\ldots,N_D$ are assumed to be independent circularly symmetric complex Gaussian random variables with zero mean and unit variance and with coherence interval of duration at least T_1+T_2, (iii) the relay nodes have no channel state information and the destination has the knowledge of all channel gains f_j, $g_{j,l}$ and (iv) the transmissions from the relay nodes to the destination are synchronized at symbol level.

Let the source transmit with power $\pi_1 P$ and each of the relays transmit with power $\pi_2 P$. The real numbers $\pi_1, \pi_2 > 0$ are chosen such that $\pi_1 T_1 + \pi_2 R T_2 = T_1 + T_2$. Thus the average transmission power used by the network is P. In each transmission cycle, the source transmits K real information symbols, $x_i,\ i = 1, \ldots, K$. The source is equipped with a finite subset $\mathcal{A} \subset \mathbb{R}^K$ called the *signal set* and K complex vectors $\{\nu_1, \ldots, \nu_K\} \subset \mathbb{C}^{T_1}$, that are linearly independent over $\mathbb{R}$. The information vector $x = [x_1, \ldots, x_K]^T$ assumes values from $\mathcal{A}$.

During the broadcast phase, the source synthesizes the vector $z = \sum_{i=1}^{K} x_i \nu_i \in \mathbb{C}^{T_1}$ and transmits $\sqrt{\pi_1 P} z$ to all the relays. The signal set $\mathcal{A}$ and the vectors ν_i are chosen in such a way that $\mathsf{E}\left(||z||_F^2\right) = T_1$. The vector received by the j^{th} relay is $r_j = f_j \sqrt{\pi_1 P} z + v_j,\ j = 1, \ldots, N$. Here, v_j is the additive white Gaussian noise vector at the j^{th} relay and it has zero mean and covariance I_{T_1}.

In the cooperation phase, the j^{th} relay transmits a linearly processed version of either r_j or r_j^*. The subset $\mathcal{S} \subset \{1, \ldots, N\}$ denotes the set of indices of the relays that process r_j^*. Let for any indexed set of N matrices or vectors $\{C_1, \ldots, C_N\}$, $\overline{C}_j = C_j^*$ if $j \in \mathcal{S}$, and $\overline{C}_j = C_j$ else. The j^{th} relay is equipped with a matrix $B_j \in \mathbb{C}^{T_2 \times T_1}$. In the cooperation phase, the j^{th} relay transmits $t_j = \sqrt{\frac{\pi_2 P}{\pi_1 P + 1}} \overline{B_j r_j}$

$$= \sqrt{\frac{\pi_1 \pi_2 P^2}{\pi_1 P + 1}} \overline{f}_j \overline{B_j z} + \sqrt{\frac{\pi_2 P}{\pi_1 P + 1}} \overline{B_j v_j}.$$

The matrices B_j are chosen in such a way that the average energy transmitted by each of the relays during the cooperation phase is $\pi_2 P$. The signal received by the l^{th} antenna, $l = 1, \ldots, N_D$, at the destination during cooperation phase is $y_l = \sum_{j=1}^{N} g_{j,l} t_j + w_l$

$$= \sum_{j=1}^{N} \left(g_{j,l} \overline{f_j} \sqrt{\frac{\pi_1 \pi_2 P^2}{\pi_1 P + 1}} \overline{B_j z} + \sqrt{\frac{\pi_2 P}{\pi_1 P + 1}} g_{j,l} \overline{B_j v_j} \right) + w_l.$$

Here, w_l is the additive circularly symmetric complex Gaussian noise at the l^{th} receiver. It has zero mean and covariance I_{T_2}. The $T \times N_D$ received matrix $Y = [y_1\ y_2 \cdots y_{N_D}]$ satisfies

$$Y = \sqrt{\frac{\pi_1 \pi_2 P^2}{\pi_1 P + 1}} X H + U, \tag{1}$$

where

$$X = [\overline{B_1 z}\ \overline{B_2 z} \cdots \overline{B_N z}] \in \mathbb{C}^{T_2 \times N} \tag{2}$$

is the codeword matrix,

$$H = F \mathcal{G} \tag{3}$$

is the channel matrix with $F = diag(\overline{f_1}, \ldots, \overline{f_N})$ and the $(j, l)^{th}$ entry of the matrix $\mathcal{G} \in \mathbb{C}^{N \times N_D}$ being $g_{j,l}$. The matrix $U \in \mathbb{C}^{T_2 \times N_D}$ is the total noise seen by the receiver. If we denote the columns of U by u_l, $l = 1, \ldots, N_D$, then we have

$$u_l = \sum_{j=1}^{N} \sqrt{\frac{\pi_2 P}{\pi_1 P + 1}} g_{j,l} \overline{B_j v_j} + w_l. \tag{4}$$

Let $vec(A)$ denote the vectorization of a matrix A and for any random vector. The noise vector $vec(U)$ is zero mean circularly symmetric complex Gaussian. The following proposition gives Γ', the covariance matrix of $vec(U)$.

Proposition 1: Let $\Gamma' = \begin{bmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,N_D} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,N_D} \\ \vdots & \vdots & \ddots & \vdots \\ C_{N_D,1} & C_{N_D,2} & \cdots & C_{N_D,N_D} \end{bmatrix}$, where the submatrices $C_{l_1,l_2} \in \mathbb{C}^{T_2 \times T_2},\ 1 \leq l_1, l_2 \leq N_D$. Then we have,

$$C_{l_1,l_2} = \frac{\pi_2 P}{\pi_1 P + 1} \sum_{j=1}^{N} g_{j,l_1} g_{j,l_2}^* \bar{B}_j \bar{B}_j^H + \mathbf{1}\{l_1 = l_2\} I_{T_2},$$

for $1 \leq l_1, l_2 \leq N_D$.

Proof: Let $1 \leq l_1, l_2 \leq N_D$, then $C_{l_1,l_2} = \mathsf{E}\left(u_{l_1} u_{l_2}^H\right)$. Expanding with the help of (4) and using the fact that the Gaussian vectors $v_j,\ j = 1, \ldots, N$, and $w_l,\ l = 1, \ldots, N_D$ are of zero mean and mutually independent we have, C_{l_1,l_2}

$$= \mathsf{E}\left\{ \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \frac{\pi_2 P}{\pi_1 P + 1} g_{j_1,l_1} g_{j_2,l_2}^* \bar{B}_{j_1} \bar{v}_{j_1} \bar{v}_{j_2}^H \bar{B}_{j_2}^H + w_{l_1} w_{l_2}^H \right\}$$
$$= \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \frac{\pi_2 P}{\pi_1 P + 1} g_{j_1,l_1} g_{j_2,l_2}^* \bar{B}_{j_1} \mathsf{E}\left(\bar{v}_{j_1} \bar{v}_{j_2}^H\right) \bar{B}_{j_2}^H$$
$$+ \mathsf{E}\left(w_{l_1} w_{l_2}^H\right).$$

Using the fact that $\mathsf{E}\left(\bar{v}_{j_1} \bar{v}_{j_2}^H\right) = \mathbf{1}\{j_1 = j_2\} I_{T_1}$ and $\mathsf{E}\left(w_{l_1} w_{l_2}^H\right) = \mathbf{1}\{l_1 = l_2\} I_{T_2}$, we get the desired result. ■

Since $z = \sum_{i=1}^{K} x_i \nu_i$, it is clear that each entry of the codeword matrix $X = [\overline{B_1 z}\ \overline{B_2 z} \cdots \overline{B_N z}]$ is a linear combination of the information symbols $x_i,\ i = 1, \ldots, K$. Thus, there exist matrices $A_i \in \mathbb{C}^{T_2 \times N},\ i = 1, \ldots, K$, such that the codeword $X = \sum_{i=1}^{K} x_i A_i$. The matrices A_i are called *linear dispersion* or *weight matrices*. The finite set of matrices $\mathcal{C} = \{\sum_{i=1}^{K} x_i A_i | [x_1, \ldots, x_K]^T \in \mathcal{A}\}$ is the DSTBC. The rate of the DSTBC $\mathcal{C}$ in cspcu is $R = \frac{K}{2(T_1+T_2)}$, and in bits per channel use (bpcu) is $\frac{log_2|\mathcal{A}|}{T_1+T_2}$. Note that each column of the codeword matrix X in (2) is either a linear combination of the vector z or its conjugate z^*. Such designs are said to be *conjugate linear* [8]. Not all linear designs are conjugate linear. In this paper, we only consider DSTBCs that are obtainable from conjugate linear designs.

III. Partial Interference Cancellation decoding and full-diversity criterion

In this section we first give the method of PIC and PIC-SIC decoding a DSTBC. We then give sufficient conditions for a conjugate linear DSTBC to yield full diversity under PIC, PIC-SIC, ZF and ZF-SIC decoding. We then show that if a DSTBC satisfies the proposed design criteria and if a subset of relay nodes in the network fail, then the residual diversity gain from the remaining relay nodes is guaranteed. This ensures resistance towards relay node failures.

A. PIC decoding of DSTBCs

Consider a DSTBC in K real symbols. A *grouping scheme* [9] is a partition $\mathcal{I}_1, \ldots, \mathcal{I}_g$ of the set $\{1, \ldots, K\}$, where $\mathcal{I}_k$ are called *groups*. There is a corresponding partition of the information symbols into g vectors, where for $k = 1, \ldots, g$, the k^{th} vector of information symbols is $x_{\mathcal{I}_k} = [x_{i_{k,1}}, x_{i_{k,2}}, \ldots, x_{i_{k,|\mathcal{I}_k|}}]^T$, where $\mathcal{I}_k = \{i_{k,1}, i_{k,2}, \ldots, i_{k,|\mathcal{I}_k|}\}$ with $i_{k,1} < i_{k,2} < \cdots < i_{k,|\mathcal{I}_k|}$. Let the g groups of information symbols be encoded independently of each other, i.e., the DSTBC

$$\mathcal{C} = \left\{ \sum_{i=1}^{K} x_i A_i | x_{\mathcal{I}_k} \in \mathcal{A}_{\mathcal{I}_k},\ k = 1, \ldots, g \right\}, \tag{5}$$

for some finite subsets $\mathcal{A}_{\mathcal{I}_k} \subset \mathbb{R}^{|\mathcal{I}_k|}$, $k = 1, \ldots, g$.

For a complex matrix A, let $\widetilde{vec}(A) = [vec(A_{Re})^T\ vec(A_{Im})^T]^T$. The received matrix Y in (1) can be rewritten as

$$\begin{aligned} y' = \widetilde{vec}(Y) &= \sqrt{\frac{\pi_1 \pi_2 P^2}{\pi_1 P + 1}} \sum_{i=1}^{K} x_i \widetilde{vec}(A_i H) + \widetilde{vec}(U) \\ &= G' x + \widetilde{vec}(U), \end{aligned}$$

where

$$G' = \sqrt{\frac{\pi_1 \pi_2 P^2}{\pi_1 P + 1}} [\widetilde{vec}(A_1 H) \cdots \widetilde{vec}(A_K H)] \in \mathbb{R}^{2 N_D T_2 \times K}, \tag{6}$$

and $x = [x_1, x_2, \ldots, x_K]^T$. Let Γ' be the covariance matrix of $vec(U)$ as given in Proposition 1. Then, it is known that $\widetilde{vec}(U)$ is a real Gaussian vector with zero mean and covariance

$$\Gamma = \frac{1}{2} \begin{bmatrix} \Gamma'_{Re} & -\Gamma'_{Im} \\ \Gamma'_{Im} & \Gamma'_{Re} \end{bmatrix}. \tag{7}$$

Consider

$$y = \Gamma^{-\frac{1}{2}} y' = G x + n, \tag{8}$$

where, $G = \Gamma^{-\frac{1}{2}} G'$ and $n = \Gamma^{-\frac{1}{2}} \widetilde{vec}(U)$ is a zero mean real Gaussian vector with covariance $I_{2 N_D T_2}$.

Let $G = [g_1\ g_2 \cdots g_K]$, where g_i, $i = 1, \ldots, K$, are the column vectors of G. For any nonempty subset $\mathcal{I} = \{i_1, \ldots, i_{|\mathcal{I}|}\} \subset \{1, \ldots, K\}$, with $i_1 < i_2 < \cdots < i_{|\mathcal{I}|}$, let $G_{\mathcal{I}} = [g_{i_1}\ g_{i_2} \cdots g_{i_{\mathcal{I}}}]$. Let $V_{\mathcal{I}_k}$ be the column space of the matrix $G_{\mathcal{I}_k^c}$ and $P_{\mathcal{I}_k}$ be the matrix that projects a vector onto the subspace $V_{\mathcal{I}_k}^{\perp}$, the orthogonal complement of the subspace $V_{\mathcal{I}_k}$. Also, let $\tilde{\mathcal{I}}_k = \cup_{\ell > k} \mathcal{I}_\ell$, $\tilde{V}_{\mathcal{I}_k}$ be the column space of the matrix $G_{\tilde{\mathcal{I}}_k}$ and $\tilde{P}_{\mathcal{I}_k}$ be the matrix that projects a vector onto the subspace $\tilde{V}_{\mathcal{I}_k}^{\perp}$.

The PIC decoding of the DSTBC is performed as

$$\hat{x}_{\mathcal{I}_k} = arg\ \ min_{x_{\mathcal{I}_k} \in \mathcal{A}_{\mathcal{I}_k}} ||P_{\mathcal{I}_k} y - P_{\mathcal{I}_k} G_{\mathcal{I}_k} x_{\mathcal{I}_k}||_F^2. \tag{9}$$

The PIC-SIC decoding of the DSTBC is performed as given by the following algorithm. The decoder is initialized with $k = 1$ and $y_1 = y$.

- Step 1: Decode the k^{th} vector of information symbols as
$$\hat{x}_{\mathcal{I}_k} := arg\ \ min_{x_{\mathcal{I}_k} \in \mathcal{A}_{\mathcal{I}_k}} ||\tilde{P}_{\mathcal{I}_k} y_k - \tilde{P}_{\mathcal{I}_k} G_{\mathcal{I}_k} x_{\mathcal{I}_k}||_F^2. \tag{10}$$
- Step 2: Assign
$$y_{k+1} := y_k - G_{\mathcal{I}_k} \hat{x}_{\mathcal{I}_k} \tag{11}$$
and then assign $k := k + 1$.
- Step 3: If $k > g$, stop. Else, go to Step 1.

When rotated lattice constellations are used for encoding the information symbol vectors, sphere decoders [17] can be used to implement (9) and (10).

B. Full-diversity criteria

Let $\mathcal{I} = \{i_1, \ldots, i_{|\Gamma|}\}$ be any non-empty subset of $\{1, \ldots, K\}$ with $i_1 < i_2 < \cdots < i_{|\Gamma|}$. For any $u = [u_1, \ldots, u_{|\mathcal{I}|}]^T \in \mathbb{R}^{|\mathcal{I}|}$, define $X_{\mathcal{I}}(u) = \sum_{j=1}^{|\mathcal{I}|} u_i A_{i_j}$. For any set of vectors $\mathcal{A}$, let $\Delta\mathcal{A} = \{a_1 - a_2 | a_1, a_2 \in \mathcal{A}\}$.

Theorem 1: Full-diversity under PIC decoding: PIC decoding of the DSTBC $\mathcal{C}$ in (5) with the grouping scheme $\mathcal{I}_1, \ldots, \mathcal{I}_g$ achieves a diversity of

- $N\left(1 - \frac{log(logP)}{logP}\right)$ if $N_D = 1$ and
- N if $N_D > 1$,

if the following condition is satisfied for every $k = 1, \ldots, g$: for every $a_k \in \Delta\mathcal{A}_{\mathcal{I}_k} \setminus \{0\}$ and every $u \in \mathbb{R}^{|\mathcal{I}_k^c|}$, the rank of $\left(X_{\mathcal{I}_k}(a_k) + X_{\mathcal{I}_k^c}(u)\right)$ is N.

Proof: Proof is given in Appendix A. ■

Theorem 2: Full-diversity under PIC-SIC decoding: PIC-SIC decoding of the DSTBC $\mathcal{C}$ in (5) with the grouping scheme $\mathcal{I}_1, \ldots, \mathcal{I}_g$ achieves a diversity of

- $N\left(1 - \frac{log(logP)}{logP}\right)$ if $N_D = 1$ and
- N if $N_D > 1$,

if the following condition is satisfied for every $k = 1, \ldots, g$: for every $a_k \in \Delta\mathcal{A}_{\mathcal{I}_k} \setminus \{0\}$ and every $u \in \mathbb{R}^{|\tilde{\mathcal{I}}_k|}$, the rank of $\left(X_{\mathcal{I}_k}(a_k) + X_{\tilde{\mathcal{I}}_k}(u)\right)$ is N.

Proof: Proof is given in Appendix B. ■

The class of PIC and PIC-SIC decoders contains the ZF and ZF-SIC decoders as special cases. When each real symbol x_i, $i = 1, \ldots, K$, forms a group by itself, the PIC decoder reduces to the ZF decoder and the PIC-SIC decoder reduces to the ZF-SIC decoder.

Corollary 1: Full-diversity under ZF and ZF-SIC decoding: The DSTBC $\mathcal{C}$ in (5) achieves a diversity of

- $N\left(1 - \frac{log(logP)}{logP}\right)$ if $N_D = 1$ and
- N if $N_D > 1$,

with ZF decoding and ZF-SIC decoding with any ordering, if the rank of $\sum_{i=1}^{K} u_i A_i$ is N for every $u = [u_1, \ldots, u_K]^T \in \mathbb{R}^K \setminus \{0\}$.

Proof: It is straightforward to show that the criteria of Theorems 1 and 2 are satisfied for the grouping scheme corresponding to ZF and ZF-SIC decoders under the hypothesis of this theorem. ■

In [3], diversity analysis of DSTBCs in relay networks with optimal, i.e., ML, decoding was given. It was shown in [3] that via proper design of DSTBCs a diversity of $N\left(1-\frac{log(logP)}{logP}\right)$ when $N_D = 1$ and a diversity of N when $N_D > 1$ can be achieved via ML decoding when the number of transmit antennas at the source is 1. From Theorems 1 and 2, we see that the same diversities can be achieved by employing less complex PIC and PIC-SIC decoders at the receivers when the DSTBC is appropriately designed. Thus, there is no loss in terms of the achievable diversity while switching the ML decoder at the destination to PIC/PIC-SIC decoders as long as the DSTBC is designed to satisfy the criteria in Theorems 1 and 2. We thus say that the DSTBCs satisfying the conditions in Theorems 1 and 2 achieve *full-diversity* under PIC and PIC-SIC decoding respectively.

The criteria in Theorems 1 and 2 are the same as the criteria given in [11] for an STBC to achieve full-diversity in a point-to-point MIMO channel with PIC/PIC-SIC decoding. Further, these are equivalent to the criteria given in [9], [10] for achieving full-diversity in a point-to-point MIMO channel. Thus, all known full-diversity codes designed for the point-to-point MIMO channel, that are available in the literature and are conjugate linear can be used as DSTBCs to achieve full-diversity in a relay network.

Example 1: Overlapped Alamouti Codes: These codes were constructed in [16] for point-to-point MIMO channels with any number of transmit antennas. These codes are conjugate linear and are known to achieve full-diversity with ZF and ZF-SIC decoding in point-to-point MIMO channels. Hence, these codes can be employed in relay networks and they give full-diversity with ZF and ZF-SIC decoding at the destination.

C. Resistance to relay node failures

Consider a DSTBC $\mathcal{C}$, as in (5), designed for a relay network with N relays and which satisfies the full-diversity condition in Theorem 1 or Theorem 2. Suppose a number of relay nodes stop participating in the cooperative protocol, for some $a \in \{1,\dots,N\}$. This may happen when the nodes leave the network or are switched off. Also, let the destination be aware of the nodes that are currently participating in the cooperative transmission. Then, the DSTBC $\mathcal{C}'$ seen by the destination is the code $\mathcal{C}$ with the a columns corresponding to the failed relay nodes dropped from each codeword matrix. One would like the new DSTBC $\mathcal{C}$ seen by the destination to provide full-diversity in the relay network with $N-a$ relays. This ensures that good error performance is maintained in the network with minimum protocol overhead when a subset of relay nodes stop participating. For $i = 1,\dots,K$, let A'_i be the $T_2 \times (N-a)$ matrix formed by dropping the a columns corresponding to the failed relay nodes from the matrix A_i.

Proposition 2: Let $\mathcal{C}$ satisfy the full-diversity criterion of Theorem 1 (Theorem 2) and let the destination be employed with a PIC (PIC-SIC) decoder. Then the new DSTBC $\mathcal{C}'$, seen by the receiver when a set of a relay nodes, $1 \leq a \leq N$, stop participating in the cooperative transmission, provides full-diversity with PIC (PIC-SIC) decoding for the modified network with $N-a$ relays.

Proof: We give the proof for the case when the destination employs a PIC decoder. The proof for PIC-SIC decoder is similar. Let the grouping scheme be $\mathcal{I}_1,\dots,\mathcal{I}_g$. The DSTBC for the modified network satisfies

$$\mathcal{C}' = \left\{ \sum_{i=1}^{K} x_i A'_i | x_{\mathcal{I}_k} \in \mathcal{A}_{\mathcal{I}_k},\ k=1,\dots,g \right\},$$

which is obtained from the design $\mathbf{X}' = \sum_{i=1}^{K} x_i A'_i$. Also, for every $k = 1,\dots,g$ the following is true: for every $a_k \in \Delta\mathcal{A}_{\mathcal{I}_k} \setminus \{0\}$ and every $u \in \mathbb{R}^{|\mathcal{I}_k^c|}$, we have that the rank of $X_{\mathcal{I}_k}(a_k) + X_{\mathcal{I}_k^c}(u)$ is N. The N columns of the matrix $X_{\mathcal{I}_k}(a_k) + X_{\mathcal{I}_k^c}(u)$ are linearly independent over $\mathbb{C}$. Thus, the $N-a$ columns of the matrix $X'_{\mathcal{I}_k}(a_k) + X'_{\mathcal{I}_k^c}(u)$, formed by dropping the a columns from $X_{\mathcal{I}_k}(a_k) + X_{\mathcal{I}_k^c}(u)$, are also linearly independent. Hence, for every $k = 1,\dots,g$ the following is true: for every $a_k \in \Delta\mathcal{A}_{\mathcal{I}_k} \setminus \{0\}$ and every $u \in \mathbb{R}^{|\mathcal{I}_k^c|}$, we have that the rank of $X'_{\mathcal{I}_k}(a_k) + X'_{\mathcal{I}_k^c}(u)$ is $N-a$. Thus, from Theorem 1, the DSTBC $\mathcal{C}'$ achieves full-diversity in the modified network with $N-a$ relays and PIC decoding. ■

Proposition 2 also tells us that new full-diversity DSTBCs for relay networks with $N-a$ relays can be obtained by simply dropping any set of a columns from a known full-diversity DSTBC for a network with N relays. The rate of the new code and the old code are identical, both in cspcu and bpcu.

IV. A NEW CLASS OF DSTBCS WITH FULL-DIVERSITY PIC/PIC-SIC DECODING

In this section, we construct a new class of DSTBCs with full-diversity PIC/PIC-SIC decoding for all even number of relays N and number of symbols per decoding group $\lambda \leq \frac{N}{2}$. These codes can achieve rates upto $\frac{\lambda}{\lambda+1}$ cspcu for any N. This class of codes includes a family of codes from [11], codes in [14] and the 4 antenna code of [12]. We then show that another family of codes given in [11] for any $N \geq 1$ and $\lambda \leq N$ can be used as DSTBCs with full-diversity PIC/PIC-SIC decoding. These codes too can achieve rates upto $\frac{\lambda}{\lambda+1}$ and they include a class of codes in [13] and the 2 antenna code of [12] as special cases. We also show that the Toeplitz Codes [15] can be used as DSTBCs with full-diversity ZF and ZF-SIC decoding.

A. A new class of codes

Let the number of relay nodes N be even and the number of symbols per group, $\lambda \leq \frac{N}{2}$. Codes for odd number of relays can be obtained by deleting appropriate number of columns from the new codes constructed for even values of N. Let $n \geq 1$ be an integer and $T_2 = N + 2(n-1)$. Number of groups $g = 4n$ and number of real symbols $K = 4n\lambda$. For $k = 1,\dots,g$, the k^{th} group is

$$\mathcal{I}_k = \{(k-1)\lambda+1, (k-1)\lambda+2, \dots, k\lambda\}, \qquad (12)$$

i.e., the first λ symbols form the first group, the second λ symbols form the second group and so on. The symbols x_i,

$i = 1, \ldots, K$, are encoded independently using a regular PAM constellation. For $k = 1, \ldots, g$, define

$$s_{\mathcal{I}_k} = [s_{(k-1)\lambda+1}, \ldots, s_{k\lambda}]^T = Q x_{\mathcal{I}_k},$$

where $Q \in \mathbb{R}^{\lambda \times \lambda}$ is a full-diversity rotation matrix [18], [19]. For $1 \le m \le n$ and $1 \le \ell \le \lambda$, define $\mathbb{A}(m,\ell)$ as in (13), at the top of this next page. Note that $\mathbb{A}(m,\ell)$ is an Alamouti block in real symbols $s_{(4m-4)\lambda+\ell}$, $s_{(4m-3)\lambda+\ell}$, $s_{(4m-2)\lambda+\ell}$ and $s_{(4m-1)\lambda+\ell}$. For $\lambda < \ell < \frac{N}{2}$ and $1 \le m \le n$, $\mathbb{A}(m,\ell)$ is recursively given by $\mathbb{A}(m,\ell) = \mathbb{A}(m,\ell-\lambda)$. The proposed DSTBC is

$$\begin{bmatrix} \mathbb{A}(1,1) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbb{A}(2,1) & \mathbb{A}(1,2) & \cdots & \mathbf{0} \\ \vdots & \mathbb{A}(2,2) & \ddots & \mathbf{0} \\ \vdots & \vdots & \ddots & \mathbb{A}(1,\frac{N}{2}) \\ \vdots & \vdots & \cdots & \mathbb{A}(2,\frac{N}{2}) \\ \vdots & \vdots & \cdots & \vdots \\ \mathbb{A}(n,1) & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbb{A}(n,2) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbb{A}(n,\frac{N}{2}) \end{bmatrix}, \tag{14}$$

where each $\mathbf{0}$ is a 2×2 all zero matrix.

Let $D_\ell = [\mathbb{A}(1,\ell)^T, \mathbb{A}(2,\ell), \ldots, \mathbb{A}(n,\ell)^T]^T$ for $1 \le \ell \le \lambda$ and let $D = [D_1^T, D_2^T, \ldots, D_\lambda^T]^T \in \mathbb{C}^{2n\lambda \times 2}$. The matrix D contains $n\lambda$ Alamouti blocks, placed one below the other. Because of the Alamouti structure, the second column of D is composed of complex variables that are conjugates of the complex variables $s_p \pm i s_q$ appearing in the first column of D. Further, the first column of D contains all the $2n\lambda$ complex symbols $s_p \pm i s_q$ appearing in the design (14). Note that all the entries appearing in the odd columns of (14) are contained in the first column of D and all the entries in the even columns of (14) are contained in the second column of D. If we choose z as the first column of D, then the j^{th} column of the design $j = 1, 3, \ldots, N-1$, can be expressed as $B_j z$ for some $B_j \in \mathbb{C}^{T_2 \times 2n\lambda}$. Similarly, the j^{th} column for $j = 2, 4, \ldots, N$ can be expressed as $B_j^* z^*$ for some $B_j \in \mathbb{C}^{T_2 \times 2n\lambda}$. Thus, the design (14) is conjugate linear and $T_1 = 2n\lambda$ is the length of the vector z that the source transmits to the relays during the broadcast phase. The rate of the DSTBC (14) is

$$R = \frac{K}{2(T_1+T_2)} = \frac{2n\lambda}{2n\lambda + N + 2(n-1)} = \frac{\lambda}{\lambda + 1 + \frac{N-2}{2n}} \text{ cspcu.}$$

By increasing n, rates arbitrarily close to $\frac{\lambda}{\lambda+1}$ can be achieved. However, increasing n also increases the delay parameters T_1 and T_2.

For $\lambda = 1$, we get single-real symbol decodable codes with rates arbitrarily close to $\frac{1}{2}$ for any even number of relays $N \ge 2$. With $\lambda = 2$, we get single-complex symbol (double-real symbol) decodable codes with rates close to $\frac{2}{3}$ for any even number of relays $N \ge 4$.

Proposition 3: The new DSTBCs of this subsection, along with the grouping scheme (12), achieve full-diversity with PIC-SIC decoding.

Proof: We will prove the result for the first group, i.e, $k = 1$. Using a similar argument for each $k = 2, \ldots, g$, we can show that the DSTBC satisfies the hypothesis of Theorem 2 for the grouping scheme (12) and hence achieves full diversity with PIC-SIC decoding. The λ coordinates of the rotated information symbol vector $s_{\mathcal{I}_1}$ act as one of the 4 real symbols in each of the $\frac{N}{2}$ Alamouti blocks $\mathbb{A}(1,1)$, $\mathbb{A}(1,2)$,…,$\mathbb{A}(1,\frac{N}{2})$. Since Q is a full-diversity rotation for the integer lattice, for any $x_{\mathcal{I}_1} \in \Delta\mathcal{A}_{\mathcal{I}_k} \setminus \{0\}$, each of the λ coordinates of $s_{\mathcal{I}_1}$ is non-zero. Hence, each of the matrices $\mathbb{A}(1,1)$, $\mathbb{A}(1,2)$,…,$\mathbb{A}(1,\frac{N}{2})$ is of full-rank. The determinant of the submatrix of $X_{\mathcal{I}_1}(z_{\mathcal{I}_1}) + X_{\tilde{\mathcal{I}}_1}(u)$ for any $u \in \mathbb{R}^{K-\lambda}$ consisting of the first N rows and all the N columns is the product $\prod_{l=1}^{\frac{N}{2}} det(\mathbb{A}(1,l)) \neq 0$. Hence, the matrix $X_{\mathcal{I}_1}(z_{\mathcal{I}_1}) + X_{\tilde{\mathcal{I}}_1}(u)$ is of rank N for any $u \in \mathbb{R}^{K-\lambda}$. ∎

Proposition 4: For $n = 1, 2$, the new DSTBCs of this subsection along with the grouping scheme (12) achieve full-diversity with PIC decoding.

Proof: The proof of this proposition is similar to the proof of Proposition 3, but uses the result of Theorem 1 instead of Theorem 2. ∎

Proposition 5: The subclass of the new DSTBCs of this subsection corresponding to $\lambda = 1$ yield full diversity with ZF and ZF-SIC decoding at the destination.

Proof: The class of codes with $\lambda = 1$ satisfy $\mathbb{A}(m,1) = \mathbb{A}(m,2) = \cdots = \mathbb{A}(m,\frac{N}{2}) = D_m$ for each $m = 1, \ldots, n$. It is enough to show that the DSTBCs corresponding to $\lambda = 1$ satisfy the criterion of Corollary 1. Consider any $u \in \mathbb{R}^K \setminus \{0\}$. The symbol vectors $s_{\mathcal{I}_k} = Q u_{\mathcal{I}_k}$. Since $\lambda = 1$, we have $Q = [1]$ and $s_i = u_i$ for $i = 1, \ldots, K$. The matrices D_m, $m = 1, \ldots, n$ are Alamouti matrices and since at least one s_i, $i = 1, \ldots, K$ is non-zero, there exists an $l \in \{1, \ldots, n\}$ such that $D_m = \mathbf{0}$ for $m = 1, \ldots, l-1$ and $D_l \neq \mathbf{0}$. Note that any non-zero Alamouti matrix is full-ranked and thus $det(D_l) \neq 0$. Thus, the first $l-1$ block diagonals of the matrix (14) are all zeros and the l^{th} block diagonal is composed of D_l, which is full-ranked. The determinant of the submatrix of (14) composed of all the N columns and N consecutive rows starting from the $(2l+1)^{th}$ row is block upper triangular and has determinant $det(D_l)^N \neq 0$. Thus the rank of $X(u)$ is N. This completes the proof. ∎

The new class of codes includes the family of codes given in [11], [14] in the context of point-to-point MIMO channel as a special case. The codes in [11], [14] are exactly the subset of the new codes corresponding to $\lambda = \frac{N}{2}$. When $N = 4$ and $n = 2$, we get the 4 antenna code reported in [12] in the context of point-to-point MIMO channel as special case. Thus, the codes in [12], [14] can be used as DSTBCs to achieve full-diversity in relay channels.

$$\mathbb{A}(m,\ell) = \begin{bmatrix} s_{(4m-4)\lambda+\ell} + is_{(4m-3)\lambda+\ell} & s_{(4m-2)\lambda+\ell} + is_{(4m-1)\lambda+\ell} \\ -s_{(4m-2)\lambda+\ell} + is_{(4m-1)\lambda+\ell} & s_{(4m-4)\lambda+\ell} - is_{(4m-3)\lambda+\ell} \end{bmatrix}. \quad (13)$$

B. A family of codes from [11]

In [11], a family of codes were constructed which give full-diversity with PIC-SIC decoding when used in a point-to-point MIMO channel. We now show that these codes can be used as DSTBCs and they give full-diversity in a rely network also.

Let number of relays in the network be any integer $N \geq 1$ and the number of real symbols per groups $\lambda \leq N$. Let $n \geq 1$ be an integer. The number of groups $g = 2n$ and the number of real symbols in the design is $K = 2n\lambda$. Let the k^{th} group, $k = 1, \ldots, g$, be

$$\mathcal{I}_k = \{(k-1)\lambda + 1, (k-1)\lambda + 2, \ldots, k\lambda\}, \quad (15)$$

i.e., the first λ symbols form the first group, the second λ symbols form the second group and so on. The real symbols x_i, $i = 1, \ldots, K$, are encoded independently of each other using a regular PAM constellation. For $k = 1, \ldots, g$, define

$$s_{\mathcal{I}_k} = [s_{(k-1)\lambda+1}, \ldots, s_{k\lambda}]^T = Qx_{\mathcal{I}_k},$$

where $Q \in \mathbb{R}^{\lambda\times\lambda}$ is a full-diversity rotation matrix. Define a set of doubly indexed variables $v_{m,\ell}$, $1 \leq m \leq n$ and $1 \leq \ell \leq N$ as follows. For $m = 1, \ldots, n$ and $\ell = 1, \ldots, \lambda$, define $v_{m,\ell} = s_{(2m-2)\lambda+\ell} + is_{(2m-1)\lambda+\ell}$. For $m = 1, \ldots, n$ and $\ell = \lambda+1, \ldots, N$, define the variables $v_{m,\ell}$ recursively as $v_{m,\ell} = v_{m,\ell-\lambda}$. The vector $[v_{m,1}, v_{m,2}, \ldots, v_{m,N}]^T$ encodes the two symbol vectors $x_{\mathcal{I}_{2m-1}}$ and $x_{\mathcal{I}_{2m}}$. The code proposed in [11] is

$$\begin{bmatrix} v_{1,1} & 0 & 0 & \cdots & 0 \\ v_{2,1} & v_{1,2} & 0 & \cdots & 0 \\ v_{3,1} & v_{2,2} & v_{1,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \cdots & v_{1,N} \\ v_{n,1} & v_{n-1,2} & \cdots & \cdots & \vdots \\ 0 & v_{n,2} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & v_{n,N} \end{bmatrix}. \quad (16)$$

It is clear that $T_2 = N + n - 1$. All the entries in the design (16) are of the form $s_{(2m-2)\lambda+\ell} + is_{(2m-1)\lambda+\ell}$ and no conjugates of these variables appear in the design. Let $t_m = [v_{m,1}, v_{m,2}, \ldots, v_{m,\lambda}]^T$, $m = 1, \ldots, n$ and let the vector transmitted by the source be $z = [t_1^T \; t_2^T \cdots t_\lambda]^T$. Since z contains all the complex symbols $s_{(2m-2)\lambda+\ell} + is_{(2m-1)\lambda+\ell}$ that appear in the design (16), there exist matrices $B_j \in \mathbb{C}^{T_2\times n\lambda}$, $j = 1, \ldots, N$, such that the design (16) equals $[B_1 z \; B_2 z \cdots B_N z]$. Thus, (16) is a conjugate linear design.

The length of the vector z transmitted by the source is $T_1 = n\lambda$. The rate of the DSTBC is

$$R = \frac{K}{2(T_1 + T_2)} = \frac{n\lambda}{n\lambda + N + n - 1} = \frac{\lambda}{\lambda + 1 + \frac{N-1}{n}} \text{ cspcu.}$$

By increasing n, rates close to $\frac{\lambda}{\lambda+1}$ can be achieved. The codes of this subsection have lower rates than the codes of Section IV-A for identical delay $T_1 + T_2$. However, the class of DSTBCs of this subsection include codes for all values of $\lambda = 1, \ldots, N$ and any $N \geq 1$, whereas the codes in Section IV-A are only for $\lambda = 1, \ldots, \frac{N}{2}$ and even values of N.

In [11], in the context of STBCs for point-to-point MIMO channels, it was shown that the class of codes of this subsection satisfies the criterion in Theorem 2 and the subclass of codes corresponding to $n = 1, 2$ satisfy the criterion in Theorem 1. Thus, the class of DSTBCs of this subsection give full-diversity with PIC-SIC decoding under the grouping scheme (15) and the subclass of codes corresponding to $n = 1, 2$ give full-diversity under PIC decoding with grouping scheme (15).

The family of DSTBCs of this subsection include a class of codes from [13] with a lower complexity grouping scheme [11] (corresponding to $\lambda = N$) and the Toeplitz Codes [15] (corresponding to $\lambda = 1$) as special cases. For $N = 2$ and $n = 2$, we get the 2 antenna code given in [12] in the context of point-to-point MIMO channels as special case.

Proposition 6: The Toeplitz codes give full-diversity as DSTBCs when a ZF or ZF-SIC receiver is used at the destination.

Proof: The Toeplitz codes correspond to the case $\lambda = 1$, i.e., $v_{m,1} = v_{m,2} = \cdots = v_{m,N} = w_m$ for each $m = 1, \ldots, n$. It is enough to show that the Toeplitz codes satisfy the hypothesis of Corollary 1. In order to prove the hypothesis of Corollary 1, it is enough to show that for any set of values $[w_1, \ldots, w_n]^T \in \mathbb{C}^n \setminus \{0\}$, the resulting matrix (16) is full-ranked. Since $[w_1, \ldots, w_n]^T$ is a non-zero vector, there exists an $l \in \{1, \ldots, n\}$ such that $w_m = 0$ for $m = 1, \ldots, l-1$ and $w_l \neq 0$. Thus, the first $l-1$ diagonal layers of (16) are zero and all the entries of the l^{th} diagonal are non-zero. This implies that the resulting matrix has linearly independent columns and hence is full-ranked. ■

In [20], it was shown that the Toeplitz codes yield full diversity with a ZF or MMSE receiver when $N_D = 1$. Proposition 6 says that the Toeplitz codes give full diversity for any $N_D \geq 1$ with ZF and ZF-SIC decoding at the destination.

V. Comparison with Mulitgroup ML decodable full-diversity DSTBCs

In this section, we first compare the rates achievable by the full-diversity PIC/PIC-SIC decodable codes of Section IV with that of the multigroup ML decodable full-diversity DSTBCs available in the literature. It is shown that, the proposed codes in Section IV achieve higher rates than the multigroup ML decodable codes of same decoding complexities. The higher rates achieved by the proposed codes can lead to better Bit Error Rate (BER) performance. In the second half of this section, we compare the BER performance of the new codes of Section IV-A with multigroup ML decodable DSTBCs of same decoding complexity for a few specific network configurations. The simulation results show that the new codes have a better BER performance than the known low decoding complexity DSTBCs in the literature.

A. Comparison of achievable rates

The decoding complexity of a multigroup ML decoding DSTBC or a PIC/PIC-SIC decodable DSTBC is determined by the number of real symbols per decoding group, λ. If a sphere decoder is used at the destination to decode the DSTBC, the dimension of the sphere decoding algorithm will be equal to λ. Thus, it is desirable that a DSTBC have a high R and a low λ. Further, two DSTBCs having the same value of λ have similar decoding complexity. Table I summarizes the comparison of the proposed codes in this paper with other low complexity DSTBCs available in the literature.

(i) $\lambda = 1$*:* Single real symbol ML decodable DSTBCs called Distributed Orthogonal Space-Time Block Codes (DOSTBCs) were constructed in [4] for any number of relays $N \geq 2$. The rate of these codes is upper bounded by $\frac{2}{2+N}$ cspcu. As the number of relays increases, the rate decreases rapidly. In [5], single real symbol ML decodable DSTBCs were constructed for any number of relays N with rate $\frac{1}{4}$ cspcu. The new codes of Section IV-A and the codes in Section IV-B corresponding to $\lambda = 1$ can achieve rates upto $\frac{1}{2}$ cspcu, which is twice the maximum rate reported in the literature so far for single real symbol decodable full-diversity DSTBCs.

(ii) $\lambda = 2$*:* Single complex (double real) symbol ML decodable codes for any number of relays $N \geq 4$ with rate at the most $\frac{4}{4+N}$ cspcu were constructed in [7]. The single complex symbol PIC-SIC decodable codes of Section IV achieve rates upto $\frac{2}{3}$ cspcu irrespective of the number of relays. The codes in [7] have rates less than $\frac{1}{2}$ cspcu and the rate decreases with increase in the number of relay nodes.

(iii) $\lambda = \frac{N}{2}$*:* In [8], 4-group ML decodable DSTBCs were constructed for any number of relays $N = 2m$, $m \geq 1$, with rate $\frac{1}{2}$ cspcu. The number of real symbols per ML decoding group is $\frac{N}{2}$. For $\lambda = \frac{N}{2}$, the codes in Sections IV-A and IV-B can achieve rates arbitrarily close to $\frac{N}{N+2}$. Thus, the new codes have higher rate than the codes in [8] when the number of relays $N > 2$. For $N = 2$, the code in Section IV-A with $n = 1$ and $\lambda = 1$ gives a rate of $\frac{1}{2}$ cspcu and this code is same as the Alamouti code [21].

(iv) $\lambda = N$*:* Using commuting set of matrices from Division Algebras [22], 2-group ML decodable, rate $\frac{1}{2}$ DSTBCs with $\lambda = N$ were constructed in [23] for even number of relays N. The codes in Section IV-B with $\lambda = N$ achieve rates upto $\frac{N}{N+1}$ and hence have higher rate than the codes in [23] for all $N > 1$ number of relays.

For same value of λ and equal delay $T_1 + T_2$, the new codes in Section IV-A have higher rate than the codes from Section IV-B. However, the codes in Section IV-A are only for $1 \leq \lambda \leq \frac{N}{2}$, whereas the codes in Section IV-B are for $1 \leq \lambda \leq N$.

B. Simulation Results

For all the codes we use the power allocation $\pi_1 = 1$ and $\pi_2 = \frac{1}{R}$. This is the optimal power allocation when the destination uses an ML decoder to decode a DSTBC [3]. We first compare the performance of the new single real symbol PIC-SIC decodable, i.e., ZF-SIC decodable code of Section IV-A with the single real symbol ML decodable code

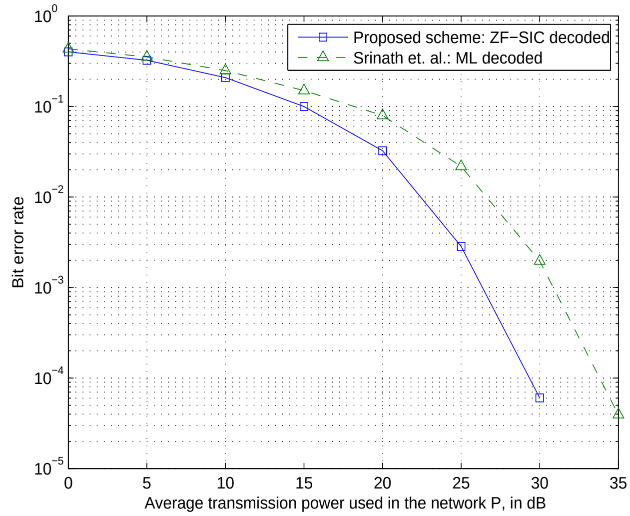

Fig. 2. BER performance at 2 bpcu for $\lambda = 1$, $N = 8$ and $N_D = 1$

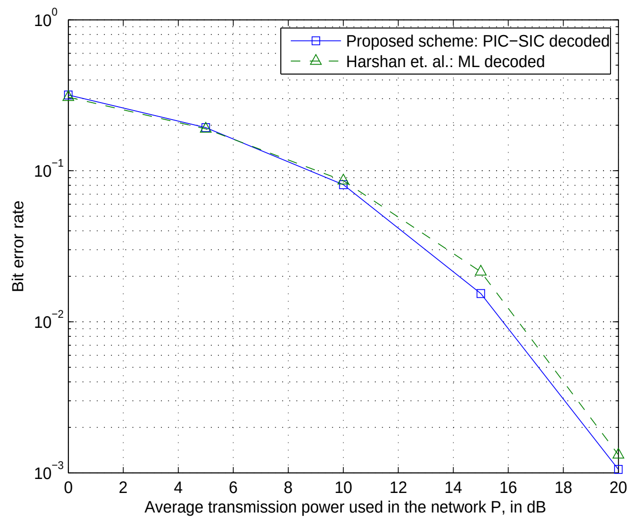

Fig. 3. BER performance at 2 bpcu for $\lambda = 2$, $N = 6$ and $N_D = 4$

TABLE I
COMPARISON OF FULL-DIVERSITY, LOW DECODING COMPLEXITY DSTBCS

	DOSTBCs [4]	Srinath et. al [5]	Harshan et. al. [7]	Rajan et. al. [8]	Kiran et. al [23]	**Codes in Sec. IV-A**	**Codes in Sec.IV-B**
Number of relays, N	≥ 2	≥ 1	≥ 4	$2m,\, m \geq 1$	$2m,\, m \geq 1$	$\mathbf{2m,\, m \geq 1}$	$\geq \mathbf{1}$
Real symbols per group, λ	1	1	2	$\frac{N}{2}$	N	$\leq \frac{\mathbf{N}}{\mathbf{2}}$	$\leq \mathbf{N}$
Rate, R	$\frac{2}{2+N}$ †	$\frac{1}{4}$	$\frac{4}{4+N}$ †	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{\lambda}{\lambda+1}$ ‡	$\frac{\lambda}{\lambda+1}$ ‡
Full-diversity decoding method	ML	ML	ML	ML	ML	**PIC-SIC**	**PIC-SIC**

† Upper bound on achievable rate. ‡Supremum of achievable rates.

of Srinath et. al [5] for number of relays $N = 8$ and number of antennas at the destination $N_D = 1$. The new code has parameters $n = 3$, $\lambda = 1$ and $R = \frac{1}{3}$ cspcu. The real symbols are encoded using regular 8-PAM. The code from [5] has rate $\frac{1}{4}$ cspcu and the symbols are encoded using regular 16-PAM. Both the DSTBCs have an information rate of 2 bpcu and Gray mapping is used to map bits to symbols for both codes. The new code is ZF-SIC decoded whereas the code from [5] is ML decoded. The BER performance is shown in Fig. 2. It is seen that the new code performs considerably better than the code in [5]. At a BER of 10^{-4} the new code beats the code from [5] by about 5 dB.

The comparison of performance of single complex decodable code given by Harshan et. al. [7] with the new code in Section IV-A for $N = 6$ relays and $N_D = 4$ antennas at the destination is shown in Fig. 3. The new code has parameters $\lambda = 2$, $n = 2$ and $R = \frac{1}{2}$ cspcu. The symbols are encoded using regular 4-PAM. The code from [7] has a rate of $\frac{1}{3}$ cspcu and the symbols are encoded pairwise using rotated 64-QAM constellation. Both the DSTBCs have an information rate of 2 bpcu and Gray mapping is used in both the cases to map information bits to symbols. The new code is PIC-SIC decoded whereas the code from [7] is ML decoded. From Fig. 3, we see that the new code performs better than the code from [7].

VI. DISCUSSION

In this paper, we have derived full-diversity criteria for a DSTBC to achieve full diversity with PIC/PIC-SIC decoding performed at the destination for a wireless relay network with single antenna at the source and the relays. We have also proposed DSTBCs with low complexity PIC/PIC-SIC decoders that achieve higher rates and perform better than the best known low complexity DSTBCs available in the literature. The following questions remain open.

- What is the full diversity criterion for PIC/PIC-SIC decoding in the case of relay networks in which all the nodes are employed with multiple antennas?
- What is the criterion to maximize the coding gain when the destination performs PIC/PIC-SIC decoding?
- What is the full diversity criterion for PIC/PIC-SIC decoding for multihop relay networks with amplify and forward cooperation?
- In this paper, the full-diversity criterion was derived only for conjugate linear DSTBCs. Does this condition hold for a general DSTBC?
- What are the optimal power allocation factors π_1 and π_2 for PIC/PIC-SIC decoding?
- In this paper, we have considered relay networks that are coherent (destination has full channel state information) and synchronous (transmissions from relays to destination are synchronized at symbol level). Finding a full-diversity criterion and constructing high-rate, full-diversity DSTBCs with low PIC/PIC-SIC decoding complexity for asynchronous and non-coherent relay networks is an interesting direction for future work.

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APPENDIX A
PROOF OF THEOREM 1

For any two square matrices A and B of same dimension, the notation $A \preceq B$ implies that $B - A$ is a positive semidefinite (PSD) matrix. Let $\beta = max\left\{||\bar{B}_j||_F^2|\ j = 1, \ldots, N\right\}$ and $\alpha = T_2 N_D + \frac{\beta \pi_2 P}{\pi_1 P + 1} \sum_{j=1}^{N} \sum_{l=1}^{N_D} |g_{j,l}|^2$. We use the following proposition to prove Theorem 1.

Proposition 7: The covariance matrix of $\widetilde{vec}(U)$, Γ satisfies $\Gamma \preceq \alpha I_{2T_2N_D}$.

Proof: Since Γ is a covariance matrix, it is a PSD matrix and hence has a complete set of eigenvalues. If λ_i, $i = 1, \ldots, 2T_2N_D$, are the eigenvalues of Γ, then we have: $\Gamma \preceq \sum_{i=1}^{2T_2N_D} \lambda_i I = Tr(\Gamma) I$. From (7), $Tr(\Gamma) = Tr(\Gamma'_{Re})$. Note that Γ' is itself a covariance matrix and hence is Hermitian. Thus, all of its diagonal entries are real and hence, $Tr(\Gamma'_{Re}) = Tr(\Gamma')$. From Proposition 1, we have: $Tr(\Gamma) = \sum_{l=1}^{N_D} Tr(C_{l,l})$, where $C_{l,l}$ is given in Proposition 1. Thus, $Tr(\Gamma)$

$$= \sum_{l=1}^{N_D} \left[\frac{\pi_2 P}{\pi_1 P + 1} \sum_{j=1}^{N} |g_{j,l}|^2 Tr(\bar{B}_j \bar{B}_j^H) + T_2 \right]$$

$$= \frac{\pi_2 P}{\pi_1 P + 1} \sum_{j=1}^{N} \sum_{l=1}^{N_D} |g_{j,l}|^2 ||\bar{B}_j||_F^2 + T_2 N_D$$

$$\leq \frac{\pi_2 P}{\pi_1 P + 1} \sum_{j=1}^{N} \sum_{l=1}^{N_D} |g_{j,l}|^2 \beta + T_2 N_D = \alpha.$$

The desired result follows. ∎

Consider the use of $\mathcal{C}$ in (5) as an STBC for the point-to-point MIMO channel with N transmit antennas and N_D receive antennas,

$$Y_{MIMO} = \sqrt{\rho} XH + W. \tag{17}$$

Here, Y_{MIMO} is the $T \times N_D$ received matrix, X is the $T \times N$ codeword matrix, H is the $N \times N_D$ channel gain matrix, W is the $T \times N_D$ noise matrix and ρ is the transmit power. The received matrix can be vectorized as

$$\begin{aligned} y_{MIMO} &= \widetilde{vec}(Y_{MIMO}) = \sum_{i=1}^{K} x_i \sqrt{\rho} A_i H + \widetilde{vec}(W) \\ &= G'x + \widetilde{vec}(W), \end{aligned}$$

where,

$$G' = \sqrt{\rho}[\widetilde{vec}(A_1 H) \cdots \widetilde{vec}(A_K H)] = [g'_1 \ g'_2 \cdots g'_K], \tag{18}$$

and is similar to (6). For any nonempty subset $\mathcal{I} = \{i_1, \ldots, i_{|\mathcal{I}|}\} \subset \{1, \ldots, K\}$, with $i_1 < i_2 < \cdots < i_{|\mathcal{I}|}$, let $G'_{\mathcal{I}} = [g'_{i_1} \ g'_{i_2} \cdots g'_{i_{\mathcal{I}}}]$. For $k = 1, \ldots, g$, let $V'_{\mathcal{I}_k}$ be the column space of the matrix $G'_{\mathcal{I}_k^c}$ and $P'_{\mathcal{I}_k}$ be the matrix that projects a vector onto the subspace ${V'_{\mathcal{I}_k}}^{\perp}$.

Theorem 3 ([10]): If the following two conditions hold:

1) for any $X_1, X_2 \in \mathcal{C}$ and $X_1 \neq X_2$ we have $rank(X_1 - X_2)$ is N and
2) for every $k = 1, \ldots, g$, every $H \neq \mathbf{0}$ and every $a_k \in \Delta\mathcal{A}_{\mathcal{I}_k} \setminus \{0\}$, we have $G'_{\mathcal{I}_k} a_k \notin V'_{\mathcal{I}_k}$,

then, there exists a real number $c > 0$ such that, for any $k = 1, \ldots, g$, any $a_k \in \Delta\mathcal{A}_{\mathcal{I}_k} \setminus \{0\}$ and any channel realization $H \neq \mathbf{0}$, we have: $||P'_{\mathcal{I}_k} G'_{\mathcal{I}_k} a_k||_F^2 > c\rho ||H||_F^2$.

The results of Theorem 3 are independent of the statistics of the channel H or the noise W. The matrix G' in Theorem 3 is identical to the matrix (6), which arises during the PIC decoding of the DSTBC $\mathcal{C}$ in a relay network with N relays and N_D antennas at the destination, when $\rho = \frac{\pi_1 \pi_2 P^2}{\pi_1 P + 1}$. Hence, the result of Theorem 3 can be used to prove diversity results for the relay network if the two conditions in the theorem are satisfied. Now, the criterion in the hypothesis of Theorem 1 is same as the criterion given in [11] for the STBC $\mathcal{C}$ to achieve full-diversity in the point-to-point MIMO channel (17) with PIC decoding under the grouping scheme $\mathcal{I}_1, \ldots, \mathcal{I}_g$. Further, it is shown in [11] that this criterion is equivalent to the sufficient condition given in Theorem 3. Thus, for any $k = 1, \ldots, g$, any $a_k \in \Delta\mathcal{A}_{\mathcal{I}_k} \setminus \{0\}$ and any channel realization (3) $H \neq \mathbf{0}$ of the relay network (1) , we have:

$$||P'_{\mathcal{I}_k} G'_{\mathcal{I}_k} a_k||_F^2 > c\rho ||H||_F^2, \tag{19}$$

where $\rho = \frac{\pi_1 \pi_2 P^2}{\pi_1 P + 1}$.

Let $k \in \{1, \ldots, g\}$. We are interested in deriving an upper bound on the pairwise error probability during PIC decoding (9) of the k^{th} symbol vector $x_{\mathcal{I}_k}$. the received vector y in (8) satisfies:

$$y = Gx + n = \sum_{k=1}^{g} G_{\mathcal{I}_k} x_{\mathcal{I}_k} + n.$$

Since $P_{\mathcal{I}_k}$ is the projection onto the subspace which is orthogonal to the subspace spanned by the column vectors of $G_{\mathcal{I}_\ell}$, $1 \leq \ell \leq g$ and $\ell \neq k$, we have that $P_{\mathcal{I}_k} G_{\mathcal{I}_\ell} = \mathbf{0}$ for $1 \leq \ell \leq g$ and $\ell \neq k$. Thus,

$$P_{\mathcal{I}_k} y = \sum_{\ell=1}^{g} P_{\mathcal{I}_k} G_{\mathcal{I}_\ell} x_{\mathcal{I}_\ell} + P_{\mathcal{I}_k} n = P_{\mathcal{I}_k} G_{\mathcal{I}_k} x_{\mathcal{I}_k} + P_{\mathcal{I}_k} n.$$

The PIC decoder (9) is

$$\begin{aligned} & arg \ \ min_{\tilde{x}_{\mathcal{I}_k} \in \mathcal{A}_{\mathcal{I}_k}} ||P_{\mathcal{I}_k} y - P_{\mathcal{I}_k} G_{\mathcal{I}_k} \tilde{x}_{\mathcal{I}_k}||_F^2 \\ &= arg \ \ min_{\tilde{x}_{\mathcal{I}_k} \in \mathcal{A}_{\mathcal{I}_k}} ||P_{\mathcal{I}_k} G_{\mathcal{I}_k} x_{\mathcal{I}_k} + P_{\mathcal{I}_k} n - P_{\mathcal{I}_k} G_{\mathcal{I}_k} \tilde{x}_{\mathcal{I}_k}||_F^2 \\ &= arg \ \ min_{\tilde{x}_{\mathcal{I}_k} \in \mathcal{A}_{\mathcal{I}_k}} ||P_{\mathcal{I}_k} G_{\mathcal{I}_k} (x_{\mathcal{I}_k} - \tilde{x}_{\mathcal{I}_k}) + P_{\mathcal{I}_k} n||_F^2. \end{aligned}$$

The Gaussian noise vector n is white and has zero mean. Since $P_{\mathcal{I}_k}$ is the projection onto $V_{\mathcal{I}_k}^{\perp}$, the noise vector $P_{\mathcal{I}_k} n$ has no components in the subspace $V_{\mathcal{I}_k}$ and the component of $P_{\mathcal{I}_k} n$ along the subspace $V_{\mathcal{I}_k}^{\perp}$ is a white Gaussian noise with zero mean and unit covariance. The difference signal vector $P_{\mathcal{I}_k} G_{\mathcal{I}_k} (x_{\mathcal{I}_k} - \tilde{x}_{\mathcal{I}_k})$ also lies in the subspace $V_{\mathcal{I}_k}^{\perp}$. Thus, the probability that the PIC decoder will decide in favor of $\tilde{x}_{\mathcal{I}_k}$ when the symbol $x_{\mathcal{I}_k}$ is transmitted, given the channel realization H, is $\mathsf{PEP}(x_{\mathcal{I}_k} \to \tilde{x}_{\mathcal{I}_k} | H) = \mathcal{Q}\left(\frac{||P_{\mathcal{I}_k} G_{\mathcal{I}_k} a_k||_F}{2}\right)$, where $\mathcal{Q}(\cdot)$ is the Gaussian tail function and $a_k = x_{\mathcal{I}_k} - \tilde{x}_{\mathcal{I}_k}$. Using the Chernoff bound on the $\mathcal{Q}$ function, we have

$$\mathsf{PEP}(x_{\mathcal{I}_k} \to \tilde{x}_{\mathcal{I}_k} | H) \leq exp\left(\frac{-||P_{\mathcal{I}_k} G_{\mathcal{I}_k} a_k||_F^2}{4}\right). \tag{20}$$

In order to derive a lower bound on $||P_{\mathcal{I}_k} G_{\mathcal{I}_k} a_k||_F^2$, we want to express $P_{\mathcal{I}_k}$ and $G_{\mathcal{I}_k}$ in terms of $P'_{\mathcal{I}_k}$ and $G'_{\mathcal{I}_k}$. Let $A = \Gamma^{-\frac{1}{2}}$ denote the square root of Γ^{-1}. Since Γ and Γ^{-1} are PSD and symmetric, A can be chosen to be the unique PSD symmetric square root of Γ^{-1} [24]. From Proposition 7, we have $A \succeq \frac{1}{\sqrt{\alpha}} I_{2T_2 N_D}$. Let $Q_{\mathcal{I}_k}^T$ be a matrix whose columns form an orthonormal basis of $V_{\mathcal{I}_k}^{\perp}$ and $Q'^T_{\mathcal{I}_k}$ be a matrix whose columns form an orthonormal basis of $V'^{\perp}_{\mathcal{I}_k}$. Thus, $P_{\mathcal{I}_k} = Q_{\mathcal{I}_k}\left(Q_{\mathcal{I}_k}^T Q_{\mathcal{I}_k}\right)^{-1} Q_{\mathcal{I}_k}^T$ and $P'_{\mathcal{I}_k} = Q'_{\mathcal{I}_k}\left(Q'^T_{\mathcal{I}_k} Q'_{\mathcal{I}_k}\right)^{-1} Q'^T_{\mathcal{I}_k}$. Also, for any vector $v \in \mathbb{R}^{2T_2 N_D}$, we have $||P_{\mathcal{I}_k} v||_F = ||Q_{\mathcal{I}_k} v||_F$ and $||P'_{\mathcal{I}_k} v||_F = ||Q'_{\mathcal{I}_k} v||_F$. Since $G = AG'$, it is clear that

$$G_{\mathcal{I}_k} = AG'_{\mathcal{I}_k}, \tag{21}$$

and $V_{\mathcal{I}_k} = AV'_{\mathcal{I}_k} = \{Av | v \in V'_{\mathcal{I}_k}\}$.

Proposition 8: For each $k = 1, \ldots, g$, we have $V_{\mathcal{I}_k}^{\perp} = A^{-1} V'^{\perp}_{\mathcal{I}_k}$.

Proof: We have,

$$\begin{aligned} V_{\mathcal{I}_k}^{\perp} &= (AV'_{\mathcal{I}_k})^{\perp} = \{w | w^T A v = 0 \ \forall v \in V'_{\mathcal{I}_k}\} \\ &= \{w | (A^T w)^T v = 0 \ \forall v \in V'_{\mathcal{I}_k}\}. \end{aligned}$$

Replacing $A^T w = Aw$ by u, we have,

$$\begin{aligned} V_{\mathcal{I}_k}^{\perp} &= \{A^{-1} u | u^T v = 0 \ \forall v \in V'_{\mathcal{I}_k}\} \\ &= A^{-1}\{u | u^T v = 0 \ \forall v \in V'_{\mathcal{I}_k}\} = A^{-1} V'^{\perp}_{\mathcal{I}_k}. \end{aligned}$$

This completes the proof. ∎

From Proposition 8, it is clear that $V_{\mathcal{I}_k}^{\perp}$ is spanned by the column vectors of the matrix $A^{-1}Q_{\mathcal{I}_k}^{\prime T}$. Thus,

$$\begin{aligned} P_{\mathcal{I}_k} &= A^{-1}Q_{\mathcal{I}_k}^{\prime T}\left(Q'_{\mathcal{I}_k}A^{-1^T}A^{-1}Q_{\mathcal{I}_k}^{\prime T}\right)^{-1}Q'_{\mathcal{I}_k}A^{-1^T} \\ &= A^{-1}Q_{\mathcal{I}_k}^{\prime T}\left(Q'_{\mathcal{I}_k}A^{-2}Q_{\mathcal{I}_k}^{\prime T}\right)^{-1}Q'_{\mathcal{I}_k}A^{-1}. \end{aligned} \tag{22}$$

Proposition 9: For any $k = 1, \ldots, g$ and any $a_k \in \Delta\mathcal{A}_{\mathcal{I}_k} \setminus \{0\}$, we have: $||P_{\mathcal{I}_k}G_{\mathcal{I}_k}a_k||_F^2 > \frac{c\rho||H||_F^2}{\alpha}$.

Proof: Consider $||P_{\mathcal{I}_k}G_{\mathcal{I}_k}a_k||_F^2 = a_k^T G_{\mathcal{I}_k}^T P_{\mathcal{I}_k}^T P_{\mathcal{I}_k}G_{\mathcal{I}_k}a_k$. Using (21) and (22), we get $||P_{\mathcal{I}_k}G_{\mathcal{I}_k}a_k||_F^2$

$$\begin{aligned} &= a_k^T G_{\mathcal{I}_k}^{\prime T}Q_{\mathcal{I}_k}^{\prime T}\left(Q'_{\mathcal{I}_k}A^{-2}Q_{\mathcal{I}_k}^{\prime T}\right)^{-1}Q'_{\mathcal{I}_k}G'_{\mathcal{I}_k}a_k \\ &= ||\left(Q'_{\mathcal{I}_k}A^{-2}Q_{\mathcal{I}_k}^{\prime T}\right)^{-\frac{1}{2}}Q'_{\mathcal{I}_k}G'_{\mathcal{I}_k}a_k||_F^2 \end{aligned}$$

Since $A \succeq \frac{1}{\sqrt{\alpha}}I_{2T_2N_D}$ and the rows of $Q'_{\mathcal{I}_k}$ are orthonormal, we have $\left(Q'_{\mathcal{I}_k}A^{-2}Q_{\mathcal{I}_k}^{\prime T}\right)^{-\frac{1}{2}} \succeq \frac{1}{\sqrt{\alpha}}(Q'_{\mathcal{I}_k}Q_{\mathcal{I}_k}^{\prime T})^{-\frac{1}{2}} = \frac{1}{\sqrt{\alpha}}I$. Thus, $||P_{\mathcal{I}_k}G_{\mathcal{I}_k}a_k||_F^2$

$$\begin{aligned} &\geq ||\frac{1}{\sqrt{\alpha}}IQ'_{\mathcal{I}_k}G'_{\mathcal{I}_k}a_k||_F^2 = \frac{1}{\alpha}||Q'_{\mathcal{I}_k}G'_{\mathcal{I}_k}a_k||_F^2 \\ &= \frac{1}{\alpha}||P'_{\mathcal{I}_k}G'_{\mathcal{I}_k}a_k||_F^2 > \frac{c\rho||H||_F^2}{\alpha}. \end{aligned}$$

The last step follows from (19). This completes the proof. ■

Using the bound from Proposition 9 with (20) we get

$$\mathsf{PEP}(x_{\mathcal{I}_k} \to \tilde{x}_{\mathcal{I}_k}|H) \leq exp\left(-\frac{c\rho||H||_F^2}{4\alpha}\right).$$

From (3), we have $||H||_F^2 = \sum_{j=1}^{N}|f_j|^2\left(\sum_{l=1}^{N_D}|g_{j,l}|^2\right)$. The squared absolute values of the channel gains $|f_j|^2$ and $|g_{j,l}|^2$ are all independent of each other and are exponential random variables with unit mean. Let $t_j = \sum_{l=1}^{N_D}|g_{j,l}|^2$, for $j = 1, \ldots, N$. Then, the random variables $|f_j|^2 t_j$, $j = 1, \ldots, N_D$ are independent of each other. Further, $exp\left(-\frac{c\rho||H||_F^2}{4\alpha}\right) = \prod_{j=1}^{N} exp\left(-\frac{c\rho|f_j|^2 t_j}{4\alpha}\right)$. Since $|f_j|^2$ is exponentially distributed with unit mean, for any $s > 0$, we have $\mathsf{E}(exp\left(-s|f_j|^2\right)) = \frac{1}{1+s}$, for $j = 1, \ldots, N$. Thus, the average pairwise error probability $\mathsf{PEP}(x_{\mathcal{I}_k} \to \tilde{x}_{\mathcal{I}_k})$

$$\begin{aligned} &\leq \mathsf{E}\left(exp\left(-\frac{c\rho||H||_F^2}{4\alpha}\right)\right) = \mathsf{E}\left(\prod_{j=1}^{N} exp\left(-\frac{c\rho|f_j|^2 t_j}{4\alpha}\right)\right) \\ &= \mathsf{E}\left(\prod_{j=1}^{N}\frac{1}{1+\frac{c\rho t_j}{4\alpha}}\right) = \mathsf{E}\left(\prod_{j=1}^{N}\left(1+\frac{c\rho t_j}{4\alpha}\right)^{-1}\right). \end{aligned}$$

Substituting the values for ρ and α as $\rho = \frac{\pi_1\pi_2P^2}{\pi_1P+1}$ and $\alpha = T_2N_D + \frac{\beta\pi_2P}{\pi_1P+1}\sum_{j'=1}^{N}t_{j'}$, we get $1 + \frac{c\rho t_j}{4\alpha}$

$$= 1 + \frac{c\pi_1\pi_2P^2t_j}{4\left[(\pi_1P+1)T_2N_D + \beta\pi_2P\sum_{j'=1}^{N}t_{j'}\right]}.$$

For large P, $\pi_1P + 1 \approx \pi_1P$. Using this approximation and on further simplification, we get $\mathsf{PEP}(x_{\mathcal{I}_k} \to \tilde{x}_{\mathcal{I}_k})$

$$\leq \mathsf{E}\left(\prod_{j=1}^{N}\left[1+\frac{c\pi_2P}{4T_2N_D}\frac{t_j}{1+\frac{\beta\pi_2}{\pi_2T_2N_D}\sum_{j'=1}^{N}t_{j'}}\right]^{-1}\right). \tag{23}$$

In Theorem 4 of [3] an upper bound for a more general expression is given. The result in [3] for the special case (23) is as follows.

Theorem 4 ([3]): For large P, the pairwise error probability (23) can be upper bounded by c_0P^{-d}, where c_0 is a positive real number and

- $d = N\left(1 - \frac{log(logP)}{logP}\right)$ if $N_D = 1$ and
- $d = N$ if $N_D > 1$.

This completes the proof of Theorem 1.

Appendix B
Proof of Theorem 2

Let $d = \mathbf{1}\{N_D = 1\}N\left(1 - \frac{log(logP)}{logP}\right) + \mathbf{1}\{N_D > 1\}N$ and $\mathsf{P}(\cdot)$ denote the probability of an event. For $k = 1, \ldots, g$, let E_k denote the event that the k^{th} information symbol vector $x_{\mathcal{I}_k}$ is erroneously decoded by the PIC-SIC decoder. We want to prove that $\mathsf{P}(E_1 \cup \cdots \cup E_g) \leq c_0P^{-d}$, for large P and for some positive real number c_0. For $k = 1, \ldots, g$, we have

$$\begin{aligned} \mathsf{P}(E_k) &= \mathsf{P}(E_k|E_1^c \cap \cdots \cap E_{k-1}^c)\mathsf{P}(E_1^c \cap \cdots \cap E_{k-1}^c) \\ &\quad + \mathsf{P}(E_k|E_1 \cup \cdots \cup E_{k-1})\mathsf{P}(E_1 \cup \cdots \cup E_{k-1}) \\ &\leq \mathsf{P}(E_k|E_1^c \cap \cdots \cap E_{k-1}^c)\cdot 1 + 1\cdot\mathsf{P}(E_1 \cup \cdots \cup E_{k-1}) \\ &\leq \mathsf{P}(E_k|E_1^c \cap \cdots \cap E_{k-1}^c) + \sum_{k'=1}^{k-1}\mathsf{P}(E_{k'}). \end{aligned} \tag{24}$$

It is enough to show that $\mathsf{P}(E_k|E_1^c \cap \cdots \cap E_{k-1}^c) \leq c_kP^{-d}$ for $k = 1, \ldots, g$ and some set of positive real numbers $\{c_k\}$. Then, from (24), it can be shown using recursion that

$$\mathsf{P}(E_1 \cup \cdots \cup E_g) \leq \sum_{k=1}^{g}\mathsf{P}(E_k) \leq c_0P^{-d},$$

for some $c_0 > 0$. We now derive the upper bound for $\mathsf{P}(E_k|E_1^c \cap \cdots \cap E_{k-1}^c)$, the probability of erroneously decoding the k^{th} symbol vector when all the previous symbol vectors have been decoded correctly.

Let G' be as defined in (18). For $k = 1, \ldots, g$, let $\tilde{V}'_{\mathcal{I}_k}$ be the column space of the matrix $G'_{\tilde{\mathcal{I}}_k}$ and $\tilde{P}'_{\mathcal{I}_k}$ be the matrix that projects a vector onto the subspace $\tilde{V}_{\mathcal{I}_k}^{\prime\perp}$.

Theorem 5 ([10]): If the following two conditions hold:

1) for any $X_1, X_2 \in \mathcal{C}$ and $X_1 \neq X_2$ we have $rank(X_1 - X_2)$ is N and
2) for every $k = 1, \ldots, g$, every $H \neq \mathbf{0}$ and every $a_k \in \Delta\mathcal{A}_{\mathcal{I}_k} \setminus \{0\}$, we have $G'_{\tilde{\mathcal{I}}_k}a_k \notin \tilde{V}'_{\mathcal{I}_k}$,

then, there exists a real number $c > 0$ such that, for any $k = 1, \ldots, g$, any $a_k \in \Delta\mathcal{A}_{\mathcal{I}_k} \setminus \{0\}$ and any channel realization $H \neq \mathbf{0}$, we have: $||\tilde{P}'_{\mathcal{I}_k}G'_{\mathcal{I}_k}a_k||_F^2 > c\rho||H||_F^2$.

The criterion in the hypothesis of Theorem 2 is same as the criterion given in [11] for the STBC $\mathcal{C}$ to achieve full-diversity in the point-to-point MIMO channel (17) with PIC-SIC decoding under the grouping scheme $\mathcal{I}_1, \ldots, \mathcal{I}_g$. Further, it is shown in [11] that this criterion is equivalent to the sufficient condition given in Theorem 5. Thus, for any $k = 1, \ldots, g$, any $a_k \in \Delta\mathcal{A}_{\mathcal{I}_k} \setminus \{0\}$ and any channel realization (3) $H \neq \mathbf{0}$ of the relay network (1), we have: $||\tilde{P}'_{\mathcal{I}_k} G'_{\mathcal{I}_k} a_k||_F^2 > c\rho ||H||_F^2$, where $\rho = \frac{\pi_1 \pi_2 P^2}{\pi_1 P + 1}$.

Consider the k^{th} iteration in the PIC-SIC decoding algorithm given in Section III-A in the situation where all the previous symbol vectors $x_{\mathcal{I}_1}, \ldots, x_{\mathcal{I}_{k-1}}$ have been decoded correctly. Due to the successive interference cancellation 11, the signal y_k will only have a noisy version of the contributions from symbol vectors $x_{\mathcal{I}_k}, \ldots, x_{\mathcal{I}_g}$, i.e.,

$$y_k = \sum_{\ell=k}^{g} G_{\mathcal{I}_\ell} x_{\mathcal{I}_\ell} + n.$$

Since $\tilde{P}_{\mathcal{I}_k}$ is the projection onto the subspace which is orthogonal to the subspace spanned by the column vectors of $G_{\mathcal{I}_\ell}$, $k < \ell \leq g$, we have that $\tilde{P}_{\mathcal{I}_k} G_{\mathcal{I}_\ell} = \mathbf{0}$ for $k < \ell \leq g$. Thus,

$$\tilde{P}_{\mathcal{I}_k} y_k = \sum_{\ell=k}^{g} \tilde{P}_{\mathcal{I}_k} G_{\mathcal{I}_\ell} x_{\mathcal{I}_\ell} + \tilde{P}_{\mathcal{I}_k} n = \tilde{P}_{\mathcal{I}_k} G_{\mathcal{I}_k} x_{\mathcal{I}_k} + \tilde{P}_{\mathcal{I}_k} n.$$

The PIC-SIC decoder in the k^{th} iteration is

$$\begin{aligned} & arg \quad min_{\tilde{x}_{\mathcal{I}_k} \in \mathcal{A}_{\mathcal{I}_k}} ||\tilde{P}_{\mathcal{I}_k} y_k - \tilde{P}_{\mathcal{I}_k} G_{\mathcal{I}_k} \tilde{x}_{\mathcal{I}_k}||_F^2 \\ = & arg \quad min_{\tilde{x}_{\mathcal{I}_k} \in \mathcal{A}_{\mathcal{I}_k}} ||\tilde{P}_{\mathcal{I}_k} G_{\mathcal{I}_k} x_{\mathcal{I}_k} + \tilde{P}_{\mathcal{I}_k} n - \tilde{P}_{\mathcal{I}_k} G_{\mathcal{I}_k} \tilde{x}_{\mathcal{I}_k}||_F^2 \\ = & arg \quad min_{\tilde{x}_{\mathcal{I}_k} \in \mathcal{A}_{\mathcal{I}_k}} ||\tilde{P}_{\mathcal{I}_k} G_{\mathcal{I}_k} (x_{\mathcal{I}_k} - \tilde{x}_{\mathcal{I}_k}) + \tilde{P}_{\mathcal{I}_k} n||_F^2. \end{aligned}$$

Using an argument similar to the proof of Theorem 1, the probability that the PIC-SIC decoder will decide in favor of $\tilde{x}_{\mathcal{I}_k}$ when the symbol $x_{\mathcal{I}_k}$ is transmitted, given the channel realization H, can be shown to be $\mathsf{PEP}(x_{\mathcal{I}_k} \rightarrow \tilde{x}_{\mathcal{I}_k} | H, E_1^c \cap \cdots \cap E_{k-1}^c) = \mathcal{Q}\left(\frac{||\tilde{P}_{\mathcal{I}_k} G_{\mathcal{I}_k} a_k||_F}{2}\right)$, where $a_k = x_{\mathcal{I}_k} - \tilde{x}_{\mathcal{I}_k}$. Using the Chernoff bound on the $\mathcal{Q}$ function, we have

$$\mathsf{PEP}(x_{\mathcal{I}_k} \rightarrow \tilde{x}_{\mathcal{I}_k} | H, E_1^c \cap \cdots \cap E_{k-1}^c) \leq exp\left(\frac{-||\tilde{P}_{\mathcal{I}_k} G_{\mathcal{I}_k} a_k||_F^2}{4}\right).$$

Using an argument similar to the proof of Theorem 1, it can be shown that

$$||\tilde{P}_{\mathcal{I}_k} G_{\mathcal{I}_k} a_k||_F^2 \geq \frac{1}{\alpha} ||\tilde{P}'_{\mathcal{I}_k} G'_{\mathcal{I}_k} a_k||_F^2 > \frac{c\rho ||H||_F^2}{\alpha},$$

and then the average pairwise error probability, $\mathsf{PEP}(x_{\mathcal{I}_k} \rightarrow \tilde{x}_{\mathcal{I}_k} | E_1^c \cap \cdots \cap E_{k-1}^c)$ can be shown to be upper bounded by $b_k P^{-d}$ for some $b_k > 0$. This completes the proof.

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