q-BERNSTEIN POLYNOMIALS ASSOCIATED WITH q-STIRLING NUMBERS AND CARLITZ'S q-BERNOULLI NUMBERS

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Abstract Recently, T. Kim([4]) introduced q-Bernstein polynomials which are different q-Bernstein polynomials of Phillips([12]). In this paper, we give p-adic q-integral representation for Kim's q-Bernstein polynomials and investigate some interesting identities of q-Bernstein polynomials associated with q-extension of binomial distribution, q-Stirling numbers and Carlitz's q-Bernoulli numbers.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$.

When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$, and if $q \in \mathbb{C}_p$, one normally assumes $|1-q|_p < 1$.

The q-bosonic natural numbers are defined by $[n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+\cdots+q^{n-1}$ for $n \in \mathbb{N}$, and the q-factorial is defined by $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$. For the q-extension of binomial coefficient, we use the following notation in the form of

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q!}.$$

Let $C[0,1]$ denote the set of continuous functions on $[0,1](\subset \mathbb{R})$. Then Bernstein operator for $f \in C[0,1]$ is defined by

$$\mathbb{B}_n(f|x) = \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k} x^k(1-x)^{n-k} = \sum_{k=0}^{n} f(\frac{k}{n}) B_{k,n}(x),$$

where $n, k \in \mathbb{Z}_+$. The polynomials $B_{k,n}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ are called Bernstein polynomials of degree n (see [1]). For $f \in C[0,1]$, Kim's q-Bernstein operator of order n for f is defined by

$$\mathbb{B}_{n,q}(f|x) = \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}[x]_q^k[1-x]_{\frac{1}{q}}^{n-k} = \sum_{k=0}^{n} f(\frac{k}{n}) B_{k,n}(x,q),$$

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where $n, k \in \mathbb{Z}_+$. Here $B_{k,n}(x,q) = \binom{n}{k}[x]_q^k[1-x]_{\frac{1}{q}}^{n-k}$ are called the Kim's q-Bernstein polynomials of degree n (see [4]).

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and write $f \in UD(\mathbb{Z}_p)$, if the difference quotient $F_f(x,y) = \frac{f(x)-f(y)}{x-y}$ has a limit $f'(a)$ as $(x,y) \to (a,a)$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, \quad \text{(see [6])}.$$

Carlitz's q-Bernoulli numbers can be represented by p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} [x]_q^n q^x = \beta_{n,q}, \quad \text{(see [6, 7])}. \tag{1}$$

The k-th order factorial of the q-number $[x]_q$, which is defined by

$$[x]_{k,q} = [x]_q[x-1]_q \cdots [x-k+1]_q = \frac{(1-q^x)(1-q^{x-1})\cdots(1-q^{x-k+1})}{(1-q)^k},$$

is called the q-factorial of x of order k (see [6]).

In this paper, we give p-adic q-integral representation for Kim's q-Bernstein polynomials and derive some interesting identities for the Kim's q-Bernstein polynomials associated with q-extension of binomial distribution, q-Stirling numbers and Carlitz's q-Bernoulli numbers.

2. q-Bernstein polynomials

In this section, we assume that $0 < q < 1$. Let $\mathbb{P}_q = \{\sum_i a_i[x]_q^i \mid a_i \in \mathbb{R}\}$ be the space of q-polynomials of degree less than or equal to n.

For $f \in C[0,1]$ and $n, k \in \mathbb{Z}_+$, Kim's q-Bernstein operator of order n for f is defined by

$$\mathbb{B}_{n,q}(f|x) = \sum_{k=0}^{n} f(\frac{k}{n})B_{k,n}(x,q). \tag{2}$$

Here $B_{k,n}(x,q) = \binom{n}{k}[x]_q^k[1-x]_{\frac{1}{q}}^{n-k}$ are the Kim's q-Bernstein polynomials of degree n (see [4]).

Kim's q-Bernstein polynomials of degree n is a basis for the space of q-polynomials of degree less than or equal to n. That is, Kim's q-Bernstein polynomials of degree n is a basis for $\mathbb{P}_q$.

We see that Kim's q-Bernstein polynomials of degree n span the space of q-polynomials. That is, any q-polynomials of degree less than or equal to n can be written as a linear combination of the Kim's q-Bernstein polynomials of degree n. For $n, k \in \mathbb{Z}_+$ and $x \in [0,1]$, we have

$$B_{k,n}(x,q) = \sum_{l=k}^{n} \binom{n}{l}\binom{l}{k}(-1)^{l-k}[x]_q^l, \quad \text{(see [4])}. \tag{3}$$

If there exist constants $C_0, C_1, \ldots, C_n$ such that $C_0B_{0,n}(x,q) + C_1B_{1,n}(x,q) + \cdots + C_nB_{n,n}(x,q) = 0$ holds for all x, then we can derive the following equation from (3):

$$\begin{aligned}
0 &= C_0B_{0,n}(x,q) + C_1B_{1,n}(x,q) + \cdots + C_nB_{n,n}(x,q) \\
&= C_0\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{i}{0}[x]_q^i + C_1\sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}\binom{i}{1}[x]_q^i \\
&\qquad\qquad + \cdots + C_n\sum_{i=n}^{n}(-1)^{i-n}\binom{n}{i}\binom{i}{n}[x]_q^i \\
&= C_0 + \{\sum_{i=0}^{1}C_i(-1)^{i-1}\binom{n}{1}\binom{1}{i}\}[x]_q + \cdots + \{\sum_{i=0}^{n}C_i(-1)^{i-n}\binom{n}{n}\binom{n}{i}\}[x]_q^n.
\end{aligned}$$

Since the power basis is a linearly independent set, it follows that

$$\begin{aligned}
C_0 &= 0, \\
\sum_{i=0}^{1}C_i(-1)^{i-1}\binom{n}{1}\binom{1}{i} &= 0, \\
\vdots \qquad & \quad \vdots \\
\sum_{i=0}^{n}C_i(-1)^{i-n}\binom{n}{n}\binom{n}{i} &= 0,
\end{aligned}$$

which implies that $C_0 = C_1 = \cdots = C_n = 0$ (C_0 is clearly zero, substituting this in the second equation gives $C_1 = 0$, substituting these two into the third equation gives $C_2 = 0$, and so on).

Let us consider a q-polynomial $P_q(x) \in \mathbb{P}_q$ which is written by a linear combination of Kim's q-Bernstein basis functions as follows:

$$P_q(x) = C_0B_{0,n}(x,q) + C_1B_{1,n}(x,q) + \cdots + C_nB_{n,n}(x,q). \tag{4}$$

It is easy to write (4) as a dot product of two values.

$$P_q(x) = (B_{0,n}(x,q), B_{1,n}(x,q), \ldots, B_{n,n}(x,q))\begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_n \end{pmatrix}. \tag{5}$$

From (5), we can derive the following equation:

$$P_q(x) = (1, [x]_q, \ldots, [x]_q^n)\begin{pmatrix} b_{00} & 0 & 0 & \cdots & 0 \\ b_{10} & b_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & b_{n2} & \ldots & b_{nn} \end{pmatrix}\begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_n \end{pmatrix},$$

where the b_{ij} are the coefficients of the power basis that are used to determine the respective Kim's q-Bernstein polynomials. We note that the matrix in this case is lower triangular.

From (2) and (3), we note that

$$\begin{aligned} B_{0,2}(x,q) &= [1-x]_{\frac{1}{q}}^2 = \sum_{l=0}^{2}\binom{2}{l}(-1)^l[x]_q^l = 1-2[x]_q+[x]_q^2, \\ B_{1,2}(x,q) &= \binom{2}{1}[x]_q[1-x]_{\frac{1}{q}} = 2[x]_q-2[x]_q^2, \\ B_{2,2}(x,q) &= \binom{2}{2}[x]_q^2 = [x]_q^2. \end{aligned}$$

In the quadratic case ($n=2$), the matrix representation is

$$P_q(x) = (1,\, [x]_q,\, [x]_q^2)\begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix}.$$

In the cubic case ($n=3$), the matrix representation is

$$P_q(x) = (1,\, [x]_q,\, [x]_q^2,\, [x]_q^3)\begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}\begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix}.$$

In many applications of q-Bernstein polynomials, a matrix formulation for the Kim's q-Bernstein polynomials seems to be useful.

3. q-Bernstein polynomials, q-Stirling numbers and q-Bernoulli numbers

In this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$.

For $f \in UD(\mathbb{Z}_p)$, let us consider the p-adic analogue of Kim's q-Bernstein type operator of order n on $\mathbb{Z}_p$ as follows:

$$\mathbb{B}_{n,q}(f|x) = \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}[x]_q^k[1-x]_{\frac{1}{q}}^{n-k} = \sum_{k=0}^{n} f(\frac{k}{n})B_{k,n}(x,q).$$

Let $(Eh)(x) = h(x+1)$ be the shift operator. Then the q-difference operator is defined by

$$\Delta_q^n := (E-I)_q^n = \prod_{i=1}^{n}(E-q^{i-1}I), \tag{6}$$

where $(Ih)(x) = h(x)$. From (6), we derive the following equation:

$$\Delta_q^n f(0) = \sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\binom{k}{2}} f(n-k), \quad \text{(see [7])}. \tag{7}$$

By (7), we easily see that

$$f(x) = \sum_{n\geq 0}\binom{x}{n}_q \Delta_q^n f(0), \quad \text{(see [6, 7])}.$$

The q-Stirling number of the first kind is defined by

$$\prod_{k=1}^{n}(1+[k]_q z)=\sum_{k=0}^{n} S_{1,q}(n,k)z^k, \quad \text{(see [5, 6])}, \tag{8}$$

and the q-Stirling number of the second kind is also defined by

$$\prod_{k=1}^{n}(\frac{1}{1+[k]_q z})=\sum_{k=0}^{n} S_{2,q}(n,k)z^k, \quad \text{(see [5])}. \tag{9}$$

By (6), (7), (8) and (9), we get

$$\begin{aligned} S_{2,q}(n,k) &= \frac{q^{-\binom{k}{2}}}{[k]_q!}\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}}\binom{k}{j}_q [k-j]_q^n \\ &= \frac{q^{-\binom{k}{2}}}{[k]_q!}\Delta_q^k 0^n, \end{aligned}$$

for $n,k\in\mathbb{Z}_+$ (see [6]).

Let us consider Kim's q-Bernstein polynomials of degree n on $\mathbb{Z}_p$ as follows:

$$B_{k,n}(x,q)=\binom{n}{k}[x]_q^k[1-x]_{\frac{1}{q}}^{n-k},$$

for $n,k\in\mathbb{Z}_+$ and $x\in\mathbb{Z}_p$. Thus, we easily see that

$$\int_{\mathbb{Z}_p} B_{k,n}(x,q)d\mu_q(x)=\sum_{l=0}^{n-k}\binom{n-k}{l}\binom{n}{k}(-1)^l\int_{\mathbb{Z}_p}[x]_q^{l+k}d\mu_q(x). \tag{10}$$

By (1) and (10), we obtain the following proposition.

Proposition 1. *For $n,k\in\mathbb{Z}_+$, we have*

$$\int_{\mathbb{Z}_p} B_{k,n}(x,q)d\mu_q(x)=\sum_{l=0}^{n-k}\binom{n-k}{l}\binom{n}{k}(-1)^l\beta_{l+k,q},$$

where $\beta_{l+k,q}$ are the $(l+k)$-th Carlitz's q-Bernoulli numbers.

From the definition of Kim's q-Bernstein polynomial, we note that

$$\sum_{k=i}^{n}\frac{\binom{k}{i}}{\binom{n}{i}}B_{k,n}(x,q)=\sum_{k=0}^{i} q^{\binom{k}{2}}\binom{x}{k}_q [k]_q! S_{2,q}(k,i-k), \tag{11}$$

where $i\in\mathbb{N}$. From the definition of q-binomial coefficient, we have

$$\binom{n+1}{k}_q=\binom{n}{k-1}_q+q^k\binom{n}{k}_q=q^{n-k}\binom{n}{k-1}_q+\binom{n}{k}_q. \tag{12}$$

By (12), we see that

$$\int_{\mathbb{Z}_p}\binom{x}{n}_q d\mu_q(x)=\frac{(-1)^n}{[n+1]_q}q^{(n+1)-\binom{n+1}{2}}, \quad \text{(see [6, 7])}. \tag{13}$$

From (1), (11) and (13), we obtain the following theorem.

Theorem 2. *For $n, k \in \mathbb{Z}_+$ and $i \in \mathbb{N}$, we have*

$$\begin{aligned}&\sum_{k=i}^{n}\sum_{l=0}^{n-k}\frac{\binom{k}{i}}{\binom{n}{i}}\binom{n-k}{l}\binom{n}{k}(-1)^l\beta_{l+k,q}\\ &=\sum_{k=0}^{i}q^{\binom{k}{2}}[k]_q!S_{2,q}(k,i-k)\frac{(-1)^k}{[k+1]_q}q^{(k+1)-\binom{k+1}{2}}.\end{aligned}$$

It is easy to see that for $i \in \mathbb{N}$,

$$\sum_{k=i}^{n}\frac{\binom{k}{i}}{\binom{n}{i}}B_{k,n}(x,q)=[x]_q^i. \tag{14}$$

By (11) and (14), we easily get

$$[x]_q^i=\sum_{k=0}^{i}q^{\binom{k}{2}}\binom{x}{k}_q[k]_q!S_{2,q}(k,i-k), \quad \text{(see [6])}.$$

Thus, we have

$$\begin{aligned}\int_{\mathbb{Z}_p}[x]_q^i d\mu_q(x)&=\sum_{k=0}^{i}q^{\binom{k}{2}}[k]_q!S_{2,q}(k,i-k)\int_{\mathbb{Z}_p}\binom{x}{k}_q d\mu_q(x) \\ &=q\sum_{k=0}^{i}[k]_q!S_{2,q}(k,i-k)\frac{(-1)^k}{[k+1]_q}.\end{aligned} \tag{15}$$

By (1) and (15), we obtain the following corollary.

Corollary 3. *For $n, k \in \mathbb{Z}_+$ and $i \in \mathbb{N}$, we have*

$$\beta_{i,q}=q\sum_{k=0}^{i}[k]_q!S_{2,q}(k,i-k)\frac{(-1)^k}{[k+1]_q}.$$

It is known that

$$S_{2,q}(n,k)=\frac{1}{(1-q)^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k+n}{k-j}\binom{j+n}{j}_q, \quad \text{(see [6])}, \tag{16}$$

and

$$\binom{n}{k}_q=\sum_{j=0}^{n}\binom{n}{j}(q-1)^{j-k}S_{2,q}(k,j-k).$$

By simple calculation, we have that

$$\begin{aligned}q^{nx}&=\sum_{k=0}^{n}(q-1)^k q^{\binom{k}{2}}\binom{n}{k}_q[x]_{k,q} \\ &=\sum_{m=0}^{n}\{\sum_{k=m}^{n}(q-1)^k\binom{n}{k}_q S_{1,q}(k,m)\}[x]_q^m\end{aligned} \tag{17}$$

and

$$q^{nx}=\sum_{m=0}^{n}\binom{n}{m}(q-1)^m[x]_q^m. \tag{18}$$

From (17) and (18), we note that

$$\binom{n}{m} = \sum_{k=m}^{n} (q-1)^{-m+k} \binom{n}{k}_q S_{1,q}(k,m), \quad \text{(see [6])}.$$

Thus, we obtain the following proposition.

Proposition 4. *For $n, k \in \mathbb{Z}_+$, we have*

$$B_{k,n}(x,q) = \binom{n}{k} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k} = \sum_{m=k}^{n} (q-1)^{-k+m} \binom{n}{m}_q S_{1,q}(m,k)[x]_q^k [1-x]_{\frac{1}{q}}^{n-k}.$$

From the definition of the q-Stirling numbers of the first kind, we get

$$q^{\binom{n}{2}} \binom{x}{n}_q [n]_q! = [x]_{n,q}\, q^{\binom{n}{2}} = \sum_{k=0}^{n} S_{1,q}(n,k)[x]_q^k. \tag{19}$$

By (11) and (19), we obtain the following theorem.

Theorem 5. *For $n, k \in \mathbb{Z}_+$ and $i \in \mathbb{N}$, we have*

$$\sum_{k=i}^{n} \frac{\binom{k}{i}}{\binom{n}{i}} B_{k,n}(x,q) = \sum_{k=0}^{i} \sum_{l=0}^{k} S_{1,q}(k,l) S_{2,q}(k, i-k)[x]_q^l.$$

By (14) and Theorem 5, we obtain the following corollary.

Corollary 6. *For $i \in \mathbb{Z}_+$, we have*

$$\beta_{i,q} = \sum_{k=0}^{i} \sum_{l=0}^{k} S_{1,q}(k,l) S_{2,q}(k, i-k) \beta_{l.q}.$$

The q-Bernoulli polynomials of order $k \in \mathbb{Z}_+$ are defined by

$$\beta_{n,q}^{(k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^i q^{ix} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{l=1}^{k} (k-l+i)x_l} d\mu_q(x_1) \cdots d\mu_q(x_k). \tag{20}$$

Thus, we have

$$\beta_{n,q}^{(k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \frac{(-1)^i \binom{n}{i} (i+k) \cdots (i+1)}{[i+k]_q \cdots [i+1]_q} q^{ix}, \quad \text{(see [6])}.$$

The inverse q-Bernoulli polynomials of order k are defined by

$$\beta_{n,q}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \frac{(-1)^i \binom{n}{i} q^{ix}}{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{l=1}^{k} (k-l+i)x_l} d\mu_q(x_1) \cdots d\mu_q(x_k)}, \quad \text{(see [6])}. \tag{21}$$

In the special case $x = 0$, $\beta_{n,q}^{(k)}(0) = \beta_{n,q}^{(k)}$ are called the n-th q-Bernoulli numbers of order k and $\beta_{n,q}^{(-k)}(0) = \beta_{n,q}^{(-k)}$ are also called the inverse q-Bernoulli numbers of order k.

From (21), we have

$$\begin{aligned}\beta_{k,q}^{(-n)} &= \frac{1}{(1-q)^k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{[j+n]_q\cdots[j+1]_q}{(j+n)\cdots(j+1)} \\ &= \frac{1}{(1-q)^k}\sum_{j=0}^{k}(-1)^j\frac{\binom{k+n}{n-j}}{\binom{k+n}{n}}\binom{j+n}{n}_q\frac{[n]_q!}{n!} \\ &= \frac{[n]_q!}{\binom{k+n}{n}n!}\{\frac{1}{(1-q)^k}\sum_{j=0}^{k}(-1)^j\binom{k+n}{n-j}\binom{j+n}{n}_q\}.\end{aligned} \tag{22}$$

By (16) and (22), we get

$$\frac{n!}{[n]_q!}\binom{k+n}{n}\beta_{k,q}^{(-n)} = S_{2,q}(n,k). \tag{23}$$

Therefore, by (11) and (23), we obtain the following theorem.

Theorem 7. *For $i,n,k \in \mathbb{Z}_+$, we have*

$$\sum_{k=i}^{n}\frac{\binom{k}{i}}{\binom{n}{i}}B_{k,n}(x,q) = \sum_{k=0}^{i}q^{\binom{k}{2}}k!\binom{i}{k}\binom{x}{k}_q\beta_{i-k,q}^{(-k)}.$$

It is easy to show that

$$\begin{aligned}q^{\binom{n}{2}}\binom{x}{n}_q &= \frac{1}{[n]_q!}\prod_{k=0}^{n-1}([x]_q-[k]_q) \\ &= \frac{1}{[n]_q!}\sum_{k=0}^{n}(-1)^k[x]_q^{n-k}S_{1,q}(n-1,k).\end{aligned}$$

Thus, we have that

$$\sum_{k=i}^{n}\frac{\binom{k}{i}}{\binom{n}{i}}B_{k,n}(x,q) = \sum_{k=0}^{i}\sum_{j=0}^{k}(-1)^j[x]_q^{k-j}S_{1,q}(k-1,j)\frac{k!}{[k]_q!}\binom{i}{k}\beta_{i-k,q}^{(-k)},$$

where $n,k,i \in \mathbb{Z}_+$.

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