CURVATURE ESTIMATES FOR SUBMANIFOLDS IN WARPED PRODUCTS

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Dedicated to Keti Tenenblat on occasion of her 65th anniversary, with admiration.

Abstract. We give estimates on the intrinsic and the extrinsic curvature of manifolds that are isometrically immersed as cylindrically bounded submanifolds of warped products. We also address extensions of the results in the case of submanifolds of the total space of a Riemannian submersion.

1. Introduction

An important problem in submanifold theory is the isometric immersion problem, this is for given two complete Riemannian manifolds (M, g_M) and (N, g_N), $\dim(M) < \dim(N)$ whether there exists an isometric immersion $\varphi\colon M \hookrightarrow N$. When N is the Euclidean space, the Nash Isometric Embedding Theorem answers affirmatively this question, provided the codimension $\dim(N) - \dim(M)$ is sufficiently high, see [8]. For small codimension, here in this paper meaning that $\dim(N) - \dim(M) \leq \dim(M) - 1$, the answer, in general, depends on the geometries of M and N. For instance, on the sectional curvatures of M and N, e.g., a classical result of C. Tompkins [15] extended in a series of papers, by Chern and Kuiper [3], Moore, [7], O'Neill [9], Otsuki [11] and Stiel [14] can be summarized as follows.

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Theorem 1.1. *Let $\varphi\colon M^m \hookrightarrow N^n$, $n \leq 2m-1$, be an isometric immersion of a compact Riemannian m-manifold M into a Cartan-Hadamard n-manifold N. Then the sectional curvatures of M and N satisfies*

$$\sup_M K_M > \inf_N K_N. \tag{1.1}$$

Theorem 1.1 was extended, in a seminal paper [5], by L. Jorge and D. Koutrofiotis to bounded, complete submanifolds with scalar curvature bounded below where they introduced the Omori-Yau maximum principle to tackle this immersion problem. Due to a much better version of the Omori-Yau maximum principle, Pigola-Rigoli-Setti [13], extended Jorge-Kourtofiotis result to to bounded, complete submanifolds with scalar curvature satisfying

$$s_M(x) \geq -\rho_M^2(x)\cdot \prod_{j=1}^{k}\left[\log^{(j)}(\rho_M(x))\right]^2,\ \rho_M(x)\gg 1 \tag{1.2}$$

where ρ_M is the distance function on M to a fixed point and $\log^{(j)}$ is the j-th iterate of the logarithm.

Theorem 1.2 (Jorge-Koutrofiotis-Pigola-Rigoli-Setti)**.** *Let $\varphi\colon M \hookrightarrow N$ be an isometric immersion of a complete m-dimensional Riemannian manifold M into an n-dimensional Riemannian manifold N, $n \leq 2m-1$. Let $B_N(r)$ be a geodesic ball of N centered at $p = \varphi(q)$ with radius $r < \min\{\mathrm{inj}_N(p), \pi/2\sqrt{b}\}$, where the radial sectional curvatures K_N^{rad} along the radial geodesics issuing from p are bounded as $K_N^{\mathrm{rad}} \leq b$ in $B_N(r)$ and where $\pi/2\sqrt{b}$ is replaced by $+\infty$ if $b \leq 0$. Suppose that the scalar curvature of M satisfies* (1.2) *and that $\varphi(M) \subset B_N(r)$. Then*

$$\sup_M K_M \geq C_b^2(r) + \inf_{B_N(r)} K_N. \tag{1.3}$$

Where

$$C_b(t) = \begin{cases} \sqrt{b}\cot(\sqrt{b}\,t) & \text{if } \ b>0,\\ 1/t & \text{if } \ b=0,\\ \sqrt{-b}\coth(\sqrt{-b}\,t) & \text{if } \ b<0.\end{cases}$$

Very recently, Theorem 1.2 was extended to the class of cylindrically bounded submanifolds by Alias-Bessa-Montenegro [2].

Theorem 1.3 (Alias-Bessa-Montenegro)**.** *Let $\varphi\colon M^m \hookrightarrow N^n \times \mathbb{R}^\ell$ be an isometric immersion of a complete Riemannian m-manifold M into the product $N^n\times\mathbb{R}^\ell$, $n+2\ell \leq 2m-1$, where N^n is a Riemannian n-manifold. Let $B_N(r)$ be a geodesic ball of N centered at $p = \pi_1(\varphi(q))$[1] with radius $r < \min\{\mathrm{inj}_N(p), \pi/2\sqrt{b}\}$, where the radial sectional curvatures K_N^{rad} along the radial geodesics issuing from p are bounded as $K_N^{\mathrm{rad}} \leq b$ in $B_N(r)$ and $\pi/2\sqrt{b}$ is replaced by $+\infty$ if $b \leq 0$. Suppose that $\varphi(M) \subset B_N(r)\times\mathbb{R}^\ell$ and one of these two conditions below holds,*

[1] $\pi_1\colon N\times\mathbb{R}^\ell \to N$ is the projection on the first factor.

i. *φ is proper and* $\sup_{B_N(r)\times B_{\mathbb{R}^\ell}(t)} \|\alpha\| \leq G(t)$, *where* $G\colon [0,\infty) \to (0,\infty)$ *with* $1/G \notin L^1(0,\infty)$.

ii. *The scalar curvature of M satisfies (1.2).*

then

$$\sup_M K_M \geq C_b^2(r) + \inf_{B_N(r)} K_N. \tag{1.4}$$

The purpose of this paper is to show that these results above can be extended naturally to isometric immersion into warped product manifolds $\mathcal{M} = \mathcal{X} \times \mathcal{V}$ endowed with the metric $g^{\mathcal{M}} = g^{\mathcal{X}} + \psi^2 g^{\mathcal{V}}$, where $(\mathcal{X}, g^{\mathcal{X}})$ and $(\mathcal{V}, g^{\mathcal{V}})$ are Riemannian manifolds and $\psi\colon \mathcal{X} \to \mathbb{R}^+$ is a smooth positive function on $\mathcal{X}$. Set $n_{\mathcal{X}} = \dim(\mathcal{X})$, $n_{\mathcal{V}} = \dim(\mathcal{V})$ and $n_{\mathcal{M}} = n_{\mathcal{X}} + n_{\mathcal{V}} = \dim(\mathcal{M})$. We will assume in our main result a certain bound on the radial curvature along geodesics issuing from a point x_0 in the base manifold $\mathcal{X}$, see (4.1).

Our first result is the following (see Theorem 4.2).

Theorem A. *Let (M, g^M) be a complete Riemannian n_M-manifold for which the weak Omori–Yau principle for the Hessian*[2] *holds and let $\varphi : M \to \mathcal{M}$ be an isometric immersion. Assume that the following hypotheses are satisfied*

(1) *$\pi_{\mathcal{X}}(\varphi(M)) \subset B_{\mathcal{X}}(r)$, a geodesic ball in $\mathcal{X}$ with center at some $x_0 \in \mathcal{X}$ and $r \in (0, \mathrm{inj}_{\mathcal{X}}(x_0))$.*
(2) *Assumption* (4.1) *holds.*
(3) $2n_M \geq 2n_{\mathcal{V}} + n_{\mathcal{X}} + 1$.

Then,

$$\sup_M K_M \geq C_b(r)^2 + \inf_{B_{\mathcal{X}}(x_0;r)} K_{\mathcal{X}}. \tag{1.5}$$

Our second main result gives an estimate on the mean curvature of cylindrically bounded submanifolds of warped product (see Theorem 4.4).

Theorem B. *Let (M, g^M) be a complete Riemannian manifold for which the weak Omori–Yau principle for the Laplacian holds, and let $\varphi : M \to \mathcal{M}$ be an isometric immersion. Assume that the following hypotheses are satisfied.*

(1) *$\pi_{\mathcal{X}}(\varphi(M)) \subset B_{\mathcal{X}}(x_0; r)$ a geodesic ball in $\mathcal{X}$ with center at some $x_0 \in \mathcal{X}$ and $r \in (0, \mathrm{inj}_{\mathcal{X}}(x_0))$.*
(2) *Assumption* (4.1) *holds.*

Then, denoting by $\vec{H}^\varphi$ the mean curvature vector of φ, one has the following estimate on the supremum of $|\vec{H}^\varphi|_{\mathcal{M}}$.

$$\sup_M \big|\vec{H}^\varphi\big|_{\mathcal{M}} \geq (n_M - n_{\mathcal{V}})\, C_b(r) - n_{\mathcal{V}} \Psi_0, \tag{1.6}$$

[2] See Section 3 for the details of the weak Omori-Yau maximum principles for the Hessian and for the Laplacian

where

$$\Psi_0 = \sup_{\mathrm{dist}(x,x_0)\le r} \Big|\frac{\mathrm{grad}^{\mathcal{X}}\psi(x)}{\psi(x)}\Big|_{\mathcal{X}}. \tag{1.7}$$

In view of the above results, it is an interesting question to establish geometric conditions for the validity of the weak Omori-Yau maximum principles in submanifolds of warped products. We study this question in Section 3, see Theorem 3.4.

We also observe that the results of Theorems A and B can be generalized to the more general situation of cylindrically bounded isometric immersions into the total space of a Riemannian submersion. These generalizations are discussed in Section 5, see Theorem 5.2 and Theorem 5.3. In this situation, the curvature estimates are given in terms of the norms of the characteristic tensors of the submersion.

2. Generalities on warped products

Let $(\mathcal{X}, g^{\mathcal{X}})$ and $(\mathcal{V}, g^{\mathcal{V}})$ be Riemannian manifolds and let $\psi\colon \mathcal{X} \to \mathbb{R}^+$ be a smooth positive function on $\mathcal{X}$. Set $n_{\mathcal{X}} = \dim(\mathcal{X})$, $n_{\mathcal{V}} = \dim(\mathcal{V})$ and $n_{\mathcal{M}} = n_{\mathcal{X}} + n_{\mathcal{V}} = \dim(\mathcal{M})$. The product manifold $\mathcal{M} = \mathcal{X} \times \mathcal{V}$ endowed with the metric $g^{\mathcal{M}} = g^{\mathcal{X}} + \psi^2 g^{\mathcal{V}}$ is the *warped product of* $\mathcal{X}$ *and* $\mathcal{V}$, with *warping function* ψ. This is also denoted with the symbol $\mathcal{M} = \mathcal{X} \times_\psi \mathcal{V}$. The projection $\pi_{\mathcal{X}} : \mathcal{M} \to \mathcal{X}$ is a Riemannian submersion, while the projection $\pi_{\mathcal{V}} : \mathcal{M} \to \mathcal{V}$ is not (unless $\psi \equiv 1$). In fact, warped products are special cases of Riemannian submersions, characterized by the property of having integrable horizontal distribution with totally geodesic leaves and with totally umbilical fibers, see Section 5. We will borrow some terminology from Riemannian submersions, and we will call $\mathcal{X}$ the *base* and $\mathcal{V}$ the *fiber* of $\mathcal{M}$. Moreover, vectors that are in the kernel of $\mathrm{d}\pi_{\mathcal{X}}$ are called *vertical*, while vectors in the kernel of $\mathrm{d}\pi_{\mathcal{V}}$ are called *horizontal*.

Vector fields $X \in \mathfrak{X}(\mathcal{X})$ will be identified with vector fields in $\mathcal{M}$ that "do not depend on the second variable", i.e., $X(x, v) = X(x)$ for all $v \in \mathcal{V}$. This type of horizontal vector fields will be called *h-basic*. Similarly, vector fields $V \in \mathfrak{X}(\mathcal{V})$ will be identified with vector fields in $\mathcal{M}$ that do not depend on the first variable; they will be called *v-basic*. Note that

$$g^{\mathcal{M}}(X, Y) = g^{\mathcal{X}}(X, Y) \tag{2.1}$$

for every h-basic X and Y, while

$$g^{\mathcal{M}}(V, W) = \psi^2 g^{\mathcal{V}}(V, W) \tag{2.2}$$

for every v-basic V and W. We will use consistently the notation X, Y, Z for h-basic fields, and U, V, W for v-basic fields on $\mathcal{M}$. Observe that the Lie bracket $[X, Y]$ of h-basic vector fields is h-basic, the Lie bracket $[V, W]$ of v-basic fields is v-basic, while the Lie bracket $[X, V]$ of an h-basic and a v-basic field is zero.

2.1. **Riemannian differential operators.** The symbols $\nabla^{\mathcal{X}}$ and $\nabla^{\mathcal{V}}$ will denote the Levi–Civita connections of $(\mathcal{X}, g^{\mathcal{X}})$ and of $(\mathcal{V}, g^{\mathcal{V}})$ respectively. Due to well known invariance properties, the Levi–Civita connection $\nabla^{\mathcal{M}}$ of $(\mathcal{M}, g^{\mathcal{M}})$ will be uniquely determined by the values of $\nabla^{\mathcal{M}}_A B$, where A and B are basic. Since $[X, V] = 0$, then

$$\nabla^{\mathcal{M}}_X V = \nabla^{\mathcal{M}}_V X, \tag{2.3}$$

for all h-basic X and v-basic V. The following formulas are easily computed

$$\nabla^{\mathcal{M}}_X Y = \nabla^{\mathcal{X}}_X Y, \tag{2.4}$$

for every h-basic fields X and Y. It follows in particular that $\mathcal{X} \times \{v\}$ is a totally geodesic submanifold of $\mathcal{M}$ for all $v \in \mathcal{V}$; moreover, the curvature tensor $R^{\mathcal{M}}$ of horizontal vector is given by

$$R^{\mathcal{M}}(X, Y)Z = R^{\mathcal{X}}(X, Y)Z.$$

Thus, the sectional curvature $K_{\mathcal{M}}(X, Y)$ of the plane in $T\mathcal{M}$ spanned by horizontal vectors X and Y coincides with the sectional curvature in the base manifold

$$K_{\mathcal{M}}(X, Y) = K_{\mathcal{X}}(X, Y). \tag{2.5}$$

As to the covariant derivative of mixed terms

$$\nabla^{\mathcal{M}}_V X \overset{\text{by (2.3)}}{=} \nabla^{\mathcal{M}}_X V = \frac{X(\psi)}{\psi}\, V, \tag{2.6}$$

$$\nabla^{\mathcal{M}}_V W = \nabla^{\mathcal{V}}_V W - g^{\mathcal{M}}(V, W)\, \frac{\mathrm{grad}^{\mathcal{X}} \psi}{\psi}, \tag{2.7}$$

for all h-basic X and all v-basic V and W. The second fundamental form of the fibers $\{x\} \times \mathcal{V}$ is

$$\mathcal{S}^{\mathcal{V}}(V, W) = -g^{\mathcal{M}}(V, W)\, \frac{\mathrm{grad}^{\mathcal{X}} \psi}{\psi}. \tag{2.8}$$

Therefore, critical points of ψ correspond to totally geodesic fibers. By taking trace in (2.8), we get the following expression for the mean curvature vector $\vec{H}$ of the fibers

$$\vec{H} = -n_{\mathcal{V}} \frac{\mathrm{grad}^{\mathcal{X}} \psi}{\psi}. \tag{2.9}$$

For an h-basic vector field X we have

$$\mathrm{div}^{\mathcal{M}}(X) = \mathrm{div}^{\mathcal{X}}(X) + n_{\mathcal{V}}\, \frac{X(\psi)}{\psi}, \tag{2.10}$$

while for a v-basic field V

$$\mathrm{div}^{\mathcal{M}}(V) = \mathrm{div}^{\mathcal{V}}(V). \tag{2.11}$$

Let $F:\mathcal{X}\to\mathbb{R}$ be a smooth function and denote by $F^{\mathrm{h}}=F\circ\pi_{\mathcal{X}}:\mathcal{M}\to\mathbb{R}$ the lifting of F to $\mathcal{M}$. It is easily seen that the gradient $\mathrm{grad}^{\mathcal{M}}F^{\mathrm{h}}$ is horizontal. The gradient of F^{h} is the h-basic field

$$\mathrm{grad}^{\mathcal{M}}F^{\mathrm{h}}=\mathrm{grad}^{\mathcal{X}}F. \tag{2.12}$$

Similarly, if $G:\mathcal{V}\to\mathbb{R}$ is a smooth function, and $G^{\mathrm{v}}=G\circ\pi_{\mathcal{V}}$ is its lifting to $\mathcal{M}$, then the gradient $\mathrm{grad}^{\mathcal{M}}G^{\mathrm{v}}$ is vertical, but not v-basic

$$\mathrm{grad}^{\mathcal{M}}G^{\mathrm{v}}=\frac{1}{\psi^2}\,\mathrm{grad}^{\mathcal{V}}G. \tag{2.13}$$

The Laplacian $\Delta^{\mathcal{M}}$ of the functions F^{h} and G^{v} is given by

$$\Delta^{\mathcal{M}}F^{\mathrm{h}}=\Delta^{\mathcal{X}}F+n_{\mathcal{V}}\cdot g^{\mathcal{X}}\big(\mathrm{grad}^{\mathcal{X}}F,\frac{\mathrm{grad}^{\mathcal{X}}\psi}{\psi}\big). \tag{2.14}$$

$$\Delta^{\mathcal{M}}G^{\mathrm{v}}=\frac{1}{\psi^2}\,\Delta^{\mathcal{V}}G. \tag{2.15}$$

As to the Hessian of the functions F^{h} and G^{v}, the following formulas can be computed easily:

$$\mathrm{Hess}^{\mathcal{M}}F^{\mathrm{h}}(X,X)=\mathrm{Hess}^{\mathcal{X}}F(X,X), \tag{2.16}$$

$$\mathrm{Hess}^{\mathcal{M}}F^{\mathrm{h}}(V,V)=g^{\mathcal{X}}\big(\mathrm{grad}^{\mathcal{X}}F,\frac{\mathrm{grad}^{\mathcal{X}}\psi}{\psi}\big)\cdot g^{\mathcal{M}}(V,V), \tag{2.17}$$

$$\mathrm{Hess}^{\mathcal{M}}F^{\mathrm{h}}(X,V)=0, \tag{2.18}$$

$$\mathrm{Hess}^{\mathcal{M}}G^{\mathrm{v}}(X,X)=0, \tag{2.19}$$

$$\mathrm{Hess}^{\mathcal{M}}G^{\mathrm{v}}(V,V)=\mathrm{Hess}^{\mathcal{V}}G(V,V), \tag{2.20}$$

$$\mathrm{Hess}^{\mathcal{M}}G^{\mathrm{v}}(X,V)=-\frac{X(\psi)}{\psi}\,V(G). \tag{2.21}$$

2.2. **Isometric immersions into warped products.** Let us now consider an immersion $\varphi:M\hookrightarrow\mathcal{M}$. Assume that M is endowed with the pull-back metric $g^M=\varphi^*(g^{\mathcal{M}})$. If $L:\mathcal{M}\to\mathbb{R}$ is a smooth function, then setting $f=L\circ\varphi:M\to\mathbb{R}$, one computes[3]

$$g^M(\mathrm{grad}^M f,e)=g^{\mathcal{M}}\big(\mathrm{grad}^{\mathcal{M}}L,\mathrm{e}\big), \tag{2.22}$$

for all $e\in TM$ and

$$\mathrm{Hess}^M f(e,e)=\mathrm{Hess}^{\mathcal{M}}L\big(\mathrm{e},\mathrm{e}\big)+g^{\mathcal{M}}\big(\mathrm{grad}^{\mathcal{M}}L,\mathcal{S}^{\varphi}(e,e)\big), \tag{2.23}$$

for all $e\in TM$. Here, $\mathcal{S}^{\varphi}$ is the second fundamental form of φ and we identified e with $d\varphi(e)$. In particular, for $L=G^{\mathrm{v}}$, in the notation above,

[3]Obviously, formulas (2.22) and (2.23) hold for an isometric immersion $\varphi:M\to\mathcal{M}$ into *any* ambient manifold $\mathcal{M}$, not just in warped products.

using (2.13), (2.19), (2.21) and (2.20) one has

$$\begin{aligned}(2.24)\quad \mathrm{Hess}^M f(e,e) = -2\, g^{\mathcal{X}}\big(\frac{\mathrm{grad}^{\mathcal{X}}\psi}{\psi}, \mathrm{e}^{\mathrm{hor}}\big)\, g^{\mathcal{V}}\big(\mathrm{grad}^{\mathcal{V}} G, \mathrm{e}^{\mathrm{ver}}\big)\\ + \mathrm{Hess}^{\mathcal{V}} G\big(\mathrm{e}^{\mathrm{ver}}, \mathrm{e}^{\mathrm{ver}}\big) + g^{\mathcal{V}}\big(\mathrm{grad}^{\mathcal{V}} G, \mathcal{S}^{\varphi}(e,e)^{\mathrm{ver}}\big).\end{aligned}$$

Here, $\mathrm{e}^{\mathrm{hor}}$ and $\mathrm{e}^{\mathrm{ver}}$ are respectively the horizontal and the vertical components of e. Similarly, for $L = F^{\mathrm{h}}$, using (2.12), (2.16), (2.17) and (2.18) one computes

$$\begin{aligned}(2.25)\quad \mathrm{Hess}^M f(e,e) = g^{\mathcal{X}}\big(\mathrm{grad}^{\mathcal{X}} F, \frac{\mathrm{grad}^{\mathcal{X}}\psi}{\psi}\big)\cdot g^{\mathcal{M}}\big(\mathrm{e}^{\mathrm{ver}}, \mathrm{e}^{\mathrm{ver}}\big)\\ + \mathrm{Hess}^{\mathcal{X}} F\big(\mathrm{e}^{\mathrm{hor}}, \mathrm{e}^{\mathrm{hor}}\big) + g^{\mathcal{X}}\big(\mathrm{grad}^{\mathcal{X}} F, \mathcal{S}^{\varphi}(e,e)^{\mathrm{hor}}\big).\end{aligned}$$

3. On the Omori-Yau Maximum Principle

Definition 3.1 (Pigola-Rigoli-Setti)**.** Let (M, g^M) be a Riemannian manifold. We say that the *Omori–Yau Maximum Principle for the Hessian holds in* (M, g^M) if for every smooth function $f\colon M \to \mathbb{R}$ with $\sup_M f < +\infty$ there exists a sequence $(p_n)_{n\in\mathbb{N}}$ in M such that

(a) $\lim_{n\to\infty} f(p_n) = \sup_M f$,

(b) $\big\|\mathrm{grad}^M f(p_n)\big\| \le \dfrac{1}{n}$,

(c) $\mathrm{Hess}^M f(p_n)(e,e) \le \dfrac{1}{n}\, g^M(e,e)$ for all $e \in T_{p_n}M$,

for all n. Similarly, the *Omori–Yau Maximum Principle for the Laplacian holds in* (M, g^M) if the above properties hold, with (c) replaced by the condition

(c') $\Delta^M f(p_n) \le \dfrac{1}{n}$.

We say that the *Weak Omori–Yau Principle for the Hessian* (*Laplacian*) in (M, g^M) if for every smooth function $f\colon M \to \mathbb{R}$ with $\sup_M f < +\infty$ there exists a sequence $(p_n)_{n\in\mathbb{N}}$ in M satisfying (a) and (c) ((a) and (c')) above.

The following Theorem, due to Pigola, Rigoli and Setti [13], gives sufficient conditions for the Omori-Yau Maximum Principle to hold in a Riemannian manifold.

Theorem 3.2 (Pigola-Rigoli-Setti)**.** *Let* (M, g^M) *be a Riemannian manifold. Assume that there exist smooth functions*

$$h\colon\ [0,+\infty) \to [0,+\infty) \quad \textit{and} \quad \gamma\colon M \to [0,+\infty)$$

such that

(1) $h(0) > 0$ *and* $h'(t) \ge 0$ *for all* $t \ge 0$,

(2) $\limsup_{t\to+\infty} t\cdot h\big(\sqrt{t}\big)/h(t) < +\infty$,

(3) $\displaystyle\int_0^{+\infty} \mathrm{d}t/\sqrt{h(t)} = +\infty$,

(4) γ *is proper,*

(5) $\left|\mathrm{grad}^M\gamma\right| \leq c\cdot\sqrt{\gamma}$ *for some* $c > 0$ *outside a compact subset of* M*,*

(6) $\mathrm{Hess}^M\gamma \leq c'\cdot\sqrt{\gamma\cdot h(\sqrt{\gamma})}$ *for some* $c' > 0$ *outside a compact subset of* M*.*

Then the Omori–Yau Maximum Principle for the Hessian holds in (M, g^M)*. A totally analogous statement holds in the case of the Omori–Yau principle for the Laplacian, with assumption* (6) *replaced by*

(6') $\triangle^M\gamma \leq c'\cdot\sqrt{\gamma\cdot h(\sqrt{\gamma})}$ *for some* $c' > 0$ *outside a compact subset of* M*.*

Note that any function h satisfying (1) and (2) is unbounded

$$\lim_{t\to+\infty} h(t) = +\infty.$$

Definition 3.3. A pair of functions (h, γ) satisfying (1)—(6) of Theorem 3.2 will be called an *OY-pair* for the Hessian in (M, g^M). Similarly, a pair (h, γ) satisfying (1)—(5) and (6') is called an *OY-pair* for the Laplacian in (M, g^M).

We will now assume that (M, g^M) is isometrically immersed in a warped product $\mathcal{X}\times_\psi\mathcal{V}$, and we want to give conditions that guarantee the validity of the Omori-Yau Maximum Principle on (M, g^M) in terms of the corresponding property of $(\mathcal{V}, g^{\mathcal{V}})$ and the geometry of the immersion.

Theorem 3.4. *Let* $\varphi\colon M\to\mathcal{M} = \mathcal{X}\times_\psi\mathcal{V}$ *be an isometric immersion, and let* (h,Γ) *be an OY-pair for the Hessian in* $(\mathcal{V}, g^{\mathcal{V}})$*. Set* $\gamma = \Gamma^{\mathrm{v}}\circ\varphi\colon M\to[0,+\infty)$*. Assume the following hypothesis.*

(a) φ *is proper,*

(b) $\pi_{\mathcal{X}}\big(\varphi(M)\big)$ *is contained in a compact subset* K *of* $\mathcal{X}$*,*

(c) $\left\|\mathcal{S}^\varphi\right\| \leq \alpha\sqrt{h(\sqrt{\gamma})}$ *for*[4] *some* $\alpha > 0$*, outside a compact subset of* M*.*

Then, (h,γ) *is an OY-pair for the Hessian in* (M, g^M)*. An analogous statement holds in the case of OY-pairs for the Laplacian, with (c) replaced by*

(c') $\left|\vec{H}^\varphi\right| \leq \alpha\sqrt{h(\sqrt{\gamma})}$ *for some* $\alpha > 0$*, outside a compact subset of* M*.*

[4] i.e., $\left\|\mathcal{S}^\varphi(e,e)\right\|_{\mathcal{M}} \leq \alpha\sqrt{h(\sqrt{\gamma})}|e|^2_M$ for all $e\in TM$

Proof. The function h by hypothesis satisfies the conditions (1)–(3) of Theorem 3.2 so we only need to show that the function γ satisfies the conditions (4)–(6). The assumptions (a) and (b) clearly imply that γ is proper. For if $p_n \in M$ is a divergent sequence, $\mathrm{dist_M(p_n, p_0)} \to \infty$ as $n \to \infty$, then $\mathrm{dist}_{\mathcal{M}}(\varphi(\mathrm{p_n}), \varphi(\mathrm{p_0})) \to \infty$ and since $\pi_{\mathcal{X}}\big(\varphi(M)\big)$ is contained in a compact subset K of $\mathcal{X}$ we have that $\mathrm{dist_M}(\Gamma^{\mathrm{v}} \circ \varphi(\mathrm{p_n}), \Gamma^{\mathrm{v}} \circ \varphi(\mathrm{p_0})) \to \infty$ as $n \to \infty$ since φ and Γ^{v} are proper. This proves that γ is proper, (condition (4)).

For $\xi \in T_{\varphi(p)}\mathcal{M}$, we write $\xi = \xi^{\mathrm{t}} + \xi^{\perp}$, where $\xi^{\mathrm{t}} \in \mathrm{Im}\big(\mathrm{d}\varphi(p)\big)$ and $\xi^{\perp} \in \mathrm{Im}\big(\mathrm{d}\varphi(p)\big)^{\perp}$. Given $p \in M$, we have using (2.22), (2.13) that

$$\big|\mathrm{grad}^M \gamma(p)\big|_M \le \frac{1}{\psi^2}\big|\mathrm{grad}^{\mathcal{V}}\Gamma\big(\pi_{\mathcal{V}}(\varphi(p))\big)\big|_{\mathcal{V}} \le c\sqrt{\Gamma\big(\pi_{\mathcal{V}}(\varphi(p))\big)} = c\sqrt{\gamma(p)}$$

This shows condition (5) of Theorem 3.2. Moreover, let $p \in M$ and $e \in T_pM$

$$\begin{aligned}
\mathrm{Hess}^M\gamma(e,e) &\overset{\text{by (2.24)}}{=} -2\,g^{\mathcal{X}}\Big(\frac{\mathrm{grad}^{\mathcal{X}}\psi}{\psi}, \mathrm{e}^{\mathrm{hor}}\Big)\cdot g^{\mathcal{V}}\big(\mathrm{grad}^{\mathcal{V}}\Gamma, \mathrm{e}^{\mathrm{ver}}\big) \\
&\quad + \mathrm{Hess}^{\mathcal{V}}\Gamma\big(\mathrm{e}^{\mathrm{ver}}, \mathrm{e}^{\mathrm{ver}}\big) + g^{\mathcal{V}}\big(\mathrm{grad}^{\mathcal{V}}\Gamma, \mathcal{S}^{\varphi}(e,e)^{\mathrm{ver}}\big) \\
&\le 2A_0\big|\mathrm{grad}^{\mathcal{V}}\Gamma\big|_{\mathcal{V}}\cdot|e|^2_M + c'\sqrt{\gamma\cdot h(\gamma)^{\frac12}}\cdot\big|\mathrm{e}^{\mathrm{ver}}\big|^2_{\mathcal{V}} \\
&\qquad + \big|\mathrm{grad}^{\mathcal{V}}\Gamma\big|_{\mathcal{V}}\cdot\big|\mathcal{S}^{\varphi}(e,e)^{\mathrm{ver}}\big|_{\mathcal{V}} \\
&\le 2A_0\big|\mathrm{grad}^{\mathcal{V}}\Gamma\big|_{\mathcal{V}}\cdot|e|^2_M + c'\sqrt{\gamma\cdot h(\gamma)^{\frac12}}\cdot\frac{1}{\psi^2}|\mathrm{e}|^2_{\mathcal{M}} \\
&\qquad + \big|\mathrm{grad}^{\mathcal{V}}\Gamma\big|_{\mathcal{V}}\cdot\frac{1}{\psi}\big|\mathcal{S}^{\varphi}(e,e)\big|_{\mathcal{M}} \\
&\le \left[2A_0\big|\mathrm{grad}^{\mathcal{V}}\Gamma\big|_{\mathcal{V}} + \frac{c'}{B_0^2}\sqrt{\gamma\cdot h(\sqrt{\gamma})} + \frac{\big|\mathrm{grad}^{\mathcal{V}}\Gamma\big|_{\mathcal{V}}}{B_0}\sqrt{h(\sqrt{\gamma})}\right]|e|^2_M \\
&\le \left[2A_0\sqrt{\gamma} + \left(\frac{c'}{B_0^2} + \frac{1}{B_0}\right)\sqrt{\gamma\cdot h(\sqrt{\gamma})}\right]|e|^2_M,
\end{aligned}$$

where

$$A_0 = \max_K\Big|\frac{\mathrm{grad}^{\mathcal{X}}\psi}{\psi}\Big|, \qquad B_0 = \min_K\psi.$$

Since h is unbounded and γ is proper, then outside a compact subset of M the inequality $\gamma \le \gamma\cdot h\big(\sqrt{\gamma}\big)$ holds. Hence, from the last inequality we get that there exists a positive constant c'' such that, outside a compact set of M:

$$\mathrm{Hess}^M\gamma \le c''\sqrt{\gamma\cdot h(\sqrt{\gamma})}.$$

This proves that (h,γ) is an OY-pair for the Hessian in (M, g^M). The statement for the Laplacian is proved similarly. □

Remark 3.5. Observe that for the last statement of Theorem 3.4, concerning the validity of the Omori–Yau principle for the Laplacian in (M, g^M), it is necessary to assume that (h,Γ) is an OY-pair for the Hessian in $(\mathcal{V},g^{\mathcal{V}})$. We also observe that assumption (c') can be weakened by requiring that only the vertical component of $\vec{H}^{\varphi}$ has norm less than or equal to $\alpha\sqrt{h(\sqrt{\gamma})}$ outside some compact set.

Corollary 3.6. *Under the assumptions of Theorem 3.4, the Omori–Yau Maximum Principle holds for* (M,g^M).

4. Curvature estimates

We will generalize the results of [1] and [2] to the case of isometric immersions into warped products. For this, let us consider an isometric immersion $\varphi\colon M\to\mathcal{M}=\mathcal{X}\times_{\psi}\mathcal{V}$ of the Riemannian manifold (M,g^M) into a warped product manifold $\mathcal{M}$. Set $n_M=\dim(M)$ and suppose that $n_M\geq n_{\mathcal{V}}+1$. We will assume that there exists a point $x_0\in\mathcal{X}$, a real number b and a positive number $r<\mathrm{inj}_{\mathcal{X}}(x_0)$ such that the radial sectional curvatures $K_{\mathcal{X}_{x_0}}$ along the radial geodesics issuing from x_0 satisfies

$$K_{\mathcal{X}_{x_0}}\leq b,\quad \text{in } B_{\mathcal{X}}(x_0;r). \tag{4.1}$$

Here $B_{\mathcal{X}}(x_0;r)$ is the geodesic ball in $\mathcal{X}$ centered at x_0 and of radius $r>0$ and $\mathrm{inj}_{\mathcal{X}}(x_0)$ is the *injectivity radius* of $\mathcal{X}$ at x_0. Our estimates will be given in terms of the function C_b, defined by

$$C_b(t)=\begin{cases}\sqrt{b}\,\cot\big(\sqrt{b}\,t\big), & \text{if } b>0 \text{ and } t\in(0,\pi/2\sqrt{b})\\ \dfrac{1}{t} & \text{if } b=0 \text{ and } t>0\\ \sqrt{-b}\coth\big(\sqrt{-b}\,t\big) & \text{if } b<0 \text{ and } t>0.\end{cases} \tag{4.2}$$

Observe that C_b is strictly decreasing in its domain. Denote by $\rho:\mathcal{X}\to\mathbb{R}$ the function

$$\rho(x)=\mathrm{dist}_{\mathcal{X}}(x_0,x);$$

this is a smooth function in $B_{\mathcal{X}}(x_0;r)$. The gradient of ρ satisfies

$$\big|\mathrm{grad}^{\mathcal{X}}\rho\big|_{\mathcal{X}}=\big|\mathrm{grad}^{\mathcal{M}}\rho^{\mathrm{h}}\big|_{\mathcal{M}}\equiv 1. \tag{4.3}$$

By the *Hessian Comparison Theorem* (see for instance Ref. [4]), given $x\in B_{\mathcal{X}}(x_0;r)$ and a vector $X\in T_x\mathcal{X}$ orthogonal to $\mathrm{grad}^{\mathcal{X}}\rho(x)$, then

$$\mathrm{Hess}^{\mathcal{X}}\rho(X,X)\geq C_b\big(\rho(x)\big)|X|^2_{\mathcal{X}}; \tag{4.4}$$

on the other hand, if $Y\in T_x\mathcal{X}$ is parallel to $\mathrm{grad}^{\mathcal{X}}\rho(x)$

$$\mathrm{Hess}^{\mathcal{X}}\rho(X,Y)=\mathrm{Hess}^{\mathcal{X}}\rho(Y,Y)=0. \tag{4.5}$$

4.1. **Sectional curvature estimates.** We will first generalize the main result in [2] to the case of isometric immersions into warped products. As above, let $\varphi\colon M \hookrightarrow \mathcal{M} = \mathcal{X} \times_\psi \mathcal{V}$ be an isometric immersion, let x_0 be a point in $\mathcal{X}$, denote by $\rho\colon \mathcal{X} \to \mathbb{R}$ the function $\rho(x) = \mathrm{dist}^{\mathcal{X}}(x_0, x)$, and define $F\colon \mathcal{X} \to \mathbb{R}$ by

$$F = \phi_b \circ \rho,$$

where ϕ_b is the function

$$\phi_b(t) = \begin{cases} 1 - \cos\big(\sqrt{b}\, t\big), & \text{if } b > 0 \text{ and } t \in (0, \pi/2\sqrt{b}) \\ t^2, & \text{if } b = 0 \text{ and } t > 0 \\ \cosh\big(\sqrt{-b}\, t\big), & \text{if } b < 0 \text{ and } t > 0. \end{cases} \tag{4.6}$$

This is a strictly increasing function, as $\phi_b'(t) > 0$ for all t in its domain, and it satisfies the following ordinary differential equation

$$\phi_b''(t) - \phi_b'(t) C_b(t) = 0. \tag{4.7}$$

We assume that x belongs to a sufficiently small neighborhood $\mathcal{U}$ of x_0, so that F is a smooth function, and that the image $\pi_{\mathcal{X}}\big(\varphi(M)\big)$ is contained in such neighborhood. The value of the parameter b is chosen in such a way that inequality (4.1) holds in $\mathcal{U}$. Thus we have a smooth function $f\colon M \to \mathbb{R}$ defined by

$$f = F^{\mathrm{h}} \circ \varphi. \tag{4.8}$$

Given $p \in M$, set $\varphi(p) = (x, v) \in \mathcal{M}$; the gradient of f at p is computed easily from the formula

$$\phi_b'\big(\rho(x)\big)\, \mathrm{grad}^{\mathcal{X}} \rho(x) = \mathrm{grad}^{\mathcal{M}} F^{\mathrm{h}}\big(\varphi(p)\big) = \mathrm{grad}^{M} f(p) + \mathrm{grad}^{\mathcal{M}} F^{\mathrm{h}}\big(\varphi(p)\big)^{\perp}. \tag{4.9}$$

Moreover, using (2.25), for $e \in T_pM$ one computes the Hessian

$$\begin{aligned} \mathrm{Hess}^{M} f(e,e) &= \mathrm{Hess}^{\mathcal{X}}(\phi_b \circ \rho)\big(\mathrm{e}^{\mathrm{hor}}, \mathrm{e}^{\mathrm{hor}}\big) \\ &\quad + \phi_b'\big(\rho(x)\big)\, g^{\mathcal{M}}\big(\mathrm{grad}^{\mathcal{X}}\rho, \mathcal{S}^{\varphi}(e,e)\big) \\ &\quad + \phi_b'\big(\rho(x)\big)\, g^{\mathcal{X}}\big(\mathrm{grad}^{\mathcal{X}}\rho, \tfrac{\mathrm{grad}^{\mathcal{X}}\psi}{\psi}\big)\, g^{\mathcal{M}}\big(\mathrm{e}^{\mathrm{ver}}, \mathrm{e}^{\mathrm{ver}}\big). \end{aligned} \tag{4.10}$$

Moreover,

$$\begin{aligned} &\mathrm{Hess}^{\mathcal{X}}(\phi_b \circ \rho)\big(\mathrm{e}^{\mathrm{hor}}, \mathrm{e}^{\mathrm{hor}}\big) \\ &\quad = \phi_b''(\rho)\, g^{\mathcal{X}}\big(\mathrm{grad}^{\mathcal{X}}\rho, \mathrm{e}^{\mathrm{hor}}\big)^2 + \phi_b'(\rho)\, \mathrm{Hess}^{\mathcal{X}}\rho\big(\mathrm{e}^{\mathrm{hor}}, \mathrm{e}^{\mathrm{hor}}\big) \\ &\overset{\text{by (4.7)}}{=} \phi_b'(\rho) C_b(\rho) g^{\mathcal{X}}\big(\mathrm{grad}^{\mathcal{X}}\rho, \mathrm{e}^{\mathrm{hor}}\big)^2 + \phi_b'(\rho)\, \mathrm{Hess}^{\mathcal{X}}\rho\big(\mathrm{e}^{\mathrm{hor}}, \mathrm{e}^{\mathrm{hor}}\big) \\ &\quad = \phi_b'(\rho) \Big[C_b(\rho) g^{\mathcal{X}}\big(\mathrm{grad}^{\mathcal{X}}\rho, \mathrm{e}^{\mathrm{hor}}\big)^2 + \mathrm{Hess}^{\mathcal{X}}\rho\big(\mathrm{e}^{\mathrm{hor}}, \mathrm{e}^{\mathrm{hor}}\big) \Big]. \end{aligned} \tag{4.11}$$

Let us now recall the following result, known in the literature as *Otsuki's Lemma.*

Lemma 4.1. *Let $\beta : V \times V \to W$ be a symmetric bilinear form, where V and W are finite dimensional vector spaces. Assume that $\beta(v, v) \neq 0$ for all $v \neq 0$, and that $\dim(W) < \dim(V)$. Then, there exist linearly independent vectors $v_1, v_2 \in V$ such that $\beta(v_1, v_1) = \beta(v_2, v_2)$ and $\beta(v_1, v_2) = 0$.*

Proof. See for instance [6, page 28]. □

We can now state our first main result (Theorem A in the Introduction).

Theorem 4.2. *Let (M, g^M) be a complete Riemannian n_M-manifold for which the Weak Omori–Yau principle for the Hessian holds and let $\varphi : M \to \mathcal{M} = \mathcal{X} \times_\psi \mathcal{V}$ be an isometric immersion. Assume that the following hypotheses are satisfied*

(1) $\pi_{\mathcal{X}}\big(\varphi(M)\big) \subset B_{\mathcal{X}}(x_0; r)$ *for some* $x_0 \in \mathcal{X}$ *and* $r \in (0, \mathrm{inj}_{\mathcal{X}}(x_0))$.

(2) *Assumption* (4.1) *holds.*

(3) $2n_M \geq 2n_{\mathcal{V}} + n_{\mathcal{X}} + 1$.

Then,

$$\sup_M K_M \geq C_b(r)^2 + \inf_{B_{\mathcal{X}}(x_0;r)} K_{\mathcal{X}}. \tag{4.12}$$

Proof. The assumption (3) together with the natural dimension bound $n_M \leq n_{\mathcal{M}} - 1$ implies that

$$2n_{\mathcal{V}} + n_{\mathcal{X}} + 1 \leq 2n_M \leq 2n_{\mathcal{X}} + 2n_{\mathcal{V}} - 2.$$

And that gives

$$n_{\mathcal{X}} \geq 3. \tag{4.13}$$

Using again assumption (3) and (4.13), together with the fact that $\mathrm{d}\varphi$ is injective, we have that for all $p \in M$ there exists a subspace $\Pi_p \subset T_pM$ with dimension

$$\dim(\Pi_p) \geq n_M - n_{\mathcal{V}} \geq \frac{1}{2}\big(n_{\mathcal{X}} + 1\big) \geq 2, \tag{4.14}$$

such that $\mathrm{d}\varphi(\Pi_p)$ is horizontal. Thus, for all $e \in \Pi_p$, $\mathrm{e}^{\mathrm{ver}} = 0$, and

$$\mathrm{e} = \mathrm{e}^{\mathrm{hor}} = g^{\mathcal{X}}\big(\mathrm{e}^{\mathrm{hor}}, \mathrm{grad}^{\mathcal{X}}\rho(p)\big)\mathrm{grad}^{\mathcal{X}}\rho + \mathrm{e}^{\perp}.$$

Using Hessian's comparison theorem, we get

$$\begin{aligned} \mathrm{Hess}^{\mathcal{X}}\rho\big(\mathrm{e}^{\mathrm{hor}}, \mathrm{e}^{\mathrm{hor}}\big) &= \mathrm{Hess}^{\mathcal{X}}\rho\big(\mathrm{e}^{\perp}, \mathrm{e}^{\perp}\big) \\ &\geq C_b(\rho)\, g^{\mathcal{X}}\big(\mathrm{e}^{\perp}, \mathrm{e}^{\perp}\big) \\ &= C_b(\rho)\Big[\big|\mathrm{e}\big|^2_{\mathcal{M}} - g^{\mathcal{X}}\big(\mathrm{e}^{\mathrm{hor}}, \mathrm{grad}^{\mathcal{X}}\rho\big)^2\Big]. \end{aligned} \tag{4.15}$$

From (4.11), we obtain that for all $e \in \Pi_p$

$$\mathrm{Hess}^{\mathcal{X}}(\phi_b \circ \rho)\big(\mathrm{e}^{\mathrm{hor}}, \mathrm{e}^{\mathrm{hor}}\big) \geq \phi_b'(\rho)\, C_b(\rho)\, \big|\mathrm{e}\big|^2_{\mathcal{M}}. \tag{4.16}$$

Moreover, for the function $f : M \to \mathbb{R}$ defined in (4.8), from (4.10) and (4.16), we obtain

$$\mathrm{Hess}^M f(e,e) \geq \phi_b'(\rho)\Big[C_b(\rho)\, \big|\mathrm{e}\big|^2_{\mathcal{M}} - \big|\mathcal{S}^{\varphi}(e,e)\big|_{\mathcal{M}}\Big], \tag{4.17}$$

for all $p \in M$ and all $e \in \Pi_p$. Here, we have used the equalities $\big|\mathrm{grad}^{\mathcal{X}}\rho\big|_{\mathcal{X}} = 1$ and $\mathrm{e}^{\mathrm{ver}} = 0$. We will now apply the Omori–Yau principle for the Hessian to the function f, which is smooth and bounded by assumption (1). Let $p_n \in M$ be a sequence satisfying

(a) $f(p_n) > \sup_M f - \dfrac{1}{n}$;

(b) $\mathrm{Hess}^M f(p_n) < \dfrac{1}{n}$,

for all n. Choose $e \in \Pi_{p_n}$, and recall that $|e|_M = \big|\mathrm{e}\big|_{\mathcal{M}} = \big|\mathrm{e}^{\mathrm{hor}}\big|_{\mathcal{M}}$. By (b) and (4.17), we have

$$\frac{1}{n}|e|^2_M > \mathrm{Hess}^M f(p_n)(e,e) \geq \phi_b'(s_n)\left(C_b(s_n)\, |e|^2_M - \big|\mathcal{S}^{\varphi}(e,e)\big|_{\mathcal{M}}\right),$$

where

$$s_n = \rho^{\mathrm{h}}\big(\varphi(p_n)\big).$$

Hence

$$\big|\mathcal{S}^{\varphi}(e,e)\big|_{\mathcal{M}} \geq \left(C_b(s_n) - \frac{1}{n\,\phi_b'(s_n)}\right)|e|^2_M \geq \left(C_b(r) - \frac{1}{n\,\phi_b'(s_n)}\right)|e|^2_M. \tag{4.18}$$

We now observe that assumption (3) gives $n_M > n_{\mathcal{V}}$; this implies that the image $\varphi(M)$ is not contained in the vertical fiber $\{x_0\} \times \mathcal{V}$, and in particular that $\sup_M \rho^{\mathrm{h}} \circ \varphi > 0$. For all b, the function ϕ_b is increasing and positive, and therefore $\sup_M f > 0$; this says that $\pi_{\mathcal{X}}(p_n)$ stays away from x_0, i.e., there exists $\delta > 0$ such that $s_n \geq \delta$. Therefore

$$\lim_{n\to\infty} \frac{1}{n\,\phi_b'(s_n)} = 0, \tag{4.19}$$

and so for n sufficiently large, we have $C_b(s_n) - \dfrac{1}{n\,\phi_b'(s_n)} > 0$, which implies in particular that $\mathcal{S}^{\varphi}(e,e) \neq 0$ for all $e \in \Pi_{p_n} \setminus \{0\}$. We can invoke Otsuki's Lemma, applied to the symmetric bilinear form given by the restriction of $\mathcal{S}^{\varphi}$ to $\Pi_{p_n} \times \Pi_{p_n}$, which takes values in the space $\mathrm{Im}\big(\mathrm{d}\varphi(p_n)\big)^{\perp}$. Note that, by assumption (3)

$$\dim\Big[\mathrm{Im}\big(\mathrm{d}\varphi(p_n)\big)^{\perp}\Big] = n_{\mathcal{X}} + n_{\mathcal{V}} - n_M < n_M - n_{\mathcal{V}} = \dim(\Pi_{p_n}).$$

Thus, there exist linearly independent vectors $e^1, e^2 \in \Pi_{p_n}$ such that:

$$\mathcal{S}^{\varphi}(e^1,e^1) = \mathcal{S}^{\varphi}(e^2,e^2), \quad \mathcal{S}^{\varphi}(e^1,e^2) = 0.$$

We can assume $|e^1|_M \geq |e^2|_M > 0$. We will now compare the sectional curvature $K_M(e^1, e^2)$ with the sectional curvature $K_{\mathcal{M}}\big(\mathrm{d}\varphi(e^1), \mathrm{d}\varphi(e^2)\big)$. The plane spanned by $\mathrm{d}\varphi(e^1)$, $\mathrm{d}\varphi(e^2)$ is horizontal, and recalling (2.5), we have:

$$K_{\mathcal{M}}\big(\mathrm{d}\varphi(e^1), \mathrm{d}\varphi(e^2)\big) = K_{\mathcal{X}}\big(\mathrm{d}\varphi(e^1), \mathrm{d}\varphi(e^2)\big). \tag{4.20}$$

Then, using Gauss equation we have

$$\begin{aligned} K_M(e^1, e^2) - K_{\mathcal{X}}\big(\mathrm{d}\varphi(e^1), \mathrm{d}\varphi(e^2)\big) &= K_M(e^1, e^2) - K_{\mathcal{M}}\big(\mathrm{d}\varphi(e^1), \mathrm{d}\varphi(e^2)\big) \\ = \frac{g^{\mathcal{M}}\big(\mathcal{S}(e^1, e^1), \mathcal{S}(e^2, e^2)\big) - \big|\mathcal{S}(e^1, e^2)\big|^2_{\mathcal{M}}}{|e^1|^2_M |e^2|^2_M - g^M(e^1, e^2)^2} &= \frac{\big|\mathcal{S}(e^1, e^1)\big|^2_{\mathcal{M}}}{|e^1|^2_M |e^2|^2_M - g^M(e^1, e^2)^2} \\ \geq \left[\frac{\big|\mathcal{S}(e^1, e^1)\big|_{\mathcal{M}}}{|e^1|^2_M}\right]^2 &\overset{\text{by (4.18)}}{\geq} \left(C_b(r) - \frac{1}{n\, \phi_b'(s_n)}\right)^2. \end{aligned}$$

Hence

$$\sup_M K_M - \inf_{B_{\mathcal{X}}(x_0; r)} K_{\mathcal{X}} \geq \left(C_b(r) - \frac{1}{n\, \phi_b'(s_n)}\right)^2.$$

Taking the limit as $n \to \infty$ and recalling (4.19) we get (4.12). □

4.2. **Mean curvature estimates.** We start with an elementary result.

Lemma 4.3. *Let $\big(W_i, \langle\ ,\ \rangle_i\big)$, $i = 1, 2$, be finite dimensional vector spaces with inner product, with dimensions n_i, $i = 1, 2$, and let $T : W_1 \to W_2$ be a linear map with the property that there exists an orthogonal decomposition $W_1 = W \oplus W'$ such that $T|_W : W \to W_2$ is a surjective isometry and $T|_{W'} = 0$. Then, for every orthonormal basis $\xi_1, \ldots, \xi_{n_1}$ of W_1, the following equality holds.*

$$\sum_{i=1}^{n_1} \Big|T\xi_i\Big|_2^2 = n_2. \tag{4.21}$$

In particular, if $\eta_1, \ldots, \eta_n$ is any orthonormal family in W_1, then $\sum\limits_{i=1}^{n} \Big|T\eta_i\Big|_2^2 \leq n_2$.

Proof. The left-hand side of (4.21) does not depend[5] on the orthonormal basis; it is the *Hilbert–Schmidt* squared norm of the linear map T. The equality is verified easily using an orthonormal basis of W_1 consisting of the union of an orthonormal basis of W and an orthonormal basis of W'. □

We can now prove the following (Theorem B in the Introduction):

[5]Namely, if $\eta_1', \ldots, \eta_{n_1}'$ is another orthonormal basis, then there exists an orthogonal $n_1 \times n_1$ matrix $A = (a_{ij})$ such that $\eta_i' = \sum_j a_{ij}\eta_j$ for all i. The orthogonality of A means that $(A^*A)_{jk} = \sum_i a_{ij}a_{ik} = \delta_{jk}$ for all j, k. Then:

$$\sum_i |T\eta_i'|_2^2 = \sum_{i,j,k} a_{ij}a_{ik}\langle T\eta_j, T\eta_k\rangle_2 = \sum_{j,k} \delta_{jk}\langle T\eta_j, T\eta_k\rangle_2 = \sum_j |T\eta_j|_2^2.$$

Theorem 4.4. *Let (M, g^M) be a complete Riemannian manifold for which the Weak Omori–Yau principle for the Laplacian holds, and let $\varphi : M \to \mathcal{M} = \mathcal{X} \times_\psi \mathcal{V}$ be an isometric immersion. Assume that the following hypotheses are satisfied.*

(1) $\pi_{\mathcal{X}}\big(\varphi(M)\big) \subset B_{\mathcal{X}}(x_0; r)$ *for some* $x_0 \in \mathcal{X}$ *and* $r \in (0, \mathrm{inj}_{\mathcal{X}}(x_0))$.

(2) *Assumption* (4.1) *holds.*

Then, denoting by $\vec{H}^\varphi$ the mean curvature vector of φ, one has the following estimate on the supremum of $|\vec{H}^\varphi|_{\mathcal{M}}$.

$$\sup_M \big|\vec{H}^\varphi\big|_{\mathcal{M}} \geq (n_M - n_{\mathcal{V}})\, C_b(r) - n_{\mathcal{V}} \Psi_0, \tag{4.22}$$

where

$$\Psi_0 = \sup_{\mathrm{dist}(x,x_0)\leq r} \Big|\frac{\mathrm{grad}^{\mathcal{X}} \psi(x)}{\psi(x)}\Big|_{\mathcal{X}}. \tag{4.23}$$

Proof. Inequality (4.22) is proved applying the Weak Omori–Yau principle to the function $f : M \to \mathbb{R}$ defined in (4.8). Let us give an estimate for the Laplacian of f as follows. From (4.10) and (4.11), given $p \in M$ and $e \in T_pM$ we have

$$\begin{aligned}\mathrm{Hess}^M f(e,e) \;=\; & \phi_b'(\rho)\Big[C_b(\rho) g^{\mathcal{X}}\big(\mathrm{grad}^{\mathcal{X}}\rho, \mathrm{e}^{\mathrm{hor}}\big)^2 + \mathrm{Hess}^{\mathcal{X}}\rho\big(\mathrm{e}^{\mathrm{hor}}, \mathrm{e}^{\mathrm{hor}}\big)\Big] \\ & + \phi_b'(\rho)\, g^{\mathcal{X}}\big(\mathrm{grad}^{\mathcal{X}}\rho, \frac{\mathrm{grad}^{\mathcal{X}}\psi}{\psi}\big)\, g^{\mathcal{M}}\big(\mathrm{e}^{\mathrm{ver}}, \mathrm{e}^{\mathrm{ver}}\big) \\ & + \phi_b'(\rho)\, g^{\mathcal{M}}\big(\mathrm{grad}^{\mathcal{X}}\rho, \mathcal{S}^\varphi(e,e)\big).\end{aligned} \tag{4.24}$$

Let us write $\mathrm{e} = \mathrm{e}^{\mathrm{hor}} + \mathrm{e}^{\mathrm{ver}}$ and $e^{\mathrm{hor}} = \mathrm{e}^\rho + \mathrm{e}^\perp$, where

$$\mathrm{e}^\rho = g^{\mathcal{X}}\big(\mathrm{e}^{\mathrm{hor}}, \mathrm{grad}^{\mathcal{X}}\rho\big)\, \mathrm{grad}^{\mathcal{X}}\rho.$$

Observe

$$\big|\mathrm{e}^{\mathrm{hor}}\big|^2_{\mathcal{M}} = \big|\mathrm{e}^\rho\big|^2_{\mathcal{M}} + \big|\mathrm{e}^\perp\big|^2_{\mathcal{M}} = g^{\mathcal{X}}\big(\mathrm{e}^{\mathrm{hor}}, \mathrm{grad}^{\mathcal{X}}\rho\big)^2 + \big|\mathrm{e}^\perp\big|^2_{\mathcal{M}}$$

Using the Hessian Comparison Theorem (4.4), we obtain

$$\begin{aligned}\mathrm{Hess}^{\mathcal{X}}\rho\big(\mathrm{e}^{\mathrm{hor}}, \mathrm{e}^{\mathrm{hor}}\big) \;&=\; \mathrm{Hess}^{\mathcal{X}}\rho\big(\mathrm{e}^\perp, \mathrm{e}^\perp\big) \\ &\geq\; C_b(\rho)\, g^{\mathcal{X}}\big(\mathrm{e}^\perp, \mathrm{e}^\perp\big) \\ &=\; C_b(\rho)\big|\mathrm{e}^\perp\big|^2_{\mathcal{M}}.\end{aligned} \tag{4.25}$$

From (4.24) and (4.25) we get to the following inequality.

$$
\begin{aligned}
\mathrm{Hess}^M f(e,e) \;&\geq\; \phi_b'(\rho)\, C_b(\rho)\, \Big[g^{\mathcal{X}}\big(\mathrm{e}^{\mathrm{hor}}, \mathrm{grad}^{\mathcal{X}}\rho\big)^2 + \big|\mathrm{e}^{\perp}\big|^2_{\mathcal{M}} \Big] \\
&\quad + \phi_b'(\rho) g^{\mathcal{M}}\big(\mathrm{grad}^{\mathcal{X}}\rho, \frac{\mathrm{grad}^{\mathcal{X}}\psi}{\psi}\big)\, \big|\mathrm{e}^{\mathrm{ver}}\big|^2_{\mathcal{M}} \\
&\quad + \phi_b'(\rho) g^{\mathcal{M}}\big(\mathrm{grad}^{\mathcal{X}}\rho, \mathcal{S}^{\varphi}(e,e)\big) \\
&= \phi_b'(\rho)\, C_b(\rho) \big|\mathrm{e}^{\mathrm{hor}}\big|^2_{\mathcal{M}} \\
&\quad + \phi_b'(\rho) g^{\mathcal{M}}\big(\mathrm{grad}^{\mathcal{X}}\rho, \frac{\mathrm{grad}^{\mathcal{X}}\psi}{\psi}\big)\, \big|\mathrm{e}^{\mathrm{ver}}\big|^2_{\mathcal{M}} \\
&\quad + \phi_b'(\rho) g^{\mathcal{M}}\big(\mathrm{grad}^{\mathcal{X}}\rho, \mathcal{S}^{\varphi}(e,e)\big)
\end{aligned} \tag{4.26}
$$

Let $(e_i)_{i=1}^{n_M}$ be an orthonormal basis of T_pM; from (4.26), we get

$$
\begin{aligned}
\triangle^M f \;&=\; \sum_{i=1}^{n_M} \mathrm{Hess}^M f(e_i, e_i) \\
&\geq \phi_b'(\rho)\, C_b(\rho) \sum_{i=1}^{n_M} \big|\mathrm{d}\varphi(e_i)^{\mathrm{hor}}\big|^2_{\mathcal{M}} \\
&\quad + \phi_b'(\rho) g^{\mathcal{M}}\big(\mathrm{grad}^{\mathcal{X}}\rho, \frac{\mathrm{grad}^{\mathcal{X}}\psi}{\psi}\big) \sum_{i=1}^{n_M} \big|\mathrm{d}\varphi(e_i)^{\mathrm{ver}}\big|^2_{\mathcal{M}} \\
&\quad + \phi_b'(\rho) g^{\mathcal{M}}\big(\mathrm{grad}^{\mathcal{X}}\rho, \vec{H}^{\varphi}\big) \\
&\geq \phi_b'(\rho)\, C_b(\rho) \sum_{i=1}^{n_M} \Big(1 - \big|\mathrm{d}\varphi(e_i)^{\mathrm{ver}}\big|^2_{\mathcal{M}}\Big) \\
&\quad - \phi_b'(\rho) \big|\frac{\mathrm{grad}^{\mathcal{X}}\psi}{\psi}\big|_{\mathcal{X}} \sum_{i=1}^{n_M} \big|\mathrm{d}\varphi(e_i)^{\mathrm{ver}}\big|^2_{\mathcal{M}} - \phi_b'(\rho) \big|\vec{H}^{\varphi}\big|_{\mathcal{M}} \\
&= \phi_b'(\rho) \Big[C_b(\rho) \Big(n_M - \sum_{i=1}^{n_M} \big|\mathrm{d}\varphi(e_i)^{\mathrm{ver}}\big|^2_{\mathcal{M}} \Big) \\
&\quad - \big|\frac{\mathrm{grad}^{\mathcal{X}}\psi}{\psi}\big|_{\mathcal{X}} \sum_{i=1}^{n_M} \big|\mathrm{d}\varphi(e_i)^{\mathrm{ver}}\big|^2_{\mathcal{M}} - \big|\vec{H}^{\varphi}\big|_{\mathcal{M}} \Big].
\end{aligned} \tag{4.27}
$$

Now, we claim that the following inequality holds

$$\sum_{i=1}^{n_M} \left|e_i^{\mathrm{ver}}\right|^2_{\mathcal{M}} \leq n_{\mathcal{V}}. \tag{4.28}$$

This follows from Lemma 4.3 applied to the linear map

$$\mathrm{d}\pi_{\mathcal{V}}\big(\varphi(p_n)\big) : T_{\varphi(p_n)}\mathcal{M} \longrightarrow T_{\pi(\varphi(p_n))}\mathcal{V},$$

where the space $T_{\pi(\varphi(p_n))}\mathcal{V}$ is endowed with the inner product $\psi^2 \cdot g^{\mathcal{V}}$, considering the orthonormal family $e_1, \dots, e_{n_M}$ in $T_{\varphi(p_n)}\mathcal{M}$.

Thus, (4.27) gives

$$\triangle^M f \geq \phi_b'(\rho)\Big[C_b(\rho)\big(n_M - n_{\mathcal{V}}\big) - n_{\mathcal{V}}\,\Psi_0 - \big|\vec{H}^{\varphi}\big|_{\mathcal{M}}\Big]. \tag{4.29}$$

The Weak Omori–Yau Principle for the Laplacian yields the existence of a sequence p_n in M such that

(a) $f(p_n) > \sup_M f - \dfrac{1}{n}$.

(b) $\Delta^M f(p_n) < \dfrac{1}{n}$.

Set $s_n = \rho\big(\varphi(p_n)\big)$. The inequality (4.29) gives

$$\phi_b'(s_n)\Big[C_b(s_n)\big(n_M - n_{\mathcal{V}}\big) - n_{\mathcal{V}}\,\Psi_0 - \big|\vec{H}^{\varphi}\big|_{\mathcal{M}}\Big] < \frac{1}{n} \tag{4.30}$$

for all n. Arguing as in the proof of Theorem 4.2, the sequence s_n is bounded away from 0, and so is the quantity $\phi_b'(s_n)$. Moreover, since C_b is decreasing, it is $C_b(s_n) \geq C_b(r)$ for all n. Taking the limit as $n \to \infty$ in (4.30), we obtain

$$C_b(r)\big(n_M - n_{\mathcal{V}}\big) - n_{\mathcal{V}}\,\Psi_0 - \sup\big|\vec{H}^{\varphi}\big|_{\mathcal{M}} \leq 0,$$

which is our thesis. □

In Theorem 4.4, the hypothesis on the validity of the weak Omori–Yau principle in (M, g^M) can be omitted by assuming instead that the fiber of the warped product has an OY-pair and that φ is proper.

Corollary 4.5. *Assume that*

- $(\mathcal{V}, g^{\mathcal{V}})$ *has an OY-pair for the Hessian,*
- φ *is proper,*
- *the ball* $B_{\mathcal{X}}(x_0; r)$ *has compact closure in* $\mathcal{X}$ *(for instance, if* $(\mathcal{X}, g^{\mathcal{X}})$ *is complete).*

Then the conclusion of Theorem 4.4 holds.

Proof. We argue by contradiction. If (4.22) does not hold, then the norm of the mean curvature vector $\big|\vec{H}^{\varphi}\big|_{\mathcal{M}}$ is bounded, and we can apply Proposition 3.4 to deduce that the (strong) Omori–Yau principle for the Laplacian holds in (M, g^M). Thus, *a fortiori*, (4.22) holds. □

5. Isometric immersions into Riemannian submersions

Our curvature estimates for isometric immersions into warped product can be partially extended to the far more general case of immersions into the total space of Riemannian submersions. Given Riemannian smooth manifolds $(\mathcal{M}, g^{\mathcal{M}})$ and $(\mathcal{X}, g^{\mathcal{X}})$, a Riemannian submersion is a smooth surjective map $\pi : \mathcal{M} \to \mathcal{X}$ such that the differential $\mathrm{d}\pi$ has everywhere maximal rank, and it is an isometry when restricted to horizontal vectors, i.e., $|X|_{\mathcal{M}} = \big|\mathrm{d}\pi(X)|_{\mathcal{X}}$ for all X orthogonal to the kernel of $\mathrm{d}\pi$. The manifold $\mathcal{M}$ is called the *total space* of the submersion, $\mathcal{X}$ is the base, and for all $x \in \mathcal{X}$, the *fiber* $\mathcal{V}_x$ is the smooth embedded submanifold of $\mathcal{M}$ given by $\pi^{-1}(x)$.

A horizontal vector field $X \in \mathfrak{X}(\mathcal{M})$ is *basic* if it is π-related to some vector field $X_* \in \mathfrak{X}(\mathcal{X})$. If X and Y are basic vector fields, then the horizontal component $(\nabla^{\mathcal{M}}_X Y)^{\mathrm{h}}$ of the covariant derivative $\nabla^{\mathcal{M}}_X Y$ is basic, and it is π-related to $\nabla^{\mathcal{X}}_{X_*} Y_*$, see [10, Lemma 1]. Having this in mind, it is easy to prove the following

Lemma 5.1. *Let $F : \mathcal{X} \to \mathbb{R}$ be a smooth function; set $F^{\mathrm{h}} = F \circ \pi \colon \mathcal{M} \to \mathbb{R}$. Then, the gradient* $\mathrm{grad}^{\mathcal{M}} F^{\mathrm{h}}$ *is basic. Given $p \in \mathcal{M}$ and horizontal vectors $X, Y \in T_p\mathcal{M}$, then* $\mathrm{Hess}^{\mathcal{M}} F^{\mathrm{h}}(X,Y)$ *is equal to* $\mathrm{Hess}^{\mathcal{X}} F\big(\mathrm{d}\pi_p(X), \mathrm{d}\pi_p(Y)\big)$. □

Let us recall that the geometry of a Riemannian submersion $\pi : \mathcal{M} \to \mathcal{X}$ is described by the *fundamental tensors* T and A, introduced by O'Neill see [10], defined by the following formulas:

$$T_\xi(\eta) = \big(\nabla^{\mathcal{M}}_{\xi^{\mathrm{ver}}}(\eta^{\mathrm{ver}})\big)^{\mathrm{hor}} + \big(\nabla^{\mathcal{M}}_{\xi^{\mathrm{ver}}}(\eta^{\mathrm{hor}})\big)^{\mathrm{ver}},$$

$$A_\xi(\eta) = \big(\nabla^{\mathcal{M}}_{\xi^{\mathrm{hor}}}(\eta^{\mathrm{hor}})\big)^{\mathrm{ver}} + \big(\nabla^{\mathcal{M}}_{\xi^{\mathrm{hor}}}(\eta^{\mathrm{ver}})\big)^{\mathrm{hor}},$$

for $\xi, \eta \in T\mathcal{M}$. Restricted to vertical vectors, T is the second fundamental form of the fibers of the submersion. On horizontal fields, A is essentially the integrability tensor of the horizontal distribution of the submersion.

5.1. **Sectional curvature estimates.** The tensor A is related to the sectional curvature of horizontal planes, as follows. Let $p \in \mathcal{M}$ and $X, Y \in T_p\mathcal{M}$ be (linearly independent) horizontal vectors; set $X_* = \mathrm{d}\pi_p(X)$ and $Y_* = \mathrm{d}\pi_p(Y)$. Then:

$$K_{\mathcal{M}}(X,Y) = K_{\mathcal{X}}(X_*, Y_*) - \frac{3\,\big|A_X Y\big|^2_{\mathcal{M}}}{\big|X\big|^2_{\mathcal{M}}\,\big|Y\big|^2_{\mathcal{M}} - g^{\mathcal{M}}(X,Y)^2}. \tag{5.1}$$

Given $p \in \mathcal{M}$, let us introduce the following notation:

$$\sec^{\mathcal{M}}_{\mathrm{hor}}(p) = \min\Big\{K_{\mathcal{M}}(\Pi) : \Pi \subset T_p\mathcal{M} \text{ horizontal 2-plane}\Big\}.$$

When the horizontal distribution is integrable, i.e., when the tensor A vanishes identically on horizontal vectors, then by (5.1) the sectional curvature of horizontal planes in $\mathcal{M}$ coincides with the curvature of the corresponding plane in the base manifold $\mathcal{X}$.

Theorem 5.2. *Let (M, g^M) be a Riemannian manifold in which the Omori–Yau Principle for the Hessian holds, and let $\varphi : M \to \mathcal{M}$ be an isometric immersion of M into the total space of a Riemannian submersion $\pi : \mathcal{M} \to \mathcal{X}$. Denote by $n_{\mathcal{V}}$ the dimension of the fibers of π. Assume that the following hypotheses are satisfied.*

(1) $\pi\big(\varphi(M)\big) \subset B_{\mathcal{X}}(x_0; r)$ *for some* $x_0 \in \mathcal{X}$ *and* $r \in (0, \mathrm{inj}_{\mathcal{X}}(x_0))$

(2) *assumption* (4.1) *holds,*

(3) $2n_M \geq 2n_{\mathcal{V}} + n_{\mathcal{X}} + 1$.

Then,

$$\sup_M K_M \geq C_b(r)^2 + \inf_{\pi^{-1}(B_{\mathcal{X}}(x_0;r))} \mathrm{sec}^{\mathcal{M}}_{\mathrm{hor}}. \tag{5.2}$$

If the horizontal distribution of π is integrable, then

$$\sup_M K_M \geq C_b(r)^2 + \inf_{B_{\mathcal{X}}(x_0;r)} K_{\mathcal{X}}. \tag{5.3}$$

Proof. In the proof of Theorem 4.2, the Omori–Yau principle is used for evaluating the Hessian of smooth functions on the base $\mathcal{X}$, in the directions of horizontal vectors. By Lemma 5.1, also in the case of Riemannian submersions the value of the Hessian in horizontal directions coincides with the Hessian on the base manifolds. Thus, the proof of Theorem 4.2 can be repeated *verbatim*, with the exception of formula (4.20), which is replaced by (5.1). This yields the estimate (5.2). When the horizontal distribution is integrable, then also formula (4.20) holds for the Riemannian submersion $\pi : \mathcal{M} \to \mathcal{X}$, and the conclusion is exactly the same as in Theorem 4.2. □

5.2. **Mean curvature estimates.** In order to extend to Riemannian submersions the result of Theorem 4.4, we need a generalization of formula (2.25). This is obtained easily using the expressions for the Levi–Civita connection of the total space of a Riemannian submersions given, for instance, in [10, Lemma 3]. If X is basic, and V is vertical, then the horizontal component of the covariant derivative $\nabla^{\mathcal{M}}_V X$ is given by

$$\big(\nabla^{\mathcal{M}}_V X\big)^{\mathrm{h}} = \big(\nabla^{\mathcal{M}}_X V\big)^{\mathrm{h}} = A_X V; \tag{5.4}$$

similarly, the vertical component of $\nabla^{\mathcal{M}}_V X$ is

$$\big(\nabla^{\mathcal{M}}_V X\big)^{\mathrm{v}} = T_V X. \tag{5.5}$$

Let $F : \mathcal{X} \to \mathbb{R}$ be a smooth map, $F^{\mathrm{h}} = F \circ \pi : \mathcal{M} \to \mathbb{R}$, $\varphi : M \to \mathcal{M}$ an isometric immersion and $f = F^{\mathrm{h}} \circ \varphi : M \to \mathbb{R}$. For $p \in M$ and $e \in T_pM$, let us set $\xi = \mathrm{d}\varphi_p(e) \in T_{\varphi(p)}\mathcal{M}$, and let $\mathcal{S}^{\varphi}$ denote the second fundamental form of φ.. Using the fact that $\mathrm{d}F^{\mathrm{h}}$ is basic, π-related to $\mathrm{grad}^{\mathcal{X}} F$, and using formulas (2.23), (5.4) and (5.5), we obtain the following expression for the Hessian of f.

$$
\begin{aligned}
\text{(5.6)}\quad \operatorname{Hess}^M f(e,e) &= \operatorname{Hess}^{\mathcal{M}} F^{\mathrm{h}}(\xi,\xi) + g^{\mathcal{M}}\big(\operatorname{grad}^{\mathcal{M}} F^{\mathrm{h}}, \mathcal{S}^{\varphi}(e,e)\big)\\
&= \operatorname{Hess}^{\mathcal{M}} F^{\mathrm{h}}(\xi^{\mathrm{hor}},\xi^{\mathrm{hor}}) + 2\operatorname{Hess}^{\mathcal{M}} F^{\mathrm{h}}(\xi^{\mathrm{hor}},\xi^{\mathrm{ver}})\\
&\quad + \operatorname{Hess}^{\mathcal{M}} F^{\mathrm{h}}(\xi^{\mathrm{ver}},\xi^{\mathrm{ver}}) + g^{\mathcal{M}}\big(\operatorname{grad}^{\mathcal{M}} F^{\mathrm{h}}, \mathcal{S}^{\varphi}(e,e)\big)\\
&= \operatorname{Hess}^{\mathcal{X}} F(\xi_*^{\mathrm{hor}},\xi_*^{\mathrm{hor}}) + 2\,g^{\mathcal{M}}\big(A_{\xi^{\mathrm{hor}}}(\operatorname{grad}^{\mathcal{M}} F^{\mathrm{h}}),\xi^{\mathrm{ver}}\big)\\
&\quad + g^{\mathcal{M}}\big(T_{\xi^{\mathrm{ver}}}(\operatorname{grad}^{\mathcal{M}} F^{\mathrm{h}}),\xi^{\mathrm{ver}}\big) + g^{\mathcal{M}}\big(\operatorname{grad}^{\mathcal{M}} F^{\mathrm{h}}, \mathcal{S}^{\varphi}(e,e)\big).
\end{aligned}
$$

Denote by $\mathcal{S}^{\mathcal{V}}$ the second fundamental form of the fibers; the term containing the fundamental tensor T can be rewritten in terms of $\mathcal{S}^{\mathcal{V}}$ as

$$
\text{(5.7)}\qquad g^{\mathcal{M}}\big(T_{\xi^{\mathrm{ver}}}(\operatorname{grad}^{\mathcal{M}} F^{\mathrm{h}}),\xi^{\mathrm{ver}}\big) = -g^{\mathcal{M}}\big(\mathcal{S}^{\mathcal{V}}(\xi^{\mathrm{ver}},\xi^{\mathrm{ver}}),\operatorname{grad}^{\mathcal{M}} F^{\mathrm{h}}\big).
$$

Comparing (5.6) with (2.25), the reader will observe that the term in (5.7) corresponds to the last term in (2.25), see (2.8). The new term here is the one containing the fundamental tensor A, which vanishes in the case of warped products. Thus, it is easy to formulate the following extension of Theorem 4.4:

Theorem 5.3. *Let (M, g^M) be a Riemannian manifold in which the Omori–Yau principle for the Laplacian holds, and let $\varphi : M \to \mathcal{M}$ be an isometric immersion into the total space of a Riemannian submersion $\pi : \mathcal{M} \to \mathcal{X}$. Assume that the following hypotheses are satisfied.*

1. *$\pi\big(\varphi(M)\big) \subset B_{\mathcal{X}}(x_0; r)$ for some $x_0 \in \mathcal{X}$ and $r \in\,]0, \operatorname{inj}_{\mathcal{X}}(x_0)[$*
2. *assumption (4.1) holds;*

Then, denoting by $\vec{H}^{\varphi}$ the mean curvature vector of φ, and by T, A the fundamental tensors of the Riemannian submersion, one has the following estimate on the supremum of $\|\vec{H}^{\varphi}\|_{\mathcal{M}}$:

$$
\text{(5.8)}\qquad \sup_M \big\|\vec{H}^{\varphi}\big\|_{\mathcal{M}} \geq (n_M - n_{\mathcal{V}})\, C_b(r) - n_M\,\alpha_0 - n_{\mathcal{V}}\,\tau_0,
$$

where

$$
\text{(5.9)}\qquad \tau_0 = \sup_{\pi^{-1}(B_{\mathcal{X}}(x_0;r))} \big|T\big|,
$$

and

$$
\text{(5.10)}\qquad \alpha_0 = \sup_{\pi^{-1}(B_{\mathcal{X}}(x_0;r))} \big|A\big|.
$$

Proof. It suffices to repeat the proof of Theorem 4.4, keeping into consideration also the contribution of the term

$$
2\,g^{\mathcal{M}}\big(A_{\xi^{\mathrm{hor}}}(\operatorname{grad}^{\mathcal{M}} F^{\mathrm{h}}),\xi^{\mathrm{ver}}\big),
$$

which is estimated as follows

$$
2\,g^{\mathcal{M}}\big(A_{\xi^{\mathrm{hor}}}(\operatorname{grad}^{\mathcal{M}} F^{\mathrm{h}}),\xi^{\mathrm{ver}}\big) \geq -2\,\alpha_0\,|\xi^{\mathrm{hor}}|_{\mathcal{M}}\,|\xi^{\mathrm{ver}}|_{\mathcal{M}}.
$$

When $|\xi|^2_{\mathcal{M}} = 1$, then $|\xi^{\mathrm{hor}}|_{\mathcal{M}}\,|\xi^{\mathrm{ver}}|_{\mathcal{M}} \le \frac{1}{2}$. The conclusion follows readily. □

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