MULTISUMMABILITY OF UNFOLDINGS OF TANGENT TO THE IDENTITY DIFFEOMORPHISMS

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Abstract

We prove the multisummability of the infinitesimal generator of unfoldings of finite codimension tangent to the identity 1-dimensional local complex analytic diffeomorphisms. We also prove the multisummability of Fatou coordinates and extensions of the Ecalle-Voronin invariants associated to these unfoldings. The quasi-analytic nature is related to the parameter variable. As an application we prove an isolated zeros theorem for the analytic conjugacy problem.

The proof is based on good asymptotics of Fatou coordinates and the introduction of a new auxiliary tool, the so called multi-transversal flows. They provide the estimates and the combinatorics of sectors typically associated to summability. The methods are based on the study of the infinitesimal stability properties of the unfoldings.

1. Introduction

We study unfoldings of non-linearizable resonant complex analytic diffeomorphisms. The group of 1-dimensional unfoldings of elements of $\mathrm{Diff}\,(\mathbb{C},0)$ is

$$\mathrm{Diff}_p(\mathbb{C}^2,0) = \{\varphi(x,y) \in \mathrm{Diff}\,(\mathbb{C}^2,0) \ : \ x \circ \varphi = x\}.$$

Most of the time we work in the set $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ composed by the elements φ of $\mathrm{Diff}_p(\mathbb{C}^2,0)$ such that $\varphi_{|x=0}$ is tangent to the identity (i.e. $j^1\varphi_{|x=0} = Id$) but $\varphi_{|x=0} \neq Id$. The main goal of the paper is providing a rigorous formulation and then proving the following statement:

Principle. *The infinitesimal generator of an element φ of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ is multi-summable in the x-variable.*

A natural way of studying unfoldings $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ of tangent to the identity diffeomorphisms is comparing the dynamics of φ and $\exp(X)$ where X is a vector field whose time 1 flow "approximates" φ. This point of view has been developed by Glutsyuk [6]. In this way extensions of the Ecalle-Voronin invariants to some sectors in the parameter space are obtained. On the one hand they are uniquely defined. On the other hand the sectors have to avoid a finite set of directions, typically associated (but not exclusively) to small divisors phenomena.

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A different point of view was introduced by Shishikura for codimension 1 unfoldings [18]. The idea is constructing appropriate fundamental domains bounded by two curves with common ends at singular points: one curve is the image of the other one. Pasting the boundary curves by the dynamics yields (by quasiconformal surgery) a Riemann surface that is conformally equivalent to the Riemann sphere. The logarithm of an appropriate affine complex coordinate on the sphere induces a Fatou coordinate for φ. These ideas were generalized to higher codimension unfoldings by Oudkerk [12]. In this approach the first curve is a phase curve of an appropriate vector field transversal to the real flow of X. In both cases the Fatou coordinates provide Lavaurs vector fields X^{φ} such that $\varphi = \exp(X^{\varphi})$ [7]. The Shishikura's approach was used by Mardesic, Roussarie and Rousseau to provide a complete system of invariants for unfoldings of codimension 1 tangent to the identity diffeomorphisms [9]. The analytic classification for the finite codimension case was completed in [16] by using the Oudkerk's point of view. On the one hand the constructions are applied to sectors whose union is a neighborhood of the origin in the x-variable. On the other hand the extensions of Fatou coordinates, Lavaurs vector fields and Ecalle-Voronin invariants depend on the choices in the construction. One of the goals of this paper is explaining how all these objects are intrinsic and can be interpreted as different sectorial sums of a quasi-analytic formal object.

1.1. **Construction of multi-transversal flows.** In order to study the properties of the infinitesimal generator of $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ we construct transversal flows defined in sectorial domains in the variable x. They are of the form $\Re(\aleph^* X)$ where $\aleph^* : ([0,\delta)e^{i[u_0,u_1]} \times B(0,\epsilon)) \setminus Sing(X) \to \mathbb{S}^1 \setminus \{-1, 1\}$ is a continuous function; let us explain how.

The main ideas of the construction can be found in [16]. We use the dynamical splitting, we express a neighborhood of the origin in $\mathbb{C}^2$ as a union of basic sets that are associated to φ by a desingularization process of the fixed points set $Fix(\varphi)$ of φ. There are two types of basic sets, namely exterior and compact-like basic sets. The exterior sets are dynamically simple and the restriction of $\Re(\mu X)$ to an exterior set is either a parametrized Fatou flower or truncated Fatou flower for any $\mu \in \mathbb{S}^1$ (see figure (3)). Thus the dynamics of $\Re(\mu X)$ in a neighborhood of the origin is determined by the dynamics of $\Re(\mu X)$ in the compact-like sets $\mathcal{C}_1, \ldots, \mathcal{C}_q$. We can associate an exponent $e_j \in \mathbb{N}$ and a polynomial vector field $P_j(w)\partial/\partial w$ such that the dynamics of $Re(\mu X)$ in $\mathcal{C}_j$ is orbitally equivalent to the dynamics of $\Re(|x|^{e_j} \lambda^{e_j} \mu P_j(w)\partial/\partial w)$ for $1 \leq j \leq q$ where $x = |x|\lambda$ (see figure (2)). We define

$$\mathcal{U}_X^j = \{(\lambda, \mu) \in \mathbb{S}^1 \times \mathbb{S}^1 : \Re(\lambda^{e_j} \mu P_j(w)\partial/\partial w) \text{ is not stable}\}$$

The definition of stability is borrowed from Douady, Estrada and Sentenac [4] (see [16]). The real flow of a vector field $P(w)\partial/\partial w \in \mathbb{C}[w]$ is stable if $Re(\mu P(w)\partial/\partial w)$ is orbitally conjugated to $Re(P(w)\partial/\partial w)$ for any $\mu \in \mathbb{S}^1$ in a neighborhood of 1. It turns out that $\Re(\mu P(w)\partial/\partial w)$ is stable except for finitely many directions $\mu \in \mathbb{S}^1$. Then $\Re(\mu X)$ is stable in a neighborhood of the direction $\lambda\mathbb{R}^+$ in the x-space if $(\lambda, \mu) \notin \mathcal{U}_X^j$; in other words there exists a sector $S = (0,\delta)\lambda e^{i[-u,u]}$ for some $\delta, u \in \mathbb{R}^+$ such that $\Re(\mu X)_{|x=x_0}$ is orbitally conjugated to $\Re(\mu X)_{|x=x_1}$ in $\mathcal{C}_j$ for any $(x_0, x_1) \in S \times S$. The stability of transversal flows is an important part of our approach since it guarantees that the objects constructed (Fatou coordinates,...) depend holomorphically on both variables.

Given a continuous function $\mu_j : e^{i[u_0,u_1]} \to \mathbb{S}^1 \setminus \{-1,1\}$ such that $(\lambda, \mu_j(\lambda)) \notin \mathcal{U}_X^j$ for any $\lambda \in e^{i[u_0,u_1]}$ it is natural to consider $\Re(\aleph^* X)$ such that

$$\Re(\aleph^* X)_{|\mathcal{C}_j \cap \{x=|x_0|\lambda_0\}} = \Re(\mu_j(\lambda_0)X)_{|\mathcal{C}_j \cap \{x=|x_0|\lambda_0\}}$$

for $0 < |x_0| < \delta$ and $\lambda_0 \in e^{i[u_0,u_1]}$. In this way we define $\Re(\aleph^* X)_{|\mathcal{C}_j}$ for $1 \leq j \leq q$. Such a vector field would be stable in every compact-like set. Since compact-like sets collapse when approaching $x = 0$ (we have $\mathcal{C}_j \cap \{x = 0\} = \{(0,0)\}$) we are requiring conditions of infinitesimal stability for $\Re(\aleph^* X)$. The exterior sets are dynamically simple, we can use them to interpolate the transversal flows defined in different compact-like sets. We obtain in this way a multi-transversal flow. Roughly speaking it is a stable flow transversal to $\Re(X)$ that is of the form $\Re(\mu_j(x/|x|)X)$ by restriction to any compact-like set $\mathcal{C}_j$. Let us remind that in [16] the functions $\aleph^*$ are constant and that it is not difficult to generalize the constructions there for functions $\aleph^* = \aleph^*(x/|x|)$.

Our objects (Fatou coordinates,...) are defined in regions. A region is a connected component of the subset T of $([0,\delta)e^{i[u_0,u_1]} \times B(0,\epsilon)) \setminus Sing(X)$ obtained as the union of the trajectories of $\Re(\aleph^* X)$ whose α and ω limits are both singular points. The multi-transversal flows have two important properties:

- The infinitesimal stability properties allow us to use the same ideas in [16] to find Fatou coordinates ψ_H^φ of φ defined in regions H of $\Re(\aleph^* X)$ such that $\psi_H^\varphi - \psi^X$ is continuous in $\overline{H}$ and holomorphic in $\dot{H}$ where ψ^X is a Fatou coordinate of X. In particular the function $\psi_H^\varphi - \psi^X$ is bounded.
- The dynamics of multi-transversal flows and the transitions of the dynamics between different multi-transversal flows can be described in a combinatorial way.

Let us explain succinctly how to use the previous properties to deduce multi-summability of Fatou coordinates, Lavaurs vector fields and Ecalle-Voronin invariants. Consider a petal L_j of $\varphi_{|x=0}$. There exists a unique region H_j of $\Re(\aleph^* X)$ containing $L_j \cap T$. Consider the region $\tilde{H}_j$ obtained in an analogous way for a multi-transversal flow $\Re(\tilde{\aleph}^* X)$ defined in $[0,\delta)e^{i[\tilde{u}_0,\tilde{u}_1]} \times B(0,\epsilon)$. The first property implies roughly speaking that $\psi_{H_j}^\varphi - \psi_{\tilde{H}_j}^\varphi$ is bounded in $H_j \cap \tilde{H}_j$. Clearly $\psi_{H_j}^\varphi - \psi_{\tilde{H}_j}^\varphi$ is constant in orbits of φ. As a consequence the function

$$(\psi_{H_j}^\varphi - \psi_{\tilde{H}_j}^\varphi) \circ (x, e^{2\pi i \psi_{H_j}^\varphi})^{-1}$$

is well-defined and bounded in a domain of the form

$$\{(x,z) \in [0,\delta)((e^{i[u_0,u_1]}) \cap (e^{i[\tilde{u}_0,\tilde{u}_1]})) \times \mathbb{C} : e^{\frac{-C}{|x|^e}} < |z| < e^{\frac{C}{|x|^e}}\}.$$

The exponent e is deduced from the combinatorial study of multi-transversal flows. Hence we obtain that up to an additive function of x the function $\psi_{H_j}^\varphi - \psi_{\tilde{H}_j}^\varphi$ is a $O(e^{K/|x|^e})$ by using Cauchy's integral formula. The combinatorics provides the exponentially small estimates and the right sectors to obtain multi-summability. Multi-summability of Lavaurs vector fields and Ecalle-Voronin invariants is deduced from the analogous property for Fatou coordinates.

Since a multi-summable power series in a direction is a sum of summable ones, the multi-summability levels of of Fatou coordinates have to appear in an independent way. The multi-summability is related to the nature of φ in compact-like sets, indeed the multi-summability levels are contained in the set $\{e_1, \ldots, e_j\}$. Imposing all the μ_j functions ($\Re(\aleph^* X)_{|\mathcal{C}_j} = \Re(\mu_j X)_{|\mathcal{C}_j}$) to be equal would result in too small sectors too obtain a multi-summable object.

Let us remark that the summability of Ecalle-Voronin invariants for generic families unfolding a codimension 1 parabolic or resonant diffeomorphism is proved in [17] and [2]. The methods are different. In particular it is used the so called compatibility condition that establishes whether different invariants correspond to the same diffeomorphism via a translation to Glutsyuk invariants. We do not need such a condition, the good estimates on the asymptotic of Fatou coordinates suffice to prove the multi-summability for Ecalle-Voronin invariants in all finite codimension unfoldings. We also prove the multi-summability of Fatou coordinates and Lavaurs vector fields. This last object is specially interesting to us since one of the goals of this paper is interpreting the sectorial Lavaurs vector fields as sums of the formal infinitesimal generator.

1.2. **Intrinsic nature of sectorial objects.** As we said earlier we could make other choices of flows transversal to $\Re(X)$. Consider a case such that the fixed points set $Fix(\varphi)$ of φ is a union of curves of the form $y = \gamma_j(x)$ for $j \in \{1, \ldots, p\}$ and $(\partial \gamma_j / \partial x)(0) \neq (\partial \gamma_k / \partial x)(0)$ for $j \neq k$. This is one of the cases providing summable formal objects. Then we only consider multi-transversal flows that are deformed versions of the imaginary flow $\Re(iX)$. More precisely our multi-transversal flows $\Re(\aleph^* X)$ are defined in sectors of the form $[0, \delta) e^{i[u_0, u_1]} \times B(0, \epsilon)$ and there exists $\lambda \in e^{i(u_0, u_1)}$ such that $(\aleph^*)_{|[0,\delta)\lambda \times B(0,\epsilon)} \equiv i$. The situation is slightly different in the multi-summable case where the richer combinatorics makes us consider other kind of multi-transversal flows. Anyway, the imaginary flow is again the basic ingredient that is present in all multi-transversal flows. For instance, the directions of non-summability (Stokes directions) are contained in $\{\lambda \in \mathbb{S}^1 : (\lambda, i) \in \cup_{j=1}^{q} \mathcal{U}_X^j\}$, i.e. the sets of directions that are unstable for the restriction of the imaginary flow to some of the compact-like basic sets. In short, on the one hand all the multi-transversal flows are related to the imaginary flow $\Re(iX)$. On the other hand we need to consider more general vector fields since in this way the sums of the multi-summable objects can be realized in wide enough sectors.

1.3. **Infinitesimal generator.** The use of normal forms $\exp(X)$ to study unfoldings $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ is classical (see Martinet's paper [11]). Vector fields are used to model the diffeomorphisms even if it is clear that generic unfoldings do not behave as nicely as flows. This paper is one step forward in the direction of justifying such an approach. The diffeomorphism φ is embedded in the "formal flow" of its infinitesimal generator. We show that such an object is of geometric nature and that its sectorial sums provide analytic vector fields whose time 1 flow coincides with φ. The complexity of the diffeomorphism can be interpreted in a cohomological way since the Lavaurs vector fields do not coincide when their domains of definition overlap. Our results allow to obtain Lavaurs vector fields from the infinitesimal generator and vice versa.

Let us explain in what sense the infinitesimal generator of φ is multi-summable in the variable x. Consider a petal L_j of $\varphi_{|x=0}$ and the region H_j of $\Re(\aleph^* X)$ as defined above. There exists a unique analytic vector field (the Lavaurs vector field) $X_{H_j}^{\varphi} = g_{H_j}^{\varphi}(x, y) \partial / \partial y$ defined in H_j such that $X_{H_j}^{\varphi}(\psi_{H_j}^{\varphi}) \equiv 1$. By fixing L_j but varying $\Re(\aleph^* X)$ we obtain a family of functions $\{g_{H_j}^{\varphi}(x, y)\}$ that is multi-summable in the variable x. The common asymptotic development of the family $\{X_{H_j}^{\varphi}\}$ is of the form

$$\hat{X}_{L_j}^{\varphi} = \left(\sum_{k=0}^{\infty} g_{j,k}^{\varphi}(y) x^k \right) \frac{\partial}{\partial y}$$

where $g_{j,k}^{\varphi}$ is defined in L_j for any $k \in \mathbb{N} \cup \{0\}$. Now we fix $k \geq 0$. The family $\{g_{j,k}^{\varphi}\}$ is parametrized by the set of petals of $\varphi_{|x=0}$. Moreover the functions in the family $\{g_{j,k}^{\varphi}\}$ are sums of a ν summable power series $\hat{g}_k^{\varphi}$ where 2ν is the number of petals of $\varphi_{|x=0}$. As a result of this two step process we recover

$$\log \varphi = \left(\sum_{k=0}^{\infty} \hat{g}_k^{\varphi}(y) x^k \right) \frac{\partial}{\partial y}$$

the infinitesimal generator of φ.

Let us remark that the estimates in this paper for Fatou coordinates are the generalizations of those in [16]. Thus, they provide

- Asymptotic developments of the Lavaurs vector fields X_H^{φ} until the first non-zero term in the neighborhood of the fixed points [16].
- Gevrey asymptotics in the neighborhood of the bifurcation set $x = 0$.

1.4. **Isolated zeros theorem for analytic conjugacy.** Given $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$, let us relate the class of analytic conjugacy of φ with the classes of analytic conjugacy of the 1-dimensional germs in the family $\{\varphi_{|x=x_n}\}$ for some sequence $x_n \to 0$. This is an application of the multi-summability of the Ecalle-Voronin invariants.

Let $\varphi, \eta \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$. We denote $\varphi \sim \eta$ if there exists $\sigma \in \text{Diff}(\mathbb{C}^2, 0)$ conjugating φ and η such that $\sigma_{|Fix(\varphi)} \equiv Id$.

Theorem 1.1. *Let $\varphi, \eta \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ with $Fix(\varphi) = Fix(\eta)$. Suppose there exist $s \in \mathbb{R}^+$ and a sequence $x_n \to 0$ contained in $B(0, \delta) \setminus \{0\}$ such that for any $n \in \mathbb{N}$ the restrictions $\varphi_{|x=x_n}$ and $\eta_{|x=x_n}$ are conjugated by an injective holomorphic mapping κ_n defined in $B(0, s)$ and fixing the points in $Fix(\varphi) \cap \{x = x_n\}$. Then we obtain $\varphi \sim \eta$.*

The previous theorem is the corollary 7.1 of subsection 7.3. Indeed we prove the more general theorem 7.3 that analyzes what is the minimum domain of definition $B(0, s_n)$ of the mappings κ_n such that the theorem is still true. The theorem can be extended to codimension finite resonant diffeomorphisms (remark 7.1). Theorem 1.1 is a generalization of the main theorem of [16] where the κ mappings where required to exist for any parameter x in a pointed neighborhood of 0.

The theorem can be interpreted as an isolated zeros theorem, i.e. if the set of parameters x_0 such that $\varphi_{|x=x_0}$ and $\eta_{|x=x_0}$ are conjugated by a mapping defined in $B(0, s)$ acumulates the origin then it contains a neighborhood of the origin. This analytic type property is a consequence of the quasi-analytic nature of Ecalle-Voronin invariants. The possibility of having flat Ecalle-Voronin invariants when $x \to 0$ would be an obstruction to the extension of conjugations even if defined in open sets of parameters.

1.5. **Remarks.** The codimension ∞ case can be studied with the techniques in this paper. Indeed all the transforms of φ in the desingularization process of $Fix(\varphi)$ are codimension ∞ unfoldings. We still obtain exponentially small estimates when comparing different Fatou coordinates, Lavaurs vector fields or Ecalle-Voronin invariants. These objects are not multi-summable since in general they are not bounded in the neighborhood of $x = 0$.

Given un unfolding $\varphi(x, y) = (x, f(x, y))$ of a resonant difeomorphism $\phi \in \text{Diff}\,(\mathbb{C}, 0)$ (i.e. $(\partial\phi/\partial y)(0)$ is a root of the unit of order p) it admits a Jordan decomposition

$$\varphi = \varphi_s \circ \varphi_u = \varphi_u \circ \varphi_s.$$

In fact φ_s, φ_u are formal diffeomorphisms such that φ_s is semisimple (or equivalently formally linearizable) and φ_u is unipotent ($j^1\varphi_u$ is unipotent). Since $j^1\varphi$ is p periodic then φ_s is also p periodic. Thus we obtain $\varphi^p = \varphi_u^p = \exp(p \log \varphi_u)$. We deduce that $\log \varphi_u$ is multi-summable since $\varphi^p \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$. The semisimple part φ_s is the unique formal diffeomorphism such that $\varphi_s^p \equiv Id$, it commutes with φ_u and $(\partial(y \circ \varphi_s)/\partial y)(0, 0) = (\partial(y \circ \varphi)/\partial y)(0, 0)$. Therefore φ_s is totally determined by φ_u (prop. 5.4 in [16]). It is possible to use the multi-summability of $\log \varphi_u$ to deduce the multi-summability of φ_s.

2. Notations and definitions

Let $\text{Diff}\,(\mathbb{C}^n, 0)$ be the group of complex analytic germs of diffeomorphisms at $0 \in \mathbb{C}^n$. Denote $Fix(\varphi)$ the set of fixed points of an element φ of $\text{Diff}\,(\mathbb{C}^n, 0)$.

Definition 2.1. *Let $\varphi : U \to V$ be a holomorphic mapping where U and V are open sets of $\mathbb{C}^n$. We say that a holomorphic $\psi : U \to \mathbb{C}$ is a Fatou coordinate of φ if $\psi \circ \varphi \equiv \psi + 1$ in $U \cap \varphi^{-1}(U)$.*

We say that $\varphi \in \text{Diff}\,(\mathbb{C}^2, 0)$ is a parametrized diffeomorphism if $\varphi(x, y)$ is of the form $(x, f(x, y))$. Equivalently φ is an unfolding of $\varphi_{|x=0} \in \text{Diff}\,(\mathbb{C}, 0)$. We denote $\text{Diff}_p(\mathbb{C}^2, 0)$ the group of parametrized diffeomorphisms. Let $\text{Diff}_1(\mathbb{C}, 0)$ be the subgroup of $\text{Diff}\,(\mathbb{C}, 0)$ of germs whose linear part is the identity.

Definition 2.2. *We define the set*

$$\text{Diff}_{p1}(\mathbb{C}^2, 0) = \{\varphi \in \text{Diff}_p(\mathbb{C}^2, 0) : \varphi_{|x=0} \in \text{Diff}_1(\mathbb{C}, 0) \setminus \{Id\}\}.$$

Then $\text{Diff}_{p1}(\mathbb{C}^2, 0)$ is the set of one dimensional unfoldings of one dimensional tangent to the identity germs of diffeomorphisms (excluding the identity). We consider $\text{Diff}_{tp1}(\mathbb{C}^2, 0)$ the subset of $\text{Diff}_{p1}(\mathbb{C}^2, 0)$ such that $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$ if all the irreducible components of $Fix(\varphi)$ are of the form $y = g(x)$.

We define a formal vector field $\hat{X}$ as a derivation of the maximal ideal of the ring $\mathbb{C}[[x_1, \ldots, x_n]]$. We also express $\hat{X}$ in the more conventional form

$$\hat{X} = \hat{X}(x_1)\partial/\partial x_1 + \ldots + \hat{X}(x_n)\partial/\partial x_n.$$

We denote $\hat{\mathcal{X}}(\mathbb{C}^n, 0)$ the set of formal vector fields. We denote $\mathcal{X}(\mathbb{C}^n, 0)$ the Lie algebra of germs of analytic vector fields in a neighborhood of $0 \in \mathbb{C}^n$. A formal vector field X belongs to $\mathcal{X}(\mathbb{C}^n, 0)$ if and only if $X(\mathbb{C}\{x_1, \ldots, x_n\}) \subset \mathbb{C}\{x_1, \ldots, x_n\}$.

Definition 2.3. *Let X be a holomorphic vector field defined in an open set U of $\mathbb{C}^n$. We say that a holomorphic $\psi : U \to \mathbb{C}$ is a Fatou coordinate of X if $X(\psi) \equiv 1$.*

Definition 2.4. *We denote $\mathcal{X}_{tp1}(\mathbb{C}^2,0)$ the subset of $\mathcal{X}(\mathbb{C}^2,0)$ of vector fields of the form*
$$X = v(x,y)(y-\gamma_1(x))^{s_1}\dots(y-\gamma_p(x))^{s_p}\partial/\partial y$$
where $v,\gamma_1,\dots,\gamma_p \in \mathbb{C}\{x,y\}$, $v(0)\neq 0=\gamma_1(0)=\dots=\gamma_p(0)$ and $s_1+\dots+s_p\geq 2$. We denote $\hat{\mathcal{X}}_{tp1}(\mathbb{C}^2,0)$ the set of formal vector fields that are of the previous form but allowing $\hat{v}\in\mathbb{C}[[x,y]]$.

Given a vector field X defined in a domain $U\subset\mathbb{C}^n$ we denote $\Re(X)$ the real flow of X, namely the two dimensional vector field on $\mathbb{R}^{2n}=\mathbb{C}^n$ defined by X.

Suppose that $X\in\mathcal{X}(\mathbb{C}^n,0)$ is singular at 0. We denote $\exp(tX)$ the flow of the vector field X, it is the unique solution of the differential equation
$$\frac{\partial}{\partial t}\exp(tX) = X(\exp(tX))$$
with initial condition $\exp(0X)=Id$. We define the exponential $\exp(X)$ of X as $\exp(1X)$. We can define the exponential operator for a nilpotent $\hat{X}\in\hat{\mathcal{X}}(\mathbb{C}^n,0)$ and in particular for $\hat{X}\in\hat{\mathcal{X}}_{tp1}(\mathbb{C}^2,0)$ as
$$\begin{array}{rccc}\exp(\hat{X}): & \mathbb{C}[[x_1,\dots,x_n]] & \to & \mathbb{C}[[x_1,\dots,x_n]] \\ & g & \to & \sum_{j=0}^{\infty}\frac{\hat{X}^{\circ(j)}}{j!}(g).\end{array}$$
Moreover the definition coincides with the previous one if $\hat{X}$ is convergent, i.e. $(\exp(X))(g)=g\circ\exp(X)$ for any $g\in\mathbb{C}[[x,y]]$. By the properties of the exponential mapping given a unipotent $\varphi\in\text{Diff}\,(\mathbb{C}^n,0)$ (i.e. $j^1\varphi$ is unipotent) there exists a unique formal nilpotent vector field $\log\varphi\in\hat{\mathcal{X}}(\mathbb{C}^n,0)$ such that $\varphi=\exp(\log\varphi)$ (see [5] and [10]). We say that $\log\varphi$ is the *infinitesimal generator* of φ.

Definition 2.5. *Let X be a holomorphic vector field defined in a connected domain $U\subset\mathbb{C}$ such that $X\not\equiv 0$. Consider $P\in SingX$. There exists a unique meromorphic differential form ω in U such that $\omega(X)=1$. We denote $Res(X,P)$ the residue of ω at the point P.*

Definition 2.6. *Let $Y=f(x,y)\partial/\partial y\in\mathcal{X}_{tp1}(\mathbb{C}^2,0)$. Given $(x^0,y^0)\in SingY$ we define $Res(Y,(x^0,y^0))=Res(f(x^0,y)\partial/\partial y,y^0)$.*

3. Dynamical splitting

We introduce a dynamical splitting $\digamma$ associated to an element of $\mathcal{X}_{tp1}(\mathbb{C}^2,0)$ along with some notation. Most of the concepts were already introduced in [16].

Let $X\in\mathcal{X}_{tp1}(\mathbb{C}^2,0)$. We say that $T_0=\{(x,y)\in B(0,\delta)\times\overline{B(0,\epsilon)}\}$ is a *seed*. We provide a method to divide the set T_0. At each step of the process we have a vector $\beta=(0,\beta_1,\dots,\beta_k)\in\{0\}\times\mathbb{C}^k$ with $k\geq 0$ and a seed $T_\beta=\{(x,t)\in B(0,\delta)\times\overline{B(0,\eta)}\}$ in coordinates (x,t) canonically associated to T_β. We either decide not to split T_β or we divide it in sets $\mathcal{E}_\beta$, $M_\beta=\mathcal{C}_\beta\cup\cup_{\zeta\in S_\beta}T_{\beta,\zeta}$ where S_β is a finite subset of $\mathbb{C}$. The seeds $T_{\beta,\zeta}$ for $(\beta,\zeta)\in\{0\}\times\mathbb{C}^{k+1}$ with $\zeta\in S_\beta$ are divided in ulterior steps of the process. The sets T_β, M_β, $\mathcal{E}_\beta$ and $\mathcal{C}_\beta$ are defined by induction on k. Every set M_β is called a *magnifying glass set*. The sets $\mathcal{E}_\beta$ are called *exterior basic sets* whereas the sets $\mathcal{C}_\beta$ are called *compact-like basic sets*. At the first step of the process we consider $\beta=0$, $k=0$ and the coordinates (x,y) in T_0.

Suppose also that
$$X = x^{e(\mathcal{E}_\beta)}v(x,t)(t-\gamma_1(x))^{s_1}\dots(t-\gamma_p(x))^{s_p}\partial/\partial t \tag{1}$$

in T_β where $\gamma_1(0)=\ldots=\gamma_p(0)=0$ and $\{v=0\}\cap T_\beta=\emptyset$. We denote

$$\partial_e \mathcal{E}_\beta = \{(x,t)\in B(0,\delta)\times \partial B(0,\eta)\} \text{ and } \nu(\mathcal{E}_\beta)=s_1+\ldots+s_p-1.$$

For $p=1$ we define the *terminal exterior set* $\mathcal{E}_\beta=T_\beta$, we do not split the terminal seed T_β. We say that $\partial_e\mathcal{E}_\beta$ is the *exterior boundary* of $\mathcal{E}_\beta$ and $e(\mathcal{E}_\beta)$ is the *exterior exponent* of $\mathcal{E}_\beta$. We define $\iota(\mathcal{E}_\beta)=e(\mathcal{E}_\beta)$ the *interior exponent* of $\mathcal{E}_\beta$. Suppose $p>1$. We define $t=xw$ and the sets $\mathcal{E}_\beta=T_\beta\cap\{|t|\geq|x|\rho\}$, $\tilde{\mathcal{E}}_\beta=T_\beta\cap\{|t|>|x|2\rho\}$ and $M_\beta=\{(x,w)\in B(0,\delta)\times\overline{B(0,\rho)}\}$ for some $\rho>>0$.

Definition 3.1. *Given an exterior set $\mathcal{E}_\beta$ we define $X_{\mathcal{E}_\beta}$ as the vector field defined in T_β such that $X=x^{e(\mathcal{E}_\beta)}X_{\mathcal{E}_\beta}$.*

We define

$$\partial_I\mathcal{E}_\beta=\{(x,t)\in B(0,\delta)\times\overline{B}(0,\eta):|t|=|x|\rho\}$$

of $\mathcal{E}_\beta$. The sets $\partial_e\mathcal{E}_\beta$ and $\partial_I\mathcal{E}_\beta$ are the *exterior* and *interior boundaries* of $\mathcal{E}_\beta$ respectively. We say that the coordinates (x,t) are *adapted* to T_β and $\mathcal{E}_\beta$. We have

$$X=x^{e(\mathcal{E}_\beta)+s_1+\ldots+s_p-1}v(x,xw)(w-\gamma_1(x)/x)^{s_1}\ldots(w-\gamma_p(x)/x)^{s_p}\partial/\partial w.$$

We denote $S_\beta=\{(\partial\gamma_1/\partial x)(0),\ldots,(\partial\gamma_p/\partial x)(0)\}$. We define

$$\mathcal{C}_\beta=\{(x,w)\in B(0,\delta)\times(\overline{B}(0,\rho)\setminus\cup_{\zeta\in S_\beta}B(\zeta,\eta_{\beta,\zeta}))\}$$

where $\eta_{\beta,\zeta}>0$ is small enough for any $\zeta\in S_\beta$. We denote

$$\partial_e\mathcal{C}_\beta=\{(x,w)\in B(0,\delta)\times\partial B(0,\rho)\},\ \partial_I\mathcal{C}_\beta=\{(x,w)\in B(0,\delta)\times\cup_{\zeta\in S_\beta}\partial B(\zeta,\eta_{\beta,\zeta})\}.$$

We define

$$\nu(\mathcal{C}_\beta)=\nu(\mathcal{E}_\beta) \text{ and } \iota(\mathcal{E}_\beta)=e(\mathcal{C}_\beta)=\iota(\mathcal{C}_\beta)=e(\mathcal{E}_\beta)+\nu(\mathcal{E}_\beta).$$

We say that $e(\mathcal{E}_\beta)$ and $\iota(\mathcal{E}_\beta)$ are the *exterior* and *interior exponents* of $\mathcal{E}_\beta$ respectively whereas $e(\mathcal{C}_\beta)$ and $\iota(\mathcal{C}_\beta)$ are the *exterior* and *interior exponents* of $\mathcal{C}_\beta$.

Definition 3.2. *Given a compact-like set $\mathcal{C}_\beta$ we define $X_{\mathcal{C}_\beta}$ as the vector field defined in M_β such that $X=x^{e(\mathcal{C}_\beta)}X_{\mathcal{C}_\beta}$.*

Definition 3.3. *We define the polynomial vector field*

$$X_\beta(\lambda)=\lambda^{e(\mathcal{C}_\beta)}v(0,0)(w-(\partial\gamma_1/\partial x)(0))^{s_1}\ldots(w-(\partial\gamma_p/\partial x)(0))^{s_p}\partial/\partial w$$

for $\lambda\in\mathbb{S}^1$ (see the equation (1), note that $t=xw$) associated to X, T_β and $\mathcal{C}_\beta$.

Fix $\zeta\in S_\beta$. We define the seed $T_{\beta,\zeta}=\{(x,t')\in B(0,\delta)\times\overline{B(0,\eta_{\beta,\zeta})}\}$ where t' is the coordinate $w-\zeta$. By definition (x,t') is the set of adapted coordinates associated to $T_{\beta,\zeta}$. We say that the seed $T_{\beta,\zeta}$ is a *son* of the seed T_β. We have

$$X=x^{e(\mathcal{E}_{\beta,\zeta})}h(x,t')\prod_{(\partial\gamma_j/\partial x)(0)=\zeta}(t'-(\gamma_j(x)/x-\zeta))^{s_j}\partial/\partial w$$

where $e(\mathcal{E}_{\beta,\zeta})=e(\mathcal{C}_\beta)$. We just introduced a method to divide $|y|\leq\epsilon$ in a union of exterior and compact-like sets.

Example: Consider $X=y(y-x^2)(y-x)\partial/\partial y$. Denote $w=y/x$. The vector field X has the form $x^2w(w-x)(w-1)\partial/\partial w$ in coordinates (x,w). The polynomial vector field $X_0(\lambda)$ associated to the seed $T_0=B(0,\delta)\times\overline{B}(0,\epsilon)$ is equal to $\lambda^2w^2(w-1)\partial/\partial w$.

The exterior and compact-like sets associated to T_0 are $\mathcal{E}_0=T_0\cap\{|y|\geq\rho_0|x|\}$ and $\mathcal{C}_0=\{|y|\leq\rho_0|x|\}\setminus(\{|w|<\eta_{00}\}\cup\{|w-1|<\eta_{01}\})$ respectively. The sons of

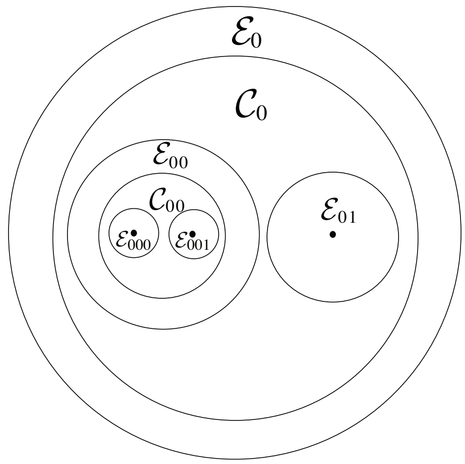


FIGURE 1. Splitting for $X = y(y - x^2)(y - x)\partial/\partial y$ in a line $x = x_0$

T_0 are the seeds $T_{00} = \{|w| \leq \eta_{00}\}$ and $\mathcal{E}_{01} = T_{01} = \{|w - 1| \leq \eta_{01}\}$. The seed T_{01} is terminal since it only contains one irreducible component of $SingX$.

Denote $w' = w/x$. We have $X = x^3 w'(w' - 1)(xw' - 1)\partial/\partial w'$ in coordinates (x, w'). Thus $-\lambda^3 w'(w' - 1)\partial/\partial w'$ is the polynomial vector field $X_{00}(\lambda)$ associated to T_{00}. The seed T_{00} contains an exterior set $\mathcal{E}_{00} = T_{00} \cap \{|w| \geq \rho_{00}|x|\}$ for $\rho_{00} >> 1$, a compact-like set $\mathcal{C}_{00} = \{|w'| \leq \rho_{00}\} \setminus (\{|w'| < \eta_{000}\} \cup \{|w' - 1| < \eta_{001}\})$ and two terminal seeds $\mathcal{E}_{000} = T_{000} = \{|w'| \leq \eta_{000}\}$ and $\mathcal{E}_{001} = T_{001} = \{|w' - 1| \leq \eta_{001}\}$ for some $0 < \eta_{000}, \eta_{001} << 1$. We have $e(\mathcal{E}_0) = 0$, $\iota(\mathcal{E}_0) = e(\mathcal{C}_0) = e(\mathcal{E}_{00}) = e(\mathcal{E}_{01}) = 2$ and $\iota(\mathcal{E}_{00}) = e(\mathcal{C}_{00}) = e(\mathcal{E}_{000}) = e(\mathcal{E}_{001}) = 3$.

Remark 3.1. *The dynamical splitting $\digamma$ depends on the choice of the constants determining the size of the basic sets. Anyway given two dynamical splittings $\digamma_1$ and $\digamma_2$ there are bijective correspondences $\mathcal{E}_\beta^1 \leftrightarrow \mathcal{E}_\beta^2$ and $\mathcal{C}_\beta^1 \leftrightarrow \mathcal{C}_\beta^2$ between exterior and compact-like sets.*

Definition 3.4. *Given two dynamical splittings $\digamma$ and $\digamma'$ we say that $\digamma$ is a refinement of $\digamma'$ if we have $\mathcal{E}_\beta \subset \mathcal{E}_\beta'$ for any exterior set $\mathcal{E}_\beta$ of $\digamma$ and $\mathcal{C}_\beta \supset \mathcal{C}_\beta'$ for any compact-like set $\mathcal{C}_\beta$ of $\digamma$.*

Definition 3.5. *Given two dynamical splittings $\digamma_1$ and $\digamma_2$ we can consider a refinement $\digamma_1 \cup \digamma_2$ of both of them. More precisely if $\mathcal{E}_\beta^1$ and $\mathcal{E}_\beta^2$ are exterior sets associated to $\digamma_1$ and $\digamma_2$ respectively then $\mathcal{E}_\beta = \mathcal{E}_\beta^1 \cap \mathcal{E}_\beta^2$ is associated to $\digamma_1 \cup \digamma_2$. Analogously suppose that $\mathcal{C}_\beta^1$ and $\mathcal{C}_\beta^2$ are compact-like sets associated to $\digamma_1$ and $\digamma_2$ respectively. Then $\mathcal{C}_\beta = \mathcal{C}_\beta^1 \cup \mathcal{C}_\beta^2$ is associated to $\digamma_1 \cup \digamma_2$.*

4. MULTI-TRANSVERSAL FLOWS

Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. This section is intended to define multi-transversal flows and describe their main properties. Roughly speaking a multi-transversal flow is of the form $\Re(\aleph_{\mathcal{B}} X)$ by restriction to any basic set $\mathcal{B}$ of the dynamical splitting. The function $\aleph_{\mathcal{B}}(r\lambda, y)$ is continuous, takes values in $\mathbb{S}^1 \setminus \{1, -1\}$ and depends only on $\lambda \in \mathbb{S}^1$. The choice of $\aleph_{\mathcal{B}}$ is related to make the dynamics of $\Re(\aleph_{\mathcal{B}} X)$ stable with respect to the direction $\lambda\mathbb{R}^+$ in the parameter space.

The dynamical behavior of a transversal flow $\Re(\mu X)_{|\mathcal{E}\cap(\{x\}\times B(0,\epsilon))}$ does not depend on $x \in B(0,\delta)\setminus\{0\}$ or $\mu \in \mathbb{S}^1 \setminus \{1,-1\}$ for any exterior basic set $\mathcal{E}$. Indeed it basically depends on $\nu(\mathcal{E})$. Therefore in order to study the stability properties of transversal or multi-transversal flows we can focus on compact-like sets where we can use the associated polynomial vector fields defined in subsection 3. The notion of stability for polynomial vector fields, namely the absence of homoclinic trajectories, was introduced in [4]. We can define an analogous concept for any compact-like set. Since the intersection of a compact-like set and the line $x = 0$ contains only the origin our concept of stability is of infinitesimal type. It imposes a set of restrictions in compact-like sets. Another way of making sense of the infinitesimal label in stability is by noticing that there are as many compact-like sets as steps in a minimal desingularizion of the singular set of X via blow-ups.

The subsection 4.1 is devoted to introduce some of the concepts and definitions that are required to define infinitesimal stability for a multi-transversal flow. The definition of $\aleph_{\mathcal{B}}$ is explained in subsections 4.2 and 4.4. The construction of the multi-transversal flows is introduce in subsection 4.4. The subsection 4.3 reviews the properties of transversal flows that are described in [16] and whose analogues for multi-transversal flows are studied in subsections 4.5 and 4.6. Subsections 4.7 and 4.8 deal with some quantitative properties of the constructions that will be used later on.

4.1. **Polynomial vector fields.** A vector field $X \in \mathcal{X}_{tp1}(\mathbb{C}^2,0)$ is of the form

$$X = u(x,y)(y-g_1(x))^{n_1}\dots(y-g_p(x))^{n_p}\frac{\partial}{\partial y}$$

where $u \in \mathbb{C}\{x,y\}$ is a unit. Given a non-degenerate element $h(x,y)\partial/\partial y$ of $\mathcal{X}_{p1}(\mathbb{C}^2,0)$ there exists $k \in \mathbb{N}$ such that $h(x^k,y)\partial/\partial y$ belongs to $\mathcal{X}_{tp1}(\mathbb{C}^2,0)$.

Definition 4.1. *Let $Y = P(w)\partial/\partial w$ be a polynomial vector field. We define $\nu(Y) = \deg(P) - 1$.*

Definition 4.2. *Let $Y = P(w)\partial/\partial w \in \mathbb{C}[w]\partial/\partial w$ with $\nu(Y) \geq 1$. We define $Tr_{\to\infty}(Y)$ as the set of trajectories $\gamma : (c,d) \to \mathbb{C}$ of $\Re(Y)$ with $c \in \mathbb{R}\cup\{-\infty\}$ and $d \in \mathbb{R}$ such that $\lim_{\zeta\to d}\gamma(\zeta) = \infty$. Analogously we define $Tr_{\leftarrow\infty}(Y) = Tr_{\to\infty}(-Y)$.*

Definition 4.3. *We say that $\Re(Y)$ has ∞-connections or homoclinic trajectories if $Tr_{\to\infty}(Y)\cap Tr_{\leftarrow\infty}(Y) \neq \emptyset$. Then there exists a trajectory $\gamma : (c_-,c_+) \to \mathbb{C}$ of $\Re(Y)$ such that $c_-,c_+ \in \mathbb{R}$ and $\lim_{\zeta\to c_-}\gamma(\zeta) = \infty = \lim_{\zeta\to c_+}\gamma(\zeta)$. This notion has been introduced in [4] for the study of deformations of elements of $\mathrm{Diff}_1(\mathbb{C},0)$ (see also [16]).*

Remark 4.1. *Let $Y = P(w)\partial/\partial w$ be a polynomial vector field with $\deg(P) \geq 2$. We define the set $S \subset \mathbb{S}^1$ defined by $\mu \in S$ if $\Re(\mu Y)$ and $\Re(\mu' Y)$ are orbitally equivalent for any $\mu' \in \mathbb{S}^1$ in a neighborhood of μ. The set S is the set of directions μ in which the dynamics of $\Re(\mu X)$ is stable with respect to μ. It turns out that S coincides with the set $\{\mu \in \mathbb{S}^1 : \Re(\mu X)$ has no homoclinic trajectories$\}$ [4].*

Definition 4.4. *We denote $\mathcal{X}_\infty(\mathbb{C},0)$ the set of polynomial vector fields in $\mathcal{X}(\mathbb{C},0)$ such that $\nu(Y) \geq 1$ and $2\pi i\sum_{P\in S} Res(Y,P) \notin \mathbb{R}\setminus\{0\}$ for any subset S of $SingY$.*

Remark 4.2. *Let $Y \in \mathcal{X}_\infty(\mathbb{C},0)$. The vector field $\Re(Y)$ has no homoclinic trajectories. Thus $Re(1\cdot Y)$ is stable at 1 (see [4] or [16]).*

Definition 4.5. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider the compact-like sets $\mathcal{C}_1, \ldots, \mathcal{C}_q$ associated to X. Let $X_j(\lambda) = \lambda^{e(\mathcal{C}_j)} P_j(w) \frac{\partial}{\partial w}$ be the polynomial vector field associated to $\mathcal{C}_j$ and X for $1 \leq j \leq q$. We define*

$$\mathcal{U}_X^j = \left\{ (\lambda, \mu) \in \mathbb{S}^1 \times \mathbb{S}^1 : \lambda^{e(\mathcal{C}_j)} \mu P_j(w) \frac{\partial}{\partial w} \not\in \mathcal{X}_\infty(\mathbb{C}, 0) \right\}$$

The vector field $\mu X_j(\lambda)$ has no homoclinic trajectories for $(\lambda, \mu) \not\in \mathcal{U}_X^j$ [4] *(see also* [16]*). We define $\Xi_X^j = \{\lambda \in \mathbb{S}^1 : (\lambda, i) \in \mathcal{U}_X^j\}$ for $1 \leq j \leq q$. We enumerate $\tilde{e}_1 < \tilde{e}_2 < \ldots < \tilde{e}_{\tilde{q}}$ the elements of $\cup_{\{j \in \{1,\ldots,q\}: \ \mathcal{U}_X^j \neq \emptyset\}} \{e(\mathcal{C}_j)\}$. By convention we denote $\tilde{e}_0 = 0$ and $\tilde{e}_{\tilde{q}+1} = \infty$. We define*

$$\tilde{\mathcal{U}}_X^k = \cup_{\{j \in \{1,\ldots,q\}: \ e(\mathcal{C}_j) = \tilde{e}_k\}} \mathcal{U}_X^j \quad \text{and} \quad \tilde{\Xi}_X^k = \cup_{\{j \in \{1,\ldots,q\}: \ e(\mathcal{C}_j) = \tilde{e}_k\}} \Xi_X^j$$

for $1 \leq k \leq \tilde{q}$. We define $\mathcal{U}_X = \cup_{j=1}^q \mathcal{U}_X^j = \cup_{k=1}^{\tilde{q}} \tilde{\mathcal{U}}_X^k$.

The notations in the previous definition are fixed from now on.

Definition 4.6. *We say that $\tilde{\Xi}_X^k$ is the set of singular directions of level $\tilde{e}_k$. Clearly the set $\tilde{\Xi}_X^k$ is finite for any $1 \leq k \leq \tilde{q}$.*

Later on we will see that given $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ we can associate a vector field $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$, namely a normal form. We will see that $\{\tilde{e}_1, \ldots, \tilde{e}_{\tilde{q}}\}$ is the set of levels of multi-summability of the infinitesimal generator of φ. Moreover we will prove that the set of singular directions of level $\tilde{e}_k$ is contained in $\tilde{\Xi}_X^k$ for any $1 \leq k \leq \tilde{q}$.

4.2. **Stable multi-directions.** Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. As a generalization of transversal flows of the form $\Re(\mu X)$ for $\mu \in \mathbb{S}^1 \setminus \{1, -1\}$ we are going to construct transversal flows of the form $\Re(\aleph^* X)$ where $\aleph^* : (B(0,\delta) \times B(0,\epsilon)) \setminus Sing X \to e^{i(0,\pi)}$ is a C^∞ function. We need them to capture the multiple summability levels associated to Fatou coordinates of elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$.

Consider the notations at the beginning of the section.

Definition 4.7. *We say that a multi-direction $(\tilde{\mu}_1, \ldots, \tilde{\mu}_{\tilde{q}}) \in (e^{i(0,\pi)})^{\tilde{q}}$ is stable at a direction $\lambda \mathbb{R}^+$ in the parameter space x if $(\lambda, \tilde{\mu}_k) \not\in \tilde{\mathcal{U}}_X^k$ for any $1 \leq k \leq \tilde{q}$. We say that a multi-direction $(\mu_1, \ldots, \mu_q) \in (e^{i(0,\pi)})^q$ is stable at a direction $\lambda \mathbb{R}^+$ if $(\lambda, \mu_j) \not\in \mathcal{U}_X^j$ for any $1 \leq j \leq q$.*

A stable multi-direction $(\tilde{\mu}_1, \ldots, \tilde{\mu}_{\tilde{q}})$ induces a stable multi-direction $(\mu_1, \ldots, \mu_q)$. We define $\mu_j = \tilde{\mu}_k$ if $e(\mathcal{C}_j) = \tilde{e}_k$ and $\mu_j = i$ if $e(\mathcal{C}_j) \not\in \{\tilde{e}_1, \ldots, \tilde{e}_{\tilde{q}}\}$.

Definition 4.8. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Let I a closed arc $e^{i[u_0,u_1]}$ of $\mathbb{S}^1$. We say that a function $\aleph : I \to (e^{i(0,\pi)})^{\tilde{q}}$ is a stable multi-direction at I if*

$$\aleph(e^{iu}) \equiv (e^{i\tilde{\theta}_1(u)}, \ldots, e^{i\tilde{\theta}_{\tilde{q}}(u)})$$

for some continuous decreasing functions $\tilde{\theta}_1, \ldots, \tilde{\theta}_{\tilde{q}} : [u_0, u_1] \to (0, \pi)$ and $\aleph(\lambda)$ is stable at $\lambda \mathbb{R}^+$ for any $\lambda \in e^{i[u_0,u_1]}$.

Definition 4.9. *Consider $\lambda \in \mathbb{S}^1$ and $\upsilon \in \mathbb{R}^+ \cup \{0\}$. We define*

$$I_j(\lambda, \upsilon) = \lambda e^{i[-\frac{\pi}{2\tilde{e}_j} - \upsilon, \frac{\pi}{2\tilde{e}_j} + \upsilon]}$$

for $1 \leq j \leq \tilde{q}$ and $I_0(\lambda, \upsilon) = \mathbb{S}^1$.

Definition 4.10. *We define* $\mathcal{M} \subset (\mathbb{S}^1)^{\tilde{q}}$ *as the set whose elements* $(\lambda_1, \dots, \lambda_{\tilde{q}})$ *satisfy* $\lambda_1 \not\in \tilde{\Xi}_X^1$, …, $\lambda_{\tilde{q}} \not\in \tilde{\Xi}_X^{\tilde{q}}$ *and* $I_{j+1}(\lambda_{j+1}, 0) \subset I_j(\lambda_j, 0)$ *for any* $1 \leq j < \tilde{q}$.

In the language of summability $(\tilde{e}_1, \dots, \tilde{e}_{\tilde{q}})$ and $(\lambda_1, \dots, \lambda_{\tilde{q}})$ are admissible parameter vectors.

Let $\Lambda = (\lambda_1, \dots, \lambda_{\tilde{q}}) \in \mathcal{M}$. Since $\lambda_j = e^{i\theta_j} \not\in \tilde{\Xi}_X^j$, given $\upsilon > 0$ small enough there exists a continuous function $\tilde{\mu}_j : I_j(\lambda_j, \upsilon) \to e^{i(0,\pi)}$ such that $\tilde{\mu}_j(\lambda_j) = i$ and

$$(\lambda, \tilde{\mu}_j(\lambda)) \not\in \tilde{\mathcal{U}}_X^j \quad \forall \lambda \in I_j(\lambda_j, \upsilon).$$

Moreover, if we choose a lift $\tilde{\theta}_j : \theta_j + [-\pi/(2\tilde{e}_j) - \upsilon, \pi/(2\tilde{e}_j) + \upsilon] \to (0, \pi)$ of $\tilde{\mu}_j$, i.e. $e^{i\tilde{\theta}_j(\theta)} = \tilde{\mu}_j(e^{i\theta})$ we choose $\tilde{\mu}_j$ such that $\tilde{\theta}_j$ is a decreasing (maybe non-strictly decreasing) function. By considering a smaller $\upsilon > 0$, if necessary, we obtain

$$(\lambda', \mu) \not\in \tilde{\mathcal{U}}_X^j \quad \forall 1 \leq j \leq \tilde{q} \;\; \forall \lambda \in I_j(\lambda_j, 0) \;\; \forall \lambda' \in \lambda e^{i[-\upsilon, \upsilon]} \;\; \forall \mu \in \tilde{\mu}_j(\lambda e^{i[-\upsilon,\upsilon]}).$$

By taking a smaller $\upsilon > 0$, we obtain that for any $1 \leq j \leq \tilde{q}$ there exist compact sets $I^{j,1}, \dots, I^{j,s_j} \subset \mathbb{S}^1$ and complex numbers $\mu_{j,1}, \dots, \mu_{j,s_j} \in e^{i[\pi/4, 3\pi/4]}$ such that

- $I^{j,1} \cup \ldots \cup I^{j,s_j} = \mathbb{S}^1$.
- $(\lambda', \mu_{j,l}) \not\in \tilde{\mathcal{U}}_X^j$ for all $\lambda' \in I^{j,l} e^{i[-\upsilon,\upsilon]}$ and $l \in \{1, \dots, s_j\}$.

Consider $1 \leq j \leq \tilde{q}$ and $\lambda \in \mathbb{S}^1$. We choose $1 \leq l \leq s_j$ such that $\lambda \in I^{j,l}$. We define $\tilde{\mu}_j^*(\lambda) = \mu_{j,l}$. Denote $\upsilon_\Lambda = \upsilon$.

Definition 4.11. *Let* $\Lambda = (\lambda_1, \dots, \lambda_{\tilde{q}}) \in \mathcal{M}$. *Consider* $\lambda \in I_k(\lambda_k, \upsilon_\Lambda)$ *if* $k \neq 0$ *or* $\lambda \in \mathbb{S}^1$ *if* $k = 0$. *The formula*

$$\aleph_{k,\Lambda,\lambda}(\lambda') = (\tilde{\mu}_1(\lambda'), \dots, \tilde{\mu}_k(\lambda'), \tilde{\mu}_{k+1}^*(\lambda), \dots, \tilde{\mu}_{\tilde{q}}^*(\lambda))$$

defines a stable multi-direction $\aleph_{k,\Lambda,\lambda} : \lambda e^{i[-\upsilon_\Lambda, \upsilon_\Lambda]} \to (e^{i(0,\pi)})^{\tilde{q}}$ *at* $\lambda e^{i[-\upsilon_\Lambda, \upsilon_\Lambda]}$. *We denote* $\aleph_{k,\Lambda,\lambda}^j$ *the projection in the* j *coordinate of the image of* $\aleph_{k,\Lambda,\lambda}$.

Remark 4.3. *Every multi-direction in* $\aleph_{k,\Lambda,\lambda}^1 \times \ldots \times \aleph_{k,\Lambda,\lambda}^{\tilde{q}}$ *is stable at* $\lambda' \mathbb{R}^+$ *for any* $\lambda' \in \lambda e^{i[-\upsilon_\Lambda, \upsilon_\Lambda]}$. *Therefore* $\mu'_j X_j(\lambda')$ *belongs to* $\mathcal{X}_\infty(\mathbb{C}, 0)$ *for all* $\mu'_j \in \aleph_{k,\Lambda,\lambda}^j$ *and* $\lambda' \in \lambda e^{i[-\upsilon_\Lambda, \upsilon_\Lambda]}$.

4.3. **Dynamics of transversal flows in basic sets.** Let us remind the reader some properties of transversal flows before defining multi-transversal flows. We will adapt these properties to the multi-transversal setting. Further details and proofs can be found in [16].

Let us introduce some notations.

Definition 4.12. *We consider coordinates* $(x, y) \in \mathbb{C} \times \mathbb{C}$ *or* $(r, \lambda, y) \in \mathbb{R}_{\geq 0} \times \mathbb{S}^1 \times \mathbb{C}$ *in* $\mathbb{C}^2$. *Given a set* $F \subset \mathbb{C}^2$ *we denote* $F(x_0)$ *the set* $F \cap \{x = x_0\}$ *and by* $F(r_0, \lambda_0)$ *the set* $F \cap \{(r, \lambda) = (r_0, \lambda_0)\}$.

Definition 4.13. *Let* $\gamma_P(s)$ *be the trajectory of* $\Re(Z)$ *such that* $\gamma_P(0) = P$. *We define* $\mathcal{I}(Z, P, F)$ *the maximal interval where* $\gamma_P(s)$ *is well-defined and belongs to* F *for any* $s \in \mathcal{I}(Z, P, F)$ *whereas* $\gamma_P(s)$ *belongs to* F° *for any* $s \neq 0$ *in the interior of* $\mathcal{I}(Z, P, F)$. *We denote* $\Gamma(Z, P, F) = \gamma_P(\mathcal{I}(Z, P, F))$. *We define*

$$\partial \mathcal{I}(Z, P, F) = \{\inf(\mathcal{I}(Z, P, F)), \sup(\mathcal{I}(Z, P, F))\} \subset \mathbb{R} \cup \{-\infty, \infty\}.$$

We denote $\Gamma(Z, P, F)(s) = \gamma_P(s)$.

Definition 4.14. *Let* $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. *Let* $\mathcal{E}$ *be an exterior set associated to* X. *We say that* $\mathcal{E}$ *is parabolic if* $\nu(\mathcal{E}) \geq 1$. *Every non-terminal exterior set is parabolic.*

The qualitative behavior of a transversal flow $\Re(\mu X)$ ($\mu \in \mathbb{S}^1 \setminus \{-1,1\}$) in an exterior set $\mathcal{E}$ depends on the nature of the set of tangent points between $\Re(\mu X)$ and $\partial\mathcal{E}$. Since we want to reproduce the same ideas for multi-transversal flows we introduce these concepts.

Definition 4.15. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider an exterior set*

$$\mathcal{E} = \{(x,t) \in B(0,\delta) \times \mathbb{C} : \eta \geq |t| \geq \rho |x|\}$$

associated to X with $0 < \eta << 1$ and $\rho \geq 0$. We define $T\mathcal{E}^{\eta}_{\mu X}(r,\lambda)$ the set of tangent points between $|t| = \eta$ and $\Re(\lambda^{e(\mathcal{E})}\mu X_{\mathcal{E}})_{|x=r\lambda}$ for $(r,\lambda,\mu) \in \mathbb{R}_{\geq 0} \times \mathbb{S}^1 \times \mathbb{S}^1$. We denote $T^{\epsilon}_{\mu X}(r\lambda) = T\mathcal{E}^{\epsilon}_{\mu X}(r,\lambda)$ for the particular case $\mathcal{E} = \mathcal{E}_0$.

Definition 4.16. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider a compact-like set*

$$\mathcal{C} = \{(x,w) \in B(0,\delta) \times (\overline{B}(0,\rho) \setminus \cup_{\zeta \in S_{\mathcal{C}}} B(\zeta, \eta_{\mathcal{C},\zeta}))\}$$

associated to X. We denote $T\mathcal{C}^{\rho}_{\mu X}(r,\lambda)$ the set of tangent points between $|w| = \rho$ and $\Re(\lambda^{e(\mathcal{C})}\mu X_{\mathcal{C}})_{|x=r\lambda}$.

Definition 4.17. *Let $\mathcal{B}$ a basic set. We say that a point $y_0 \in T\mathcal{B}_{\mu X}(r,\lambda)$ is convex if the germ of trajectory of $\Re(\lambda^{e(\mathcal{B})}\mu X_{\mathcal{B}})_{|x=r\lambda}$ passing through y_0 is contained in $\mathcal{B}$.*

Lemma 4.1. [16] *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ and an exterior set $\mathcal{E} = \{\eta \geq |t| \geq \rho|x|\}$ associated to X with $0 < \eta << 1$ and $\rho \geq 0$. Then the set $T\mathcal{E}^{\eta}_{\mu X}(r,\lambda)$ is composed of $2\nu(\mathcal{E})$ convex points for all $(\lambda,\mu) \in \mathbb{S}^1 \times \mathbb{S}^1$ and r close to 0. Each connected component of $\{t \in \partial B(0,\eta)\} \setminus T\mathcal{E}^{\eta}_{\mu X}(r,\lambda)$ contains a unique point of $T\mathcal{E}^{\eta}_{\mu' X}(r,\lambda)$ for any $\mu' \in \mathbb{S}^1 \setminus \{-\mu,\mu\}$.*

Lemma 4.2. [16] *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ and a compact-like set*

$$\mathcal{C} = \{(x,w) \in B(0,\delta) \times (\overline{B}(0,\rho) \setminus \cup_{\zeta \in S_{\mathcal{C}}} B(\zeta, \eta_{\mathcal{C},\zeta}))\}$$

associated to X with $\rho >> 0$. Then $T\mathcal{C}^{\rho}_{\mu X}(r,\lambda)$ is composed of $2\nu(\mathcal{C})$ convex points for all $(\lambda,\mu) \in \mathbb{S}^1 \times \mathbb{S}^1$ and r close to 0. Moreover each connected component of $\{|w| = \rho\} \setminus T\mathcal{C}^{\rho}_{\mu X}(r,\lambda)$ contains a unique point of $T\mathcal{C}^{\rho}_{\mu' X}(r,\lambda)$ for any point $\mu' \in \mathbb{S}^1 \setminus \{-\mu,\mu\}$.

4.3.1. *Parabolic exterior sets.* Let us analyze the quantitative and the qualitative dynamical behavior of a transversal flow $\Re(\mu X)$ ($\mu \in \mathbb{S}^1 \setminus \{-1,1\}$) in a parabolic exterior set $\mathcal{E}$. We want to apply the same ideas in the more general setting of multi-transversal flows.

Remark 4.4. *The qualitative behavior of $\Re(\mu X)$ ($\mu \in \mathbb{S}^1 \setminus \{-1,1\}$) in a parabolic exterior set $\mathcal{E} = \mathcal{E}_\beta$ is described in propositions 6.1 and 6.2 and corollary 6.1 of* [16]*. It is a truncated Fatou flower (see figure (3)). The proof is based on:*

- *Tangent points between $\Re(\mu X)$ and $\partial\mathcal{E}$ are convex.*
- *$\sharp T\mathcal{E}^{\eta}_{\mu X}(r,\lambda) = \nu(\mathcal{E})$ for any $(r,\lambda) \in [0,\delta) \times \mathbb{S}^1$.*
- *Suppose $\mathcal{E}$ is non-terminal and denote $\mathcal{C} = \mathcal{C}_\beta$. Then we have*

$$\sharp T\mathcal{E}^{\eta}_{\mu X}(r,\lambda) = \sharp T\mathcal{C}^{\rho}_{\mu X}(r,\lambda) = \nu(\mathcal{E})$$

  for any $(r,\lambda) \in [0,\delta) \times \mathbb{S}^1$.

The previous properties are guaranteed by lemmas 4.1 and 4.2. The results hold true for $0 < \eta \leq \eta^0$ for some $\eta^0 \in \mathbb{R}^+$. We also need $\rho \geq \rho^0$ for some $\rho^0 \in \mathbb{R}^+$ if $\mathcal{E}$ is non-terminal. These properties are preserved by refinement of the dynamical splitting $\digamma$. Let us remark that the choice of η^0 and ρ^0 does not depend on $\mu \in \mathbb{S}^1$.

Next, we introduce the other ingredient in [16] to describe the dynamics in exterior sets. It is of quantitative nature.

Definition 4.18. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Let $\mathcal{E} = \{\eta \geq |t| \geq \rho|x|\}$ be a parabolic exterior set associated to a seed T. The vector field $X_{\mathcal{E}}$ is of the form*

$$X_{\mathcal{E}} = v(x,t)(t-\gamma_1(x))^{s_1} \dots (t-\gamma_p(x))^{s_p} \partial/\partial t$$

where v is a function never vanishing in T. Denote $\gamma_{\mathcal{E}} = \gamma_1$. Denote $\psi_{\mathcal{E}}^0$ a holomorphic integral of the time form of $v(0, t-\gamma_{\mathcal{E}}(x))(t-\gamma_{\mathcal{E}}(x))^{\nu(\mathcal{E})+1}\partial/\partial t$ defined in the neighborhood of $\mathcal{E} \setminus SingX$.

Remark 4.5. *The function $\psi_{\mathcal{E}}^0$ is of the form*

$$\psi_{\mathcal{E}}^0 = \frac{-1}{\nu(\mathcal{E})v(0,0)} \frac{1}{(t-\gamma_{\mathcal{E}}(x))^{\nu(\mathcal{E})}} + Res(X_{\mathcal{E}}, (0,0)) \ln(t-\gamma_{\mathcal{E}}(x)) + h(t-\gamma_{\mathcal{E}}(x)) + b(x)$$

where $h(z)$ is a $O(1/z^{\nu(\mathcal{E})-1})$ meromorphic function and $b(x)$ is a holomorphic function in the neighborhood of 0. Thus given $\zeta > 0$ there exists $C_\zeta \in \mathbb{R}^+$ such that

$$\frac{1}{C_\zeta} \frac{1}{|t-\gamma_{\mathcal{E}}(x)|^{\nu(\mathcal{E})}} \leq |\psi_{\mathcal{E}}^0|(x,t) \leq C_\zeta \frac{1}{|t-\gamma_{\mathcal{E}}(x)|^{\nu(\mathcal{E})}}$$

in $\mathcal{E} \cap \{t-\gamma_{\mathcal{E}}(x) \in \mathbb{R}^+ e^{i[-\zeta,\zeta]}\} \cap \{x \in B(0,\delta(\zeta))\}$.

Definition 4.19. *Let $\mathcal{E} = \{\eta \geq |t| \geq \rho|x|\}$ be a parabolic exterior set associated to $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Denote $\psi_{\mathcal{E}}$ a holomorphic integral of the time form of $X_{\mathcal{E}}$ defined in the neighborhood of $\mathcal{E} \setminus SingX$ such that $\psi_{\mathcal{E}}(0,y) \equiv \psi_{\mathcal{E}}^0(0,y)$. The function $\psi_{\mathcal{E}}$ is multi-valued.*

Lemma 4.3. *(lemma 6.5 [16]) Let $\mathcal{E} = \{(x,t) \in B(0,\delta) \times \mathbb{C} : \eta \geq |t| \geq \rho|x|\}$ be a parabolic exterior set associated to $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Let $\upsilon > 0$, $\zeta > 0$. Suppose $\mathcal{E}$ is terminal. Then $|\psi_{\mathcal{E}}/\psi_{\mathcal{E}}^0 - 1| \leq \upsilon$ in $\mathcal{E} \cap \{t-\gamma_{\mathcal{E}}(x) \in \mathbb{R}^+ e^{i[-\zeta,\zeta]}\} \cap \{x \in B(0,\delta(\upsilon,\zeta))\}$ for some $\delta(\upsilon,\zeta) \in \mathbb{R}^+$. The same inequality is true for a non-terminal $\mathcal{E}$ if $\rho > 0$ is big enough.*

Remark 4.6. *The lemma 4.3 implies that the qualitative behavior of*

$$\Re(\mu X) \text{ and } \Re(\mu x^{e(\mathcal{E})} v(0, t-\gamma_{\mathcal{E}}(x))(t-\gamma_{\mathcal{E}}(x))^{\nu(\mathcal{E})}\partial/\partial t)$$

is very similar for all exterior set $\mathcal{E}$ and $\mu \in \mathbb{S}^1$. We say that the properties in remark 4.4 and lemma 4.3 are the stability properties for the behavior of transversal flows in parabolic exterior sets. They are preserved by refinement. Moreover they do not depend on the transversal flow $\Re(\mu X)$ whose dynamics we are studying.

4.3.2. *Non-parabolic exterior sets.* A non-parabolic exterior set $\mathcal{E} = \mathcal{E}_{\beta,\zeta}$ is terminal. We have

$$\mathcal{E} = \mathcal{E}_{\beta,\zeta} = \{(x,t) \in B(0,\delta) \times \mathbb{C} : |t| \leq \eta\}$$

and $\mathcal{C}_\beta = \mathcal{C}_j$ for some $1 \leq j \leq q$. Consider a compact set $I \subset \mathbb{S}^1$.

Remark 4.7. *The stability properties associated to a transversal flow $\Re(\mu X)$ and a non-parabolic exterior set $\mathcal{E}$ are:*

- $(\lambda, \mu) \notin \mathcal{U}_X^j$ *for any* $\lambda \in I$.

- $\Re(\mu X)$ *is transversal to* $\partial_e \mathcal{E}$ *in* $\{(x,t) \in (0,\delta)I \times \partial B(0,\eta)\}$.

The first property implies the second one for $0 < \eta << 1$ by the argument in subsection (6.4.2) of [16]. We can choose $0 < \eta \leq \eta_0$ for some $\eta_0 \in \mathbb{R}^+$. The choice of $\eta_0 > 0$ depends on I and μ. Given I, we can choose the same $\eta_0 > 0$ for any $\mu' \in \mathbb{S}^1$ in a neighborhood of μ. There exists a dynamical splitting $\mathcal{F}'$ whose every refinement $\mathcal{F}$ satisfies $\mathcal{E}_{\beta,\zeta} \subset \{(x,t) \in B(0,\delta) \times \mathbb{C} : |t| \leq \eta_0\}$.

Let us remark that the unique singular point $(x_0, \gamma_{\mathcal{E}}(x_0))$ in $\mathcal{E}(x_0)$ is attracting or repelling for $\Re(\mu X)_{|x=x_0}$ and any $x_0 \in (0,\delta)I$ (subsection 6.4.2 of [16]).

4.3.3. *Compact-like sets.* Let

$$\mathcal{C} = \mathcal{C}_j = \mathcal{C}_\beta = \{(x,w) \in B(0,\delta) \times (\overline{B}(0,\rho) \setminus \cup_{\zeta \in S_\beta} B(\zeta, \eta_{\beta,\zeta}))\}$$

be a compact-like set. Consider a compact set $I \subset \mathbb{S}^1$.

Remark 4.8. *The stability properties associated to a transversal flow* $\Re(\mu X)$ *and a compact-like set* $\mathcal{C}$ *are:*

- $(\lambda,\mu) \notin \mathcal{U}_X^j$ *for any* $\lambda \in I$.
- $\sharp T\mathcal{C}^{\rho}_{\mu' X}(r,\lambda) = \nu(\mathcal{C})$ and $\sharp T(\mathcal{E}_{\beta,\zeta})^{\eta_{\beta,\zeta}}_{\mu' X}(r,\lambda) = \nu(\mathcal{E}_{\beta,\zeta})$ $\forall \zeta \in S_\beta$ $\forall \mu' \in \mathbb{S}^1$.
- *If there is a trajectory* γ *of* $\Re(\mu X_j(\lambda))$ *for some* $\lambda \in I$ *whose* α *and* ω *limits are singletons contained in* $Sing(X_j(1))$ *then there exists a trajectory* γ' *of* $\Re(\mu X_j(\lambda))$ *contained in* $B(0,\rho)$ *such that* $\alpha(\gamma) = \alpha(\gamma')$ *and* $\omega(\gamma) = \omega(\gamma')$.

The second property can be obtained by choosing $\rho \geq \rho^0 > 0$ and $0 < \eta_{\beta,\zeta} \leq \eta_{\beta,\zeta}^0$ for any $\zeta \in S_\beta$. The above properties imply that the behavior of $\Re(\mu X_j(\lambda))$ in $\mathbb{C}$ and $B(0,\rho)$ are analogous (see equation (4) and proposition 6.6 in [16]). They are the ingredients that we use to describe the behavior of $\Re(\mu X)$ in the compact-like set $\mathcal{C}_\beta$. In fact the dynamics of $\Re(\mu X)$ in $\mathcal{C}_\beta$ is analogous to the dynamics of the stable polynomial vector field $\Re(\mu X_j(\lambda))$ (see section 6.5 in [16] and figure (2)). Given I we can consider the same choice of η_0 and $\{\eta_{\beta,\zeta}^0\}_{\zeta \in S_\beta}$ for any $\mu' \in \mathbb{S}^1$ in the neighborhood of μ.

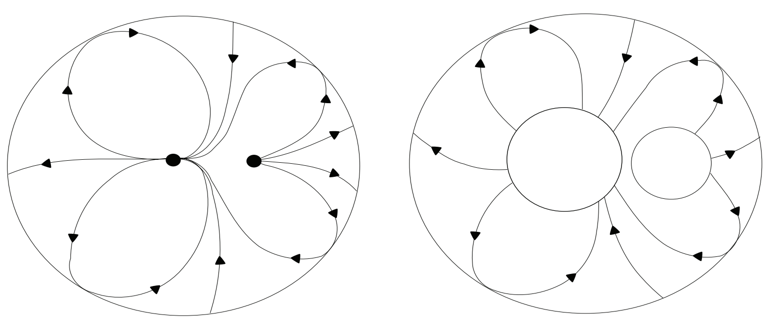


FIGURE 2. Dynamics of $\Re(X_0(1))_{|B(0,\rho)}$ and $\Re(X)_{|\mathcal{C}_0(x_0)}$ for x_0 in $\mathbb{R}^+$ and $X = y^2(y-x)\partial/\partial y$

Remark 4.9. *The stability properties that we demand to exterior and compact-like sets are compatible since they are preserved by refinement.*

Next remark is basically lemma 6.13 in [16]. It is helpful to make pictures of the dynamics of $\Re(\mu X)$ in $\mathcal{C}_j$.

Remark 4.10. *Consider $P = (r, \lambda, w) \in \partial_e \mathcal{C}_j$ with $\lambda \in I$ and $(\lambda, \mu) \notin \mathcal{U}_X^j$. If $\Re(\mu X)$ points towards the interior of $\mathcal{C}_j$ at P or $\Re(\mu X)$ is tangent to $\partial_e \mathcal{C}_j$ at P then $\Gamma(s) \in \partial_I \mathcal{C}_j$ where $\Gamma = \Gamma(\lambda^{e(\mathcal{C}_j)} \mu X_{\mathcal{C}_j}, P, \mathcal{C}_j)$ and $s = \sup(\mathcal{I}(\Gamma))$. A possible dynamical behavior for the trajectories of $\Re(\aleph X)_{|\mathcal{C}_j}$ is represented in figure (2).*

4.4. **Construction of the multi-transversal flow.** Consider $\lambda \in \mathbb{S}^1$ and a stable multi-direction $\aleph = (\tilde{\mu}_1, \ldots, \tilde{\mu}_{\tilde{q}}) : e^{i[u_0,u_1]} \to (e^{i(0,\pi)})^{\tilde{q}}$. We denote

$$(\mu_1, \ldots, \mu_q) : e^{i[u_0,u_1]} \to (e^{i(0,\pi)})^q$$

the multi-direction induced by $\aleph$. We define

- $\aleph_{\mathcal{C}_j} \equiv \mu_j$ for a a compact-like set $\mathcal{C}_\beta = \mathcal{C}_j$.
- $\aleph_{\mathcal{E}_0} \equiv i$ for the first exterior set.
- $\aleph_{\mathcal{E}} \equiv \aleph_{\mathcal{C}_\beta}$ for an exterior seed $\mathcal{E}$ whose father we denote T_β.

We defined a function $\aleph_{\mathcal{B}} : e^{i[u_0,u_1]} \to e^{i(0,\pi)}$ for any basic set $\mathcal{B}$.

We say that a dynamical splitting $\digamma$ is associated to $\aleph$ if

- $\Re(\mu X)$ is stable at $\mathcal{E}$ for any $\mu \in \mathbb{S}^1$ and any parabolic exterior set $\mathcal{E}$.
- $\Re(\aleph_{\mathcal{E}} X)$ is stable at $\cup_{(r,\theta) \in [0,\delta) \times [u_0,u_1]} \mathcal{E}(r, e^{i\theta})$ for any non-parabolic exterior set $\mathcal{E}$.
- $\Re(\aleph_{\mathcal{C}} X)$ is stable at $\cup_{(r,\theta) \in [0,\delta) \times [u_0,u_1]} \mathcal{C}(r, e^{i\theta})$ for any compact-like set $\mathcal{C}$.

The stability properties are introduced in subsections 4.3.1, 4.3.2 and 4.3.3. We always consider associated dynamical splittings. The first condition is obtained just by considering small exterior sets (see subsection 4.3.1). Given $u \in [u_0, u_1]$ there exist a dynamical splitting and a neighborhood I^u of u in $[u_0, u_1]$ such that $\Re(\aleph_{\mathcal{B}} X)$ is stable at $\cup_{(r,\theta) \in [0,\delta) \times I^u} \mathcal{B}(r, e^{i\theta})$ for any non-parabolic basic set $\mathcal{B}$ (see subsections 4.3.2 and 4.3.3). The dynamical splitting is obtained by doing successive refinements. Since $[u_0, u_1]$ is compact we can find the same dynamical splitting satisfying the conditions above.

We want to define a continuous function

$$\aleph^* : ((\mathbb{R}^+ \cup \{0\}) e^{i[u_0,u_1]} \times B(0,\epsilon)) \setminus SingX \to e^{i(0,\pi)}.$$

We obtain a flow $\Re(\aleph^* X)$ defined in $[0,\delta) e^{i[u_0,u_1]} \times B(0,\epsilon)$. We require $\Re(\aleph^* X)$ to fulfill the following properties:

- We define $\aleph^*_{|\mathcal{C}_j} = \aleph_{\mathcal{C}_j} = \mu_j$ and $(\aleph X)_{\mathcal{C}_j} = \mu_j X_{\mathcal{C}_j}$ for any $1 \leq j \leq q$.
- Let $\mathcal{E}$ be a terminal exterior set. We define $\aleph^*_{|\mathcal{E}} = \aleph_{\mathcal{E}}$ and $(\aleph X)_{\mathcal{E}} = \aleph^*_{|\mathcal{E}} X_{\mathcal{E}}$.
- Consider a non-terminal exterior set $\mathcal{E}_\beta = \{(x,t) \in \mathbb{C}^2 : \rho|x| \leq |t| \leq \eta\}$ for some $\rho, \eta \in \mathbb{R}^+$. We require
  - $\Re(\aleph^* X)_{|\tilde{\mathcal{E}}_\beta} \equiv \Re(\aleph_{\mathcal{E}_\beta} X)$.
  - $\Re(\aleph^* X)_{|\partial_I \mathcal{E}_\beta} \equiv \Re(\aleph_{\mathcal{C}_\beta} X)$.

Denote $\aleph_{\mathcal{E}_\beta} = e^{i\theta_0}$ and $\aleph_{\mathcal{C}_\beta} = e^{i\theta_1}$ for $\theta_0, \theta_1 : e^{i[u_0,u_1]} \to (0,\pi)$. Let $\varsigma : \mathbb{R} \to [0,1]$ be a C^∞ function such that $\varsigma(-\infty, 1+1/4] = \{1\}$ and $\varsigma[2-1/4, \infty) = \{0\}$. We define

$$\aleph^*(r,\lambda,t) = \aleph_{\mathcal{E}_\beta}(\lambda) e^{i(\theta_1-\theta_0)(\lambda)\varsigma(|t|/(\rho r))} \; \forall (r,\lambda,t) \in \mathcal{E}_\beta \cap ([0,\delta) \times e^{i[u_0,u_1]} \times \mathbb{C})$$

and $(\aleph X)_{\mathcal{E}_\beta} \equiv \aleph^* X_{\mathcal{E}_\beta}$.

Definition 4.20. *Let $\aleph : I \to (e^{i(0,\pi)})^{\tilde{q}}$ be a stable multi-direction at I. We denote $\Re(\aleph X)$ the flow $\Re(\aleph^* X)$ defined in $[0,\delta) I \times B(0,\epsilon)$. In particular we can define $\Re(\aleph_{k,\Lambda,\lambda} X)$ in $[0,\delta) \lambda e^{i[-\upsilon_\Lambda, \upsilon_\Lambda]} \times B(0,\epsilon)$ for any function $\aleph_{k,\Lambda,\lambda}$. We say that $\Re(\aleph X)$ is a multi-transversal flow.*

Remark 4.11. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ and $\Lambda = (\lambda_1, \dots, \lambda_{\tilde{q}}) \in \mathcal{M}$. Any multi-direction $\aleph_{k,\Lambda,\lambda}$ is of the form*

$$\aleph_{k,l_{k+1},\dots,l_{\tilde{q}}}(\lambda') = (\tilde{\mu}_1(\lambda'), \dots, \tilde{\mu}_k(\lambda'), \mu_{k+1,l_{k+1}}, \dots, \mu_{\tilde{q},l_{\tilde{q}}})$$

where $1 \leq l_j \leq s_j$ for any $k < j \leq \tilde{q}$. We denote

$$D_{k,l_{k+1},\dots,l_{\tilde{q}}} = I_k(\lambda_k, 0) \cap \cap_{j=k+1}^{\tilde{q}} I^{j,l_j}.$$

Indeed $\aleph_{k,l_{k+1},\dots,l_{\tilde{q}}} : D_{k,l_{k+1},\dots,l_{\tilde{q}}} e^{i[-\upsilon_\Lambda,\upsilon_\Lambda]} \to (e^{i(0,\pi)})^{\tilde{q}}$ is a stable multi-direction. We construct a dynamical splitting $\digamma_{k,l_{k+1},\dots,l_{\tilde{q}}}$ associated to $\aleph_{k,l_{k+1},\dots,l_{\tilde{q}}}$. By taking a smaller $\upsilon_\Lambda > 0$ and a compactness argument we can suppose that $\digamma_{k,l_{k+1},\dots,l_{\tilde{q}}}$ is associated to the constant multi-direction $\aleph_{k,l_{k+1},\dots,l_{\tilde{q}}}(\lambda') : \lambda e^{i[-\upsilon_\Lambda,\upsilon_\Lambda]} \to (e^{i(0,\pi)})^{\tilde{q}}$ for any $\lambda \in D_{k,l_{k+1},\dots,l_{\tilde{q}}}$ and any $\lambda' \in \lambda e^{i[-\upsilon_\Lambda,\upsilon_\Lambda]}$. Since the choices of sequences $(k, l_{k+1}, \dots, l_{\tilde{q}})$ are finite there exists a dynamical splitting $\digamma_\Lambda$ associated to any $\aleph_{k,\Lambda,\lambda}$. It is obtained by taking a common refinement of every splitting $\digamma_{k,l_{k+1},\dots,l_{\tilde{q}}}$.

Remark 4.12. *Let $\lambda \in \mathbb{S}^1$ and $\mu \in e^{i(0,\pi)}$. Consider $\aleph(\lambda) = (\mu, \dots, \mu)$ such that $(\lambda, \mu) \notin \tilde{\mathcal{U}}_X^k$ for any $1 \leq k \leq \tilde{q}$. Then $\Re(\aleph(\lambda)X)$ is a multi-transversal flow at $\lambda\mathbb{R}^+$ if and only if the transversal flow $\Re(\mu X)$ is stable at $\lambda\mathbb{R}^+$ in the sense in* [16] *(which happens by definition if $(\lambda, \mu) \notin \mathcal{U}_X = \cup_{k=1}^{\tilde{q}} \tilde{\mathcal{U}}_X^k$). Multi-transversal flows are then a natural generalization of transversal flows and they share their good properties.*

4.5. **Dynamics of multi-transversal flows in basic sets.** Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider a multi-transversal flow $\Re(\aleph X)$ and a basic set $\mathcal{B}$. If $\mathcal{B}$ is a compact-like or a terminal exterior set then we have $\Re(\aleph X)_{|\mathcal{B}} \equiv \Re(\aleph_{\mathcal{B}} X)_{|\mathcal{B}}$. Thus the dynamics of a multi-transversal flow in $\mathcal{B}$ is the dynamics of a transversal flow. The dynamics of $\Re(\aleph X)$ in a terminal exterior set $\mathcal{E}$ is described in subsections 4.3.1 and 4.3.2. It is a Fatou flower dynamics in the parabolic case. Otherwise it is an attractor or a repellor.

Consider a stable direction $\aleph : I \to (e^{i(0,\pi)})^{\tilde{q}}$ at a closed arc I. Let $\mathcal{C}_1$, …, $\mathcal{C}_q$ be the compact-like sets associated to X. We denote $(\mu_1, \dots, \mu_q) : I \to (e^{i(0,\pi)})^q$ the multi-direction induced by $\aleph$. We have $\Re(\aleph X)_{|\mathcal{C}_j} \equiv \Re(\mu_j X)$. Since $(\lambda, \mu_j(\lambda)) \notin \mathcal{U}_X^j$ for any $\lambda \in I$ then the dynamics of $\Re(\aleph X)_{|\mathcal{C}_j}$ is as described in section (6.4) of [16] for any $1 \leq j \leq q$. The dynamics of $\Re(\mu X)_{|\mathcal{C}_j(r\lambda)}$ is analogous to the dynamics of a stable polynomial vector field $\Re(\mu X_j(\lambda))$ for $(\lambda, \mu) \notin \mathcal{U}_j^X$.

Next, we see that, even if $\mathcal{B}$ is a non-terminal exterior set, the dynamics of $\Re(\aleph X)_{|\mathcal{B}}$ is still analogous to the dynamics of a transversal flow, i.e. a truncated Fatou flower.

Definition 4.21. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider an exterior set*

$$\mathcal{E} = \{(x, t) \in B(0, \delta) \times \mathbb{C} : \eta \geq |t| \geq \rho|x|\}$$

associated to X with $0 < \eta << 1$ and $\rho \geq 0$. Given a multi-transversal flow $\Re(\aleph X)$ we denote $T\mathcal{E}_{\aleph X}^{\eta}(r, \lambda) = T\mathcal{E}_{\aleph_{\mathcal{E}}(\lambda)X}^{\eta}(r, \lambda)$ the set of tangent points between $\Re(\aleph X)$ and $\partial_e \mathcal{E}$. We denote $T_{\aleph X}^{\epsilon}(r\lambda) = T\mathcal{E}_{\aleph X}^{\epsilon}(r, \lambda)$ for the particular case $\mathcal{E} = \mathcal{E}_0$.

Definition 4.22. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider a compact-like set*

$$\mathcal{C} = \{(x, w) \in B(0, \delta) \times (\overline{B}(0, \rho) \setminus \cup_{\zeta \in S_{\mathcal{C}}} B(\zeta, \eta_{\mathcal{C},\zeta})\}$$

associated to X. We denote $T\mathcal{C}_{\aleph X}^{\rho}(r, \lambda) = T\mathcal{C}_{\aleph_{\mathcal{C}}(\lambda)X}^{\rho}(r, \lambda)$

Definition 4.23. *Let $\mathcal{B}$ a basic set. Given a multi-transversal flow $\Re(\aleph X)$ we say that $y_0 \in T\mathcal{B}_{\aleph X}(r,\lambda)$ is convex if the germ of trajectory of $\Re(\lambda^{e(\mathcal{B})}(\aleph X)_{\mathcal{B}})_{|x=r\lambda}$ passing through y_0 is contained in $\mathcal{B}$. Equivalently $y_0 \in T\mathcal{B}_{\aleph X}(r,\lambda)$ is convex if it is a convex point of $T\mathcal{B}_{\aleph_{\mathcal{B}(\lambda)} X}(r,\lambda)$.*

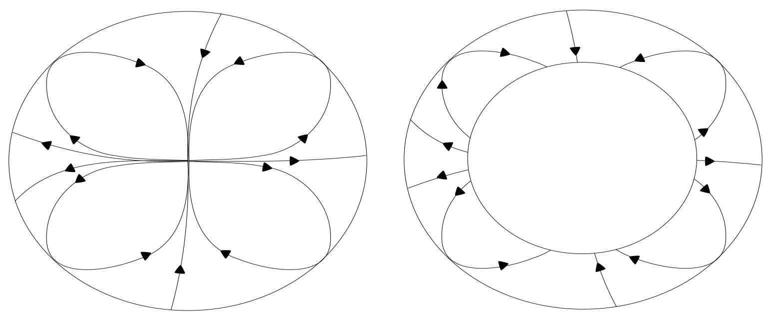


FIGURE 3. Parabolic exterior sets

Remark 4.13. *Let $\mathcal{E} = \mathcal{E}_\beta$ be a parabolic exterior set and a multi-transversal flow $\Re(\aleph X)$. We have:*

- *Tangent points between $\Re(\aleph X)$ and $\partial\mathcal{E}$ are convex.*
- *$\sharp T\mathcal{E}^{\eta}_{\aleph X}(r,\lambda) = \nu(\mathcal{E})$ for any $(r,\lambda) \in [0,\delta) \times \mathbb{S}^1$.*
- *Suppose $\mathcal{E}$ is non-terminal and denote $\mathcal{C} = \mathcal{C}_\beta$. Then we have*
$$\sharp T\mathcal{E}^{\eta}_{\aleph X}(r,\lambda) = \sharp T\mathcal{C}^{\rho}_{\aleph X}(r,\lambda) = \nu(\mathcal{E})$$
for any $(r,\lambda) \in [0,\delta) \times \mathbb{S}^1$.

The properties are a consequence of lemmas 4.1 and 4.2. Analogously as for transversal flows (see remark 4.4) the qualitative behavior of $\Re(\aleph X)$ in a parabolic exterior set $\mathcal{E} = \mathcal{E}_\beta$ is a truncated Fatou flower (see figure (3)). Lemma 4.3 is also satisfied since it does not depend on the choice of $\aleph$.

Next we see that the behavior of a multi-transversal flow in a parabolic exterior set is analogous to a Fatou flower also from a quantitative point of view. In particular we prove that the spiraling behavior is bounded in exterior basic sets.

Proposition 4.1. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2,0)$ and let $\mathcal{E} = \{\eta \geq |t| \geq \rho|x|\}$ be a parabolic exterior set associated to X. Consider a trajectory $\Gamma = \Gamma(\lambda^{e(\mathcal{E})}(\aleph X)_{\mathcal{E}}, (r,\lambda,t), \mathcal{E})$. for $r\lambda$ in a neighborhood of 0 and a transversal multi-direction $\aleph \in (e^{i(0,\pi)})^{\tilde{q}}$ at $\lambda\mathbb{R}^+$. Then Γ is contained in a sector centered at $t = \gamma_{\mathcal{E}}(r\lambda)$ (see def. 4.18) of angle less than ζ for some $\zeta > 0$ independent of r, λ, Γ and $\aleph$.*

Let us explain the statement. Consider the universal covering
$$(r,\lambda,\gamma_{\mathcal{E}}(r\lambda) + e^z) : \mathcal{E}^\flat \to \mathcal{E} \setminus SingX.$$
Let $\Gamma^\flat$ the lifting of Γ by (r,λ,e^z). We claim that the set $(Im(z))(\Gamma^\flat)$ is contained in an interval of length ζ.

Proof. We have
$$X = x^{e(\mathcal{E})} v(x,t)(t-\gamma_1(x))^{s_1} \ldots (t-\gamma_p(x))^{s_p} \partial/\partial t$$

where we consider $\gamma_{\mathcal{E}} \equiv \gamma_1$. We denote

$$\psi_{\mathcal{E}}^{00} = \frac{-1}{\nu(\mathcal{E})v(0,0)} \frac{1}{(t-\gamma_{\mathcal{E}}(x))^{\nu(\mathcal{E})}} = \frac{-1}{\nu(\mathcal{E})v(0,0)} \frac{1}{(t-\gamma_1(x))^{\nu(\mathcal{E})}}.$$

Given $\upsilon > 0$ and $\zeta_0 > 0$ we can consider $\eta > 0$ small to obtain that $|\psi_{\mathcal{E}}^0/\psi_{\mathcal{E}}^{00} - 1| < \upsilon$ in the set $\{(x,t) \in \mathcal{E} : |\arg(t-\gamma_{\mathcal{E}}(x))| \leq \zeta_0\}$ (see remark 4.5). Therefore we obtain $|\psi_{\mathcal{E}}/\psi_{\mathcal{E}}^{00} - 1| < \upsilon$ in $\{(x,t) \in \mathcal{E} : |\arg(t-\gamma_1(x))| \leq \zeta_0\}$ by considering $\eta > 0$ small enough and $\rho > 0$ big enough if $\mathcal{E}$ is not terminal (lemma 4.3).

We have $(\aleph X)_{\mathcal{E}} \equiv \aleph_{|\mathcal{E}}^* X_{\mathcal{E}}$. Since $\aleph^*(x,t) \in e^{i(0,\pi)}$ for any $(x,t) \in \mathcal{E}$ then we have that either

$$(\psi_{\mathcal{E}}/\lambda^{e(\mathcal{E})})(\Gamma) \cap (\mathbb{R}^+ \cup \{0\}) = \emptyset \text{ or } (\psi_{\mathcal{E}}/\lambda^{e(\mathcal{E})})(\Gamma) \cap (\mathbb{R}^- \cup \{0\}) = \emptyset.$$

Thus $(\psi_{\mathcal{E}}/\lambda^{e(\mathcal{E})})(\Gamma)$ lies in a sector of angle of angle 2π. Since $\psi_{\mathcal{E}}/\psi_{\mathcal{E}}^{00} \sim 1$ then Γ lies in a sector of center $t = \gamma_{\mathcal{E}}(r,\lambda)$ and angle close to $2\pi/\nu(\mathcal{E})$. □

4.6. **Dynamics of multi-transversal flows in a neighborhood of the origin.** Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2,0)$. The dynamical properties of $Re(\mu X)_{|\lambda(\mathbb{R}\cup\{0\})\times B(0,\epsilon)}$ for $(\lambda,\mu) \notin \mathcal{U}_X$ are obtained by pasting the dynamics in exterior and compact-like basic sets. The important facts are that $\Re(\mu X)_{|\mathcal{E}}$ is a Fatou flower dynamics for any parabolic exterior set $\mathcal{E}$, an attractor or a repellor for any non-parabolic exterior set and that the dynamics of $\Re(\mu X)_{|\mathcal{C}_j(r\lambda)}$ is analogous to the dynamics of a stable polynomial vector field $\Re(\mu X_j(\lambda))$ for $(\lambda,\mu) \notin \mathcal{U}_j^X$. These properties are preserved for a multi-transversal flow $\Re(\aleph X)$. Namely, the dynamics of $\Re(\aleph X)$ is a Fatou flower, an attractor or a repellor in exterior sets and since $(\lambda,\mu_j(\lambda)) \notin \mathcal{U}_j^X$ then the dynamics of $\Re(\aleph X)_{|\mathcal{C}_j(r\lambda)} = \Re(\mu_j(\lambda)X)_{|\mathcal{C}_j(r\lambda)}$ is analogous to the dynamics of a stable polynomial vector field $\Re(\mu_j(\lambda)X_j(\lambda))$.

Next we introduce the generalizations for multi-transversal flows of some definitions and theorem for transversal flows. We skip the proofs since they are straightforward generalizations of their analogue counterparts in [16].

Lemma 4.4. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2,0)$. Consider the multi-transversal flow $\Re(\aleph X)$ for some stable multi-direction $\aleph : I \to (e^{i(0,\pi)})^{\tilde{q}}$. Let $P_0 \in [0,\delta)I \times \partial B(0,\epsilon)$ such that $\Re(\aleph X)$ does not point towards $\mathbb{C} \setminus \overline{B}(0,\epsilon)$ at P_0. Denote $\Gamma = \Gamma(\aleph X, P_0, T_0)$. Then $[0,\infty)$ is contained in $\mathcal{I}(\Gamma)$ and $\lim_{\zeta\to\infty}\Gamma(\zeta) \in SingX$. Moreover the intersection of $\Gamma[0,\infty)$ with every compact-like or exterior set is connected.*

The previous lemma is a generalization of lemma 6.13 in [16]. The proof of the previous lemma is obtained by using that $\Re(\aleph X)$ is a Fatou flower in parabolic exterior sets, an attractor or a repellor in non-parabolic exterior sets and remark 4.10 in compact-like sets.

Definition 4.24. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2,0)$. Consider a multi-transversal flow $\Re(\aleph X)$ for some stable multi-direction $\aleph : I \to (e^{i(0,\pi)})^{\tilde{q}}$. We define the set of regions $Reg^*(\epsilon, \aleph X, I)$ associated to $\Re(\aleph X)$ in $B(0,\delta) \times B(0,\epsilon)$ as the set of connected components of*

$$([0,\delta)I \times B(0,\epsilon)) \setminus (SingX \cup_{x\in[0,\delta)I} \cup_{P\in T_{\aleph X}^{\epsilon}(x)} \Gamma(\aleph X, P, T_0)).$$

We define $\alpha^{\aleph X}(P)$ as the α-limit of $\Gamma(\aleph X, P, T_0)$ for any $P \in B(0,\delta) \times \overline{B(0,\epsilon)}$ such that $\mathcal{I}(\aleph X, P, |y| \leq \epsilon)$ contains $(-\infty,0)$. Otherwise we define $\alpha^{\aleph X}(P) = \infty$. We define $\omega^{\aleph X}(P)$ in an analogous way.

Figures (1) and (2) in [16] are examples of partitions in regions. The dynamical behavior of $\Re(\aleph X)$ in terminal exterior sets implies that the singular points of $\Re(\aleph X)$ in $[0,\delta)I \times B(0,\epsilon)$ are attracting, repelling or parabolic. We have:

Lemma 4.5. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2,0)$. Consider a multi-transversal flow $\Re(\aleph X)$ for some stable multi-direction $\aleph : I \to (e^{i(0,\pi)})^{\tilde{q}}$. Given $P \in B(0,\delta) \times \overline{B}(0,\epsilon)$ then either $\alpha^{\aleph X}(P) = \infty$ (resp. $\omega^{\aleph X}(P) = \infty$) or $\alpha^{\aleph X}(P)$ (resp. $\omega^{\aleph X}(P)$) is a singleton contained in $SingX$.*

The basins of attraction and repulsion of the singular points are open sets. The functions $\alpha^{\aleph X}$ and $\omega^{\aleph X}$ are locally constant in

$$(\{x\} \times B(0,\epsilon)) \setminus (SingX \cup \cup_{P \in T^{\epsilon}_{\aleph X}(x)} \Gamma(\aleph X, P, T_0))$$

for any $x \in [0,\delta)I$. Given $H \in Reg^{*}(\epsilon,\aleph X, I)$ the function $\omega^{\aleph X}_{|H}$ is either identically ∞ or $\omega^{\aleph X}(H(x))$ is a singleton $\{(x,g(x))\}$ for any $x \in [0,\delta)I$ where $y=g(x)$ is one of the irreducible components of $SingX$. We denote $\omega^{\aleph X}(H)$ the curve $y=g(x)$. An analogous property holds true for $\alpha^{\aleph X}$.

Definition 4.25. *Denote*

$$Reg_{\infty}(\epsilon,\aleph X,I) = Reg^{*}(\epsilon,\aleph X,I) \cap ((\alpha^{\aleph X})^{-1}(\infty) \cup (\omega^{\aleph X})^{-1}(\infty))$$

and $Reg(\epsilon,\aleph X,I) = Reg^{*}(\epsilon,\aleph X,I) \setminus Reg_{\infty}(\epsilon,\aleph X,I)$. We define

$$Reg_j(\epsilon,\aleph X,I) = \{H \in Reg(\epsilon,\aleph X,I) : \sharp\{\alpha^{\aleph X}(H), \omega^{\aleph X}(H)\} = j\}$$

for $j \in \{1,2\}$. We define $Reg_1^{*}(\epsilon,\aleph X,I) = Reg_1(\epsilon,\aleph X,I) \cup Reg_{\infty}(\epsilon,\aleph X,I)$ and $Reg_2^{*}(\epsilon,\aleph X,I) = Reg_2(\epsilon,\aleph X,I)$.

Remark 4.14. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2,0)$. Consider a multi-transversal flow $\Re(\aleph X)$ for some stable multi-direction $\aleph : I \to (e^{i(0,\pi)})^{\tilde{q}}$. The domains represented by points of $Reg(\epsilon,\aleph X,I)$ are $\Re(\aleph X)$ invariant.*

Remark 4.15. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2,0)$. Consider a multi-transversal flow $\Re(\aleph X)$ for some stable multi-direction $\aleph : I \to (e^{i(0,\pi)})^{\tilde{q}}$. The set $H(x)$ (see def. 4.12) is connected for $H \in Reg^{*}(\epsilon,\aleph X,I)$ and $x \in (0,\delta)I$. The set $H(0)$ is connected for $H \in Reg_1^{*}(\epsilon,\aleph X,I)$ whereas otherwise $H(0)$ has two connected components.*

Definition 4.26. *Let $H \in Reg^{*}(\epsilon,\aleph X,I)$ (see def. 4.25). We denote $\mathcal{P}(H)$ the set of connected components of $H(0)$ (see def. 4.12). Given $L \in \mathcal{P}(H)$ we define $H_L = (H \setminus H(0)) \cup L$. We define $\mathcal{P}_X^{\epsilon} = \cup_{H \in Reg(\epsilon,\aleph X,I)} \mathcal{P}(H)$. The set $\mathcal{P}_X^{\epsilon}$ does not depend on I or $\aleph$.*

Remark 4.16. *Let $H \in Reg_j^{*}(\epsilon,\aleph X,I)$. We have $\sharp\mathcal{P}(H) = j$ (see remark 4.15).*

Definition 4.27. *Let $L \in \mathcal{P}_X^{\epsilon}$. We denote $L_{iX}^{\epsilon}(x)$ the unique continuous section of T_{iX}^{ϵ} such that $L_{iX}^{\epsilon}(0) \in \overline{L}$.*

Definition 4.28. *Fix $\epsilon_0 > 0$ small enough. Let $\epsilon \in (0,\epsilon_0/2)$. Consider $L \in \mathcal{P}_X^{\epsilon}$ and the region $H \in Reg(\epsilon,\aleph X,I)$ containing L. There exist $H_0 \in Reg(\epsilon_0,\aleph X,I)$ containing H and $L_0 \in \mathcal{P}(H_0)$ containing L. Let ψ be an integral of the time form of X defined in a neighborhood of $(L_0)_{iX}^{\epsilon_0}(0)$ in $\mathbb{C}^2$ such that $\psi(x,y_0) \equiv 0$ where $(0,y_0)$ is the point $(L_0)_{iX}^{\epsilon_0}(0)$. By analytic continuation of ψ we obtain a continuous integral of the time form $\psi_{H,L}^X$ of X in H_L. We denote $\psi_{H,L}^X$ by ψ_L^X if the data ϵ, $\aleph$ and I are implicit.*

Remark 4.17. *The choice of ϵ_0 is intended to make the definition of Fatou coordinates as independent of ϵ as possible. Given $H \in Reg(\epsilon, \aleph X, I)$, $L \in \mathcal{P}(H)$ and $H' \in Reg(\epsilon', \aleph X, I)$, $L' \in \mathcal{P}(H)$ with $H \subset H'$ and $L \subset L'$ we have that $\psi_{H,L}^X \equiv (\psi_{H',L'}^X)_{|H}$.*

Definition 4.29. *There exist $2\nu(\mathcal{E}_0)$ continuous sections*

$$T_{iX}^{\epsilon,1}(r,\lambda),\ \ldots,\ T_{iX}^{\epsilon,2\nu(\mathcal{E}_0)}(r,\lambda),\ T_{iX}^{\epsilon,2\nu(\mathcal{E}_0)+1}(r,\lambda) = T_{iX}^{\epsilon,1}(r,\lambda)$$

of $T_{iX}^{\epsilon}(r,\lambda)$. The sections are ordered in counter clock wise sense. Let L_j be the element of $\mathcal{P}_X^\epsilon$ such that $T_{iX}^{\epsilon,j}(0) \in \overline{L_j}$.

Definition 4.30. *Consider $H \in Reg^*(\epsilon, \aleph X, I) \setminus Reg(\epsilon, \aleph X, I)$ and $L \in \mathcal{P}(H)$. Consider $L^1 \in \mathcal{P}_X^\epsilon$ such that $(L^1)_{iX}^\epsilon(x) \in \overline{H}$ for any $x \in [0,\delta)I$. We define ψ_L^X as the analytic continuation of $\psi_{L^1}^X$ to H. Let us remark that there are two choices of L^1 and then two possible definitions of ψ_L^X since $\sharp(T_{iX}^\epsilon(x) \cap \overline{H}) = 2$ for any $x \in [0,\delta)I$.*

Consider $H \in Reg(\epsilon, \aleph X, I)$ and $L \in \mathcal{P}(H)$. By construction ψ_L^X is holomorphic in H°. Moreover (x, ψ_L^X) is injective in H since $\psi_L^X(H_L(x))$ is simply connected for any $x \in [0,\delta)I$. By construction $|\psi_L^X|$ is bounded by below by a positive constant in H. The function $\psi_L^X + d(x)$ is also a Fatou coordinate of X in H for any $d \in \mathbb{C}\{x\}$.

We call subregion of a region $H \in Reg^*(\epsilon, \aleph X, I)$ every set of the form $H \cap \mathcal{E}$ or $H \cap \mathcal{C}$ where $\mathcal{E}$ is an exterior set and $\mathcal{C}$ is a compact-like set. We say that all the subregions of $H \in Reg_1^*(\epsilon, \aleph X, I)$ are L-subregions where $\mathcal{P}(H) = \{L\}$. Consider $H \in Reg_2^*(\epsilon, \aleph X, I) = Reg_2(\epsilon, \aleph X, I)$ with $\mathcal{P}(H) = \{L, R\}$. There exists a unique seed T_β such that the curves $\alpha^{\mu X}(H)$ and $\omega^{\mu X}(H)$ are contained in T_β but they are not contained in the same son of T_β. A subregion of H contained in M_β is a L-subregion. A subregion in the same connected component of $H \setminus (M_\beta^\circ \cup \{(0,0)\})$ as L is also a L-subregion. We define H^L the union of the L-subregions of H. Clearly we have $H = H^L \cup H^R$ by lemma 4.4.

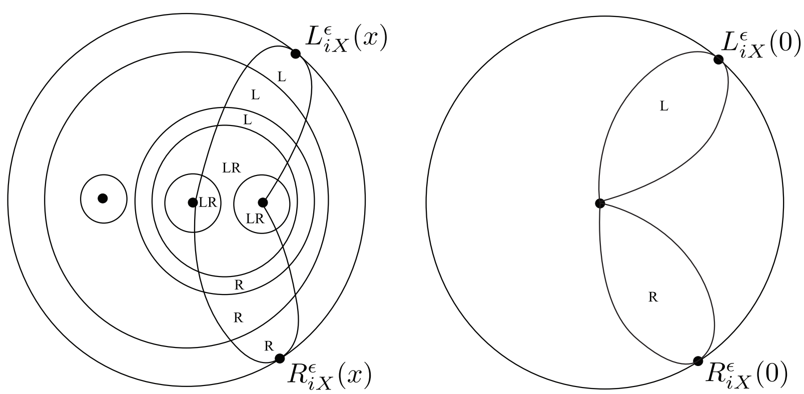


FIGURE 4. Subregions of a region H given $\mathcal{P}(H) = \{L, R\}$

4.7. **Non-spiraling properties of multi-transversal flows.** This subsection introduces some of framework leading to the proof of the summability estimates. We need them in order to prove the multi-summability of the infinitesimal generator of an element of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. The task requires an analysis of the dynamics of multi-transversal flows in basic sets. This infinitesimal study is key to capture all levels of multi-summability.

Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider a multi-transversal flow $\Re(\aleph X)$ for some stable multi-direction $\aleph : I \to (e^{i(0,\pi)})^{\tilde{q}}$. Let $H \in Reg^*(\epsilon, \aleph X, I)$. In this subsection we study the behavior of $H(r,\lambda) \cap \mathcal{B}$ when $(r,\lambda) \to (0,\lambda_0)$ for a basic set $\mathcal{B}$ and $\lambda_0 \in I$. We see that such limit exists in adapted coordinates.

Definition 4.31. *Let A be a subset of $\mathbb{C} \setminus \{0\}$. Consider the blow-up mapping $\pi : (\mathbb{R}^+ \cup \{0\}) \times \mathbb{S}^1 \to \mathbb{C}$ defined by $\pi(r,\lambda) = r\lambda$. We denote A_π the subset $\overline{\pi^{-1}(A)}$ of $(\mathbb{R}^+ \cup \{0\}) \times \mathbb{S}^1$.*

Definition 4.32. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ and let $\mathcal{E} = \{\eta \geq |t| \geq \rho|x|\}$ be an exterior set associated to X. Consider $A \subset \mathbb{C} \setminus \{0\}$ and a continuous section $\tau : A \to \partial_e \mathcal{E}$ (i.e. $\tau(x) \in \partial_e \mathcal{E}(x)$ for any $x \in A$). The map τ is of the form $(x, \tau_{\mathcal{E}}(x))$ in adapted coordinates (x,t). We say that τ is asymptotically continuous if $\tau_{\mathcal{E}}$ admits a continuous extension to A_π.*

Definition 4.33. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$ and let*

$$\mathcal{C} = \{(x,w) \in B(0,\delta) \times (\overline{B}(0,\rho) \setminus \cup_{\zeta \in S_{\mathcal{C}}} B(\zeta, \eta_{\mathcal{C},\zeta}))\}$$

be a compact-like set associated to X. Consider $A \subset \mathbb{C} \setminus \{0\}$ and a continuous section $\tau : A \to \partial_e \mathcal{C}$. The map τ is of the form $(x, \tau_{\mathcal{C}}(x))$ in adapted coordinates (x,w). We say that τ is asymptotically continuous if $\tau_{\mathcal{C}}$ admits a continuous extension to A_π.

Example: Let $X = y(y-x^2)(y-x)\partial/\partial y$. We have that $\partial_I \mathcal{E}_0 = \partial_e \mathcal{C}_0$ is of the form $\{(x,w) \in B(0,\delta) \times \partial B(0,\rho)\}$ with $w = y/x$. Consider the sections $\tau_0 : \mathbb{R}^+ \to \partial_e \mathcal{C}_0$ and $\tau_1 : \mathbb{R}^+ \to \partial_e \mathcal{C}_0$ defined by $\tau_0(x) = (x, x\rho)$ and $\tau_1(x) = (x, xe^{i/x}\rho)$ in (x,y) coordinates. Both sections admit a continuous extension to $x = 0$ by defining $\tau_0(0) = (0,0) = \tau_1(0)$. Nevertheless τ_0 is asymptotically continuous whereas τ_1 is not.

Definition 4.34. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider a multi-transversal flow $\Re(\aleph X)$ for some stable multi-direction $\aleph : I \to (e^{i(0,\pi)})^{\tilde{q}}$. Let $\mathcal{B}$ be a non-terminal basic set associated to X. Consider a continuous section $\tau : (0,\delta)I \to \partial_e \mathcal{B}$. Suppose that $\Re(\aleph X)$ does not point towards the exterior of $\mathcal{B}$ at $\tau(x)$ for any $x \in (0,\delta)I$. The dynamics of $\Re(\aleph X)$ implies $\sup \mathcal{I}(\Gamma_x) < \infty$ and $\Gamma_x(\sup \mathcal{I}(\Gamma_x)) \in \partial_I \mathcal{B}$ for $\Gamma_x = \Gamma(\aleph X, \tau(x), \mathcal{B})$ and any $x \in (0,\delta)I$. The formula $\partial\tau(x) = \Gamma_x(\sup \mathcal{I}(\Gamma_x))$ defines a continuous section $\partial\tau : (0,\delta)I \to \partial_I \mathcal{B}$ whose image is contained in a connected component of $\partial_I \mathcal{B}$. We define $\psi(\partial\tau(x)) - \psi(\tau(x))$ by considering a Fatou coordinate ψ of X defined in a neighborhood of $\Gamma_x[0, \sup \mathcal{I}(\Gamma_x)]$. The definition does not depend on the choice of ψ.*

Proposition 4.2. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider a multi-transversal flow $\Re(\aleph X)$ for some continuous function $\aleph : I \to (e^{i(0,\pi)})^{\tilde{q}}$. Let $\mathcal{B}$ be a non-terminal basic set associated to X. Consider a continuous section $\tau : (0,\delta)I \to \partial_e \mathcal{B}$. Suppose that $\Re(\aleph X)$ does not point towards the exterior of $\mathcal{B}$ at $\tau(x)$ for any $x \in (0,\delta)I$. Suppose that τ is asymptotically continuous. Then $\partial\tau$ is asymptotically continuous. Moreover the function $F : (0,\delta) \times I \to \mathbb{C}$ defined by*

$$|r|^{\iota(\mathcal{B})}(\psi(\partial\tau(r,\lambda)) - \psi(\tau(r,\lambda)))$$

admits a continuous extension to $[0,\delta) \times I$ *such that* $F(0,\lambda) \in \mathbb{H}$ *for any* $\lambda \in I$.

Proof. Suppose that $\mathcal{B}$ is a compact-like set

$$\mathcal{C} = \{(x,w) \in B(0,\delta) \times (\overline{B}(0,\rho) \setminus \cup_{\zeta \in S_{\mathcal{C}}} B(\zeta, \eta_{\mathcal{C},\zeta}))\}.$$

Denote $X^{\mathcal{C}}(\lambda)$ the polynomial vector field associated to $\mathcal{C}$. We denote

$$\Gamma_\lambda = \Gamma(\aleph_{\mathcal{C}}(\lambda) X^{\mathcal{C}}(\lambda), \tau_{\mathcal{C}}(0,\lambda), \mathcal{C}).$$

We have that $\Re(\lambda^{e(\mathcal{C})}(\aleph X)_{\mathcal{C}})$ does not point towards the exterior of $\overline{B}(0,\rho)$ at $(r,\lambda,\tau_{\mathcal{C}}(r,\lambda))$ for any $(r,\lambda) \in (0,\delta) \times I$. By continuity $\Re(\lambda^{e(\mathcal{C})}(\aleph X)_{\mathcal{C}})$ does not point towards the exterior of $\overline{B}(0,\rho)$ at $(r,\lambda,\tau_{\mathcal{C}}(r,\lambda))$ for any $(r,\lambda) \in [0,\delta) \times I$. The vector field $\Re(\aleph X)_{|\mathcal{C}}$ is of the form $\Re(|x|^{e(\mathcal{C})}\lambda^{e(\mathcal{C})}(\aleph X)_{\mathcal{C}})$. We have

$$\left(\lambda^{e(\mathcal{C})}(\aleph X)_{\mathcal{C}}\right)_{|(r,\lambda)=(0,\lambda_0)} \equiv \aleph_{\mathcal{C}}(\lambda_0) X^{\mathcal{C}}(\lambda_0) \ \ \forall \lambda_0 \in I.$$

Hence the remark (4.10) implies that $\sup(\mathcal{I}(\Gamma_\lambda)) < \infty$ and $\Gamma_\lambda(\sup(\mathcal{I}(\Gamma_\lambda))) \in \partial_I \mathcal{C}$ for any $\lambda \in I$. We deduce that $\partial\tau$ extends continuously to $((0,\delta)I)_\pi$ by defining

$$\partial\tau(0,\lambda) = \Gamma_\lambda(\sup(\mathcal{I}(\Gamma_\lambda)))$$

in (x,w) coordinates for any $\lambda \in I$. Consider a Fatou coordinate $\psi^{\mathcal{C}}$ of $X^{\mathcal{C}}(\lambda_0)$ defined in a neighborhood of $\Gamma_{\lambda_0}[0, \sup \mathcal{I}(\Gamma_{\lambda_0})]$. We obtain

$$\lim_{(r,\lambda)\to(0,\lambda_0)} |r|^{e(\mathcal{C})}(\psi(\partial\tau(r,\lambda)) - \psi(\tau(r,\lambda))) = \psi^{\mathcal{C}}(\partial\tau(0,\lambda_0)) - \psi^{\mathcal{C}}(\tau(0,\lambda_0))$$

for any $\lambda_0 \in I$. Clearly $F(0,\lambda_0)$ belongs to $\aleph_{\mathcal{C}}(\lambda_0)\mathbb{R}^+ \subset \mathbb{H}$.

Suppose now that $\mathcal{B}$ is an exterior set

$$\mathcal{E} = \{(x,t) \in B(0,\delta) \times \mathbb{C} : \eta \geq |t| \geq \rho|x|\}.$$

Let $\mathcal{C}$ be the compact-like set such that $\partial_I \mathcal{E} = \partial_e \mathcal{C}$. There exist $2\nu(\mathcal{E})$ continuous sections

$$T\mathcal{E}_{\aleph X}^{\eta,1}(r,\lambda), \ \ldots, \ T\mathcal{E}_{\aleph X}^{\eta,2\nu(\mathcal{E})}(r,\lambda), \ T\mathcal{E}_{\aleph X}^{\eta,2\nu(\mathcal{E})+1}(r,\lambda) = T\mathcal{E}_{\aleph X}^{\eta,1}(r,\lambda)$$

of $T\mathcal{E}_{\aleph X}^{\eta}(r,\lambda)$. The sections are ordered in counter clock wise sense. We denote $arc_j(r,\lambda)$ the closed arc going from $T\mathcal{E}_{\aleph X}^{\eta,j}(r,\lambda)$ to $T\mathcal{E}_{\aleph X}^{\eta,j+1}(r,\lambda)$ in counter clock wise sense. The choice of the order implies

$$arc_j(r,\lambda) \cap T\mathcal{E}_{\aleph X}^{\eta}(r,\lambda) = \{T\mathcal{E}_{\aleph X}^{\eta,j}(r,\lambda), T\mathcal{E}_{\aleph X}^{\eta,j+1}(r,\lambda)\} \ \ \forall (r,\lambda) \in [0,\delta) \times I.$$

There exists $k \in \mathbb{Z}/(2\nu(\mathcal{E})\mathbb{Z})$ such that $\tau_{\mathcal{E}}(r,\lambda) \in arc_k(r,\lambda)$ for any $(r,\lambda) \in [0,\delta) \times I$. In the rest of the proof we are going to show that $\partial\tau$ and F only depend on k.

Consider coordinates (x,w) with $t = wx$. We denote

$$\mathcal{C}(\rho_1) = \mathcal{C} \cup \{(x,w) \in B(0,\delta) \times \mathbb{C} : \rho_1 \geq |w| \geq \rho\}$$

and $T\mathcal{C}_{\aleph X}^{\rho_1}(r,\lambda) = T\mathcal{C}(\rho_1)_{\aleph_{\mathcal{E}} X}^{\rho_1}(r,\lambda)$ for $\rho_1 \geq 2\rho$. Fix $\rho_1 \geq 2\rho$, we define

$$\Gamma_P^{\rho_1} = \Gamma(\lambda^{e(\mathcal{E})}(\aleph X)_{\mathcal{E}}, P, \mathcal{E} \setminus \mathcal{C}(\rho_1)).$$

We have that $\sup \mathcal{I}(\Gamma_P^{\rho_1}) < \infty$ and $\Gamma_P^{\rho_1}(\sup \mathcal{I}(\Gamma_P^{\rho_1})) \in \partial_e \mathcal{C}(\rho_1) \setminus T\mathcal{C}_{\aleph X}^{\rho_1}(r,\lambda)$ for all $P \in arc_k(r,\lambda)$ and $(r,\lambda) \in (0,\delta) \times I$. All the points of the form $\Gamma_P^{\rho_1}(\sup \mathcal{I}(\Gamma_P^{\rho_1}))$ are in a unique connected component A_{ρ_1} of $\partial_e \mathcal{C}(\rho_1) \setminus T\mathcal{C}_{\aleph X}^{\rho_1}$. We have

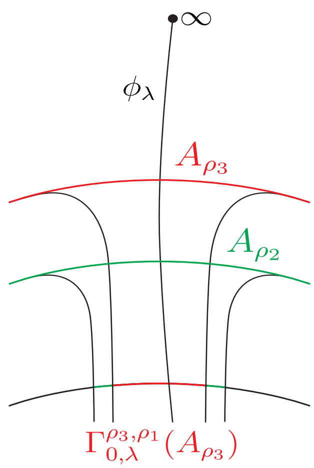
$$\left(\lambda^{e(\mathcal{C})} \aleph_{\mathcal{E}}(\lambda) X_{\mathcal{C}}\right)_{|(\mathcal{E}\setminus\mathcal{C}(\rho_1))(0,\lambda_0)} \equiv \aleph_{\mathcal{E}}(\lambda_0) X^{\mathcal{C}}(\lambda_0) \ \ \forall \lambda_0 \in I.$$

Thus $A_{\rho_1}(0,\lambda)$ is an open arc transverse to $\Re(\aleph_{\mathcal{E}}(\lambda)X^{\mathcal{C}}(\lambda))$ and whose closure contains two points of $T\mathcal{C}_{\aleph X}^{\rho_1}(0,\lambda)$ for any $\lambda \in I$. Moreover $\aleph_{\mathcal{E}}(\lambda)X^{\mathcal{C}}(\lambda)$ is conjugated to $w^{\nu(\mathcal{C})+1}\partial/\partial w$ in the neighborhood of $w=\infty$. Thus there exists a unique trajectory $\phi_\lambda : (0, s_{\rho_1,\lambda}] \to \mathbb{C}$ of $\Re(\aleph_{\mathcal{E}}(\lambda)X^{\mathcal{C}}(\lambda))$ such that $\lim_{s\to 0}\phi_\lambda(s) = \infty$, $\phi_\lambda(0, s_{\rho_1,\lambda}) \subset \mathbb{C} \setminus \overline{B}(0,\rho_1)$ and $\phi_\lambda(s_{\rho_1,\lambda}) \in A_{\rho_1}(\lambda)$ for any $\lambda \in I$. The germ of the curve ϕ_λ in the neighborhood of ∞ does not depend on ρ_1.

Consider $\rho_2 > \rho_1$ and $(r,\lambda) \in [0,\delta) \times I$. We define $\Gamma_{r,\lambda}^{\rho_2,\rho_1} : A_{\rho_2}(r,\lambda) \to A_{\rho_1}(r,\lambda)$, it is the mapping given by the formula

$$\Gamma_{r,\lambda}^{\rho_2,\rho_1}(P) = \Gamma_P^{\rho_2,\rho_1}(\sup \mathcal{I}(\Gamma_P^{\rho_2,\rho_1}))$$

where $\Gamma_P^{\rho_2,\rho_1} = \Gamma(\aleph_{\mathcal{E}}X^{\mathcal{C}}, P, \mathcal{C}(\rho_2)\setminus\mathcal{C}(\rho_1))$. Let us remark that given $\rho_2 > \rho_1$ we have $\Gamma_P^{\rho_2}(\sup \mathcal{I}(\Gamma_P^{\rho_2})) \subset A_{\rho_2}(r,\lambda)$ for all $P \in arc_k(r,\lambda)$ and $(r,\lambda) \in (0,\delta)\times I$. We deduce that $\Gamma_P^{\rho_1}(\sup \mathcal{I}(\Gamma_P^{\rho_1})) \in \Gamma_{r,\lambda}^{\rho_2,\rho_1}(A_{\rho_2}(r,\lambda))$ for all $P \in arc_k(r,\lambda)$ and $(r,\lambda) \in (0,\delta)\times I$. We have



$$\cap_{\rho_2>\rho_1}\Gamma_{0,\lambda}^{\rho_2,\rho_1}(A_{\rho_2}(0,\lambda)) = \{\phi_\lambda(s_{\rho_1,\lambda})\} \quad \forall \lambda \in I.$$

We can consider $\rho_1 = 2\rho$. We define

$$\partial\tau(0,\lambda) = (0, \tilde{\Gamma}_\lambda(\sup(\mathcal{I}(\tilde{\Gamma}_\lambda)))$$

in (x,w) coordinates for $\lambda \in I$ where $\tilde{\Gamma}_\lambda = \Gamma(\lambda^{e(\mathcal{C})}\aleph^* X_{\mathcal{C}}, (0,\phi_\lambda(s_{2\rho,\lambda})), \mathcal{E})$. We obtain in this way a continuous extension of $\partial\tau$ to $((0,\delta)I)_\pi$.

Consider a Fatou coordinate $\psi^{\mathcal{C}}$ of $X^{\mathcal{C}}(\lambda_0)$ defined in a neighborhood of the set $\tilde{\Gamma}_{\lambda_0} \cup \phi_\lambda(0, s_{2\rho,\lambda_0}] \cup \{\infty\}$. Denote $w_{r,\lambda,\rho_1} = \Gamma_{\tau(r,\lambda)}^{\rho_1}(\sup \mathcal{I}(\Gamma_{\tau(r,\lambda)}^{\rho_1}))$. We have

$$\psi(\partial\tau(r,\lambda)) - \psi(\tau(r,\lambda)) = [\psi(\partial\tau(r,\lambda)) - \psi(w_{r,\lambda,\rho_1})] + [\psi(w_{r,\lambda,\rho_1}) - \psi(\tau(r,\lambda))]$$

and

$$lim_{(r,\lambda)\to(0,\lambda_0)}|r|^{\iota(\mathcal{E})}(\psi(\partial\tau(r,\lambda)) - \psi(w_{r,\lambda,\rho_1})) = \psi^{\mathcal{C}}(\partial\tau(0,\lambda)) - \psi^{\mathcal{C}}(\phi_\lambda(s_{\rho_1,\lambda})).$$

We have $t \sim t - \gamma_{\mathcal{E}}(x)$ in $\mathcal{E}$. By using lemma 4.3 and remark 4.5 we obtain

$$|\psi_{\mathcal{E}}(w_{r,\lambda,\rho_1}) - \psi_{\mathcal{E}}(\tau(r,\lambda))| \leq \frac{C'}{\rho_1^{\nu(\mathcal{E})} r^{\nu(\mathcal{E})}} + C''$$

for some constants $C', C'' \in \mathbb{R}^+$. We can consider $\psi = \psi_{\mathcal{E}}/x^{e(\mathcal{E})}$ to get

$$|r|^{\iota(\mathcal{E})}|\psi(w_{r,\lambda,\rho_1}) - \psi(\tau(r,\lambda))| \leq \frac{C'}{\rho_1^{\nu(\mathcal{E})}} + C'' r^{\nu(\mathcal{E})}$$

whose limit when $(\rho_1, r) \to (\infty, 0)$ is equal 0. Therefore we obtain

$$\lim_{(r,\lambda)\to(0,\lambda_0)} |r|^{\iota(\mathcal{B})}(\psi(\partial\tau(r,\lambda)) - \psi(\tau(r,\lambda))) = \psi^{\mathcal{C}}(\partial\tau(0,\lambda)) - \psi^{\mathcal{C}}(\infty)$$

for any $\lambda_0 \in I$. Moreover $F(0,\lambda_0) \in \mathbb{H}$ for any $\lambda_0 \in I$ since the image of $\aleph^*$ is contained in $e^{i(0,\pi)}$. □

The previous proposition makes simpler the analysis of regions of multi-transversal flows. The proof of the following corollary is contained in the previous one.

Corollary 4.1. *Consider the setting of the previous proposition. Suppose that $\mathcal{B}$ is an exterior set. The set $\partial_\downarrow \mathcal{B}$ of points in $\partial_e \mathcal{B}$ where $\Re(\aleph X)$ does not point towards the exterior of $\mathcal{B}$ has $\nu(\mathcal{B})$ connected components. Then $(\partial\tau)_{|r=0}$ only depends on the component of $\partial_\downarrow \mathcal{B}$ containing $\tau((0,\delta)I)$.*

Corollary 4.2. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider a multi-transversal flow $\Re(\aleph X)$ for some stable multi-direction $\aleph : I \to (e^{i(0,\pi)})^{\tilde{q}}$. Denote $\Gamma_x^j = \Gamma(\aleph X, T_{\aleph X}^{\epsilon,j}(x), T_0)$ for $x \in (0,\delta)I$ and $j \in \mathbb{Z}/(2\nu(\mathcal{E}_0)\mathbb{Z})$ (see def. 4.29). Given a basic set $\mathcal{B}$ associated to X we have that either $\Gamma_x^j[0,\infty) \cap \partial_e \mathcal{B}$ is empty for any $x \in (0,\delta)I$ or $\Gamma_x^j[0,\infty) \cap \partial_e \mathcal{B}$ contains a unique point for any $x \in (0,\delta)I$. Suppose that we are in the latter case. Then the mapping $x \to \Gamma_x^j[0,\infty) \cap \partial_e \mathcal{B}$ is asymptotically continuous in $(0,\delta)I$. Moreover the function $F : (0,\delta) \times I \to \mathbb{C}$ defined by $(|r|^{e(\mathcal{B})}\psi_{L_j}^X)(\Gamma_{r,\lambda}^j[0,\infty) \cap \partial_e \mathcal{B})$ admits a continuous extension to $[0,\delta) \times I$ such that $F(0,\lambda) \in \mathbb{H}$ for any $\lambda \in I$.*

Proposition 4.3. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider a multi-transversal flow $\Re(\aleph X)$ for some stable multi-direction $\aleph : I \to (e^{i(0,\pi)})^{\tilde{q}}$. Let $\mathcal{E} = \{\eta \geq |t| \geq \rho|x|\}$ be a parabolic exterior set associated to $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider $H \in Reg^*(\epsilon, \aleph X, I)$. Then there exists $C_{\mathcal{E}} \in \mathbb{R}^+$ such that*

$$\frac{1}{C_{\mathcal{E}}} \frac{1}{|t - \gamma_{\mathcal{E}}(x)|^{\nu(\mathcal{E})}} \leq |\psi_{\mathcal{E}}|(x,t) \leq C_{\mathcal{E}} \frac{1}{|t - \gamma_{\mathcal{E}}(x)|^{\nu(\mathcal{E})}}$$

in every sub-region of H contained in $\mathcal{E}$.

Proof. We define $\Gamma_0 = \partial H \setminus SingX$. The set $\Gamma_0(x)$ is the union of the trajectories of $\Re(\aleph X)$ bounding $H(x)$ for $x \in [0,\delta)I$. The corollary 4.2 implies that $\Gamma_0 \cap \partial_e \mathcal{E}$ is the union of a finite number of asymptotically continuous sections. Therefore there exists $\zeta \in \mathbb{R}^+$ such that

$$\overline{H} \cap \partial_e \mathcal{E} \subset \{t - \gamma_{\mathcal{E}}(x) \in \mathbb{R}^+ e^{i[-\zeta,\zeta]}\}$$

by proposition 4.1. The result is a consequence of lemma 4.3 and remark 4.5. □

Proposition 4.4. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider a multi-transversal flow $\Re(\aleph X)$ for some stable multi-direction $\aleph : I \to (e^{i(0,\pi)})^{\tilde{q}}$. Consider $H \in Reg^*(\epsilon, \aleph X, I)$ and $L \in \mathcal{P}(H)$. Then there exists $C_1 \in \mathbb{R}^+$ such that*

$$\frac{1}{C_1} \frac{1}{|y|^{\nu(\mathcal{E}_0)}} \leq |\psi_L^X|(x,y) \leq C_1 \frac{1}{|y|^{\nu(\mathcal{E}_0)}} \tag{2}$$

for any (x,y) that belongs to the L-subregion contained in $\mathcal{E}_0$. The constant C_1 depends on X but it does not depend on I, $\aleph$, H or L.

Proof. We have $y \sim (y - \gamma_{\mathcal{E}_0}(x))$ in $\mathcal{E}_0$. The proposition 4.1 implies that the subregions of H contained in $H \cap \mathcal{E}_0$ are contained in $\{y \in \mathbb{R}^+ e^{i[-\zeta,\zeta]}\}$ for some $\zeta \in \mathbb{R}^+$. In fact ζ does not depend on I, $\aleph$ or H. Thus we can apply lemma 4.3 and remark 4.5 in order to obtain equation (2) for some $C_1 \in \mathbb{R}^+$ that does not depend on I, $\aleph$, H or L. □

4.8. **Size of regions.** Given $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ we define Fatou coordinates in the regions of $Reg(\epsilon, \aleph X, I)$ for convenient choices of X, I and $\aleph$ (section 5). The Fatou coordinates are the main ingredients that we use in the theorem of analytic conjugation 7.3. In order for such an approach to analytic conjugacy to work it is required that regions be somehow preserved by analytic conjugations. Very roughly speaking we have to prove that regions are not too small. This subsection is devoted to make the previous ideas more precise and to introduce the concepts and results that will be used in section 7.

Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider a subset H of T_0. Given a point $P \in H(x)$ we consider $\Gamma_P = \Gamma(X, P, H)$. We define

$$width_H(P) = length(\mathcal{I}(\Gamma_P))$$

where $length(\mathcal{I}(\Gamma_P))$ is by definition the length of the interval of definition of $\mathcal{I}(\Gamma_P)$. It can be eventually equal to ∞. We define

$$width_H^{\min}(x) = \min_{P \in H(x)} width_H(P), \quad width_H^{\max}(x) = \max_{P \in H(x)} width_H(P).$$

Definition 4.35. *Let $H \in Reg_2(\epsilon, \aleph X, I)$. There exists a seed T_β containing $\alpha^{\aleph X}(H)$ and $\omega^{\aleph X}(H)$ but such that $\alpha^{\aleph X}(H)$ and $\omega^{\aleph X}(H)$ are contained in different sons of T_β. We denote $\mathcal{C}_H = \mathcal{C}_\beta$. We define $e(H) = e(\mathcal{C}_H)$.*

Proposition 4.5. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider a multi-transversal flow $\Re(\aleph X)$ for some stable multi-direction $\aleph : I \to (e^{i(0,\pi)})^{\tilde{q}}$. Suppose $H \in Reg_1(\epsilon, \aleph X, I)$, then*

$$width_H^{\min}(x) = width_H^{\max}(x) = \infty \quad \forall x \in [0, \delta)I.$$

Suppose $H \in Reg_2(\epsilon, \aleph X, I)$, then there exists $J \in \mathbb{R}^+$ such that

$$width_H^{min}(x) \geq \frac{J}{|x|^{e(H)}} \quad \forall x \in (0, \delta)I.$$

Proof. The result is obvious if $H \in Reg_1(\epsilon, \aleph X, I)$. Suppose $H \in Reg_2(\epsilon, \aleph X, I)$. Denote $\mathcal{C} = \mathcal{C}_H$. We have $\mathcal{C} = \{(x, w) \in B(0, \delta) \times (\overline{B}(0, \rho) \setminus \cup_{\zeta \in S_{\mathcal{C}}} B(\zeta, \eta_{\mathcal{C},\zeta}))\}$. Consider the magnifying glass $M = \{(x, w) \in B(0, \delta) \times \overline{B}(0, \rho)\}$. We define

$$H_{\mathcal{C}}(x) = \{P \in H(x) : \Gamma(\aleph X, P, T_0) \subset M\}$$

for any $x \in (0, \delta)I$. We define

$$width_{\mathcal{C}}^*(P) = length(\mathcal{I}(\Gamma(\lambda^{e(H)} X_{\mathcal{C}}, P, H_{\mathcal{C}})))$$

for $P \in H_{\mathcal{C}}(r, \lambda)$ and $(r, \lambda) \in [0, \delta) \times I$. It suffices to prove that $width_{\mathcal{C}}^* > J_0$ in $H_{\mathcal{C}}$ for some $J_0 \in \mathbb{R}^+$. The rest of the proof is devoted to show this result.

We have $H_{\mathcal{C}}(r, \lambda) \cap \partial_e \mathcal{C}_H = \{T\mathcal{C}_{\aleph X}^{\rho,1}(r, \lambda), T\mathcal{C}_{\aleph X}^{\rho,2}(r, \lambda)\}$ for any $(r, \lambda) \in (0, \delta) \times I$ where $T\mathcal{C}_{\aleph X}^{\rho,1}(r, \lambda)$ and $T\mathcal{C}_{\aleph X}^{\rho,2}(r, \lambda)$ are continuous sections of $T\mathcal{C}_{\aleph X}^{\rho}(r, \lambda)$ defined for $(r, \lambda) \in [0, \delta) \times I$. Up to exchange $T\mathcal{C}_{\aleph X}^{\rho,1}$ and $T\mathcal{C}_{\aleph X}^{\rho,2}$ if necessary we suppose that $\Re(X)$ points towards $H_{\mathcal{C}}$ at $T\mathcal{C}_{\aleph X}^{\rho,1}(x)$ for any $x \in (0, \delta)I$. We claim that $H_{\mathcal{C}}(r, \lambda)$

extends continuously to $[0,\delta) \times I$, i.e. there exists $H_{\mathcal{C}}(0,\lambda_0) \subset M(0,\lambda_0) \setminus SingX$ such that

$$\lim_{(r,\lambda)\to(0,\lambda_0)} (H_{\mathcal{C}} \cup SingX_{\mathcal{C}})(r,\lambda) = (H_{\mathcal{C}} \cup SingX_{\mathcal{C}})(0,\lambda_0)$$

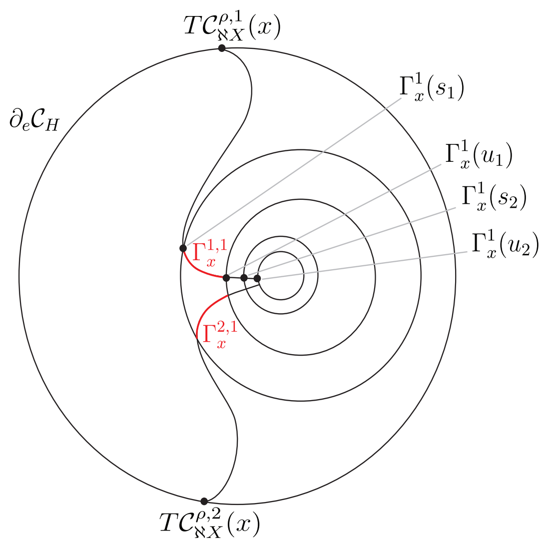
for any $\lambda_0 \in I$. The limit is considered in the Hausdorff topology for compact sets in $\{(r,\lambda,w) \in [0,\infty) \times \mathbb{S}^1 \times \mathbb{C}\}$. The claim is a consequence of the analogous property for $T\mathcal{C}^{\rho,1}_{\aleph X}(r,\lambda)$ and $T\mathcal{C}^{\rho,2}_{\aleph X}(r,\lambda)$. Consider a Fatou coordinate $\psi_{\mathcal{C}}$ of $\lambda^{e(H)} X_{\mathcal{C}}$ in $H_{\mathcal{C}}$. There exists a continuous function $c : [0,\delta) \times I \to \mathbb{R}^+$ such that we have the equality of sets

$$\psi_{\mathcal{C}}(T\mathcal{C}^{\rho,2}_{\aleph X}(r,\lambda)) + \aleph_{\mathcal{C}}(\lambda)\mathbb{R} = c(r,\lambda) + \psi_{\mathcal{C}}(T\mathcal{C}^{\rho,1}_{\aleph X}(r,\lambda)) + \aleph_{\mathcal{C}}(\lambda)\mathbb{R}$$

for any $(r,\lambda) \in [0,\delta) \times I$. Thus there exists $c_0 \in \mathbb{R}^+$ such that $c_0 \leq c(r,\lambda)$ for any $(r,\lambda) \in [0,\delta) \times I$. We denote

$$\Gamma^j_{r,\lambda} = \Gamma(\lambda^{e(H)}(\aleph X)_{\mathcal{C}}, T\mathcal{C}^{\rho,j}_{\aleph X}(r,\lambda), M)$$

for $j \in \{1,2\}$ and $(r,\lambda) \in [0,\delta) \times I$. It suffices to prove that $width^*_{\mathcal{C}}(P) > J_0$ for some constant $J_0 \in \mathbb{R}^+$ and any $P \in \cup_{x \in (0,\delta)I} \Gamma^1_x(-\infty,\infty)$. Let $\mathcal{E}_1$, …, $\mathcal{E}_a$ be the



sequence of non-terminal exterior sets intersected by $\Gamma^j_x(0,\infty)$. We have

$$\mathcal{E}_k = \{(x,t_k) \in B(0,\delta) \times \mathbb{C} : \eta_k \geq |t_k| \geq \rho_k |x|\}$$

in adapted coordinates (x,t_k) for any $1 \leq k \leq a$. Consider coordinates $(x,w_k) \in \mathbb{C}^2$ such that $t_k = x w_k$. Denote

$$\mathcal{E}^*_k = \{(x,t_k) \in B(0,\delta) \times \mathbb{C} : \rho_k \leq |w_k| \leq 2\rho_k\} \text{ and } \partial_e \mathcal{E}^*_k = \mathcal{E}_k \cap \{|w_k| = 2\rho_k\}.$$

Given $x \in (0,\delta)I$ we denote $s_k(x)$ and $u_k(x)$ the positive real numbers such that $\Gamma^1_x(s_k(x)) \in \partial_e \mathcal{E}^*_k$ and $\Gamma^1_x(u_k(x)) \in \partial_I \mathcal{E}_k$ for $1 \leq k \leq a$. By reordering the sequence $\mathcal{E}_1$, …, $\mathcal{E}_a$ we suppose $s_1 < u_1 < \ldots < s_a < u_a$. We denote $\Gamma^{1,k}_x$ the compact set $(w_k \circ \Gamma^1_x)[s_k(x), u_k(x)]$. Analogously as corollary 4.2 the mappings $w_k \circ \Gamma^1_x(s_k(x))$ and $w_k \circ \Gamma^1_x(u_k(x))$ extend continuously to $(r,\lambda) \in [0,\delta) \times I$. Therefore the function $x \to \Gamma^{1,k}_x$ defined in $(0,\delta) \times I$ can be extended continuously to $[0,\delta) \times I$. In an analogous way we obtain that $\Gamma_x^{2,k} = \Gamma_x^2(0,\infty) \cap \mathcal{E}_k^*$ extends continuously to $(r,\lambda) \in [0,\delta) \times I$. Moreover since $\Gamma_{r,\lambda}^1(0,\infty) \cap \partial_e \mathcal{E}_1$ and $\Gamma_{r,\lambda}^2(0,\infty) \cap \partial_e \mathcal{E}_1$ intersect $\partial_e \mathcal{E}_1$ in the same component of $\partial_e \mathcal{E}_1 \setminus T(\mathcal{E}_1)_{\aleph X}^{\eta_1}(r,\lambda)$ then prop. 4.2 and cor. 4.1 imply that $\Gamma_{0,\lambda}^{1,1} \equiv \Gamma_{0,\lambda}^{2,1}$ for any $\lambda \in I$. Then we obtain $\Gamma_{0,\lambda}^{1,k} \equiv \Gamma_{0,\lambda}^{2,k}$ for all $1 \leq k \leq a$ and $\lambda \in I$. The negative trajectory $\Gamma_x^1(-\infty,0)$ can be analyzed analogously.

Let us study the behavior of $width_{\mathcal{C}}^*$ in

$$\Gamma_x^1[0,s_1(x)],\ \Gamma_x^1[s_1(x),u_1(x)],\ \ldots,\ \Gamma_x^1[s_a(x),u_a(x)],\ \Gamma_x^1[u_a(x),\infty).$$

Denote $u_0 \equiv 0$ and $s_{a+1} \equiv \infty$. Let $\mathcal{C}_k$ be the compact-like set associated to the same seed as $\mathcal{E}_k$. We claim that given $x \in (0,\delta)I$ the function $width_{\mathcal{C}}^*$ is constant in $\Gamma_x^1[u_k(x),s_{k+1}(x)]$ for any $0 \leq k \leq a$. Suppose $k \geq 1$. Then $\Gamma_x^1(u_k)$ and $\Gamma_x^2(\mathbb{R}^+) \cap \partial_I \mathcal{E}_k$ (respectively $\Gamma_x^1(s_k)$ and $\Gamma_x^2(\mathbb{R}^+) \cap \partial_e \mathcal{E}_k^*$) are asymptotically continuous sections whose extensions to $r=0$ coincide. Thus we have $\aleph^* \equiv \aleph_{\mathcal{C}_k}(x)$ in $\Gamma(\lambda^{e(H)} X_{\mathcal{C}}, P, H_{\mathcal{C}})$. for any $P \in \Gamma_x^1[u_k(x),s_{k+1}(x)]$ and the claim follows. Suppose $k=0$. Let $s_{-1}(x)$ be the negative real number such that $\Gamma_x^1(s_{-1}(x)) \in \partial_e \mathcal{E}_1^*$. By arguing as in the previous case we obtain that $width_{\mathcal{C}}^*$ is constant in $\Gamma_x^1[s_{-1}(x),s_1(x)]$. Moreover $width_{\mathcal{C}}^*(\Gamma_x^1[s_{-1}(x),s_1(x)]) \subset [c_0,\infty)$ for any $x \in (0,\delta)I$. It suffices to prove that $width_{\mathcal{C}}^*$ decreases at most by a multiplicative constant in $\Gamma_x^1[s_k(x),u_k(x)]$ for all $x \in (0,\delta)I$ and $1 \leq k \leq a$.

The sets $\Gamma_{r,\lambda}^{1,k}[s_k(r,\lambda),u_k(r,\lambda)]$ and $\Gamma_{0,\lambda}^{1,k}$ for $(r,\lambda) \in (0,\delta) \times I$ are contained in trajectories of $\Re(\lambda^{\iota(\mathcal{E}_k)} \aleph^* X_{\mathcal{C}_k})$ that we denote $\tilde{\Gamma}_{r,\lambda}^{1,k}[0,v(r,\lambda)]$. The vector field $Re(\lambda^{\iota(\mathcal{E}_k)} X_{\mathcal{C}_k})$ is transversal to $\Re(\lambda^{\iota(\mathcal{E}_k)} \aleph^* X_{\mathcal{C}_k})$ in $\cup_{(r,\lambda)\in[0,\delta)\times I} \tilde{\Gamma}_{r,\lambda}^{1,k}[0,v(r,\lambda)]$. We can define an holonomy mapping in the neighborhood of $\cup_{(r,\lambda)\in[0,\delta)\times I} \tilde{\Gamma}_{r,\lambda}^{1,k}[0,v(r,\lambda)]$. More precisely consider $(r,\lambda) \in [0,\delta) \times I$ and points $h_1,h_2 \in [0,v(r,\lambda)]$. Let us define $hol_{r,\lambda,h_1,h_2}(z)$ for $z \in \mathbb{R}$ in a neighborhood of 0 such that

$$\Gamma(\lambda^{\iota(\mathcal{E}_k)} X_{\mathcal{C}_k}, \tilde{\Gamma}_{r,\lambda}^{1,k}(h_2), T_0)(hol_{r,\lambda,h_1,h_2}(z))$$

is the point of intersection of $\Gamma(\lambda^{\iota(\mathcal{E}_k)} X_{\mathcal{C}_k}, \tilde{\Gamma}_{r,\lambda}^{1,k}(h_2), T_0)$ and the trajectory of the vector field $\Re(\lambda^{\iota(\mathcal{E}_k)} \aleph^* X_{\mathcal{C}_k})$ passing through $\Gamma(\lambda^{\iota(\mathcal{E}_k)} X_{\mathcal{C}_k}, \tilde{\Gamma}_{r,\lambda}^{1,k}(h_1), T_0)(z)$. The function $hol_{r,\lambda,h_1,h_2}(z)$ is continuous in (r,λ,h_1,h_2,z) and real analytic in z. Since $\cup_{(r,\lambda)\in[0,\delta)\times I}(r,\lambda) \times [0,v(r,\lambda)]^2$ is compact then there exists $J_k \in \mathbb{R}^+$ such that

$$|hol_{r,\lambda,h_1,h_2}(z)| \geq \frac{|z|}{J_k} \tag{3}$$

for all $h_1,h_2 \in [0,v(r,\lambda)]$ and z in a neighborhood of 0 and any $(r,\lambda) \in [0,\delta) \times I$. We denote $J = c_0/\prod_{k=1}^{a} J_k$. Consider $1 \leq k \leq a$ and $\lambda_0 \in I$. The compact set $H_{\mathcal{C}}(r,\lambda) \cap \mathcal{E}_k^*$ tends to the curve $\Gamma_{0,\lambda_0}^{1,k} = \Gamma_{0,\lambda_0}^{2,k}$ when $(r,\lambda) \to (0,\lambda_0)$. Thus the property (3) on the behavior of the holonomy in a neighborhood of $\cup_{\lambda\in I}\Gamma_{0,\lambda}^{1,\lambda}$ implies $width_{\mathcal{C}}^*(\Gamma_x^1[0,\infty)) \geq J$ for any $x \in (0,\delta)I$. We do the same analysis with $\Gamma_x^1(-\infty,0]$ and consider a smaller $J>0$ if necessary to obtain $width_{\mathcal{C}}^*(\Gamma_x^1(-\infty,\infty)) \geq J$ for any $x \in (0,\delta)I$. This implies $width_H^{\min}(x) \geq J/|x|^{e(H)}$ for any $x \in (0,\delta)I$. □

Let $\varphi,\eta \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ with the same fixed points set. Suppose that $\varphi_{|x=x_0}$ is analytically conjugated to $\eta_{|x=x_0}$ by an injective mapping κ_{x_0} whose fixed points set contains the fixed points set of $\varphi_{|x=x_0}$ for any x_0 in a pointed neighborhood of 0. If κ_{x_0} is defined in some $B(0,\epsilon)$ for any $x_0 \neq 0$ and some $\epsilon>0$ independent of x_0 then φ is analytically conjugated to η (main theorem in [16]). The family κ_{x_0} is not required to depend analytically or even continually on x_0. It turns out that if we drop the hypothesis on the common domain of definition $B(0,\epsilon)$ then the result is no longer true. There are counterexamples where φ and η are not analytically conjugated and κ_{x_0} is defined in $B(0, C_0/\sqrt[\nu(\mathcal{E}_0)]{|\ln x_0|})$ for some $C_0 \in \mathbb{R}^+$ and any $x_0 \neq 0$. One of the goals of this paper is proving that such a counterexample is optimal: φ is analytically conjugated to η if the domain of definition is any "bigger" than $\{|y| < \sqrt[\nu(\mathcal{E}_0)]{|\ln x|}\}$, i.e. if it is of the form $\{|y| < s(x)\}$ where s is a $\nu(\mathcal{E}_0)$ slow decaying function (see theorem 7.3 for a precise statement).

Definition 4.36. *Let us consider a bounded function $s(x) : B(0,\delta)\setminus\{0\} \to \mathbb{R}^+$. We say that s is a slow decaying function if* $\lim_{x\to 0} s^{-1}(x)|x|^{\tau} = 0$ *for any $\tau \in \mathbb{R}^+$. Let $\nu \in \mathbb{N}$. We say that s is a ν slow decaying function if* $\lim_{x\to 0} s(x)\sqrt[\nu]{|\ln|x||} = \infty$.

Remark 4.18. *A ν slow decaying function is a slow decaying function. The constant fuctions are examples of ν slow decaying functions for any $\nu \in \mathbb{N}$. Moreover both concepts of decay are preserved if we replace $s(x)$ with $s(x)\tau$ for $\tau \in \mathbb{R}^+$.*

Definition 4.37. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider a multi-transversal flow $\Re(\aleph X)$ for some stable multi-direction $\aleph : I \to (e^{i(0,\pi)})^{\tilde{q}}$. Let s be a slow decaying function. We define $Reg(s, \aleph X, I)$ the set of connected components of*

$$\{(x,y) \in (0,\delta)I \times B(0,\epsilon) : \Gamma(\aleph X, (x,y), T_0) \subset \{x\} \times B(0, s(x))\} \setminus SingX.$$

If s satisfies $s < \epsilon$ there is a bijective correspondence between $Reg(s, \aleph X, I)$ and $Reg(\epsilon, \aleph X, I)$ and any element of $Reg(\epsilon, \aleph X, I)$ contains an element of $Reg(s, \aleph X, I)$.

The next result is used to prove the proposition 7.1. It is one of the ingredients of the proof of theorem 7.3 .

Proposition 4.6. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$. Consider a multi-transversal flow $\Re(\aleph X)$ for some stable multi-direction $\aleph : I \to (e^{i(0,\pi)})^{\tilde{q}}$. Let s be a slow decaying function with $s < \epsilon$ and $\tau \in (0,1)$. Consider $H \in Reg(\epsilon, \aleph X, I)$. Denote $H_{s\tau}$ the element of $Reg(s\tau, \aleph X, I)$ contained in H. There exists $J_0, J_1 \in \mathbb{R}^+$ such that* $\exp(tX)$ *is well-defined in $H_{s\tau}(x)$ and* $\exp(tX)(H_{s\tau}(x)) \subset H(x)$ *for all $x \in (0,\delta)I$ and $t \in B(0, J_0/(s^{\nu(\mathcal{E}_0)}(x)\tau^{\nu(\mathcal{E}_0)}) - J_1)$. In particular we obtain*

$$width^{\min}_{H\setminus H_{s\tau}}(x) \geq \frac{J_0}{s^{\nu(\mathcal{E}_0)}(x)\tau^{\nu(\mathcal{E}_0)}} - J_1 \quad \forall x \in (0,\delta)I.$$

Proof. Denote $\nu = \nu(\mathcal{E}_0)$. We have $H \in Reg_j(\epsilon, \aleph X, I)$. The set $H \setminus H_{s\tau}$ has j connected components. Consider $L \in \mathcal{P}(H)$. We denote $L_{s\tau}$ the element of $\mathcal{P}(H_{s\tau})$ such that $L_{s\tau} \subset L$. There exists a unique connected component $G_{s\tau}$ of $\overline{H} \setminus (H_{s\tau} \cup SingX)$ such that $L^{\epsilon}_{iX}(x) \in G_{s\tau}$ for any $x \in [0,\delta)I$. It suffices to prove $width^{\min}_{G_{s\tau}}(x) \geq J_0/(s(x)\tau)^{\nu} - J_1$ for any $x \in (0,\delta)I$.

We can suppose that $\Re(X)$ points towards the interior of T_0 at $L^{\epsilon}_{iX}(x)$ for any $x \in [0,\delta)I$ by replacing X with $-X$ if necessary. Denote

$$\Gamma^L_x = \Gamma(X, L^{\epsilon}_{iX}(x), \overline{B}(0,\epsilon) \setminus B(0, s(x)\tau)).$$

Denote $\psi = \psi^X_L$. The interval $\mathcal{I}(\Gamma^L_x)$ is of the form $[0, \upsilon(x)]$ for $x \in (0,\delta)I$. Let us remark that $\aleph X \equiv iX$ in $\mathcal{E}_0$. We have $C_1^{-1}/|y|^{\nu} \leq |\psi(x,y)| \leq C_1/|y|^{\nu}$ in $\mathcal{E}_0$ for some positive constant $C_1 \geq 1$ (prop. 4.4). Denote $D(x,\tau) = C_1^{-1}/(s(x)\tau)^{\nu} - C_1/\epsilon^{\nu}$. We deduce

$$\upsilon(x) = |\psi(\Gamma^L_x(\upsilon(x)) - \psi(\Gamma^L_x(0))| \geq D(x,\tau) \quad \forall x \in (0,\delta)I.$$

Denote $\mathcal{C} = \mathcal{C}_0$,

$$\Gamma_x^{\epsilon} = \Gamma(\aleph X, L_{iX}^{\epsilon}(x), T_0), \quad \Gamma_x^{s\tau} = \Gamma(\aleph X, \Gamma_x^{L}(\upsilon(x)), T_0).$$

Consider the notations in the proof of prop. 4.2. Given $\rho_1 \geq 2\rho_0$ the points $\Gamma_x^{\epsilon}(0,\infty) \cap (\{x\} \times \partial B(0, \rho_1|x|))$ and $\Gamma_x^{s\tau}(0,\infty) \cap (\{x\} \times \partial B(0, \rho_1|x|))$ belong to the same connected component A_{ρ_1} of $\partial_e \mathcal{C}(\rho_1) \setminus T\mathcal{C}_{\aleph X}^{\rho_1}$ for any $x \in (0,\delta)I$ in a neighborhood of 0. Analogously as in proposition 4.2 the sections

$$x \to \Gamma_x^{\epsilon}(0,\infty) \cap (\{x\} \times \partial B(0, \rho_1|x|)) \ \text{ and } \ x \to \Gamma_x^{s\tau}(0,\infty) \cap (\{x\} \times \partial B(0, \rho_1|x|))$$

are asymptotically continuous and their value at $(r,\lambda) = (0,\lambda_0)$ coincides and is equal to the element of the set $\cap_{\rho_2 > \rho_1} \Gamma_{0,\lambda_0}^{\rho_2,\rho_1}(A_{\rho_2}(0,\lambda_0))$ for any $\lambda_0 \in I$. An analogous result holds true for the negative trajectories $\Gamma_x^{\epsilon}(-\infty,0)$ and $\Gamma_x^{s\tau}(-\infty,0)$. We have

$$width_{G_{s\tau}}(L_{iX}^{\epsilon}(x)) \geq \upsilon(x) \geq D(x,\tau) \quad \forall x \in [0,\delta)I.$$

Holonomy arguments, analogous to those in the proof of prop. 4.5, imply that there exists $J \in \mathbb{R}^+$ such that $width_{G_{s\tau}}^{\min} \geq JD(x,\tau)$ for any $x \in [0,\delta)I$. Moreover there exists $\zeta \in \mathbb{R}^+$ such that $\aleph^*(P) \in e^{i[\zeta,\pi-\zeta]}$ for any $P \in ([0,\delta)I \times B(0,\epsilon)) \setminus SingX$. Thus we can deduce that $\exp(tX)(x,y)$ is well defined and belongs to H for all $(x,y) \in H_{s\tau}$ and $t \in B(0, J\sin(\zeta)D(x,\tau))$. □

5. Comparing $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2,0)$ and a convergent normal form

The goal of this section is constructing Fatou coordinates for $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2,0)$ and analyzing their asymptotic properties. The first step of such a project is choosing an element $X \in \mathcal{X}_{tp1}(\mathbb{C}^2,0)$ such that φ is tangent to $\exp(X)$ at a high enough order in the neighborhood of the fixed points. Given a region $H \in Reg(\epsilon, \aleph X, I)$ the space of orbits of $\exp(X)_{|H(x)}$ is biholomorphic to $\mathbb{C}^*$ for any $x \in (0,\delta)I$. Such a property also holds true for the orbit space of $\varphi_{|H(x)}$. A consequence is the possibility of building holomorphic Fatou coordinates and Lavaurs vector fields in the region H. This approach was introduced in Lavaurs thesis [7] and developped in several papers [18] [12] [9] [16].

One of the main properties proved in [16] is that the difference between Fatou coordinates of φ and $\exp(X)$ in any given region of a transversal flow is always bounded. We prove the analogue for multi-transversal flows. Such a property and the study of the intersection of regions associated to different choices of multi-transversal flows suffice to prove the exponentially small estimates required for multi-summability (see subsection 5.5). The boundness of the difference of Fatou coordinates of φ and $\exp(X)$ is proved in subsection 5.2. The main difficulty is that in order to capture the higher levels of summability the estimates have to be expressed in adapted coordinates (prop. 5.2). The shape of the regions and their intersections is studied in subsections 5.3 and 5.4. The regions associated to the same multi-transversal flow do not intersect and then a priori we can not compare Fatou coordinates defined in them. But we can acomplish that goal by extending Fatou coordinates to slightly bigger domains (subsection 5.6). Given the Lavaurs vector fields associated to φ we prove that they correspond to a multi-summable object by using a cohomological approach a la Ramis-Sibuya [14]. Anyway, in order to prove that the infinitesimal generator of φ is the asymptotic development of the Lavaurs vector fields we construct Fatou coordinates whose asymptotic development coincides with the power expansion of the infinitesimal generator up to an arbitrary order (subsection 5.7); the proof is completed by using the flatness arguments in subsection 5.5.

Remark 5.1. *The constructions in this paper can be generalized to elements* φ *of* $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. *Indeed there exists* $k \in \mathbb{N}$ *such that* $(x^{1/k},y)\circ\varphi\circ(x^k,y)$ *belongs to* $\mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$. *The results can be translated to* $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ *via a ramification in the parameter space. We chose* $\mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ *since the presentation is simpler.*

5.1. **Normal forms.** The infinitesimal generator $\log\varphi$ of $\varphi\in\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ is of the form $(y\circ\varphi-y)\hat{u}\partial/\partial y$ for some unit $\hat{u}\in\mathbb{C}[[x,y]]$ (see prop. (3.2) of [16]).

Definition 5.1. *Let* $\varphi=\exp((y\circ\varphi-y)\hat{u}\partial/\partial y)\in\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. *We say that* $\Upsilon\in\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ *is a* k*-convergent normal form of* φ *if* $\log\Upsilon=(y\circ\varphi-y)u\partial/\partial y$ *for some* $u\in\mathbb{C}\{x,y\}$ *and* $\hat{u}-u$ *belongs to the ideal* $(y\circ\varphi-y)^k$. *The last condition is equivalent to* $y\circ\varphi-y\circ\Upsilon\in(y\circ\varphi-y)^{k+1}$. *We say that* Υ *is a convergent normal form of* φ *if* Υ *is a* 1*-convergent normal form.*

Definition 5.2. *Let* $\varphi\in\mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$. *Consider a convergent normal form* Υ *of* φ. *We define* $Res(\varphi,P)=Res(\log\Upsilon,P)$ *for* $P\in Fix(\varphi)$ *(see def. 2.6). The definition does not depend on the choice of* Υ [15].

Proposition 5.1. [15] *Let* $\varphi\in\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ *and* $k\in\mathbb{N}$. *Then there exists a* k*-convergent normal form* Υ.

Let $\varphi\in\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. Denote $f=y\circ\varphi-y$. Fix a k-convergent normal form Υ of φ. Consider an integral of the time form ψ of $\log\Upsilon$. We define (see section 7.1 of [16])

$$\Delta_\varphi=\psi\circ\varphi-\psi\circ\Upsilon=\psi\circ\varphi-(\psi+1).$$

The definition depends on Υ but it does not depend on ψ. A priori the function Δ_φ is defined only outside of $Fix(\varphi)$. Nevertheless, by Taylor's formula we have

$$\Delta_\varphi\sim\left(\frac{\partial\psi}{\partial y}\circ\Upsilon\right)(y\circ\varphi-y\circ\Upsilon)=O\left(\left(\frac{1}{f}\circ\Upsilon\right)f^{k+1}\right)=O(f^k).$$

We obtain

Lemma 5.1. *Let* $\varphi\in\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ *with fixed* k*-convergent normal form. Then* Δ_φ *belongs to the ideal* $(y\circ\varphi-y)^k$ *of the ring* $\mathbb{C}\{x,y\}$.

Next we estimate Δ_φ in terms of the Fatou coordinates of regions. We translate $\Delta_\varphi=O(f^k)$ to an estimate in adapted coordinates by using the asymptotical good behavior of regions.

Proposition 5.2. *Let* $\varphi\in\mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$. *Let* $\Upsilon=\exp(X)$ *be a* k*-convergent normal form. Consider a multi-transversal flow* $\Re(\aleph X)$ *for some stable multi-direction* $\aleph: I\to(e^{i(0,\pi)})^{\tilde{q}}$. *Take* $H\in Reg^*(\epsilon,\aleph X,I)$ *and* $L\in\mathcal{P}(H)$. *Then there exists* $K\in\mathbb{R}^+$ *such that*

$$|\Delta_\varphi(x,y)|\leq\frac{K}{(1+|\psi^X_{H,L}(x,y)|)^k}\quad\forall(x,y)\in H^L. \tag{4}$$

The constant K *depends only on* Υ *but does not depend on* I, $\aleph$, H *or* L.

Proof. Denote $f=X(y)$. Let us prove the result for a L-subregion J. Lemma 4.4 implies that there exists a sequence B_0, …, $B_k=J$ of L-subregions of H such that $\beta(0)=0$ and

- $B_{2j} \subset \mathcal{E}_{\beta(2j)}$ for any $0 \leq 2j \leq k$.
- $B_{2j+1} \subset \mathcal{C}_{\beta(2j)}$ for any $0 \leq 2j+1 \leq k$.
- $\beta(2j+2) = (\beta(2j), \kappa(j))$ for some $\kappa(j) \in \mathbb{C}$ for any $0 \leq 2j+2 \leq k$.

We denote $E_{2j} = \mathcal{E}_{\beta(2j)}$ and $E_{2j+1} = \mathcal{C}_{\beta(2j)}$. We define $\partial_e B_0 = \partial_e \mathcal{E}_0 \cap \overline{B_0}$ and $\partial_e B_j = \overline{B_j} \cap \partial_I E_{j-1}$ for $j \geq 1$. As in the proof of proposition 4.3 we obtain that $\partial H \cap \partial_e B_j$ is the union of a finite number of asymptotically continuous sections $\tau_1, \ldots, \tau_b : [0,\delta) \times I \to \partial H \cap \partial_e B_j$. Consider adapted coordinates (x,t) for E_j. Corollary 4.2 implies that $|r|^{e(E_j)} \psi_{H,L}^X(r,\lambda,t)$ is continuous in $\overline{B_j} \setminus SingX$ and satisfies $|r|^{e(E_j)} |\psi_{H,L}^X| \leq A_j$ in $\partial_e B_j$ for some $A_j > 0$. Denote $\psi_j = x^{e(E_j)} \psi_{H,L}^X$.

Let $T = \{(x,t) \in B(0,\delta) \times \overline{B(0,\eta)}\}$ be the seed associated to E_j. The construction of the splitting implies that $f \in (x^{e(E_j)+[(j+1)/2]})$ in E_j for any $0 \leq j \leq k$ (let us remark that $[(j+1)/2]$ is the integer part of $(j+1)/2$).

Suppose that $E_j = \{\eta \geq |t| \geq \rho|x|\}$ is a parabolic exterior set, we obtain

$$f = O\left(\frac{x^{e(E_j)+[(j+1)/2]}}{(1+|\psi_j|)^{1+1/\nu(E_j)}}\right) = O\left(\frac{x^{e(E_j)+[(j+1)/2]}}{(1+|\psi_j|)^{1+1/\nu(\mathcal{E}_0)}}\right) \tag{5}$$

by prop. 4.3 since $\nu(E_j) \leq \nu(\mathcal{E}_0)$. We obtain

$$f = O\left(\frac{x^{e(E_j)+[(j+1)/2]-e(E_j)(1+1/\nu(\mathcal{E}_0))}}{(1+|\psi_{H,L}^X|)^{1+1/\nu(\mathcal{E}_0)}}\right) = O\left(\frac{1}{(1+|\psi_{H,L}^X|)^{1+1/\nu(\mathcal{E}_0)}}\right) \tag{6}$$

in B_j since $e(E_j) \leq [(j+1)/2]\nu(\mathcal{E}_0)$ by construction.

Suppose that $E_j = \{|t| \leq \eta\}$ is a non-parabolic exterior set, this implies $j = k$. We have

$$X = x^{e(E_k)} X_{E_k} = x^{e(E_k)} v(x,t)(t - \gamma(x))\partial/\partial t$$

where v is a holomorphic function never vanishing in E_k. The seed E_k is the son of a seed T_β. We obtain $\mathcal{C}_\beta = \mathcal{C}_l$ for any $1 \leq l \leq q$. We have $X_{\mathcal{C}_\beta} = (x, w-\zeta)^* X_{E_k}$ for some $\zeta \in \mathbb{C} \cap S_\beta$ by construction of the dynamical splitting. This implies

$$v(0,0)^{-1} = Res(X_{E_k}, (0,0)) = Res(X_{\mathcal{C}_\beta}, (0,\zeta)) = Res(X_l(1), \zeta).$$

Since $(\lambda, \aleph_{E_k}(\lambda)) \not\in \mathcal{U}_X^l$ (see section 4) for any $\lambda \in I$ we deduce that

$$\aleph_{E_k}(\lambda)\lambda^{e(E_k)} X_l(1) = \aleph_{E_k}(\lambda) X_l(\lambda) \in \mathcal{X}_\infty(\mathbb{C},0)$$

for any $\lambda \in I$. The definition of $\mathcal{X}_\infty(\mathbb{C},0)$ implies

$$\lambda^{-e(E_k)} \aleph_{E_k}(\lambda)^{-1} v(0,0)^{-1} = Res(\aleph_{E_k}(\lambda) X_l(\lambda), \zeta) \not\in i\mathbb{R}$$

for any $\lambda \in I$. The function $\lambda^{-e(E_k)} \aleph_{E_k}(\lambda)^{-1} \psi_k(r,\lambda,t)$ is an integral of the time form of $\Re(\lambda^{e(E_k)}(\aleph X)_{E_k})$. We define $D(r,\lambda) = \lambda^{-e(E_k)} \aleph_{E_k}(\lambda)^{-1} v(0,0)^{-1}$ and

$$F(r,\lambda,t) = \lambda^{-e(E_k)} \aleph_{E_k}(\lambda)^{-1} [\psi_k(r,\lambda,t) - v(r\lambda, \gamma(r\lambda))^{-1} \ln(t - \gamma(r\lambda))].$$

The function $\partial F/\partial t$ satisfies

$$\frac{\partial F}{\partial t}(r,\lambda,t) = \frac{1}{\lambda^{e(E_k)} \aleph_{E_k}(\lambda)} \left(\frac{1}{v(r\lambda,t)(t-\gamma(r\lambda))} - \frac{1}{v(r\lambda,\gamma(r\lambda))(t-\gamma(r\lambda))}\right).$$

It is bounded in B_k and then so is F. There exists $\upsilon > 0$ such that

$$\arg(D(r,\lambda)) \in (-\pi/2 + \upsilon, \pi/2 - \upsilon)$$

for any $(r, \lambda) \in [0, \delta) \times I$ if B_k is a basin of repulsion, otherwise we have that $\arg(D(r,\lambda)) \in (\pi/2 + \upsilon, 3\pi/2 - \upsilon)$ for any $(r, \lambda) \in [0, \delta) \times I$. We deduce that

$$f = O(x^{e(E_k)+[(k+1)/2]}(t - \gamma(x))) = O(x^{e(E_k)+[(k+1)/2]}e^{-A|\psi_k|})$$

in B_k for some $A > 0$. This implies equation (5) and then equation (6).

Finally suppose that E_j is a compact-like set. The asymptotically continuous character of $\partial H \cap \partial_e E_j$ implies that E_j is compact in the coordinates (r, λ, w) associated to E_j. We have that ψ_j is bounded in B_j. Hence $f = O(x^{e(E_j)+[(j+1)/2]})$ implies equation (5) and then equation (6).

We proved that there exists $K' \in \mathbb{R}^+$ such that

$$(7) \qquad |\Delta_\varphi(x, y)| \leq \frac{K'}{(1 + |\psi^X_{H,L}(x, y)|)^{k+k/\nu(\mathcal{E}_0)}} \quad \forall (x, y) \in H^L.$$

Proposition 4.4 implies

$$|\Delta_\varphi(x, y)| \leq \frac{K}{(1 + |\psi^X_{H,L}(x, y)|)^{k+k/\nu(\mathcal{E}_0)}} \quad \forall (x, y) \in H^L \cap \mathcal{E}_0$$

where K depends only on Υ. Since we have

$$\lim_{\delta' \to 0} \left(\inf_{x \in [0,\delta')I, \ (x,y) \in H^L \setminus \mathcal{E}_0} |\psi^X_{H,L}(x, y)| \right) = \infty$$

then equation (4) holds true for any $(x, y) \in H^L$ with x close to 0. □

Both in subsections 5.6 and 5.7 will be necessary to extend Fatou coordinates ψ^φ of φ by using the equation $\psi^\varphi \circ \varphi = \psi^\varphi + 1$. The next lemma assures that the asymptotic properties of Fatou coordinates are preserved when extending ψ^φ along long orbits.

Definition 5.3. *Let $\theta \in (0, \pi/2)$ and $M \in \mathbb{R}^+ \cup \{0\}$. We define the set $W_{\theta,M} \subset \mathbb{C}$ given by*

$$W_{\theta,M} = \{z \in \mathbb{C} : Re(z) > 0\} \cup \{z \in \mathbb{C} : |Im(z)| + \tan(\theta)Re(z) - M > 0\}.$$

We define $W_{\pi/2,M} = \{z \in \mathbb{C} : Re(z) > 0\}$.

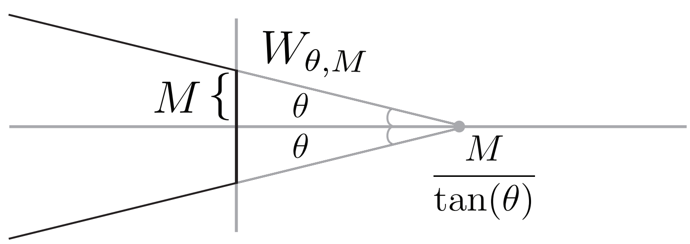


FIGURE 5. Picture of $W_{\theta,M}$

Lemma 5.2. *Let $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$. Let $\theta \in (0, \pi/2]$, $M \geq 0$ and $k \geq 2$. Let Υ be a k-convergent normal form. Consider a orbit $\vartheta = \{P, \varphi(P), \ldots, \varphi^j(P)\}$ and a function ψ defined in ϑ such that $\psi(\varphi(Q)) = \psi + 1 + \Delta_\varphi(Q)$ for any $Q \in \vartheta$. Suppose* *that $\psi(P) \in W_{\theta,M}$, $|\Delta_\varphi(Q)| \leq 1/2$ for any $Q \in \vartheta$ and $|\Delta_\varphi(Q)| \leq \sin(\theta)/2$ for any $Q \in \vartheta$ such that $|Im(\psi(Q))| \geq M$. Consider a function $\Pi : \vartheta \to \mathbb{C}$ such that*

$$|\Pi(Q)| \leq \frac{D}{(1+|\psi(Q)|)^k} \quad \forall Q \in \vartheta.$$

Then, we have

$$\sum_{l=0}^{j} |\Pi(\varphi^l(P))| \leq \frac{4^k \sqrt{2}^k Dk}{c^k(k-1)} \frac{1}{(1+|\psi(P)|)^{k-1}}$$

where $c^2 = (1-\cos(\theta))/2$.

Proof. We claim that $\psi(\vartheta)$ is contained in $W_{\theta,M}$. We denote $z = z_1 + iz_2$. The set $W_{\theta,M}$ is a union of the sets $W_{\pi/2,M}$,

$$E_1 = \{z_2 + (\tan(\theta))z_1 > M\} \text{ and } E_2 = \{-z_2 + (\tan(\theta))z_1 > M\}.$$

The distance from $Re(z) = 0$ to $Re(z) = 1$ is 1. Thus $|(\Delta_\varphi)|_\vartheta| \leq 1/2$ implies that a point $Q \in \vartheta$ such that $\psi(Q) \in W_{\pi/2,M}$ satisfies $\psi(\varphi(Q)) \in W_{\pi/2,M}$. A point $Q \in \vartheta$ such that $\psi(Q) \in E_1 \setminus W_{\pi/2,M}$ satisfies $|Im(\psi(Q))| \geq M$. Since the distance between the lines ∂E_1 and $1 + \partial E_1$ is $\sin(\theta)$ we obtain that $\psi(\varphi(Q)) \in E_1$. Analogously we prove $\psi(\varphi(Q)) \in E_2$ for $Q \in \vartheta$ such that $\psi(Q) \in E_2 \setminus W_{\pi/2,M}$. We deduce that $\psi(\vartheta)$ is contained in $W_{\theta,M}$.

Let us deal with the case of an orbit whose image by ψ does not intersect $W_{\pi/2,M}$. We denote $\tau_l = \psi(\varphi^l(P))$ for any $l \in \{0, \dots, j\}$. Take $l_1 \in \{0, \dots, j-1\}$ such that $\{\tau_0, \dots, \tau_{l_1}\}$ is contained in $(E_1 \cup E_2) \setminus W_{\pi/2,M}$. Given $\tau \in W_{\theta,0}$ and $l \in \mathbb{N} \cup \{0\}$ we have $|\tau + l| \geq \sin(\theta) l$. We obtain

$$(8) \qquad |\tau_{l_2}| \geq |\tau_0 + l_2| - \sum_{a=0}^{l_2-1} |\Delta_\varphi(\varphi^a(P))| \geq |\tau_0 + l_2| - \frac{l_2}{2}\sin(\theta) \geq \frac{|\tau_0 + l_2|}{2}$$

for any $l_2 \in \{0, \dots, l_1 + 1\}$.

Now we consider the case $\psi(\vartheta) \cap W_{\pi/2,M} \neq \emptyset$. We define

$$l_0 = \min\{l \in \{0, \dots, j\} : \tau_l \in W_{\pi/2,M}\}.$$

We have $\tau_l \in W_{\pi/2,M}$ for any $l_0 \leq l \leq j$. Fix $l_0 \leq l \leq j$. We obtain

$$\tau_l - (\tau_{l_0} + (l - l_0)) = \sum_{a=l_0}^{l-1} |\Delta_\varphi(\varphi^a(P))| \implies |\tau_l - (\tau_{l_0} + (l - l_0))| \leq \frac{l - l_0}{2}.$$

The property $|\tau_{l_0} + (l - l_0)| \geq l - l_0$ is a consequence of $Re(\tau_{l_0}) > 0$. It implies

$$|\tau_l| \geq \frac{|\tau_{l_0} + (l - l_0)|}{2} \geq \frac{\sqrt{|\tau_{l_0}|^2 + (l - l_0)^2}}{2} \geq \frac{|\tau_{l_0}| + (l - l_0)}{2\sqrt{2}}.$$

We put $l_2 = l_0$ in equation (8) and simplify the previous inequality to get

$$(9) \qquad |\tau_l| \geq \frac{|\tau_0 + l_0| + (l - l_0)}{4\sqrt{2}} \geq \frac{|\tau_0 + l|}{4\sqrt{2}}.$$

Equations (8) and (9) imply that $|\tau_l| \geq |\tau_0 + l|/(4\sqrt{2})$ for any $l \in \{0, \dots, j\}$.

We claim that $|\tau + l| \geq c(|\tau| + l)$ for all $\tau \in W_{\theta,0}$ and $l \in \mathbb{N} \cup \{0\}$. It suffices to prove that

$$|\tau|^2 + l^2 + 2|\tau| l \cos(\theta_0) = |\tau + l|^2 \geq c^2(|\tau| + l)^2$$

where θ_0 is the angle enclosed by the vectors ψ and l. Since $\theta_0 \leq \pi - \theta$ we deduce that $\cos(\theta_0) \geq -\cos(\theta)$. It suffices to prove

$$|\tau|^2 + l^2 - 2|\tau| l \cos(\theta) \geq c^2(|\tau| + l)^2 \Leftrightarrow (|\tau|^2 + l^2)(1 - c^2) \geq 2|\tau| l(\cos(\theta) + c^2).$$

The last inequality can be deduced from $c^2 = (1 - \cos(\theta))/2$ and $|\tau|^2 + l^2 \geq 2|\tau| l$. The previous properties imply

$$|\Pi(\varphi^l(P))| \leq \frac{D}{(1 + |\tau_l|)^k} \leq \frac{4^k \sqrt{2}^k D}{(1 + |\tau_0 + l|)^k} \leq \frac{4^k \sqrt{2}^k D}{c^k} \frac{1}{(|\tau_0| + l + 1)^k}.$$

We estimate the right hand side to obtain

$$\sum_{l=0}^{j} |\Pi(\varphi^l(P))| \leq \frac{4^k \sqrt{2}^k D}{c^k} \left(\frac{1}{(|\tau_0| + 1)^k} + \int_0^\infty \frac{1}{(|\tau_0| + 1 + t)^k} dt \right).$$

This leads us to

$$\sum_{l=0}^{j} |\Pi(\varphi^l(P))| \leq \frac{4^k \sqrt{2}^k D}{c^k} \left(\frac{1}{(|\tau_0| + 1)^k} + \frac{1}{k-1} \frac{1}{(|\tau_0| + 1)^{k-1}} \right)$$

and then to

$$\sum_{l=0}^{j} |\Pi(\varphi^l(P))| \leq \frac{4^k \sqrt{2}^k k D}{(k-1) c^k} \left(\frac{1}{(|\psi(P)| + 1)^{k-1}} \right)$$

as we intended to prove. □

5.2. **Defining Fatou coordinates.** The construction of Fatou coordinates of elements in $\mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ is based on building quasiconformal homeomorphisms σ in regions $H \in Reg(\epsilon, \aleph X, I)$ conjugating φ and one of its normal forms. The mapping σ induces a quasiconformal conjugation between the space of orbits of $\varphi_{|H(x)}$ and $\mathbb{C}^*$ for $x \in (0, \delta) I$. This conjugation can be turned into a holomorphic one by using the Ahlfors-Bers theorem. As a result, we obtain holomorphic Fatou coordinates.

Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$. Let $\Upsilon = \exp(X)$ be a 2-convergent normal form. Consider $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}}) \in \mathcal{M}$ and the dynamical splitting $\digamma_\Lambda$ in remark 4.11.

Definition 5.4. *Let $\lambda \in \mathbb{S}^1$. We define*

$$I_\Lambda^\lambda = \lambda e^{i[-\upsilon_\Lambda, \upsilon_\Lambda]} \text{ and } d_\Lambda^\lambda = \max\{j \in \{0, \ldots, \tilde{q}\} : \lambda \in I_j(\lambda_j, 0)\}$$

and $\aleph_{\Lambda,\lambda} = \aleph_{d_\Lambda^\lambda, \Lambda, \lambda}$ (see def. 4.11).

Given H in $Reg(\epsilon, \aleph_{\Lambda,\lambda} X, I_\Lambda^\lambda)$ we construct Fatou coordinates of φ in H. The construction is analogous to the one in [16] where detailed proofs can be found.

Let $L \in \mathcal{P}(H)$. Consider a point $P = (x_0, y_0) \in H^L$. Let γ be the trajectory of $\Re(\aleph_{\Lambda,\lambda} X)$ passing through P. The subset of H_L enclosed by γ and $\exp(X)(\gamma)$, i.e.

$$B_X(P) \stackrel{def}{=} \psi_L^X(x_0, y)^{-1}(\psi_L^X(\gamma) + [0, 1])$$

satisfies that $B_X(P) \setminus \exp(X)(\gamma)$ is a fundamental domain for $\exp(X)_{|H_L(x_0)}$.

We want to prove that the set $B_\varphi(P)$ enclosed by γ and $\varphi(\gamma)$ satisfies that $B_\varphi(P) \setminus \varphi(\gamma)$ is a fundamental domain for $\varphi_{|H_L(x_0)}$. The proof relies on showing that there exists a quasi-conformal homeomorphism σ defined in a neighborhood of $B_X(P)$ in $\{x_0\} \times B(0, \epsilon)$ conjugating $\exp(X)$, φ such that $\sigma(B_X(P)) = B_\varphi(P)$.

Step 1. Since $\Re(\aleph_{\Lambda,\lambda} X)$ is transversal to $\Re(X)$ then $\psi_L^X(\gamma) \cap \{Im(z) = b\}$ is a singleton whose element we denote $a(b) + ib$. The curve γ is parametrized by

$Im(\psi_L^X)$. Let $\varrho : \mathbb{R} \to [0,1]$ a C^∞ increasing function such that $\varrho(-\infty, 1/3] = \{0\}$ and $\varrho[2/3, \infty) = \{1\}$. We define

$$\sigma_0(z) = \sigma_0(z_1 + iz_2) = z + \varrho(z_1 - a(z_2))(\Delta_\varphi \circ (\psi_L^X(x_0, y))^{-1}(z-1))$$

and

$$\sigma(x_0, y) = (\psi_L^X(x_0, y))^{-1} \circ \sigma_0 \circ \psi_L^X(x_0, y).$$

Let us prove that σ is a q.c. diffeomorphism from a neighborhood of $B_X(P)$ onto a neighborhood of $B_\varphi(P)$ such that $\sigma \circ \exp(X) = \varphi \circ \sigma$ and $\sigma(B_X(P)) = B_\varphi(P)$.

Clearly σ is the identity in a neighborhood of γ and $\sigma = \varphi \circ \exp(X)^{-1}(x_0, y)$ in a neighborhood of $\exp(X)(\gamma)$. Thus we obtain $\sigma \circ \exp(X) = \varphi \circ \sigma$ in a neighborhood of γ. By construction σ is a C^∞ mapping.

Step 2. In order to prove that σ is a q.c. diffeomorphism we divide H^L in two parts, in one of them $\Re(\aleph_{\Lambda,\lambda} X)$ is "very transversal" to $\Re(X)$ whereas in the other one σ is very close to Id. We will use different estimates in both kind of sets in order to analyze the properties of σ.

Denote $\aleph = \aleph_{\Lambda,\lambda}$. Consider $\mathcal{E}_0 = \{(x,y) \in B(0,\delta) \times B(0,\epsilon) : |y| \geq \eta_0 |x|\}$. Next, we prove that ψ_L^X is big outside $\mathcal{E}_0' = \{(x,y) \in B(0,\delta) \times B(0,\epsilon) : |y| \geq 2\eta_0 |x|\}$. By prop. 4.4 we obtain

$$|\psi_L^X(x,y)| \geq \frac{1}{C_1 (2\eta_0)^{\nu(\mathcal{E}_0)} |x|^{\nu(\mathcal{E}_0)}} \tag{10}$$

for any $(x,y) \in H^L$ such that $|y| = 2\eta_0 |x|$. Denote $\Gamma_x = \Gamma(\aleph X, L_{iX}^\epsilon(x), \mathcal{E}_0')$ and $\mathcal{I}(\Gamma_x) = [h_1(x), h_2(x)]$. Since $\Gamma_x(h_j(x)) \in \{|y| = 2\eta_0|x|\}$ and

$$\psi_L^X(\Gamma_x(h_j(x))) - \psi_L^X(L_{iX}^\epsilon(x)) \in i\mathbb{R}$$

for all $x \in (0,\delta) I_\Lambda^\lambda$ and $j \in \{1,2\}$ then we obtain

$$|Im(\psi_L^X(\Gamma_x(h_j(x))))| \geq \frac{1}{2C_1 (2\eta_0)^{\nu(\mathcal{E}_0)} |x|^{\nu(\mathcal{E}_0)}} \tag{11}$$

for all $x \in (0,\delta) I_\Lambda^\lambda$ and $j \in \{1,2\}$ by considering a smaller $\delta > 0$ if necessary. Let $\tilde{\upsilon}_x$ be the open arc in $\partial_I \mathcal{E}_0'$ contained in H and such that $\partial \tilde{\upsilon}_x = \{\Gamma_x(h_1(x)), \Gamma_x(h_2(x))\}$. Suppose that $\Re(X)$ points towards H at $L_{iX}^\epsilon(0)$ without lack of generality. Denote υ_x the curve

$$\Gamma_x(-\infty, h_1(x)] \cup \upsilon_x \cup \Gamma_x[h_2(x), \infty).$$

Given $(x,y) \in H_L \setminus \mathcal{E}_0'$ the point $\psi_L^X(x,y)$ is to the right of the curve $\psi_L^X(\upsilon_x)$. Equations (10) and (11) imply that the ball $B(0, C_2/|x|^{\nu(\mathcal{E}_0)})$ does not intersect $\psi_L^X(\upsilon_x)$ where $C_2 = 1/(2C_1(2\eta_0)^{\nu(\mathcal{E}_0)})$. Thus $B(0, C_2/|x|^{\nu(\mathcal{E}_0)})$ is to the left of $\psi_L^X(\upsilon_x)$. We deduce that

$$|\psi_L^X(x,y)| \geq \frac{C_2}{|x|^{\nu(\mathcal{E}_0)}} \quad \forall (x,y) \in H_L \setminus \mathcal{E}_0'.$$

We obtain $\aleph_{\Lambda,\lambda}^*(x,y) = i$ for any $(x,y) \in H^L$ such that $|\psi_L^X(x,y)| < C_2/|x|^{\nu(\mathcal{E}_0)}$.

Step 3.

Proposition 5.3. *The mapping σ is a diffeomorphism between neighborhoods of $B_X(P)$ and $B_\varphi(P)$ in $\{x_0\} \times \mathbb{C}$.*

Proof. Denote

$$\Delta_0 = \Delta_\varphi \circ (\psi_L^X(x_0,y))^{-1}(z-1) \quad \text{and} \quad \mathcal{J}(h) = \begin{pmatrix} \frac{\partial Re(h)}{\partial z_1} & \frac{\partial Re(h)}{\partial z_2} \\ \frac{\partial Im(h)}{\partial z_1} & \frac{\partial Im(h)}{\partial z_2} \end{pmatrix}.$$

We have that $(\mathcal{J}(\sigma_0))(z) - Id - \varrho(z_1 - a(z_2))(\mathcal{J}(\Delta_0))(z)$ is equal to

$$\begin{pmatrix} Re(\Delta_0)(z)\frac{\partial \varrho}{\partial t}(z_1 - a(z_2)) & -Re(\Delta_0)(z)\frac{\partial \varrho}{\partial t}(z_1 - a(z_2))\frac{\partial a}{\partial t}(z_2) \\ Im(\Delta_0)(z)\frac{\partial \varrho}{\partial t}(z_1 - a(z_2)) & -Im(\Delta_0)(z)\frac{\partial \varrho}{\partial t}(z_1 - a(z_2))\frac{\partial a}{\partial t}(z_2) \end{pmatrix}.$$

By lemma 5.1 we have that $\Delta_\varphi(0,0) = 0$. By using Cauchy's integral formula

$$\frac{\partial \Delta_0}{\partial z}(z) = \frac{1}{2\pi i}\int_{|w|=1} \frac{\Delta_0(w)}{(w-z)^2}dw$$

we can suppose that $|\Delta_\varphi(Q)| < (\sup_{\mathbb{R}} |\partial\varrho/\partial t|)^{-1}/16$ and $|\partial\Delta_0/\partial z|(\psi_L^X(Q)) < 1/16$ for any Q in a neighborhood of $B_X(P)$ in $\{x_0\}\times B(0,\epsilon)$. Let us remark that we could need to consider a smaller domain of definition $B(0,\delta)\times B(0,\epsilon)$ but the domain with the good estimates only depend on φ and X (and not on P for example).

Denote $\tau(z) = \Delta_0(z)(\partial a/\partial t)(z_2)$. Let us estimate $\tau(z)$. There exist continuous functions $\theta_1, \ldots, \theta_r : I_\Lambda^\lambda \to (0,\pi)$ such that θ_j is decreasing on $\arg \lambda'$ and $\aleph_{\Lambda,\lambda}(\lambda') = (e^{i\theta_1}, \ldots, e^{i\theta_r})(\lambda')$ for any $\lambda' \in I_\Lambda^\lambda$. We define

$$\theta' = \min_{1\le j\le r,\ \lambda' \in I_\Lambda^\lambda} \min(\pi - \theta_j(\lambda'), \theta_j(\lambda')).$$

We have $\theta' > 0$. The tangent vector to $\psi_L^X(\gamma)$ at any point $\psi_L^X(Q)$ for Q in γ belongs to $\mathbb{R}^+ e^{i[\theta',\pi-\theta']}$. We deduce that $|\partial a/\partial t| \le 1/\tan(\theta')$. The equation (11) implies that $a(t)$ is constant in $(-C_2/|x_0|^{\nu(\mathcal{E}_0)}, C_2/|x_0|^{\nu(\mathcal{E}_0)})$. We obtain that $\tau(z) = 0$ if $|z_2| < C_2/|x_0|^{\nu(\mathcal{E}_0)}$. Suppose now that $|z_2| \ge C_2/|x_0|^{\nu(\mathcal{E}_0)}$. We obtain

$$|\Delta_0(z)| \le \frac{K_1}{(1+|z|)^2} \le \frac{K_1|x_0|^{2\nu(\mathcal{E}_0)}}{C_2^2} < \frac{\tan(\theta')}{16\sup_{\mathbb{R}}|\partial\varrho/\partial t|} \tag{12}$$

by considering $\delta > 0$ small enough. We obtain $|\tau(z)| < (\sup_{\mathbb{R}} |\partial\varrho/\partial t|)^{-1}/16$. The coefficients of the matrix $(\mathcal{J}(\sigma_0))(z) - Id$ belong to $(-1/8, 1/8)$ for any z in a neighborhood of $\psi_L^X(B_X(P))$. Since $||(\mathcal{J}(\sigma_0))(z) - Id|| < 1/4$ for the spectral norm then σ_0 is a diffeomorphism in the neighborhood of $\psi_L^X(B_X(P))$. We deduce that σ is a diffeomorphism in the neighborhood of $B_X(P)$. □

Remark 5.2. *Consider $Q \in \gamma$. Denote $z' = \psi_L^X(Q)+1$. Let $e^{i\theta_1}$ be the unit tangent vector to $\psi_L^X(\gamma)$ at $\psi_L^X(Q)$. The vector $(\mathcal{J}(\sigma_0)(z'))(e^{i\theta_1})$ is tangent to $\psi_L^X(\varphi(\gamma))$ at $\psi_L^X(\varphi(Q))$. We claim that*

$$(\mathcal{J}(\sigma_0)(z'))(e^{i\theta_1}) \in \mathbb{R}^+ e^{i[\theta'/2,\pi-\theta'/2]} \cup \mathbb{R}^+ i e^{i[-\theta_0,\theta_0]}$$

where $\theta_0 = \arctan(1/4)$. Indeed if $|\psi_L^X(Q)| < C_2/|x_0|^{\nu(\mathcal{E}_0)}$ we have that $\theta_1 = \pi/2$ and $|(\mathcal{J}(\sigma_0)(z'))(i) - i| \le 1/4$ imply $(\mathcal{J}(\sigma_0)(z'))(i) \in \mathbb{R}^+ i e^{i[-\theta_0,\theta_0]}$. Otherwise $|\mathcal{J}(\sigma_0) - Id| = O(|x|^{\nu(\mathcal{E}_0)})$ and we obtain $(\mathcal{J}(\sigma_0)(z'))(e^{\theta_1}) \in \mathbb{R}^+ e^{i[\theta'/2,\pi-\theta'/2]}$. Anyway $\Re(X)$ is transversal to $\varphi(\gamma)$ at Q' for any $Q' \in \varphi(\gamma)$.

Step 4. Analogously we can prove

Proposition 5.4. *The mapping σ is quasiconformal.*

Proof. We have

$$\chi_{\sigma_0} = \frac{\frac{\partial \sigma_0}{\partial \bar{z}}}{\frac{\partial \sigma_0}{\partial z}}(z) = \frac{\frac{\partial \varrho(z_1 - a(z_2))}{\partial \bar{z}} \Delta_0(z)}{1 + \varrho(z_1 - a(z_2)) \frac{\partial \Delta_0}{\partial z}(z) + \Delta_0(z) \frac{\partial \varrho(z_1 - a(z_2))}{\partial z}} \tag{13}$$

and

$$\frac{\partial \varrho(z_1 - a(z_2))}{\partial z_1} = \frac{\partial \varrho}{\partial t}(z_1 - a(z_2)), \ \frac{\partial \varrho(z_1 - a(z_2))}{\partial z_2} = -\frac{\partial \varrho}{\partial t}(z_1 - a(z_2)) \frac{\partial a}{\partial t}(z_2).$$

We obtain

$$|\chi_{\sigma_0}| = \left| \frac{\frac{\partial \sigma_0}{\partial \bar{z}}}{\frac{\partial \sigma_0}{\partial z}}(z) \right| \leq \frac{\frac{\sqrt{2}}{2} \frac{1}{16}}{1 - \frac{1}{16} - \frac{\sqrt{2}}{2} \frac{1}{16}} < \frac{1}{14}$$

in the neighborhood of $\psi_L^X(B_X(P))$. Therefore σ is a 15/13 q.c. mapping in a neighborhood of $B_X(P)$. □

Step 5. We prove now that the domain enclosed by γ and $\varphi(\gamma)$ is a fundamental domain for $\varphi_{|H_L(x_0) \cup \varphi(H_L(x_0))}$.

The vector field $\Re(X)$ is transversal to $\varphi(\gamma)$ at Q' for any $Q' \in \varphi(\gamma)$. Hence given $z_2 \in \mathbb{R}$ there exists a unique point $c(z_2) + iz_2$ in $\psi_L^X(\varphi(\gamma)) \cap \{Im(z) = z_2\}$.

We define (see Step 1 for the definition of the function a)

$$B_\varphi(P) = \{Q \in H_L(x_0) : Re(\psi_L^X(Q)) \in [a(Im(\psi_L^X(Q))), c(Im(\psi_L^X(Q)))]\}.$$

Let us prove that $B_\varphi(P) \backslash \varphi(\gamma)$ is a fundamental domain for $\varphi_{|H_L(x_0) \cup \varphi(H_L(x_0))}$. Step 3 implies that $B_\varphi(P) \setminus \varphi(\gamma)$ is a fundamental domain for φ in the neighborhood of $B_\varphi(P)$ and that orbits of $\varphi_{|H_L(x_0) \cup \varphi(H_L(x_0))}$ intersects $B_\varphi(P) \setminus \varphi(\gamma)$ at most once. It suffices to prove that every orbit of a point Q in $H_L(x_0) \cup \varphi(H_L(x_0))$ intersects $B_\varphi(P)$.

Let $Q \in H_L(x_0) \cup \varphi(H_L(x_0))$. Denote $z_1 + iz_2 = \psi_L^X(Q)$. If $z_1 \in [a(z_2), c(z_2)]$ then $Q \in B_\varphi(P)$ and there is nothing to prove. Suppose without lack of generality that $z_1 < a(z_2)$. We define

$$A = \{Q \in H_L(x_0) : Re(\psi_L^X(Q)) < a(Im(\psi_L^X(Q)))\}.$$

It suffices to prove that there exists $j \in \mathbb{N}$ such that $Q, \ldots, \varphi^{j-1}(Q) \in A$ and $\varphi^j(Q) \notin A$ since then $\varphi^j(Q)$ belongs to $B_\varphi(P)$. Let us argue by contradiction, we suppose that $\varphi^j(Q) \in A$ for any $j \in \mathbb{N}$. Denote $\psi_L^X(\varphi^j(Q)) = s_j^1 + is_j^2$. We have $|\Delta_\varphi(\varphi^j(P))| \leq 1/16$ and then $s_{j+1}^1 - s_j^1 \geq 15/16$ for any $j \geq 0$. Since $|\partial a/\partial t| \leq 1/\tan(\theta')$ then $Re(\psi_L^X)$ is bounded by above in

$$\{(x_0, y) \in A : |Im(\psi_L^X)|(x_0, y) < M\}$$

for any $M > 0$. Therefore we obtain $\lim_{j \to \infty} |Im(\psi_L^X)|(\varphi^j(Q)) = \infty$. In particular the limit $\lim_{j \to \infty} \varphi^j(Q)$ exists and is equal to a point $Z \in SingX$. We deduce that $\lim_{j \to \infty} \Delta_\varphi(\varphi^j(Q)) = 0$. We have

$$|s_{j+1}^2 - s_j^2| \leq |\Delta_\varphi(\varphi^j(Q))| \Rightarrow |a(s_{j+1}^2) - a(s_j^2)| \leq \frac{|\Delta_\varphi(\varphi^j(Q))|}{\tan(\theta')}$$

and then

$$(s_{j+1}^1 - a(s_{j+1}^2)) - (s_j^1 - a(s_j^2)) \geq \frac{15}{16} - \frac{|\Delta_\varphi(\varphi^j(Q))|}{\tan(\theta')} > 1/2$$

for any $j \in \mathbb{N}$ big enough. This is impossible since $\varphi^j(P) \in A$ implies $s_j^1 - a(s_j^2) < 0$ for any $j \in \mathbb{N}$.

Step 6. We denote $B_\varphi^*(P)$ the space of orbits of $\varphi_{|H_L(x_0)}$. We construct a biholomorphism from $B_\varphi^*(P)$ to $\mathbb{C}^*$.

The mapping $\xi = e^{2\pi i z} \circ \psi_L^X \circ \sigma^{-1}$ is a diffeomorphism from $B_\varphi^*(P)$ onto $\mathbb{C}^*$. The mapping ξ depends on the point P. The function $\psi_L^X \circ \sigma^{-1}$ is a C^∞ Fatou coordinate of φ in $B_\varphi(P)$. Equations (4) and (7) imply

$$|\Delta_0(z)| \leq \frac{K_1}{(1+|z|)^2} \text{ and } |\Delta_0(z)| \leq \frac{K'}{(1+|z|)^{2+2/\nu(\mathcal{E}_0)}} \text{ in } H^L(x_0) \tag{14}$$

where K_1 depends only on X, φ and $K' > 0$ does not depend on P or x_0 but depends on $\aleph_{\Lambda,\lambda}$. Consider the notations in Step 3. Let $z = z_1 + iz_2 \in \psi_L^X(B_X(P))$. If $|z_2| < C_2/|x_0|^{\nu(\mathcal{E}_0)}$ we obtain $(\partial a/\partial t)(z_2) = 0$; the complex dilatation χ_{σ_0} of σ_0 satisfies

$$|\chi_{\sigma_0}|(z) = \left| \frac{\frac{\partial \sigma_0}{\partial \bar{z}}}{\frac{\partial \sigma_0}{\partial z}}(z) \right| \leq \frac{\frac{\sup_{\mathbb{R}} |\partial \varrho/\partial t| K_1}{2(1+|z|)^2}}{1 - \frac{1}{16} - \frac{\sqrt{2}}{2}\frac{1}{16}} = \frac{C_3 K_1}{(1+|z|)^2}.$$

Suppose that $|z_2| \geq C_2/|x_0|^{\nu(\mathcal{E}_0)}$, we have

$$|\chi_{\sigma_0}|(z) \leq \frac{\frac{1}{2}\left(\sup_{\mathbb{R}} |\partial \varrho/\partial t| + \frac{\sup_{\mathbb{R}} |\partial \varrho/\partial t|}{\tan(\theta')}\right) \frac{K'}{K_1} \frac{1}{(1+|z|)^{2/\nu(\mathcal{E}_0)}} \frac{K_1}{(1+|z|)^2}}{1 - \frac{1}{16} - \frac{\sqrt{2}}{2}\frac{1}{16}}.$$

Consider $x_0 \in B(0,\delta)$ and $\delta > 0$ small enough. Since

$$\frac{1}{(1+|z|)^{2/\nu(\mathcal{E}_0)}} \leq \frac{1}{|z_2|^{2/\nu(\mathcal{E}_0)}} \leq \frac{|x_0|^2}{C_2^{2/\nu(\mathcal{E}_0)}}$$

the previous calculations and Step 4 lead us to

$$|\chi_{\sigma_0}|(z) \leq \min\left(\frac{C_3 K_1}{(1+|z|)^2}, \frac{1}{14}\right) \quad \forall z \in \psi_L^X(B_X(P)).$$

Since ξ^{-1} is equal to $(\psi_L^X)^{-1} \circ \sigma_0 \circ ((1/2\pi i)\ln z)$ then

Lemma 5.3.

$$|\chi_{\xi^{-1}}|(z) \leq \min\left(\frac{C_3 K_1}{(1+2^{-1}\pi^{-1}|\ln z|)^2}, \frac{1}{14}\right) \tag{15}$$

for any $z \in e^{2\pi i w} \circ \psi_L^X(B_X(P)) = \mathbb{C}^*$.

There exists a quasi-conformal homeomorphism $\tilde{\rho} : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that $\chi_{\tilde{\rho}} = \chi_{\xi^{-1}}$. Since $||\chi_{\xi^{-1}}||_\infty = \sup_{z \in \mathbb{C}^*} |\chi_{\xi^{-1}}(z)| \leq 1/14$ it is a consequence of Ahlfors-Bers theorem. The choice of $\tilde{\rho}$ is unique if we require the normalizing conditions $\tilde{\rho}(0) = 0$, $\tilde{\rho}(1) = 1$ and $\tilde{\rho}(\infty) = \infty$. By construction $\tilde{\rho} \circ \xi$ is a biholomorphism from $B_\varphi^*(P)$ to $\mathbb{C}^*$.

Step 7. Let us construct a holomorphic Fatou coordinate of φ defined in a neighborhood of $B_\varphi(P)$ in $\{x_0\} \times B(0,\epsilon)$. The construction and its properties are analogous to those in section 7.2 of [16]. In both cases the key ingredient is the equation (15). Further details can be found in [16].

We define

$$J(r) = \frac{2}{\pi} \int_{|z|<r} \frac{C_3 K_1}{(1+2^{-1}\pi^{-1}|\ln |z||)^2} \frac{1}{|z|^2} d\sigma$$

for $r \in \mathbb{R}^+$. We have that $J(r) < \infty$ for any $r \in \mathbb{R}^+$.

Lemma 5.4. *The mapping $\tilde{\rho}$ is conformal at* 0 *and at* ∞*. Moreover we have*

$$\left|\frac{\tilde{\rho}(z)}{z} - \frac{\partial \tilde{\rho}}{\partial z}(0)\right| \leq \left|\frac{\partial \tilde{\rho}}{\partial z}(0)\right| \jmath(|z|) \text{ and } \left|\frac{z}{\tilde{\rho}(z)} - \frac{\partial \tilde{\rho}}{\partial z}(\infty)^{-1}\right| \leq \left|\frac{\partial \tilde{\rho}}{\partial z}(\infty)\right|^{-1} \jmath(1/|z|)$$

where $\jmath$ is a function depending only on φ, X; it satisfies $\lim_{|z| \to 0} \jmath(|z|) = 0$*. We have*

$$min_{|z|=1} |\tilde{\rho}(z)| e^{-J(1)} \leq |\partial \tilde{\rho}/\partial z|(0), |\partial \tilde{\rho}/\partial z|(\infty) \leq max_{|z|=1} |\tilde{\rho}(z)| e^{J(1)}.$$

Since $\tilde{\rho}$ is conformal at 0 we can define $\rho = \tilde{\rho}/(\partial \tilde{\rho}/\partial z)(0)$. The q.c. mapping $\rho : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is the unique solution of $\chi_\rho = \chi_{\xi^{-1}}$ such that $\rho(0) = 0$, $\rho(\infty) = \infty$ and $(\partial \rho/\partial z)(0) = 1$. We define the function

$$\psi_{H,L,P}^{\varphi} = \frac{1}{2\pi i} \ln z \circ \rho \circ e^{2\pi i z} \circ \psi_{H,L}^{X} \circ \sigma^{-1}. \tag{16}$$

It is a holomorphic Fatou coordinate of φ defined in the neighborhood of $B_\varphi(P)$ in $\{x_0\} \times \mathbb{C}$. The set $B_\varphi(P)$ contains a fundamental domain of $\varphi_{|H_L(x_0) \cup \varphi(H_L(x_0))}$ by Step 5. Thus we can extend $\psi_{H,L,P}^{\varphi}$ to $H_L(x_0)$ by using $\psi_{H,L,P}^{\varphi} \circ \varphi \equiv \psi_{H,L,P}^{\varphi} + 1$. Consider the points $Z_\pm \in SingX$ such that $Z_\pm = \lim_{Q \in H_L(x_0),\ Im(\psi_L^X(Q)) \to \pm\infty} Q$. By construction we have

$$|\psi_L^X \circ \sigma - \psi_L^X|(x_0, y) \leq \frac{K_1}{(1 + |\psi_L^X(x_0, y)|)^2}$$

for any (x_0, y) in a neighborhood of $B_X(P)$. We deduce that

$$\lim_{Q \in B_\varphi(P),\ |Im(\psi_L^X(Q))| \to \infty} |\psi_L^X \circ \sigma^{-1} - \psi_L^X|(Q) = 0.$$

The mapping ρ is conformal at 0 and ∞, hence there exist $\kappa_+, \kappa_- \in \mathbb{C}$ such that

$$\lim_{Q \in B_\varphi(P),\ Im(\psi_L^X(Q)) \to \pm\infty} |\psi_{H,L,P}^{\varphi} - \psi_L^X|(Q) = \kappa_\pm.$$

Moreover $(\partial \rho/\partial z)(0) = 1$ implies $\kappa_+ = 0$. Since $\Delta_\varphi(Z_+) = \Delta_\varphi(Z_-) = 0$ we obtain

$$\lim_{Q \in B_\varphi(P'),\ Im(\psi_L^X(Q)) \to \pm\infty} |\psi_{H,L,P}^{\varphi} - \psi_L^X|(Q) = \kappa_\pm$$

for any $P' \in H^L(x_0)$. The function $(\psi_{H,L,P}^{\varphi} - \psi_{H,L,P'}^{\varphi}) \circ (e^{2\pi i w} \circ \psi_{H,L,P}^{\varphi})^{-1}(z)$ is a bounded holomorphic function defined in $\mathbb{C}^* = e^{2\pi i w} \circ \psi_{H,L,P}^{\varphi}(B_\varphi(P'))$ and then constant. Since its value at $z = 0$ is 0 we deduce that $\psi_{H,L,P}^{\varphi} \equiv \psi_{H,L,P'}^{\varphi}$ for all $P, P' \in H^L(x_0)$.

Definition 5.5. *We denote $\psi_{H,L}^{\varphi}$ any of the functions $\psi_{H,L,P}^{\varphi}$ defined in $H_L(x_0)$. We denote $\psi_L^{\varphi} = \psi_{H,L}^{\varphi}$ if the choice of the domain of definition and multi-transversal flow is implicit.*

Consider $H \in Reg_2(\epsilon, \aleph_{\Lambda,\lambda} X, I_\Lambda^\lambda)$. We denote $\mathcal{P}(H) = \{L, R\}$. Consider $x_0 \neq 0$. Proceeding in an analogous way as above we obtain

$$\psi_L^{\varphi} - \psi_L^X \equiv \psi_R^{\varphi} - \psi_R^X \text{ in } H_L(x_0) = H_R(x_0).$$

Definition 5.6. *We denote $\psi_H^{\varphi} - \psi_H^X$ the function defined in $H(x_0)$ and given by the formula $\psi_{H,L}^{\varphi} - \psi_{H,L}^X$ in $H_L(x_0)$ for $L \in \mathcal{P}(H)$.*

Step 8. We introduce the main results concerning Fatou coordinates.

Proposition 5.5. *Let $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$. Let Υ be a 2-convergent normal form. Consider $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}}) \in \mathcal{M}$ and $\lambda \in \mathbb{S}^1$. Let $H \in Reg(\epsilon, \aleph_{\Lambda,\lambda} X, I_\Lambda^\lambda)$ and $L \in \mathcal{P}(H)$. Then the mapping (x, ψ_L^φ) is holomorphic in H° and continuous and injective in H_L.*

We constructed holomorphic Fatou coordinates in $H_L(x)$ for any $x \in [0, \delta) I_\Lambda^\lambda$. Next, we explain why the dependence of $\psi_L^\varphi(x, y)$ on x is continuous. The holomorphic part of the statement is proved in Step 11.

Consider $P = (x_0, y_0) \in H^L(x_0)$. Given $x \in [0, \delta) I_\Lambda^\lambda$ in a neighborhood of x_0 we define the continuous section $P(x) \in H_L(x)$ such that $\psi_L^X(P(x)) \equiv \psi_L^X(P)$ and $P(x_0) = P$. We consider the trajectory $\gamma_x = \Gamma(\aleph_{\Lambda,\lambda} X, P(x), T_0)$ and then we define the function $a(x, z_2)$ as in Step 1.

We define

$$\sigma_0(x, z) = \sigma_0(x, z_1 + i z_2) = (x, z + \varrho(z_1 - a(x, z_2))(\Delta_\varphi \circ (x, \psi_L^X)^{-1}(x, z - 1)))$$

and

$$\sigma(x, y) = (x, \psi_L^X)^{-1} \circ \sigma_0 \circ (x, \psi_L^X)(x, y).$$

The functions $a(x, z_2)$ and $\partial a(x, z_2)/\partial z_2$ are continuous. Therefore the complex dilatations χ_{σ_0} and $\chi_{\xi^{-1}}$ depend continuously on (x, z) (see Step 4 and in particular equation (13)). We deduce that $\tilde{\rho}$ and ρ depend continuously on (x, z). By definition of ψ_L^φ (see equation (16)) we obtain that ψ_L^φ is continuous in $\cup_{x \in V} B_\varphi(P(x))$ for some neighborhood V of x_0 in $[0, \delta) I_\Lambda^\lambda$. As a consequence ψ_L^φ is continuous in $\cup_{x \in V} H_L(x)$ and then in H_L.

A proof for the next proposition can be found by replicating the proof of the analogous result in [16].

Proposition 5.6. *Let $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$. Let Υ be a 2-convergent normal form. Consider $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}}) \in \mathcal{M}$ and $\lambda \in \mathbb{S}^1$. Let $H \in Reg(\epsilon, \aleph_{\Lambda,\lambda} X, I_\Lambda^\lambda)$; the function $\psi_H^\varphi - \psi_H^X$ is continuous in $\overline{H}$.*

Corollary 5.1. *Let $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$. Let Υ be a 2-convergent normal form. Consider $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}}) \in \mathcal{M}$ and $\lambda \in \mathbb{S}^1$. Let $H \in Reg(\epsilon, \aleph_{\Lambda,\lambda} X, I_\Lambda^\lambda)$. There exists a unique vector field $X_H^\varphi = X_H^\varphi(y) \partial/\partial y = (e^{2\pi i \psi_H^\varphi})^*(2\pi i z \partial/\partial z)$, the so called Lavaurs vector field, which is continuous in H, holomorphic in H° and satisfies $X_H^\varphi(\psi_H^\varphi) \equiv 1$. Moreover $X_H^\varphi(y)/X(y) - 1$ is continuous in $\overline{H}$ and vanishes at $\overline{H} \cap Fix(\varphi)$.*

Proposition 5.6 implies all the statements in corollary 5.1 except the holomorphic character of X_H^φ (see [16]). This last property is a consequence of the analogous result for Fatou coordinates (prop. 5.5) and it will be proved later on.

Step 9. We prove that Fatou coordinates are holomorphic if multi-directions are constant.

Consider the notations at the beginning of this section. Let $\lambda_0 \in I_\Lambda^\lambda$. Then $\Re(\aleph_{\Lambda,\lambda}(\lambda_0) X)$ is a multi-transversal flow in $[0, \delta) I_\Lambda^\lambda$ by remark 4.3. The multi-direction $\aleph_{\Lambda,\lambda}(\lambda_0)$ associated to $\Re(\aleph_{\Lambda,\lambda}(\lambda_0) X)$ is constant.

Let $H \in Reg(\epsilon, \aleph_{\Lambda,\lambda} X, I_\Lambda^\lambda)$ and $L \in \mathcal{P}(H)$. Let H_{λ_0} be the element of the set $Reg(\epsilon, \aleph_{\Lambda,\lambda}(\lambda_0) X, I_\Lambda^\lambda)$ such that $L \subset H_{\lambda_0}$.

Let $x_0 \in (0, \delta) \lambda e^{i(-\upsilon_\Lambda, \upsilon_\Lambda)}$. Consider $P \in H_{\lambda_0}^L(x_0)$ and $\gamma = \Gamma(\aleph_{\Lambda,\lambda}(\lambda_0) X, P, T_0)$. Let $P(x)$ be the section defined in Step 8. We consider the continuous family of curves $\{\gamma_x\}_{x \in V}$ defined for an open neighborhood V of x_0 in $(0, \delta) \lambda e^{i(-\upsilon_\Lambda, \upsilon_\Lambda)}$

by $\gamma_{x_0} = \gamma$ and $\psi_L^X(\gamma_x) = \psi_L^X(\gamma)$ for any $x \in V$. Since the multi-direction of $\Re(\aleph_{\Lambda,\lambda}(\lambda_0)X)$ is constant we obtain that γ_x is contained in H_{λ_0} for any $x \in V$ (by considering a smaller V if necessary), indeed we have

$$\lim_{x \to x_0} |\psi_L^X(\gamma_x(s)) - \psi_L^X(\Gamma(\aleph_{\Lambda,\lambda}(\lambda_0)X, P(x), T_0)(s))| = 0$$

uniformly on $s \in \mathbb{R}$. This property does not hold true in general if we replace $\aleph_{\Lambda,\lambda}(\lambda_0)$ with $\aleph_{\Lambda,\lambda}$ since then for instance we could have

$$\sup_{s \in \mathbb{R}} |\psi_L^X(\Gamma(\aleph_{\Lambda,\lambda}X, P, T_0)(s)), \psi_L^X(\Gamma(\aleph_{\Lambda,\lambda}X, P(x), T_0)(s))| = \infty$$

for any $x \in V \setminus \{x_0\}$.

The function $a(x, z_2)$ (see Steps 1 and 8) does not depend on x. The complex dilation χ_{σ_0} depends holomorphically on x (see equation (13)) and then $\chi_{\xi^{-1}}$ is also holomorphic on x. As a consequence $\tilde{\rho}$ and ρ are holomorphic on x. Therefore $\psi_{H_{\lambda_0},L}^{\varphi}$ is holomorphic in a neighborhood of $\cup_{x \in V} B_{\varphi}(P(x))$. The set $B_{\varphi}(P(x))$ contains a fundamental domain of φ in $H_{\lambda_0}(x)$. By using $\psi_{H_{\lambda_0},L}^{\varphi} \circ \varphi \equiv \psi_{H_{\lambda_0},L}^{\varphi} + 1$ we obtain that $\psi_{H_{\lambda_0},L}^{\varphi}$ is holomorphic in $\cup_{x \in V} H_{\lambda_0}(x)$. Thus $\psi_{H_{\lambda_0},L}^{\varphi}$ is holomorphic in $H_{\lambda_0,L}^{\circ} = H_{\lambda_0}^{\circ}$.

Step 10. We prove that $\psi_{H_\lambda,L}^{\varphi}$ can be extended to H by iteration. We deduce that $\psi_{H_\lambda,L}^{\varphi}$ is holomorphic in H°. Step 11 is dedicating to prove $\psi_{H,L}^{\varphi} \equiv \psi_{H_\lambda,L}^{\varphi}$ in H.

The difficulty is that in general $H_{\lambda_0} \neq H$ for any $\lambda_0 \in I_\Lambda^\lambda$. We could try to consider $\cup_{\lambda_0 \in I_\Lambda^\lambda} H_{\lambda_0,L}$ since $H_L \subset \cup_{\lambda_0 \in I_\Lambda^\lambda} H_{\lambda_0,L}$ but in general (x, ψ_L^X) is not injective in $\cup_{\lambda_0 \in I_\Lambda^\lambda} H_{\lambda_0,L}$. We define

$$G(x) = \cup_{\lambda_0 \in I_\Lambda^\lambda} \psi_L^X(H_{\lambda_0,L}(x)) \text{ and } G = \cup_{x \in [0,\delta) I_\Lambda^\lambda} (\{x\} \times G(x)).$$

Consider the continuous functions $\theta_1, \ldots, \theta_r : I_\Lambda^\lambda \to (0, \pi)$ defined in Step 3. We have that $\theta_j(\lambda')$ is decreasing on $\arg \lambda'$. We denote $(\aleph_{\Lambda,\lambda}(\lambda_0))^* = e^{i\theta_{\lambda_0}}$ where

$$\theta_{\lambda_0} : ([0,\delta) I_\Lambda^\lambda \times B(0,\epsilon)) \setminus SingX \to (0, \pi)$$

is a continuous function. The construction of multi-transversal flows implies that given $P \in ([0,\delta) I_\Lambda^\lambda \times B(0,\epsilon)) \setminus SingX$ the function $\upsilon \mapsto \theta_{\lambda e^{i\upsilon}}(P)$ is decreasing in $[-\upsilon_\Lambda, \upsilon_\Lambda]$. We define

$$\Gamma_{x,\lambda_0} = \Gamma(\aleph_{\Lambda,\lambda}(\lambda_0)X, L_{iX}^{\epsilon}(x), T_0).$$

Let $\upsilon_1, \upsilon_2 \in [-\upsilon_\Lambda, \upsilon_\Lambda]$ with $\upsilon_1 < \upsilon_2$ and $x \in [0,\delta) I_\Lambda^\lambda$. Denote $\lambda_1 = \lambda e^{i\upsilon_1}$ and $\lambda_2 = \lambda e^{i\upsilon_2}$. The decreasing character of $\upsilon \mapsto \theta_{\lambda e^{i\upsilon}}(P)$ implies that there exist $t_-, t_+ \in \mathbb{R}^+ \cup \{\infty\}$ such that $\Gamma_{x,\lambda_1}(t) = \Gamma_{x,\lambda_2}(t)$ if and only if $-t_- \leq t \leq t_+$. Moreover if $t_+ < \infty$ we obtain that $\psi_L^X(\Gamma_{x,\lambda_2}(t_+, \infty))$ is to the right of $\psi_L^X(\Gamma_{x,\lambda_1})$ whereas if $t_- < \infty$ then $\psi_L^X(\Gamma_{x,\lambda_2}(-\infty, t_-))$ is to the left of $\psi_L^X(\Gamma_{x,\lambda_1})$. We deduce that $\cup_{\lambda_0 \in I_\Lambda^\lambda} \psi_L^X(\Gamma_{x,\lambda_0})$ is a simply connected subset whose boundary is the union of the curves $\psi_L^X(\Gamma_{x,\lambda e^{-i\upsilon_\Lambda}})$ and $\psi_L^X(\Gamma_{x,\lambda e^{i\upsilon_\Lambda}})$ (see figure (6)). Therefore $G(x)$ is a simply connected open set in $\mathbb{C}$. Suppose that $\Re(X)$ points towards the interior of $H(0)$ at $L_{iX}^{\epsilon}(0)$ without lack of generality. Then the curve

$$\psi_L^X(\Gamma_{x,\lambda e^{-i\upsilon_\Lambda}}[0,\infty)) \cup \psi_L^X(\Gamma_{x,\lambda e^{i\upsilon_\Lambda}}(-\infty, 0]) \tag{17}$$

is in the boundary of $G(x)$. If $H \in Reg_1(\epsilon, \aleph_{\Lambda,\lambda}X, I_\Lambda^\lambda)$ or $x = 0$ there are no other points in the boundary. Otherwise let R be the element in $\mathcal{P}(H) \setminus \{L\}$. The

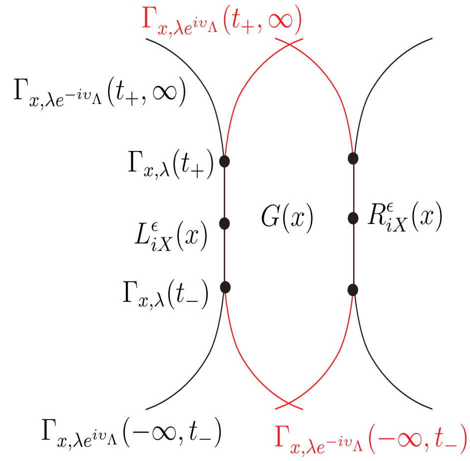


FIGURE 6. Picture of $G(x)$ in the coordinate ψ_L^X

boundary of $G(x)$ is the union of the curve (17) and a curve, passing through the point $\psi_L^X(R^\epsilon_{iX}(x))$, whose description is analogous (see figure (6)).

Since $G(x)$ is simply connected for any $x \in [0,\delta)I_\Lambda^\lambda$ we have that

$$(x, \psi_L^X)^{-1} : G \to B(0,\delta) \times B(0,\epsilon)$$

is a well-defined continuous function. It is holomorphic if we restrict the parameter x to $(0,\delta)\lambda e^{i(-\upsilon_\Lambda, \upsilon_\Lambda)}$. Consider the lift $\tilde{\varphi}$ of φ to G and $\tilde{\Delta}_\varphi = \Delta_\varphi \circ (x, \psi_L^X)^{-1}$. The diffeomorphism $\tilde{\varphi}$ is of the form

$$(x,z) \mapsto (x, z + 1 + \tilde{\Delta}_\varphi(x,z)).$$

We define

$$G^L = \cup_{x \in [0,\delta) I_\Lambda^\lambda, \ \lambda_0 \in I_\Lambda^\lambda} (\{x\} \times \psi_L^X(H^L_{\lambda_0}(x)))$$

Since I_Λ^λ is a compact set the proof of prop. 5.2 implies

$$|\tilde{\Delta}_\varphi(x,z)| \leq \frac{K}{(1+|z|)^2} \quad \forall (x,z) \in G^L. \tag{18}$$

More precisely, there are two main ingredients in the proof of prop. 5.2, namely we use that $\partial H_{\lambda_0} \cap \partial \mathcal{B}$ is the union of a finite number of asymptotically continuous sections for any basic set $\mathcal{B}$ and prop. 4.3. The proof can be adapted since the asymptotically continuous sections depend continuously on $\lambda_0 \in I_\Lambda^\lambda$ and prop. 4.3 is still valid.

Fix $x_0 = r_0 \lambda_0$ with $r_0 \in [0,\delta)$ and $\lambda_0 \in I_\Lambda^\lambda$. Suppose $\sharp \mathcal{P}(H) = 1$ or $x_0 = 0$. Then we obtain $G^L(x_0) = G(x_0)$ and

$$\lim_{z \in G(x_0), \ |z| \to \infty} \tilde{\Delta}_\varphi(x_0, z) = 0. \tag{19}$$

Suppose $\sharp \mathcal{P}(H) = 2$ and $x_0 \neq 0$. Let R be the element in $\mathcal{P}(H) \setminus \{L\}$. We can define G^R in an analogous way as G^L We obtain

$$|\Delta_\varphi \circ (x, \psi_{H,R}^X)^{-1}(x,z')| \leq \frac{K}{(1+|z'|)^2} \quad \forall (x,z') \in G^R.$$

We have $\psi_R^X \circ (\psi_L^X(x_0,y))^{-1} = z+\tau$ for some $\tau \in \mathbb{C}$. Therefore we obtain $|\tilde{\Delta}_\varphi|(x_0,z) \leq K/(1+|z+\tau|)^2$ for any $z \in G^R(x_0)$. This inequality and (18) imply equation (19). Consider the set $B_\varphi(L_{iX}^\epsilon(x_0))$ as defined in Step 5. It is enclosed by Γ_{x_0,λ_0} and $\varphi(\Gamma_{x_0,\lambda_0})$. Analogously consider the set $B_\varphi^{\lambda_1}$ enclosed by Γ_{x_0,λ_1} and $\varphi(\Gamma_{x_0,\lambda_1})$. Denote $B_\varphi^\flat = \cup_{\lambda_1 \in I_\Lambda^\lambda} \psi_L^X(B_\varphi^{\lambda_1})$. By proceeding as in Step 5 we obtain that the set $\psi_L^X(B_\varphi^{\lambda_1} \setminus \varphi(\Gamma_{x_0,\lambda_1}))$ is a fundamental domain of $\tilde{\varphi}_{|B_\varphi^\flat}$ for any $\lambda_1 \in I_\Lambda^\lambda$. Therefore we can extend $\psi_{H_{\lambda_1},L}^\varphi$ by iteration to G for any $\lambda_1 \in I_\Lambda^\lambda$. We deduce that $\psi_{H_\lambda,L}^\varphi$ is defined in H^L and holomorphic in H°.

Step 11. In order to complete the proofs of proposition 5.5 and corollary 5.1 it suffices to show $(\psi_{H,L}^\varphi)_{|x=x_0} \equiv (\psi_{H_\lambda,L}^\varphi)_{|x=x_0}$ for $x_0 = r_0\lambda_0 \neq 0$.

By arguing as in Step 2 (see equation (11)) we obtain $C_4 \in \mathbb{R}^+$ such that

$$B_\varphi^{\lambda_1} \cap \left\{ |Im(\psi_L^X)| \leq \frac{C_4}{|x_0|^{\nu(\mathcal{E}_0)}} \right\} = B_\varphi^\lambda \cap \left\{ |Im(\psi_L^X)| \leq \frac{C_4}{|x_0|^{\nu(\mathcal{E}_0)}} \right\}$$

for any $\lambda_1 \in I_\Lambda^\lambda$. The constant C_4 is independent of λ_1 and x_0.

Consider $P \in B_\varphi^{\lambda_0} \setminus B_\varphi^\lambda$. Denote $\tilde{z} = \psi_L^X(P)$. We obtain $|Im(\tilde{z})| > C_4/|x_0|^{\nu(\mathcal{E}_0)}$. There exist $j \in \mathbb{Z}$ and a orbit $\tilde{z}, \ldots, \tilde{\varphi}^j(\tilde{z})$ contained in $B_\varphi^\flat \cap \{|Im(z)| > C_4/|x|^{\nu(\mathcal{E}_0)}\}$ such that $\psi_L^X(\tilde{\varphi}^j(\tilde{z})) \in \psi_L^X(B_\varphi^\lambda)$. We have

$$\psi_{H_\lambda,L}^\varphi(\tilde{z}) - \tilde{z} = \psi_{H_\lambda,L}^\varphi(\tilde{\varphi}^j(\tilde{z})) - \tilde{\varphi}^j(\tilde{z}) - \sum_{s=0}^{|j|-1} \tilde{\Delta}_\varphi(\tilde{\varphi}^{j+s}(\tilde{z}))$$

if $j<0$ and

$$\psi_{H_\lambda,L}^\varphi(\tilde{z}) - \tilde{z} = \psi_{H_\lambda,L}^\varphi(\tilde{\varphi}^j(\tilde{z})) - \tilde{\varphi}^j(\tilde{z}) + \sum_{s=0}^{j-1} \tilde{\Delta}_\varphi(\tilde{\varphi}^{s}(\tilde{z}))$$

for $j>0$. Suppose $j>0$ without lack of generality. Since $B_\varphi^\flat$ is contained in G^L and $|Im(\tilde{\varphi}^s(\tilde{z}))| > C_4/|x|^{\nu(\mathcal{E}_0)}$ the eq. (18) implies that $|\tilde{\Delta}_\varphi(\tilde{\varphi}^s(\tilde{z}))| \leq K|x_0|^{2\nu(\mathcal{E}_0)}/C_4^2$ for any $0 \leq s < j$. Thus $|\tilde{\Delta}_\varphi(\tilde{\varphi}^s(\tilde{z}))|$ is as small as desired for $0 \leq s < j$ by considering a smaller $\delta > 0$ if necessary. By applying lemma 5.2 we obtain

$$|(\psi_{H_\lambda,L}^\varphi(\tilde{z}) - \tilde{z}) - (\psi_{H_\lambda,L}^\varphi(\tilde{\varphi}^j(\tilde{z})) - \tilde{\varphi}^j(\tilde{z}))| \leq \frac{C_5}{1+|\tilde{z}|}$$

where $C_5 \in \mathbb{R}^+$ does not depend on $\tilde{z}$ or j. Moreover we deduce

$$|Im(\tilde{\varphi}^j(\tilde{z})) - Im(\tilde{z})| \leq \frac{C_5}{1+|\tilde{z}|}.$$

By definition of $\psi_{H_\lambda,L}^\varphi$ we get

$$\lim_{\tilde{z} \in \psi_L^X(B_\varphi^{\lambda_0}),\ Im(\tilde{z}) \to \pm\infty} (\psi_{H_\lambda,L}^\varphi(\tilde{\varphi}^j(\tilde{z})) - \tilde{\varphi}^j(\tilde{z})) = \kappa_\pm$$

for $\kappa_+ = 0$ and some $\kappa_- \in \mathbb{C}$. We obtain

$$\lim_{Q \in B_\varphi^{\lambda_0},\ Im(\psi_L^X(Q)) \to \pm\infty} (\psi_{H_\lambda,L}^\varphi(Q) - \psi_{H_\lambda,L}^X(Q)) = \kappa_\pm.$$

The function $(\psi_{H,L}^\varphi - \psi_{H_\lambda,L}^\varphi) \circ (e^{2\pi i \psi_{H,L}^\varphi(x_0,y)})^{-1}(z)$ is holomorphic in $\mathbb{C}^* = e^{2\pi i z}(B_\varphi^{\lambda_0})$ and extends continuously to $\mathbb{P}^1(\mathbb{C})$. Moreover it takes the value 0 at 0. Therefore we obtain $\psi_{H,L}^\varphi \equiv \psi_{H_\lambda,L}^\varphi$. The Fatou coordinate $\psi_{H,L}^\varphi$ is holomorphic in H°.

Remark 5.3. *Apparently the constructions depend on the choice of the* 2*-convergent normal form* $\Upsilon = \exp(X)$*. Anyway, the polynomial vector fields associated to compact-like sets are the same. Then* $\mathcal{M}$ *is independent of* Υ *and so is* $\aleph_{\Lambda,\lambda}$ *for all* $\Lambda \in \mathcal{M}$ *and* $\lambda \in \mathbb{S}^1$*. It is easy to see that the dynamical splitting* $\digamma_\Lambda$ *is independent of the choice of normal form too. Of course the regions in* $Reg(\epsilon, \aleph_{\Lambda,\lambda}X, I_\Lambda^\lambda)$ *depend on* X *and then on* Υ*. Anyway, the different Fatou coordinates that we obtain in slightly different regions can be interpreted as particularizations of the same object in an analogous way as described in Steps 10 and 11.*

5.3. **Flow-convex sets.** This subsection is of technical importance for the results presented in subsections 5.4.2, 5.6 and 5.7. The language required to study properties related to the shape of subregions contained in the first exterior set is introduced below. We use those properties in order to analyze or extend Fatou coordinates.

Definition 5.7. *Let us consider a domain* $D \subset \mathbb{C}$ *such that there exists a homeomorphism* $\sigma : D \to \mathbb{D}$ *extending to a homeomorphism* $\tilde{\sigma} : \overline{D} \to \overline{\mathbb{D}}$*. Let* Z *be a holomorphic vector field defined in the neighborhood of* $\overline{D}$ *such that* $D \cap SingZ = \emptyset$*. We say that* (Z, D) *is a regular pair.*

Definition 5.8. *Let* (Z, D) *be a regular pair. We say that* D *is* $\Re(Z)$*-convex if* $\Gamma_P = \Gamma(Z, P, \overline{D})$ *satisfies that* $\{t \in \mathcal{I}(P) : \Gamma_P(t) \in D\}$ *is connected for any* $P \in D$*. In other words we have* $\Gamma(Z, P, \overline{D}) \cap D = \Gamma(Z, P, D)$ *for any* $P \in D$*.*

Definition 5.9. *Let* (Z, D) *be a regular pair and* $Q \in \partial D \setminus SingZ$*. We say that* $\Re(Z)$ *is almost transversal to* ∂D *at* Q *if there exists* $s_0 \in \mathbb{R}^+$ *such that either* $\exp(sX)(Q) \notin \overline{D}$ *for any* $s \in (0, s_0)$ *or* $\exp(-sX)(Q) \notin \overline{D}$ *for any* $s \in (0, s_0)$*. If the curve* ∂D *is smooth in the neighborhood of* Q *then transversal implies almost transversal.*

The following result is straightforward.

Lemma 5.5. *Let* (Z, D) *be a regular pair and* $P \in D$*. Suppose that* $\Re(Z)$ *is almost transversal to* ∂D *at the points in* $\Gamma(Z, P, D)(\partial \mathcal{I}(\Gamma(Z, P, D)))$*. Then we obtain* $\mathcal{I}(\Gamma(Z, P, \overline{D})) = \overline{\mathcal{I}(\Gamma(Z, P, D))}$*. In particular* D *is* $\Re(Z)$*-convex if* $\Re(Z)$ *is almost transversal to* ∂D *at* Q *for any* $Q \in \overline{D} \setminus SingZ$*.*

Definition 5.10. *Since* $\overline{D} \setminus SingZ$ *is simply connected there exists a holomorphic Fatou coordinate* ψ *of* Z *defined in a neighborhood of* $\overline{D} \setminus SingZ$*. We say that* ψ *is a Fatou coordinate of the pair* (Z, D)*.*

Lemma 5.6. *Consider a regular pair* (Z, D) *such that* D *is* $\Re(Z)$*-convex. Let* ψ *a Fatou coordinate of* (Z, D)*. Consider continuous paths* $\tau_1, \tau_2 : [0, 1] \to D$ *such that*

$$Im(\psi(\tau_1(s))) = Im(\psi(\tau_2(s))) \ \forall s \in [0, 1] \text{ and } \tau_1(1) = \tau_2(1).$$

Then $\tau_2(0)$ *belongs to* $\Gamma(Z, \tau_1(0), D)$*.*

Proof. Consider $F = \{s \in [0, 1] : \tau_2(s) \in \Gamma(Z, \tau_1(s), D)\}$. It suffices to prove $F = [0, 1]$. Suppose $F \neq [0, 1]$. Denote $s_0 = \sup\{s \in [0, 1] : s \notin F\}$. Since F is open and contains the point 1 we deduce that $s_0 < 1$, $s_0 \notin F$ and $s \in F$ for any $s \in (s_0, 1]$. We obtain

$$\tau_2(s) \in \Gamma(Z, \tau_1(s), D)(a(s)) \ \forall s \in (s_0, 1]$$

where $a(s) = \psi(\tau_2(s)) - \psi(\tau_1(s))$. Therefore we have

$$\tau_2(s_0) \in \Gamma(Z, \tau_1(s_0), \overline{D})(a(s_0)).$$

As a consequence of definition 5.8 the point $\tau_2(s_0)$ belongs to $\Gamma(Z, \tau_1(s_0), D)$ and then $s_0 \in F$. This is a contradiction. We obtain $F = [0,1]$. $\square$

Proposition 5.7. *Consider a regular pair (Z, D) such that D is $\Re(Z)$-convex. Let ψ a Fatou coordinate of (Z, D) and $P \in D$. Then there exists a continuous path $\gamma_Q : [0,1] \to D$ with $\gamma_Q(0) = P$, $\gamma_Q(1) = Q$ such that $Im(\psi \circ \gamma_Q) : [0,1] \to \mathbb{R}$ is injective for any $Q \in D \setminus \Gamma(Z, P, D)$.*

Proof. We define the set E of points $Q \in D$ satisfying that there exists a continuous path $\gamma_Q : [0,1] \to D$ with $\gamma_Q(0) = P$, $\gamma_Q(1) = Q$ such that $Im(\psi \circ \gamma_Q) : [0,1] \to \mathbb{R}$ is either an injective or a constant function. A point $Q \in E$ belongs to $\Gamma(Z, P, D)$ if $Im(\psi \circ \gamma_Q)$ is a constant function. It suffices to prove that $E = D$.

Let $\tau_1, \tau_2 : [0,1] \to D$ be continuous paths such that $\tau_1(0) = P$, $\tau_1(1) = \tau_2(0)$ and $\tau_2(1) = Q'$. We claim that if $Im(\psi \circ \tau_1)$ is constant and $Im(\psi \circ \tau_2)$ is injective or $Im(\psi \circ \tau_1)$ is injective and $Im(\psi \circ \tau_2)$ is constant then Q' belongs to E. For instance in the latter case we consider $a \in (0,1)$ near 1 and we define a path $\tau : [0,1] \to D$ such that $\tau(s) = \tau_1(s)$ for $s \in [0,a]$ and $\psi(\tau((1-s)a + s)) = (1-s)\psi(\tau_1(a)) + s\psi(Q')$ for $s \in [0,1]$. Clearly $Im(\psi \circ \tau)$ is injective.

Given $Q \in E$ any point Q' in a neighborhood of Q satisfies either $Q' \in \Gamma(Z, P, D)$ or we can find paths τ_1, τ_2 as in the previous paragraph. As a consequence the set E is open in D. Since D is connected it suffices to prove that E is closed in D.

Let $Q \in \overline{E} \cap D$. Consider a sequence $Q_n \to Q$ of points in $E \cap D$. Choose $n >> 1$. We can choose a continuous path $\tilde{\gamma}_n : [0,1] \to D$ with $\tilde{\gamma}_n(0) = Q$, $\tilde{\gamma}_n(1) = Q_n$ such that $Im(\psi \circ \tilde{\gamma}_n)$ is injective or constant. We can suppose that $Im(\psi(Q_n))$ does not belong to the closed interval whose ends are $Im(\psi(P))$ and $Im(\psi(Q))$. Otherwise the argument in the second paragraph implies $Q \in E$. Without lack of generality we can suppose $Im(\psi(Q_n)) > \max(Im(\psi(P)), Im(\psi(Q)))$. We obtain $Im(\psi(Q)) \geq Im(\psi(P))$.

Suppose $Im(\psi(P)) = Im(\psi(Q))$. Lemma 5.6 implies $Q \in \Gamma(Z, P, D)$ and then $Q \in E$. Now suppose $Im(\psi(P)) < Im(\psi(Q))$. There exists $P' \in \gamma_{Q_n}[0,1]$ such that $Im(\psi(P')) = Im(\psi(Q))$. We obtain $Q \in \Gamma(Z, P', D)$ by lemma 5.6 and then $Q \in E$ (see second paragraph). The set E is closed in D as we wanted to prove. $\square$

Corollary 5.2. *Consider a regular pair (Z, D) such that D is $\Re(Z)$-convex. Let ψ a Fatou coordinate of (Z, D) and $P, Q \in \overline{D} \setminus SingZ$. Then $Im(\psi(P)) = Im(\psi(Q))$ implies that $Q \in \Gamma(Z, P, \overline{D})$. In particular ψ is injective in $\overline{D} \setminus SingZ$.*

Proof. Proposition 5.7 implies that ψ is injective in D. Denote $E = Im(\psi)(D)$ and $E' = Im(\psi)(\overline{D} \setminus SingZ)$. We have $E' \subset \overline{E}$. The set E is an open interval.

Suppose $Im(\psi(P)) \in E$. There exists $P' \in D$ such that $Im(\psi(P')) = Im(\psi(P))$. There exists a sequence $P_n \to P$ of points in D. We choose a sequence $P'_n \to P'$ of points in D such that $Im(\psi(P'_n)) = Im(\psi(P_n))$ for any $n \in \mathbb{N}$. Proposition 5.7 implies

$$P_n \in \Gamma(Z, P'_n, D)(a_n) \ \forall n \in \mathbb{N}$$

where $a_n = \psi(P_n) - \psi(P'_n)$. We obtain $P \in \Gamma(Z, P', \overline{D})(\psi(P) - \psi(P'))$. Analogously Q belongs to $\Gamma(Z, P', \overline{D})$. We deduce $Q = \Gamma(Z, P, \overline{D})(\psi(Q) - \psi(P))$.

Suppose $Im(\psi(P)) \in E' \setminus E$. Then $Im(\psi(P))$ is either the supremum or the infimum or $Im(\psi)$ at $\overline{D} \setminus SingZ$. Suppose without lack of generality that we are in the former case. There exists a continuous path $\gamma : [0,1] \to \overline{D}$ such that $\gamma[0,1) \subset D$ and $\gamma(1) = P$. We deduce that $Im(\psi)(\gamma[0,1)) = [Im(\psi(P)) - a, Im(\psi(P)))$ for some $a \in \mathbb{R}^+$. An analogous property holds true for Q. Therefore there exist sequences $P_n \to P$ and $Q_n \to Q$ of points in D such that $Im(\psi(P_n)) = Im(\psi(Q_n))$ for any $n \in \mathbb{N}$. Since $Q_n \in \Gamma(Z, P_n, D)(\psi(Q_n) - \psi(P_n))$ for any $n \in \mathbb{N}$ we obtain $Q \in \Gamma(Z, P, \overline{D})(\psi(Q) - \psi(P))$. □

5.4. **Comparing multi-transversal flows.** Let $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$ with 2-convergent normal form $\Upsilon = \exp(X)$. Consider $\Lambda, \Lambda' \in \mathcal{M}$ and the dynamical splittings $\digamma_\Lambda$ and $\digamma_{\Lambda'}$ in remark 4.11. Let $\lambda, \lambda' \in \mathbb{S}^1$.

Definition 5.11. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$, $\Lambda \in \mathcal{M}$ and $\lambda \in \mathbb{S}^1$. We define $H_{\Lambda,j}^{\lambda}$ the element in $Reg(\epsilon, \aleph_{\Lambda,\lambda} X, I_\Lambda^\lambda)$ such that $L_j \subset H_{\Lambda,j}^{\lambda}$ (see def. 4.29). We denote $\tilde{\psi}_{j,\Lambda,\lambda}^{\varphi} = \psi_{H_{\Lambda,j}^{\lambda}, L_j}^{\varphi}$ or just $\tilde{\psi}_{j,\lambda}^{\varphi}$ if Λ is implicit.*

Definition 5.12. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2, 0)$, $\Lambda, \Lambda' \in \mathcal{M}$ and $\lambda, \lambda' \in \mathbb{S}^1$. We denote $d_{\Lambda,\Lambda'}^{\lambda,\lambda'} = 0$ if $\Lambda \neq \Lambda'$. We denote*

$$d_{\Lambda,\Lambda}^{\lambda,\lambda'} = \min(d_\Lambda^\lambda, d_\Lambda^{\lambda'}) \text{ and } I_{\Lambda,\Lambda'}^{\lambda,\lambda'} = I_\Lambda^\lambda \cap I_{\Lambda'}^{\lambda'},$$

see def. 5.4. Denote $R_{\Lambda,\Lambda',j}^{\lambda,\lambda'} = 2$ if $H_{\Lambda,j}^{\lambda} \in Reg_2(\epsilon, \aleph_{\Lambda,\lambda} X, I_\Lambda^\lambda)$ and $e(H) \leq \tilde{e}_{d_{\Lambda,\Lambda'}^{\lambda,\lambda'}}$ (see definitions 4.5 and 4.35). Otherwise we define $R_{\Lambda,\Lambda',j}^{\lambda,\lambda'} = 1$.

Remark 5.4. *The equation $R_{\Lambda,\Lambda',j}^{\lambda,\lambda'} = 2$ implies $H_{\Lambda,j}^{\lambda} \in Reg_2(\epsilon, \aleph_{\Lambda,\lambda} X, I_\Lambda^\lambda)$ and $H_{\Lambda',j}^{\lambda'} \in Reg_2(\epsilon, \aleph_{\Lambda',\lambda'} X, I_{\Lambda'}^{\lambda'})$. Moreover, we have*

$$\partial\mathcal{E}_0 \cap \overline{H_{\Lambda,j}^{\lambda}(x)} = \partial\mathcal{E}_0 \cap \overline{H_{\Lambda',j}^{\lambda'}(x)} = \{T_{iX}^{\epsilon,j}(x), T_{iX}^{\epsilon,k}(x)\}$$

for some $k \in \mathbb{Z}/(2\nu(\mathcal{E}_0)\mathbb{Z})$ and any $x \in [0, \delta)(I_\Lambda^\lambda \cap I_{\Lambda'}^{\lambda'})$. The equation $R_{\Lambda,\Lambda',j}^{\lambda,\lambda'} = 1$ is less restrictive, for instance $d_{\Lambda,\Lambda'}^{\lambda,\lambda'} = 0$ implies $R_{\Lambda,\Lambda',j}^{\lambda,\lambda'} = 1$ for any $j \in \mathbb{Z}/(2\nu(\mathcal{E}_0)\mathbb{Z})$.

We want to estimate $\tilde{\psi}_{j,\Lambda,\lambda}^{\varphi} - \tilde{\psi}_{j,\Lambda',\lambda'}^{\varphi}$. It is defined in $H_{\Lambda,j}^{\lambda} \cap H_{\Lambda',j}^{\lambda'}$ but this set is not well suited to work with since for instance it is not necessarily connected. We work with the set $H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}$ (see def. 5.15). It is contained in $H_{\Lambda,j}^{\lambda} \cap H_{\Lambda',j}^{\lambda'}$ and the restrictions of $\Re(\aleph_{\Lambda,\lambda}^* X)$ and $\Re(\aleph_{\Lambda',\lambda'}^* X)$ to $H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}$ coincide. Roughly speaking the orbit space of φ restricted to $H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}(x_0)$ is biholomorphic to the subset $B(0, \kappa_-(x_0)) \setminus \overline{B}(0, \kappa_+(x_0))$ of $\mathbb{C}^*$. The idea is studying the subregions of both $H_{\Lambda,j}^{\lambda}$ and $H_{\Lambda',j}^{\lambda'}$ contained in $H_{\Lambda,j}^{\lambda} \cap H_{\Lambda',j}^{\lambda'}$ to prove that κ_- and κ_+ are exponentially small of the right order.

Definition 5.13. *Let $En_{\Lambda,\Lambda',j}^{\lambda,\lambda'}$ be the set of basic sets $\mathcal{B}$ of $\digamma_\Lambda \cup \digamma_{\Lambda'}$ (see def. 3.5) such that $e(\mathcal{B}) \leq \iota(\mathcal{B}) < \tilde{e}_{d_{\Lambda,\Lambda'}^{\lambda,\lambda'}+1}$ and $H_{\Lambda,j}^{\lambda} \cap \mathcal{B} \neq \emptyset$.*

We have $\aleph_{\Lambda,\lambda}^* \equiv \aleph_{\Lambda',\lambda'}^*$ in $\cup_{(r,\lambda_0) \in [0,\delta) \times I_{\Lambda,\Lambda'}^{\lambda,\lambda'}} \mathcal{B}'(r, \lambda_0)$ for any $\mathcal{B}' \in En_{\Lambda,\Lambda' j}^{\lambda,\lambda'}$ by definition. We obtain

$$(H_{\Lambda,j}^{\lambda})^{L_j} \cap \mathcal{B}(r, \lambda_0) = (H_{\Lambda',j}^{\lambda'})^{L_j} \cap \mathcal{B}(r, \lambda_0) \ \forall (r, \lambda_0) \in [0, \delta) \times I_{\Lambda,\Lambda'}^{\lambda,\lambda'} \ \forall \mathcal{B} \in En_{\Lambda,\Lambda',j}^{\lambda,\lambda'}.$$

We deduce that $En_{\Lambda,\Lambda',j}^{\lambda,\lambda'} \subset En_{\Lambda',\Lambda,j}^{\lambda',\lambda}$. We can permute the roles of λ and λ' to get $En_{\Lambda,\Lambda',j}^{\lambda,\lambda'} = En_{\Lambda',\Lambda,j}^{\lambda',\lambda}$.

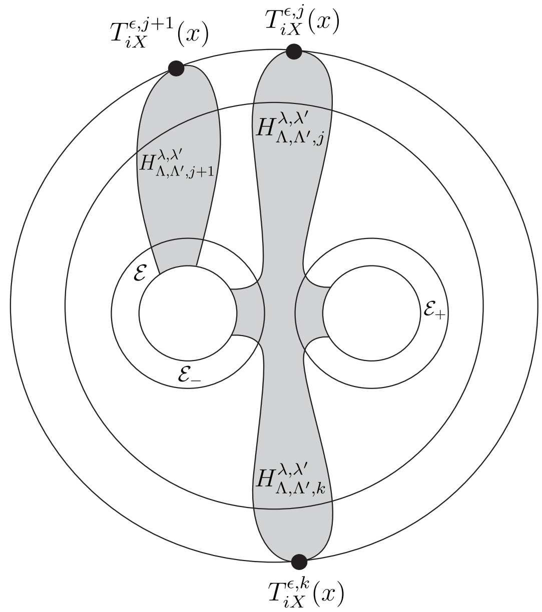


FIGURE 7. Examples: $R^{\lambda,\lambda'}_{\Lambda,\Lambda',j+1} = 1$ and $R^{\lambda,\lambda'}_{\Lambda,\Lambda',j} = 2$, $H^{\lambda,\lambda'}_{\Lambda,\Lambda',j} = H^{\lambda,\lambda'}_{\Lambda,\Lambda',k}$

Definition 5.14. *Let $Ex^{\lambda,\lambda'}_{\Lambda,\Lambda',j}$ be the set of basic sets $\mathcal{B}$ of $\digamma_\Lambda \cup \digamma_{\Lambda'}$ satisfying the properties $H^{\lambda}_{\Lambda,j} \cap \mathcal{B} \neq \emptyset$ and $e(\mathcal{B}) < \tilde{e}_{d^{\lambda,\lambda'}_{\Lambda,\Lambda'}+1} \leq \iota(\mathcal{B})$. Since $e(\mathcal{B}) < \iota(\mathcal{B})$ the elements of $Ex^{\lambda,\lambda'}_{\Lambda,\Lambda',j}$ are non-terminal exterior sets.*

Analogously as in the previous paragraph we obtain

$$(H^{\lambda}_{\Lambda,j})^{L_j} \cap \tilde{\mathcal{E}}(r,\lambda_0) = (H^{\lambda'}_{\Lambda',j})^{L_j} \cap \tilde{\mathcal{E}}(r,\lambda_0) \ \forall (r,\lambda_0) \in [0,\delta) \times I^{\lambda,\lambda'}_{\Lambda,\Lambda'} \ \forall \mathcal{E} \in Ex^{\lambda,\lambda'}_{\Lambda,\Lambda',j},$$

where $\tilde{\mathcal{E}}$ is defined in section 3. We have $\sharp Ex^{\lambda,\lambda'}_{\Lambda,\Lambda',j} \leq 2$. Moreover we obtain $Ex^{\lambda,\lambda'}_{\Lambda,\Lambda',j} \subset Ex^{\lambda',\lambda}_{\Lambda',\Lambda,j}$ and then $Ex^{\lambda,\lambda'}_{\Lambda,\Lambda',j} = Ex^{\lambda',\lambda}_{\Lambda',\Lambda,j}$.

Definition 5.15. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2,0)$, $\Lambda, \Lambda' \in \mathcal{M}$ and $\lambda, \lambda' \in \mathbb{S}^1$. We define*

$$H^{\lambda,\lambda'}_{\Lambda,\Lambda',j} = [(H^{\lambda}_{\Lambda,j})^{L_j} \cap (\cup_{\mathcal{B} \in En^{\lambda,\lambda'}_{\Lambda,\Lambda',j}} \mathcal{B} \cup \cup_{\mathcal{E} \in Ex^{\lambda,\lambda'}_{\Lambda,\Lambda',j}} \tilde{\mathcal{E}})] \cap \{x \in [0,\delta) I^{\lambda,\lambda'}_{\Lambda,\Lambda'}\}$$

if $R^{\lambda,\lambda'}_{\Lambda,\Lambda',j} = 1$. We define

$$H^{\lambda,\lambda'}_{\Lambda,\Lambda',j} = [H^{\lambda}_{\Lambda,j} \cap (\cup_{\mathcal{B} \in En^{\lambda,\lambda'}_{\Lambda,\Lambda',j}} \mathcal{B} \cup \cup_{\mathcal{E} \in Ex^{\lambda,\lambda'}_{\Lambda,\Lambda',j}} \tilde{\mathcal{E}})] \cap \{x \in [0,\delta) I^{\lambda,\lambda'}_{\Lambda,\Lambda'}\}$$

for $R^{\lambda,\lambda'}_{\Lambda,\Lambda',j} = 2$ (see figure (7)).

We have $H^{\lambda,\lambda'}_{\Lambda,\Lambda',j} \subset H^{\lambda}_{\Lambda,j} \cap H^{\lambda'}_{\Lambda',j}$ and $\Re(\aleph_{\Lambda,\lambda} X) \equiv \Re(\aleph_{\Lambda',\lambda'} X)$ in $H^{\lambda,\lambda'}_{\Lambda,\Lambda',j}$.

Definition 5.16. *Let $X \in \mathcal{X}_{tp1}(\mathbb{C}^2,0)$, $\Lambda, \Lambda' \in \mathcal{M}$ and $\lambda, \lambda' \in \mathbb{S}^1$. Denote $\Gamma_x = \Gamma(\aleph_{\Lambda,\lambda} X, T^{\epsilon,j}_{iX}(x), T_0)$ (see def. 4.29). Either $\Gamma_x[0,\infty)$ intersects an element $\mathcal{E}_+$ of $Ex^{\lambda,\lambda'}_{\Lambda,\Lambda',j}$ for any $x \in (0,\delta) I^{\lambda,\lambda'}_{\Lambda,\Lambda'}$ or $\Gamma_x \cap \mathcal{E} = \emptyset$ for all $\mathcal{E} \in Ex^{\lambda,\lambda'}_{\Lambda,\Lambda',j}$ and $x \in (0,\delta) I^{\lambda,\lambda'}_{\Lambda,\Lambda'}$. In the former case we define $e^{\lambda,\lambda',+}_{\Lambda,\Lambda',j} = \iota(\mathcal{E}_+)$. Otherwise we define*

$e_{\Lambda,\Lambda',j}^{\lambda,\lambda',+} = \infty$. *Analogously we define* $\mathcal{E}_-$ *and* $e_{\Lambda,\Lambda',j}^{\lambda,\lambda',-}$ *by considering* $\Gamma_x(-\infty,0]$ *(see figure (7)).*

Definition 5.17. *Let* $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ *with* 2*-convergent normal form* $\exp(X)$*. We define* $\mathcal{D}(\varphi) = \mathbb{Z}/(2\nu(\mathcal{E}_0)\mathbb{Z})$*. We define*

$$\mathcal{D}_1(\varphi) = \{j \in \mathbb{Z}/(2\nu(\mathcal{E}_0)\mathbb{Z}) : \Re(X) \text{ points towards } B(0,\delta)\times B(0,\epsilon) \text{ at } T_{iX}^{\epsilon,j}(0)\}$$

and $\mathcal{D}_{-1}(\varphi) = \mathcal{D}(\varphi) \setminus \mathcal{D}_1(\varphi)$*.*

Definition 5.18. *Let* $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ *with* 2*-convergent normal form* $\exp(X)$*. Let* $\Lambda,\Lambda' \in \mathcal{M}$ *and* $\lambda,\lambda' \in \mathbb{S}^1$*. We define*

$$\psi_{j,\lambda,\lambda'}^{\varphi} = (\tilde{\psi}_{j,\Lambda,\lambda}^{\varphi} - \tilde{\psi}_{j,\Lambda',\lambda'}^{\varphi}) \circ (x, e^{2\pi i \tilde{\psi}_{j,\Lambda,\lambda}^{\varphi}})^{-1}.$$

The values of Λ and Λ' in $\psi_{j,\lambda,\lambda'}^{\varphi}$ are implicit. We want to prove that $\psi_{j,\lambda,\lambda'}^{\varphi}$ is defined in the space of orbits of $\varphi_{|H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}}$ in order to estimate $\tilde{\psi}_{j,\lambda}^{\varphi} - \tilde{\psi}_{j,\lambda'}^{\varphi}$ in $H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}$.

Proposition 5.8. *Let* $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ *with* 2*-convergent normal form* $\exp(X)$*. Let* $\Lambda,\Lambda' \in \mathcal{M}$*. Consider* $\lambda,\lambda' \in \mathbb{S}^1$ *and* $j \in \mathcal{D}(\varphi)$*. Then we have*

$$(x, e^{2\pi i \tilde{\psi}_{j,\Lambda,\lambda}^{\varphi}})(H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}) \subset \cup_{(r,\tilde{\lambda})\in[0,\delta)\times I_{\Lambda,\Lambda'}^{\lambda,\lambda'}} (\{r\tilde{\lambda}\} \times [B(0,\kappa_-(r,\tilde{\lambda})) \setminus \overline{B}(0,\kappa_+(r,\tilde{\lambda}))])$$

where $\kappa_\pm(r,\tilde{\lambda}) \equiv e^{\mp C_\pm(\lambda)/|r|^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',\pm}}}$ *for some continuous function* $C_\pm : I_{\Lambda,\Lambda'}^{\lambda,\lambda'} \to \mathbb{R}^+$*. Moreover there exists* $\zeta \in \mathbb{R}^+$ *such that* $\psi_{j,\lambda,\lambda'}^{\varphi}$ *is well defined in*

$$\cup_{(r,\tilde{\lambda})\in[0,\delta)\times I_{\Lambda,\Lambda'}^{\lambda,\lambda'}} (\{r\tilde{\lambda}\} \times [B(0,\tilde{\kappa}_-(r,\tilde{\lambda})) \setminus \overline{B}(0,\tilde{\kappa}_+(r,\tilde{\lambda}))])$$

and holomorphic outside $x=0$ *where* $\tilde{\kappa}_\pm(r,\tilde{\lambda}) = e^{\mp(C_\pm(\tilde{\lambda})+\zeta)/|r|^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',\pm}}}$.

Let us remark that in the previous proposition $\tilde{\kappa}_+ \equiv \kappa_+ \equiv 0$ if $e_{\Lambda,\Lambda',j}^{\lambda,\lambda',+} = \infty$ and $\tilde{\kappa}_- \equiv \kappa_- \equiv \infty$ if $e_{\Lambda,\Lambda',j}^{\lambda,\lambda',-} = \infty$.

The subsections 5.4.2 and 5.4.1 are intended to prove proposition 5.8 in the cases $R_{\Lambda,\Lambda',j}^{\lambda,\lambda'} = 1$ and $R_{\Lambda,\Lambda',j}^{\lambda,\lambda'} = 2$ respectively. The subsection 5.4.2 is though a little more ambitious since it introduces the setup that will be used in subsections 5.6 and 5.7.

5.4.1. *Proof of proposition 5.8 in the case* $R_{\Lambda,\Lambda',j}^{\lambda,\lambda'} = 2$. The exterior sets $\mathcal{E}_-$ and $\mathcal{E}_+$ (see def. 5.16) are different if $e_{\Lambda,\Lambda',j}^{\lambda,\lambda',-} \neq \infty$ and $e_{\Lambda,\Lambda',j}^{\lambda,\lambda',+} \neq \infty$. Thus the cardinal of the set $Ex_{\Lambda,\Lambda',j}^{\lambda,\lambda'}$ coincides with the cardinal of $\{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',-}, e_{\Lambda,\Lambda',j}^{\lambda,\lambda',+}\} \setminus \{\infty\}$.

Denote $\psi = \psi_{H_{\Lambda,j}^{\lambda},L_j}^{X}$, $\aleph = \aleph_{\Lambda,\lambda}$ and $\Gamma_x = \Gamma(\aleph_{\Lambda,\lambda}X, T_{iX}^{\epsilon,j}(x), T_0)$. Suppose that $\Re(X)$ points towards $B(0,\delta)\times B(0,\epsilon)$ at $T_{iX}^{\epsilon,j}(0)$ without lack of generality.

Suppose that $e_{\Lambda,\Lambda',j}^{\lambda,\lambda',+} \neq \infty$. We have

$$\mathcal{E}_\beta = \mathcal{E}_+ = \{(x,t) \in B(0,\delta)\times\mathbb{C} : \eta_+ \geq |t| \geq \rho_+|x|\}.$$

We define

$$\mathcal{E}_+' = \{(x,t) \in B(0,\delta)\times\mathbb{C} : \eta_+ \geq |t| \geq (2-1/4)\rho_+|x|\}.$$

The construction of multi-transversal flows implies that $\Re(\aleph_{\Lambda,\lambda}X) \equiv \Re(\aleph_{\Lambda',\lambda'}X)$ in $\mathcal{E}_+' \cap \{x \in (0,\delta)I_{\Lambda,\Lambda'}^{\lambda,\lambda'}\}$. Given $k \in \{1,2\}$ we consider the continuous section $\tau_{k,+} : (0,\delta)I_{\Lambda,\Lambda'}^{\lambda,\lambda'} \to \mathcal{E}_+$ defined by

$$\tau_{k,+}(x) = \Gamma_x[0,\infty) \cap \{|t| = (2-k/16)|x|\}.$$

We can define $\tau_{1,-}$ and $\tau_{2,-}$ by replacing $\Gamma_x[0,\infty)$ with $\Gamma_x(-\infty,0]$ if $e_{\Lambda,\Lambda',j}^{\lambda,\lambda',-} \neq \infty$.

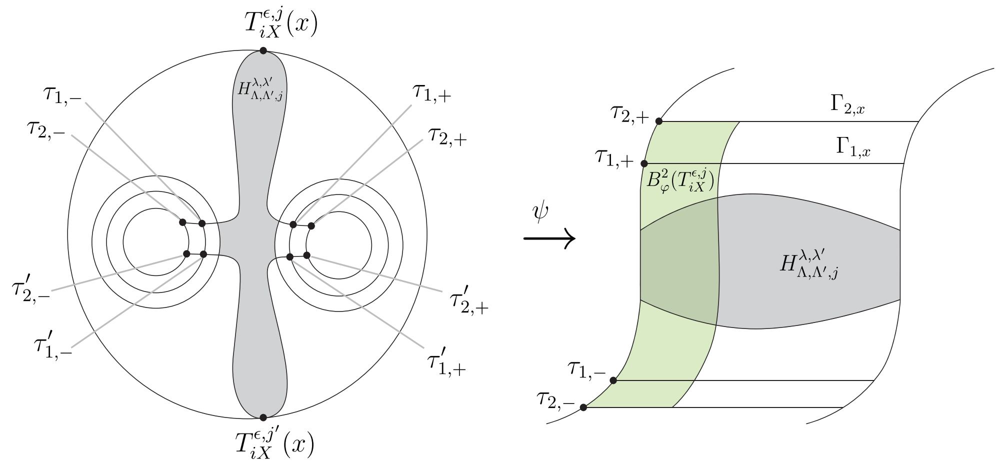


FIGURE 8. Case $R_{\Lambda,\Lambda',j}^{\lambda,\lambda'} = 2$

We claim that $H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}(r,\tilde{\lambda}) \cap \{|t| = (2-k/16)|x|\}$ adheres to the point $\tau_{k,+}(0,\lambda_0)$ when $(r,\tilde{\lambda}) \to (0,\lambda_0)$ for $k \in \{1,2\}$. Let us study the properties of $\tau_{1,+}$ and $\tau_{2,+}$. Those of $\tau_{1,-}$ and $\tau_{2,-}$ are analogous. By applying several times prop. 4.2 we obtain that $\tau_{1,+}$ and $\tau_{2,+}$ are asymptotically continuous. The points $\tau_{1,+}(0,\tilde{\lambda})$ and $\tau_{2,+}(0,\tilde{\lambda})$ belong to the same element $\gamma_{\tilde{\lambda}}$ of $Tr_{\leftarrow\infty}(\aleph_{\mathcal{E}_+}X_\beta(\tilde{\lambda}))$ for any $\tilde{\lambda} \in I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$ (see def. 3.3). Consider the tangent section $T_{iX}^{\epsilon,j'}(x)$ containing the other point in $T_{iX}^{\epsilon}(x) \cap \overline{H_{\Lambda,j}^{\lambda}}$. Then we can replace j with j' to obtain asymptotically continuous sections $\tau'_{1,+}(r,\tilde{\lambda})$ and $\tau'_{2,+}(r,\tilde{\lambda})$. Corollary 4.1 implies $\tau'_{k,+}(0,\tilde{\lambda}) = \tau_{k,+}(0,\tilde{\lambda})$ for all $\tilde{\lambda} \in I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$ and $k \in \{1,2\}$. Thus $H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}(r,\tilde{\lambda}) \cap \{|t| = (2-k/16)|x|\}$ adheres to the point $\tau_{k,+}(0,\lambda_0)$ when $(r,\tilde{\lambda}) \to (0,\lambda_0)$.

We define

$$\Gamma_{k,x} = \Gamma(X, \tau_{k,+}(x), \overline{H_{\Lambda,j}^{\lambda}}) \text{ for } k \in \{1,2\} \text{ and } x \in (0,\delta)I_{\Lambda,\Lambda'}^{\lambda,\lambda'}.$$

The vector field $\Re(X_\beta(\lambda_0))$ is transversal to γ_{λ_0} at $\tau_{1,+}(0,\lambda_0)$ and $\tau_{2,+}(0,\lambda_0)$. Hence $\Gamma_{k,r,\tilde{\lambda}}$ tends to $\tau_{k,+}(0,\lambda_0)$ when $(r,\tilde{\lambda}) \to (0,\lambda_0)$ for all $\lambda_0 \in I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$ and $k \in \{1,2\}$. Hence the trajectories $\Gamma_{1,x}$ and $\Gamma_{2,x}$ are contained in $\mathcal{E}_+' \setminus \tilde{\mathcal{E}}_+$ and then in $H_{\Lambda,j}^{\lambda} \cap H_{\Lambda',j}^{\lambda'}$ for any $x \in (0,\delta)I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. Given $k \in \{1,2\}$ we define

$$H_{j,k}^{\lambda_0}(x) = H_{\Lambda,j}^{\lambda_0}(x) \cap \{Im(\psi) \in (Im[\psi(\tau_{k,-}(x))], Im[\psi(\tau_{k,+}(x))])\}$$

for $\lambda_0 \in \{\lambda, \lambda'\}$ and $x \in (0,\delta) I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. We define $Im(\psi \circ \tau_{k,\pm}) = \pm\infty$ if $e_{\Lambda,\Lambda',j}^{\lambda,\lambda',\pm} = \infty$. The previous discussion implies $H_{\Lambda,\Lambda',j}^{\lambda,\lambda'} \subset H_{j,k}^{\lambda} = H_{j,k}^{\lambda'} \subset H_{\Lambda,j}^{\lambda} \cap H_{\Lambda',j}^{\lambda'}$ for $k \in \{1,2\}$.

We want to study the sets $Im(\tilde{\psi}_{j,\lambda}^{\varphi})(H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}(r,\lambda_0))$ for $(r,\lambda_0) \in [0,\delta) \times I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. Let $\psi_{\lambda_0}^{\beta}$ be a Fatou coordinate of $X_\beta(\lambda_0)$ defined in the neighborhood of γ_{λ_0}. Given $k \in \{1,2\}$, $l \in \{+,-\}$ consider the function $F_{k,l} : (0,\delta) \times I_{\Lambda,\Lambda'}^{\lambda,\lambda'} \to \mathbb{C}$ defined by

$$F_{k,l}(r,\tilde{\lambda}) = |r|^{\iota(\mathcal{E}_l)} \psi(\tau_{k,l}(r,\tilde{\lambda})).$$

It extends continuously to $[0,\delta) \times I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$ by cor. 4.2. Moreover $F_{k,+}(\{0\} \times I_{\Lambda,\Lambda'}^{\lambda,\lambda'}) \subset \mathbb{H}$ and $F_{k,-}(\{0\} \times I_{\Lambda,\Lambda'}^{\lambda,\lambda'}) \subset -\mathbb{H}$ for $k \in \{1,2\}$. We also have

$$(F_{2,+} - F_{1,+})(0,\lambda_0) = \psi_{\lambda_0}^{\beta}(\tau_{2,+}(0,\lambda_0)) - \psi_{\lambda_0}^{\beta}(\tau_{1,+}(0,\lambda_0)) \in \aleph_{\mathcal{E}_+}\mathbb{R}^+ \subset \mathbb{H}$$

and $(F_{1,-} - F_{2,-})(0,\lambda_0) \in \mathbb{H}$ for any $\lambda_0 \in I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. There exists $\zeta \in \mathbb{R}^+$ such that $Im(F_{2,+} - F_{1,+})(0,\lambda_0) \geq 2\zeta$ and $Im(F_{1,-} - F_{2,-})(0,\lambda_0) \geq 2\zeta$ for any $\lambda_0 \in I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. Since $H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}(r,\lambda_0) \subset H_{j,1}^{\lambda}(r,\lambda_0)$ for any $(r,\lambda_0) \in [0,\delta) \times I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$ and $\tilde{\psi}_{j,\lambda}^{\varphi} - \psi$ is bounded (prop. 5.6) we obtain

$$Im(\tilde{\psi}_{j,\lambda}^{\varphi})(H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}(r,\lambda_0)) \subset \left(\frac{Im(F_{1,-})(0,\lambda_0) - \zeta/2}{r^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',-}}}, \frac{Im(F_{1,+})(0,\lambda_0) + \zeta/2}{r^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',+}}} \right)$$

for all $\lambda_0 \in I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$ and $r \in [0,\delta)$.

Consider the domain $B_\varphi(T_{iX}^{\epsilon,j}(x))$ enclosed by Γ_x and $\varphi(\Gamma_x)$. We define

$$B_\varphi^2(T_{iX}^{\epsilon,j}(x)) = B_\varphi(T_{iX}^{\epsilon,j}(x)) \cap H_{j,2}^{\lambda}(x)$$

for $x \in (0,\delta) I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. The function $\psi_{j,\lambda,\lambda'}^{\varphi}$ is then well-defined in

$$\cup_{(r,\tilde{\lambda}) \in [0,\delta) \times I_{\Lambda,\Lambda'}^{\lambda,\lambda'}} (\{r\tilde{\lambda}\} \times e^{2\pi i \tilde{\psi}_{j,\lambda}^{\varphi}}[B_\varphi^2(T_{iX}^{\epsilon,j}(x))])$$

since $B_\varphi(T_{iX}^{\epsilon,j}(x)) \setminus \varphi(\Gamma_x)$ is a fundamental domain of $\varphi_{|H_{\Lambda,j}^{\lambda}(x)}$. Moreover we have

$$e^{2\pi i \tilde{\psi}_{j,\lambda}^{\varphi}}(B_\varphi^2(T_{iX}^{\epsilon,j}(r,\tilde{\lambda})) \supset \mathbb{S}^1 e^{-2\pi z} \left(\frac{Im(F_{2,-})(0,\tilde{\lambda}) + \zeta/2}{r^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',-}}}, \frac{Im(F_{2,+})(0,\tilde{\lambda}) - \zeta/2}{r^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',+}}} \right)$$

for all $\tilde{\lambda} \in I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$ and $r \in (0,\delta)$. It remains to show that $\psi_{j,\lambda,\lambda'}^{\varphi}$ is well-defined in $(x, e^{2\pi i \tilde{\psi}_{j,\lambda}^{\varphi}})(H_{\Lambda,\Lambda',j}^{\lambda,\lambda'})$. More precisely, given $Q = (x_0,y_0) \in H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}$ there exist $Q_0 \in B_\varphi(T_{iX}^{\epsilon,j}(x_0))$ and $k \geq 0$ such that $Q_0, \ldots, \varphi^k(Q_0) \in H_{\Lambda,j}^{\lambda}$ and $Q = \varphi^k(Q_0)$. It suffices to prove that $\{Q_0, \ldots, \varphi^k(Q_0)\}$ is contained in $H_{\Lambda',j}^{\lambda'}$ and $Q_0 \in B_\varphi^2(T_{iX}^{\epsilon,j}(x_0))$ since then $\tilde{\psi}_{j,\Lambda,\lambda}^{\varphi} - \tilde{\psi}_{j,\Lambda',\lambda'}^{\varphi}$ is constant along orbits of φ.

We have $|\tilde{\psi}_{j,\lambda}^{\varphi} - \psi| \leq M$ in $H_{\Lambda,j}^{\lambda}$ for some $M > 0$ by prop. 5.6. Thus we obtain $|\psi \circ \varphi^l(Q_0) - (\psi(Q_0) + l)| \leq 2M$ for any $0 \leq l \leq k$. Since $Q \in H_{j,1}^{\lambda}(x_0)$ and

$$\lim_{x \in (0,\delta) I_{\Lambda,\Lambda'}^{\lambda,\lambda'},\ x \to 0} Im(\psi(\tau_{2,\pm}(x))) - Im(\psi(\tau_{1,\pm}(x))) = \pm\infty$$

for $e_{\Lambda,\Lambda',j}^{\lambda,\lambda',\pm} \neq \infty$ we deduce that $Q_0, \ldots, \varphi^k(Q_0) \in H_{j,2}^{\lambda} \subset H_{\Lambda,j}^{\lambda} \cap H_{\Lambda',j}^{\lambda'}$.

5.4.2. *Case* $R^{\lambda,\lambda'}_{\Lambda,\Lambda',j} = 1$. First we introduce the setup that we use for $R^{\lambda,\lambda'}_{\Lambda,\Lambda',j} = 1$. Then we discuss the topological properties that allow to adapt the proof for the case $R^{\lambda,\lambda'}_{\Lambda,\Lambda',j} = 2$ to the new setting. Such properties are key to prove the results in subsections 5.6 and 5.7.

Step 1. Let us introduce some definitions. If $Ex^{\lambda,\lambda'}_{\Lambda,\Lambda',j} = \emptyset$ we have $H^{\lambda}_{\Lambda,j} = H^{\lambda'}_{\Lambda',j}$ and $\tilde{\psi}^{\varphi}_{j,\lambda} \equiv \tilde{\psi}^{\varphi}_{j,\lambda'}$. Otherwise we denote $\mathcal{E} = \mathcal{E}_{+} = \mathcal{E}_{-}$ and $e^{\lambda,\lambda'}_{\Lambda,\Lambda',j} = e^{\lambda,\lambda',+}_{\Lambda,\Lambda',j} = e^{\lambda,\lambda',-}_{\Lambda,\Lambda',j}$ (see def. 5.16). Denote $\aleph = \aleph_{\Lambda,\lambda}$. We have

$$\mathcal{E}_{\beta} = \mathcal{E} = \{(x,t) \in B(0,\delta) \times \mathbb{C} : \eta \geq |t| \geq \rho|x|\}.$$

Denote $\mathcal{C} = \mathcal{C}_{\beta}$. We define

$$\mathcal{E}' = \{(x,t) \in B(0,\delta) \times \mathbb{C} : \eta \geq |t| > (2-1/4)\rho|x|\}.$$

We define $\rho^0 = 2\rho$, $\rho^4 = (2-1/4)\rho$, $H^{\lambda,\lambda'}_{j,0} = H^{\lambda,\lambda'}_{\Lambda,\Lambda',j}$ and

$$H^{\lambda,\lambda'}_{j,4} = [(H^{\lambda}_{\Lambda,j})^{L_j} \cap (\mathcal{E}' \cup \cup_{\mathcal{B} \in En^{\lambda,\lambda'}_{\Lambda,\Lambda',j}} \mathcal{B})] \cap \{x \in [0,\delta) I^{\lambda,\lambda'}_{\Lambda,\Lambda'}\}.$$

We have $H^{\lambda,\lambda'}_{\Lambda,\Lambda',j} \subset H^{\lambda,\lambda'}_{j,4} \subset H^{\lambda}_{\Lambda,j} \cap H^{\lambda'}_{\Lambda',j}$. Denote $\Gamma_x = \Gamma(\aleph_{\Lambda,\lambda} X, T^{\epsilon,j}_{iX}(x), T_0)$. Let $x \in (0,\delta) I^{\lambda,\lambda'}_{\Lambda,\Lambda'}$. The boundary of $H^{\lambda,\lambda'}_{j,k}(x)$ is composed of a piece of trajectory $\Gamma_x[s^{-}_k(x), s^{+}_k(x)]$ of Γ_x and a closed arc $arc_k(x) \subset \{|t| = \rho^k |x|\}$ for $k \in \{0,4\}$. The section $\tau^l_k(r,\tilde{\lambda}) = \Gamma_x(s^l_k(r,\tilde{\lambda}))$ is asymptotically continuous in $[0,\delta) \times I^{\lambda,\lambda'}_{\Lambda,\Lambda'}$ for $k \in \{0,4\}$ and $l \in \{-,+\}$. Moreover $\tau^{-}_0(0,\lambda_0)$ and $\tau^{-}_4(0,\lambda_0)$ belong to an element γ_{-,λ_0} of $Tr_{\to\infty}(\aleph_{\mathcal{E}} X_{\beta}(\lambda_0))$ whereas $\tau^{+}_0(0,\lambda_0)$ and $\tau^{+}_4(0,\lambda_0)$ belong to an element γ_{+,λ_0} of $Tr_{\leftarrow\infty}(\aleph_{\mathcal{E}} X_{\beta}(\lambda_0))$ for any $\lambda_0 \in I^{\lambda,\lambda'}_{\Lambda,\Lambda'}$. The functions $arc_0(r,\tilde{\lambda})$ and $arc_4(r,\tilde{\lambda})$ defined in $(0,\delta) \times I^{\lambda,\lambda'}_{\Lambda,\Lambda'}$ admit a continuous extension to $[0,\delta) \times I^{\lambda,\lambda'}_{\Lambda,\Lambda'}$ (we are considering the Hausdorff topology for compact sets). The extrema of $arc_k(0,\lambda_0)$ are the points $\tau^{-}_k(0,\lambda_0)$ and $\tau^{+}_k(0,\lambda_0)$. We define

$$\gamma^k_{+,\lambda_0} = \gamma_{+,\lambda_0} \cap (\mathbb{C} \setminus B(0,\rho^k)) \text{ and } \gamma^k_{-,\lambda_0} = \gamma_{-,\lambda_0} \cap (\mathbb{C} \setminus B(0,\rho^k)).$$

Let $\tilde{H}^{\lambda,\lambda'}_{j,k}(0,\lambda_0)$ be the connected component of $\mathbb{C} \setminus [\gamma^k_{+,\lambda_0} \cup \gamma^k_{-,\lambda_0} \cup arc_k(0,\lambda_0)]$ not containing 0. We denote $\psi^{\mathcal{C}}_{\lambda_0}$ the Fatou coordinate of $X_{\beta}(\lambda_0)$ defined in the neighborhood of $\overline{\tilde{H}^{\lambda,\lambda'}_{j,k}(0,\lambda_0)}$ such that $\psi^{\mathcal{C}}_{\lambda_0}(\infty) = 0$. Denote $\psi = \psi^X_{H^{\lambda}_{\Lambda,j},L_j}$.

Step 2. We present the main properties of the sets $H^{\lambda,\lambda'}_{j,0}$ and $H^{\lambda,\lambda'}_{j,k}$. Our goal is showing that the qualitative shape of $H^{\lambda,\lambda'}_{\Lambda,\Lambda',j}$ is as in figure (9). Suppose that $\Re(X)$ points towards $B(0,\delta) \times B(0,\epsilon)$ at $T^{\epsilon,j}_{iX}(0)$ without lack of generality.

Lemma 5.7. *Let* $X \in \mathcal{X}_{tp1}(\mathbb{C}^2,0)$*. Consider* $\Lambda, \Lambda' \in \mathcal{M}$*,* $\lambda, \lambda' \in \mathbb{S}^1$ *and* $j \in \mathbb{Z}/(2\nu(\mathcal{E}_0)\mathbb{Z})$*. Assume* $R^{\lambda,\lambda'}_{\Lambda,\Lambda',j} = 1$*. Then* $H^{\lambda,\lambda'}_{j,k}(x)$ *is* $\Re(X)$*-convex for all* $k \in \{0,4\}$ *and* $x \in [0,\delta) I^{\lambda,\lambda'}_{\Lambda,\Lambda'}$*.*

Proof. If $Ex^{\lambda,\lambda'}_{\Lambda,\Lambda',j} = \emptyset$ or $x = 0$ the vector field $\Re(X)$ is transversal and then almost transversal to $\partial H^{\lambda,\lambda'}_{j,k}(x) \setminus SingX$. Thus we can suppose $Ex^{\lambda,\lambda'}_{\Lambda,\Lambda',j} \neq \emptyset$ and $x \neq 0$ by lemma 5.5. Suppose that $\Re(X)$ points towards $B(0,\delta) \times B(0,\epsilon)$ at $T^{\epsilon,j}_{iX}(0)$ without lack of generality.

The vector field $\Re(X)$ is transversal to $\partial H^{\lambda,\lambda'}_{j,k}(x) \setminus (\{\tau^{+}_k(x), \tau^{-}_k(x)\} \cup T\mathcal{C}^{\rho^k}_X(x))$ for $k \in \{0,4\}$ and $x \in (0,\delta) I^{\lambda,\lambda'}_{\Lambda,\Lambda'}$. Moreover $\Re(X)$ is almost transversal to $\partial H^{\lambda,\lambda'}_{j,k}(x)$ at

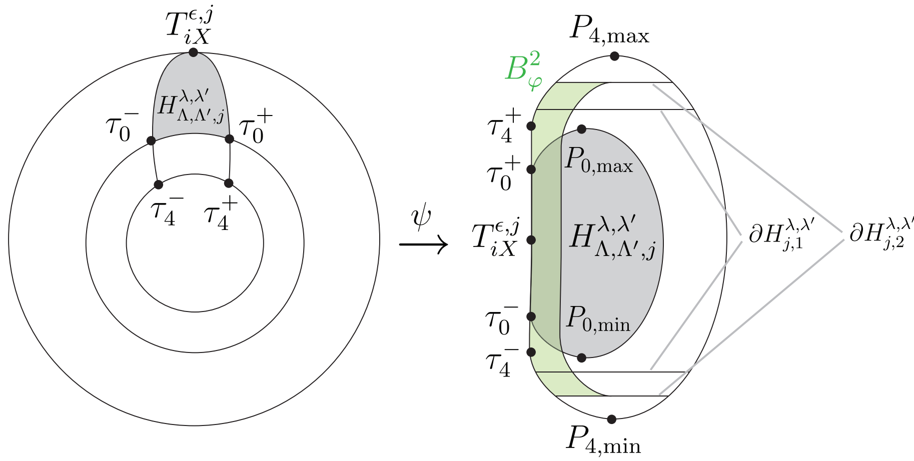


FIGURE 9. Case $R_{\Lambda,\Lambda',j}^{\lambda,\lambda'} = 1$

the points in $TC_X^{\rho^k}(x)$ since they are convex. The curve Γ_x is transversal to the set $\{|t| = \rho^k|x|\}$ at $\tau_k^+(x)$ and $\tau_k^-(x)$ for $k \in \{0,4\}$ and $x \in (0,\delta)I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$ since the points of $TC_{\aleph_{\Lambda,\lambda}X}^{\rho^k}(x)$ are convex (lemma 4.2). Moreover the curve $\psi(arc_k(x) \setminus \tau_k^\pm(x))$ is to the right of $\psi(\Gamma_x)$ in the neighborhood of $\psi(\tau_k^\pm(x))$. Thus a neighborhood of $\psi(\tau_k^\pm(x))$ in $\psi(\overline{H_{j,k}^{\lambda,\lambda'}(x)})$ does not contain points to the left of $\psi(\Gamma_x)$. The point $\psi(\exp(sX)(\tau_k^\pm(x)))$ is to the left of $\psi(\Gamma_x)$ and then $\exp(sX)(\tau_k^\pm(x)) \notin \overline{H_{j,k}^{\lambda,\lambda'}(x)}$ for any $s < 0$ in a neighborhood of 0. Therefore $\Re(X)$ is almost transversal to $\partial H_{j,k}^{\lambda,\lambda'}(x)$ at both $\tau_k^+(x)$ and $\tau_k^-(x)$ for all $k \in \{0,4\}$ and $x \in (0,\delta)I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. We obtain that $\Re(X)$ is almost transversal to $\partial H_{j,k}^{\lambda,\lambda'}(x)$ at any of its points. Lemma 5.5 implies that $H_{j,k}^{\lambda,\lambda'}(x)$ is $\Re(X)$-convex for all $k \in \{0,4\}$ and $x \in (0,\delta)I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. $\square$

The set $\overline{H_{j,k}^{\lambda,\lambda'}(x)}$ does not contain pieces of trajectories of $Re(X)$ for all $k \in \{0,4\}$ and $x \in (0,\delta)I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. Hence the maximum (resp. the minimum) of $Im(\psi)$ in $\overline{H_{j,k}^{\lambda,\lambda'}(x)}$ is attained at a unique point $P_{k,\max}(x)$ (resp. $P_{k,\min}(x)$) by corollary 5.2. Since Γ_x is transversal to $\Re(X)$ the points $P_{k,\max}(x)$ and $P_{k,\min}(x)$ belong to $\{|t| = \rho^k|x|\}$. The points in $TC_X^{\rho^k}(x)$ are convex and then they can not be accessed by trajectories of $\Re(X)$ lying outside $\{|t| \leq \rho^k|x|\}$. We deduce that

$$\overline{H_{j,k}^{\lambda,\lambda'}(x)} \cap TC_X^{\rho^k}(x) \subset \{P_{k,\min}(x), P_{k,\max}(x)\} \ \forall k \in \{0,4\} \ \forall x \in (0,\delta)I_{\Lambda,\Lambda'}^{\lambda,\lambda'}.$$

We proceed in an analogous way with $\Re(X_\beta(\lambda_0))$ and $\tilde{H}_{j,k}^{\lambda,\lambda'}(0,\lambda_0)$ for $\lambda_0 \in I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. There exists a unique point $P_{k,\max}(0,\lambda_0)$ (resp. $P_{k,\max}(0,\lambda_0)$) where is attained the maximum (resp. the minimum) of $Im(\psi_{\lambda_0}^{\mathcal{C}})$ in $\overline{\tilde{H}_{j,k}^{\lambda,\lambda'}(0,\lambda_0)}$. We obtain

$$\{P_{k,\min}(0,\lambda_0), P_{k,\max}(0,\lambda_0)\} \subset \partial B(0,\rho^k).$$

Moreover $\Re(X_\beta(\lambda_0))$ is transversal to $arc_k(0,\lambda_0)$ except maybe at $P_{k,\min}(0,\lambda_0)$ and $P_{k,\max}(0,\lambda_0)$. The sections $P_{k,\min}$ and $P_{k,\max}$ are asymptotically continuous. Given

$k \in \{0,4\}$ and $l \in \{\min, \max\}$ we define the functions

$$F_{k,l}(r,\lambda_0) = Im(|r|^{\iota(\mathcal{E})}\psi(P_{k,l}(r,\lambda_0)))$$

for $(r,\lambda_0) \in (0,\delta) \times I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. Moreover they extend continuously to $[0,\delta) \times I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$ (prop. 4.2) by defining

$$F_{k,\min}(0,\lambda_0) = Im(\psi_{\lambda_0}^{\mathcal{C}}(P_{k,\min}(0,\lambda_0))) \text{ and } F_{k,\max}(0,\lambda_0) = Im(\psi_{\lambda_0}^{\mathcal{C}}(P_{k,\max}(0,\lambda_0)))$$

for $k \in \{0,4\}$ and $\lambda_0 \in I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. We have

$$F_{4,\max}(0,\lambda_0) > F_{0,\max}(0,\lambda_0) \geq Im(\psi_{\lambda_0}^{\mathcal{C}}(\tau_0^+(0,\lambda_0))) > 0$$

for any $\lambda_0 \in I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. The last inequality is a consequence of prop. 4.2. Analogously we obtain

$$F_{4,\min}(0,\lambda_0) < F_{0,\min}(0,\lambda_0) \leq Im(\psi_{\lambda_0}^{\mathcal{C}}(\tau_0^-(0,\lambda_0))) < 0$$

for any $\lambda_0 \in I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. There exists $\zeta \in \mathbb{R}^+$ such that

$$(F_{4,\max} - F_{0,\max})(0,\lambda_0) \geq 2\zeta \leq (F_{0,\min} - F_{4,\min})(0,\lambda_0) \ \forall \lambda_0 \in I_{\Lambda,\Lambda'}^{\lambda,\lambda'}.$$

We define $\zeta_1 = 7\zeta/4$, $\zeta_2 = \zeta/4$, $\zeta_3 = \zeta/8$ and

$$H_{j,k}^{\lambda,\lambda'}(r,\lambda_0) = H_{j,4}^{\lambda,\lambda'}(r,\lambda_0) \cap \left\{ Im(\psi) \in \left(\frac{F_{4,\min}(0,\lambda_0) + \zeta_k}{r^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda'}}}, \frac{F_{4,\max}(0,\lambda_0) - \zeta_k}{r^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda'}}} \right) \right\}$$

for $(r,\lambda_0) \in (0,\delta) \times I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$ and $k \in \{1,2,3\}$. We have

$$H_{\Lambda,\Lambda',j}^{\lambda,\lambda'} = H_{j,0}^{\lambda,\lambda'} \subset H_{j,1}^{\lambda,\lambda'} \subset H_{j,2}^{\lambda,\lambda'} \subset H_{j,3}^{\lambda,\lambda'} \subset H_{j,4}^{\lambda,\lambda'} \subset H_{\Lambda,j}^{\lambda} \cap H_{\Lambda',j}^{\lambda'}.$$

Let $x \in (0,\delta) I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. The set $\overline{H_{j,3}^{\lambda,\lambda'}(x)} \cap \partial H_{j,4}^{\lambda,\lambda'}(x)$ is a union of two curves, namely a curve ϖ_x containing $T_{iX}^{\epsilon,j}(x)$ a curve ϖ_x' contained in $arc_4(x)$.

There exists $\theta' > 0$ such that the angle between $\Re(X_\beta(\lambda_0))$ and $arc_4(0,\lambda_0)$ at Q is greater than $\theta' \in \mathbb{R}^+$ for any $\lambda_0 \in I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$ and any $Q \in arc_4(0,\lambda_0)$ such that

$$Im(\psi_{\lambda_0}^{\mathcal{C}}(Q))) \in [F_{4,\min}(0,\lambda_0) + \zeta/8, F_{4,\max}(0,\lambda_0) - \zeta/8].$$

Thus the angle between $\Re(X)$ and $arc_4(x)$ at Q is greater than θ' for any x in $(0,\delta) I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$ and any $Q \in arc_4(x) \cap (\varpi_x \cup \varpi_x')$. The angle between $\Re(X)$ and Γ_x at any point $Q \in \Gamma_x$ is bounded by below by a positive constant not depending on $x \in (0,\delta) I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$ or Q. Thus the angle between $\Re(X)$ and $\varpi_x \cup \varpi_x'$ at any of its points is greater than $\theta'' \in \mathbb{R}^+$ for any $x \in (0,\delta) I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. Analogously as in Step 3 of subsection 5.2 we can prove that $\varphi(\varpi_x) \cup \varphi^{-1}(\varpi_x')$ is transversal to $\Re(X)$ at any of its points for any $x \in (0,\delta) I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$.

Step 3. Let $x \in (0,\delta) I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. Let $B_{\varphi}^2(T_{iX}^{\epsilon,j}(x))$ be the closure of the bounded connected component of the complementary of $\varpi_x \cup \varphi(\varpi_x) \cup (\partial H_{j,2}^{\lambda,\lambda'}(x) \setminus \varpi_x')$. As in subsection 5.4.1 we obtain that given $Q \in H_{j,1}^{\lambda,\lambda'}(x)$ there exist $Q_0 \in B_{\varphi}^2(T_{iX}^{\epsilon,j}(x))$ and $k \geq 0$ such that $Q_0, \ldots, \varphi^k(Q_0) \in H_{j,2}^{\lambda,\lambda'}$ and $Q = \varphi^k(Q_0)$. We can proceed as in subsection 5.4.1 to prove prop. 5.8.

5.5. **Flatness properties of Fatou coordinates.** Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$. Let $\Upsilon = \exp(X)$ be a 2-convergent normal form. Consider $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}}) \in \mathcal{M}$ and the dynamical splitting $\mathcal{F}_\Lambda$ in remark 4.11.

We want to prove that $\tilde{\psi}^{\varphi}_{j,\Lambda,\lambda} - \tilde{\psi}^{\varphi}_{j,\Lambda',\lambda'}$ is exponentially small and then flat up to an additive function of x. We use two ingredients, namely the boundness of $\tilde{\psi}^{\varphi}_{j,\Lambda,\lambda} - \psi^{X}_{H^{\lambda}_{\Lambda,j},L_j}$ (subsection 5.2) and the study of the shape of $H^{\lambda,\lambda'}_{\Lambda,\Lambda',j}$ provided by prop. 5.8. Flatness is the key property to prove multi-summability of Fatou coordinates of elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$.

Definition 5.19. *Let $\lambda \in \mathbb{S}^1$ and $j \in \mathcal{D}(\varphi)$. Denote $(0, y_0) = T^{\epsilon,j}_{iX}(0)$. We define*

$$\ddot{\psi}^{\varphi}_{j,\Lambda,\lambda}(x,y) = \tilde{\psi}^{\varphi}_{j,\Lambda,\lambda}(x,y) - \tilde{\psi}^{\varphi}_{j,\Lambda,\lambda}(x,y_0) \quad \text{for } (x,y) \in H^{\lambda}_{\Lambda,j}.$$

The function $\ddot{\psi}^{\varphi}_{j,\Lambda,\lambda}$ is a Fatou coordinate of φ in $H^{\lambda}_{\Lambda,j}$ such that $\ddot{\psi}^{\varphi}_{j,\Lambda,\lambda}(x,y_0) \equiv 0$. We denote $\ddot{\psi}^{\varphi}_{j,\lambda} = \ddot{\psi}^{\varphi}_{j,\Lambda,\lambda}$ if Λ is implicit.

Let us remark that the Fatou coordinate $\tilde{\psi}^{\varphi}_{j,\lambda}$ can be extended to a neighborhood of $([0,\delta)I^{\lambda}_{\Lambda}) \times \{y_0\}$ by using the equation $\tilde{\psi}^{\varphi}_{j,\lambda} \circ \varphi \equiv \tilde{\psi}^{\varphi}_{j,\lambda} + 1$.

We have the normalizing conditions

$$\ddot{\psi}^{\varphi}_{j,\lambda}(x,y_0) \equiv 0 \text{ and } (\tilde{\psi}^{\varphi}_{j,\lambda} - \psi^{X}_{H^{\lambda}_{\Lambda,j},L_j})(\omega^{\aleph_{\Lambda,\lambda}X}(H^{\lambda}_{\Lambda,j}(x))) \equiv 0.$$

The latter one is not a good choice to compare $\tilde{\psi}^{\varphi}_{j,\lambda}$ and $\tilde{\psi}^{\varphi}_{j,\lambda'}$ since for example we could have $\omega^{\aleph_{\Lambda,\lambda}X}(H^{\lambda}_{\Lambda,j}(x))) \neq \omega^{\aleph_{\Lambda',\lambda'}X}(H^{\lambda'}_{\Lambda',j}(x)))$ for any $x \in (0,\delta)I^{\lambda,\lambda'}_{\Lambda,\Lambda'}$.

In the next proposition we denote $e^{-K/|x|^{\infty}} \equiv 0$ by convention.

Proposition 5.9. *Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ with 2-convergent normal form $\exp(X)$. Let $\Lambda, \Lambda' \in \mathcal{M}$. Consider $\lambda, \lambda' \in \mathbb{S}^1$ and $j \in \mathcal{D}(\varphi)$. Then there exists $K \in \mathbb{R}^+$ such that*

$$|\ddot{\psi}^{\varphi}_{j,\Lambda,\lambda} - \ddot{\psi}^{\varphi}_{j,\Lambda',\lambda'}|(x,y) \leq \frac{e^{-K/|x|^{e^{\lambda,\lambda',-}_{\Lambda,\Lambda',j}}}}{2} + \frac{e^{-K/|x|^{e^{\lambda,\lambda',+}_{\Lambda,\Lambda',j}}}}{2} \leq e^{-K/|x|^{\tilde{e}_{d^{\lambda,\lambda'}_{\Lambda,\Lambda'}+1}}}$$

for any $(x,y) \in H^{\lambda,\lambda'}_{\Lambda,\Lambda',j}$.

Proof. The functions $\tilde{\psi}^{\varphi}_{j,\lambda} - \psi^{X}_{H^{\lambda}_{\Lambda,j},L_j}$ and $\tilde{\psi}^{\varphi}_{j,\lambda'} - \psi^{X}_{H^{\lambda'}_{\Lambda',j},L_j}$ are bounded by prop. 5.6. Thus there exists $M \in \mathbb{R}^+$ such that $|\psi^{\varphi}_{j,\lambda,\lambda'}| \leq M$ in

$$\cup_{(r,\tilde{\lambda}) \in [0,\delta) \times I^{\lambda,\lambda'}_{\Lambda,\Lambda'}} (\{r\tilde{\lambda}\} \times [B(0, \tilde{\kappa}_-(r,\tilde{\lambda})) \setminus \overline{B}(0, \tilde{\kappa}_+(r,\tilde{\lambda}))]),$$

see prop. 5.8. We obtain

$$\psi^{\varphi}_{j,\lambda,\lambda'}(x,z) = a_0(x) + \sum_{k \in \mathbb{N}} a_k(x) z^k + \sum_{k \in \mathbb{N}} \frac{a_{-k}(x)}{z^k}$$

by considering the Laurent series of $\psi^{\varphi}_{j,\lambda,\lambda'}$. We have

$$a_k(x) = \frac{1}{2\pi i} \int_{|z| = \tilde{\kappa}_-(x)} \frac{\psi^{\varphi}_{j,\lambda,\lambda'}(x,z)}{z^{k+1}} dz \implies |a_k(x)| \leq \frac{M}{\tilde{\kappa}_-(x)^k}$$

for all $k \in \mathbb{N}$ and $x \in (0,\delta)I^{\lambda,\lambda'}_{\Lambda,\Lambda'}$. Analogously we deduce

$$a_{-k}(x) = \frac{1}{2\pi i} \int_{|z| = \tilde{\kappa}_+(x)} \psi^{\varphi}_{j,\lambda,\lambda'}(x,z) z^{k-1} dz \implies |a_k(x)| \leq M \tilde{\kappa}_+(x)^k$$

for all $k \in \mathbb{N}$ and $x \in (0,\delta) I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. We get

$$|\tilde{\psi}_{j,\lambda}^{\varphi}(x,y) - \tilde{\psi}_{j,\lambda'}^{\varphi}(x,y) - a_0(x)| \leq M \sum_{k \in \mathbb{N}} \left(\frac{\kappa_-(x)}{\tilde{\kappa}_-(x)} \right)^k + M \sum_{k \in \mathbb{N}} \left(\frac{\tilde{\kappa}_+(x)}{\kappa_+(x)} \right)^k$$

in $H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}$. By plugging the values of κ_-, $\tilde{\kappa}_-$, κ_+ and $\tilde{\kappa}_+$ in the previous equation we obtain

$$|\tilde{\psi}_{j,\lambda}^{\varphi}(x,y) - \tilde{\psi}_{j,\lambda'}^{\varphi}(x,y) - a_0(x)| \leq M \sum_{k \in \mathbb{N}} e^{-k\zeta/|x|^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',-}}} + M \sum_{k \in \mathbb{N}} e^{-k\zeta/|x|^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',+}}}$$

and then

$$|\tilde{\psi}_{j,\lambda}^{\varphi}(x,y) - \tilde{\psi}_{j,\lambda'}^{\varphi}(x,y) - a_0(x)| \leq 2M(e^{-\zeta/|x|^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',-}}} + e^{-\zeta/|x|^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',+}}})$$

in $H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}$. Denote $(0,y_0) = T_{iX}^{\epsilon,j}(0)$ and $b_0(x) = a_0(x) - \tilde{\psi}_{j,\lambda}^{\varphi}(x,y_0) + \tilde{\psi}_{j,\lambda'}^{\varphi}(x,y_0)$. We have

$$|\ddot{\psi}_{j,\lambda}^{\varphi}(x,y) - \ddot{\psi}_{j,\lambda'}^{\varphi}(x,y) - b_0(x)| \leq 2M(e^{-\zeta/|x|^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',-}}} + e^{-\zeta/|x|^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',+}}})$$

in $H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}$. We obtain

$$|b_0(x)| \leq 2M(e^{-\zeta/|x|^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',-}}} + e^{-\zeta/|x|^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',+}}})$$

for any $x \in (0,\delta) I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$ by evaluating at (x,y_0). This property implies

$$|\ddot{\psi}_{j,\lambda}^{\varphi}(x,y) - \ddot{\psi}_{j,\lambda'}^{\varphi}(x,y)| \leq 4M(e^{-\zeta/|x|^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',-}}} + e^{-\zeta/|x|^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',+}}})$$

for any $(x,y) \in H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}$. It suffices to consider $K \in \mathbb{R}^+$ such that $K < \zeta$. □

5.6. **Extending Fatou coordinates.** Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$. Let $\Upsilon = \exp(X)$ be a k-convergent normal form. Consider $\Lambda, \Lambda' \in \mathcal{M}$. Let $\lambda, \lambda' \in \mathbb{S}^1$ and $j \in \mathcal{D}(\varphi)$. The goal of this subsection is extending $\ddot{\psi}_{j,\Lambda,\lambda}^{\varphi}$ and $\ddot{\psi}_{j,\Lambda,\lambda}^{\varphi} - \ddot{\psi}_{j,\Lambda',\lambda'}^{\varphi}$ to domains slightly bigger than $H_{\Lambda,j}^{\lambda}$ and $H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}$, namely $H_{j,\theta}^{\epsilon,\rho,\lambda}$ and $H_{j,\theta}^{\epsilon,\rho,\lambda,\lambda'}$ respectively. We intend to use such an extension to deduce properties of quasi-analytic type for the infinitesimal generator of φ.

Step 1. Fix $\theta \in (0,\pi/2]$. We define the set $H_{j,\theta}^{\epsilon,\rho}$ where $\ddot{\psi}_{j,\lambda}^{\varphi}$ can be extended.

Suppose without lack of generality that $\Re(X)$ points towards $B(0,\delta) \times B(0,\epsilon)$ at $T_{iX}^{\epsilon,j}(0)$. There exists a unique section $T_X^{\epsilon,+}$ of T_X^{ϵ} such that $T_X^{\epsilon,+}(x)$ is in the arc in $\{x\} \times \partial B(0,\epsilon)$ going from $T_{iX}^{\epsilon,j-1}(x)$ to $T_{iX}^{\epsilon,j}(x)$ in counter clockwise sense for any $x \in B(0,\delta)$ (see def. 4.29). Analogously we can define the section $T_X^{\epsilon,-}$ of T_X^{ϵ} contained in the arc going from $T_{iX}^{\epsilon,j}$ to $T_{iX}^{\epsilon,j+1}$ in counter clockwise sense. The exterior set $\mathcal{E}_0$ is of the form $\{\rho_0|x| \leq |y| \leq \epsilon\}$. Given $\rho \geq 2\rho_0$ we denote $\mathcal{E}_0^{\rho} = \{\rho|x| \leq |y| \leq \epsilon\}$. We define

$$\Gamma_{x,j,0}^{\epsilon,\rho} = \Gamma(iX, T_{iX}^{\epsilon,j}(x), \mathcal{E}_0^{\rho}),\ \Gamma_{x,j,-}^{\epsilon,\rho} = \Gamma(X, T_X^{\epsilon,-}(x), \mathcal{E}_0^{\rho}),\ \Gamma_{x,j,+}^{\epsilon,\rho} = \Gamma(X, T_X^{\epsilon,+}(x), \mathcal{E}_0^{\rho}).$$

We define $H_{j,s}^{\epsilon,\rho}(x_0)$ as the bounded component of $\{(x_0,y) \in \mathbb{C}^2 : |y| > \rho|x_0|\} \setminus \Gamma_{x_0,j,s}^{\epsilon,\rho}$ for $x_0 \in B(0,\delta)$ and $s \in \{0,+,-\}$.

The sets $\Gamma_{0,j,0}^{\epsilon,\rho} \cap \Gamma_{0,j,+}^{\epsilon,\rho}$ and $\Gamma_{0,j,0}^{\epsilon,\rho} \cap \Gamma_{0,j,-}^{\epsilon,\rho}$ are singletons. Indeed we have

$$\Gamma_{0,j,0}^{\epsilon,\rho} \cap \Gamma_{0,j,+}^{\epsilon,\rho} = \{\Gamma_{0,j,0}^{\epsilon,\rho}(h_+)\} \text{ and } \Gamma_{0,j,0}^{\epsilon,\rho} \cap \Gamma_{0,j,-}^{\epsilon,\rho} = \{\Gamma_{0,j,0}^{\epsilon,\rho}(-h_-)\}$$

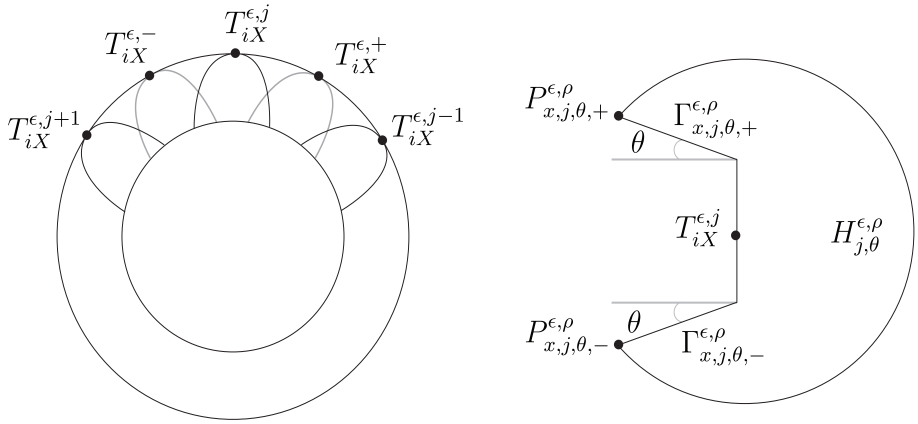


FIGURE 10. Picture of $H_{j,\theta}^{\epsilon,\rho}$ in the $\psi_{L_j}^X$ coordinate on the right

for some $h_-, h_+ \in \mathbb{R}^+$. Consider $M_0 > \max(h_-, h_+)$. Hence the point $\Gamma_{0,j,0}^{\epsilon,\rho}(-M_0)$ belongs to $H_{j,-}^{\epsilon,\rho}(0)$ whereas $\Gamma_{0,j,0}^{\epsilon,\rho}(M_0)$ belongs to $H_{j,+}^{\epsilon,\rho}(0)$. We obtain

$$\Gamma_{x,j,0}^{\epsilon,\rho}(-M_0) \in H_{j,-}^{\epsilon,\rho}(x) \text{ and } \Gamma_{x,j,0}^{\epsilon,\rho}(M_0) \in H_{j,+}^{\epsilon,\rho}(x)\ \forall x \in B(0,\delta).$$

By applying prop. 4.1 to X and $-iX$ we obtain that there exists $\theta_0 \in \mathbb{R}^+$ such that

$$H_{j,0}^{\epsilon,2\rho_0}(x) \cup H_{j,+}^{\epsilon,2\rho_0}(x) \cup H_{j,-}^{\epsilon,2\rho_0}(x) \subset \{x\} \times (0,\epsilon)e^{i[-\theta_0,\theta_0]}$$

for any $x \in B(0,\delta)$. We extend $\psi_{L_j}^X$ to $H_{j,0}^{\epsilon,2\rho_0} \cup H_{j,+}^{\epsilon,2\rho_0} \cup H_{j,-}^{\epsilon,2\rho_0}$ by analytic continuation. We can apply prop. 4.4 to X and $-iX$ to obtain

$$\frac{1}{C_6|y|^{\nu(\mathcal{E}_0)}} \leq |\psi_{L_j}^X| \leq \frac{C_6}{|y|^{\nu(\mathcal{E}_0)}} \text{ in } H_{j,0}^{\epsilon,2\rho_0} \cup H_{j,+}^{\epsilon,2\rho_0} \cup H_{j,-}^{\epsilon,2\rho_0} \tag{20}$$

for some $C_6 \geq 1$. Moreover $\Delta_\varphi = O(y^{k(\nu(\mathcal{E}_0)+1)})$ implies that there exists $K_2 \in \mathbb{R}^+$ such that

$$|\Delta_\varphi(x,y)| \leq \frac{K_2}{(1+|\psi_{L_j}^X(x,y)|)^{\frac{k(\nu(\mathcal{E}_0)+1)}{\nu(\mathcal{E}_0)}}} \quad \forall (x,y) \in H_{j,0}^{\epsilon,2\rho_0} \cup H_{j,+}^{\epsilon,2\rho_0} \cup H_{j,-}^{\epsilon,2\rho_0}. \tag{21}$$

Consider $d_0 \in \mathbb{R}^+$ such that

$$\frac{K_2}{d_0^k} < \min\left(\frac{\sin(\theta)}{2}, \frac{\tan(\theta)}{16\sup_{\mathbb{R}}|\varrho|}\right) \tag{22}$$

where ϱ is the function defined in Step 1 of subsection 5.2. Consider $M \geq M_0$ such that $Im(\psi_{L_j}^X(\Gamma_{x,j,0}^{\epsilon,\rho}(-M))) < -d_0$ and $Im(\psi_{L_j}^X(\Gamma_{x,j,0}^{\epsilon,\rho}(M))) > d_0$ for any $x \in B(0,\delta)$. We define

$$\Gamma_{x,j,\theta,\pm}^{\epsilon,\rho} = \Gamma(-e^{i\mp\theta}X, \Gamma_{x,j,0}^{\epsilon,\rho}(\pm M), \mathcal{E}_0^\rho)(\mathcal{I}(\Gamma_{x,j,\theta,\pm}^{\epsilon,\rho}))$$

where $\mathcal{I}(\Gamma_{x,j,\theta,\pm}^{\epsilon,\rho}) = \mathcal{I}(-e^{i\mp\theta}X, \Gamma_{x,j,0}^{\epsilon,\rho}(\pm M), \mathcal{E}_0^\rho) \cap (\mathbb{R}^+ \cup \{0\})$ and

$$\Gamma_{x,j,\theta}^{\epsilon,\rho} = \Gamma_{x,j,0}^{\epsilon,\rho}[-M,M] \cup \Gamma_{x,j,\theta,+}^{\epsilon,\rho} \cup \Gamma_{x,j,\theta,-}^{\epsilon,\rho}, \quad P_{x,j,\theta,\pm}^{\epsilon,\rho} = \Gamma_{x,j,\theta,\pm}^{\epsilon,\rho}(\sup \mathcal{I}(\Gamma_{x,j,\theta,\pm}^{\epsilon,\rho})).$$

The vector field $\Re(-e^{-i\theta}X)$ is transversal to $\Gamma_{x,j,+}^{\epsilon,\rho}$ and points towards $H_{j,+}^{\epsilon,\rho}$ at $T_X^{\epsilon,+}(x)$ for any $x \in B(0,\delta)$. Therefore $P_{x,j,\theta,+}^{\epsilon,\rho}$ belongs to $\{|y| = \rho|x|\}$. Analogously

$\Re(-e^{i\theta}X)$ is transversal to $\Gamma_{x,j,-}^{\epsilon,\rho}$ and points towards $H_{j,-}^{\epsilon,\rho}$ at $T_X^{\epsilon,-}(x)$ for any $x \in B(0,\delta)$. Thus $P_{x,j,\theta,-}^{\epsilon,\rho}$ belongs to $\{|y| = \rho|x|\}$.

Definition 5.20. *We define $H_{j,\theta}^{\epsilon,\rho}(x_0)$ as the bounded component of*

$$\{(x_0,y) \in \mathbb{C}^2 : |y| > \rho|x_0|\} \setminus \Gamma_{x_0,j,\theta}^{\epsilon,\rho}$$

for $x_0 \in B(0,\delta)$. We have $H_{j,\pi/2}^{\epsilon,\rho}(x) \subset H_{j,\theta}^{\epsilon,\rho}(x)$ for any choice of $\rho \geq 2\rho_0$, $\theta \in (0,\pi/2]$ and $x \in B(0,\delta)$. We denote $\tilde{H}_{j,\theta}^{\epsilon,\rho}(x_0) = H_{j,\theta}^{\epsilon,\rho}(x_0)$ for $x_0 \neq 0$.

Definition 5.21. *We define*

$$H_{j,\theta}^{\epsilon,\rho,\lambda} = H_{\Lambda,j}^{\lambda} \cup \cup_{x \in [0,\delta) I_\Lambda^\lambda} H_{j,\theta}^{\epsilon,\rho}(x), \;\; H_{j,\theta}^{\epsilon,\rho,\lambda,\lambda'} = H_{\Lambda,\Lambda',j}^{\lambda,\lambda'} \cup \cup_{x \in [0,\delta) I_{\Lambda,\Lambda'}^{\lambda,\lambda'}} H_{j,\theta}^{\epsilon,\rho}(x).$$

Step 2. The extension of Fatou coordinates is going to be obtained through iteration. In order to obtain similar asymptotic behavior as in subsection 5.2 we use lemma 5.2. Next result assures that the hypotheses of lemma 5.2 are satisfied.

Lemma 5.8. *Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$. Let $\Upsilon = \exp(X)$ be a k-convergent normal form. Let $j \in \mathcal{D}(\varphi)$ and $\theta \in (0,\pi/2]$. Consider $\rho \geq 2\rho_0$ and $\theta_1 \in [-\theta,\theta]$. Then $H_{j,\theta}^{\epsilon,\rho}(x)$ is $\Re(e^{i\theta_1}X)$-convex for any $x \in B(0,\delta)$. In particular $\psi_{L_j}^X(H_{j,\theta}^{\epsilon,\rho}(x))$ is contained in the set $\psi_{L_j}^X(T_{iX}^{\epsilon,j}(x)) + W_{\theta,M}$ (see def. 5.3).*

Proof. The result is straightforward if $x = 0$. Suppose $x \neq 0$. Suppose without lack of generality that $j \in \mathcal{D}_1(\varphi)$ (see def. 5.17). Denote $\mathcal{C} = \mathcal{C}_0$. The vector field $\Re(e^{i\theta_1}X)$ is transversal to $\partial H_{j,\theta}^{\epsilon,\rho}(x) \setminus (\Gamma_{x,j,\theta}^{\epsilon,\rho} \cup T\mathcal{C}_{e^{i\theta_1}X}^{\rho}(x))$ for any $\theta_1 \in [-\theta,\theta]$. Moreover $\Re(e^{i\theta_1}X)$ is almost transversal to $\partial H_{j,\theta}^{\epsilon,\rho}(x)$ at the set of convex points $T\mathcal{C}_{e^{i\theta_1}X}^{\rho}(x)$ for any $\theta_1 \in [-\theta,\theta]$.

The function $Im(\psi_{L_j}^X)$ is injective in $\Gamma_{x,j,\theta}^{\epsilon,\rho}$ by construction. Since $\Re(e^{i\theta_1}X)$ is transversal to $\Gamma_{x,j,0}^{\epsilon,\rho}[-M,M]$, $\Gamma_{x,j,\theta,+}^{\epsilon,\rho}$ and $\Gamma_{x,j,\theta,-}^{\epsilon,\rho}$ at any of their points we deduce that $Im(\psi_{L_j}^X e^{-i\theta_1})$ is injective in $\Gamma_{x,j,\theta}^{\epsilon,\rho}$ for any $\theta_1 \in (-\theta,\theta)$. As a consequence given $\theta_1 \in (-\theta,\theta)$ the vector field $\Re(e^{i\theta_1}X)$ is almost transversal to $\Gamma_{x,j,\theta}^{\epsilon,\rho}$ at any point in $\Gamma_{x,j,\theta,-}^{\epsilon,\rho} \setminus \{P_{x,j,\theta,+}^{\epsilon,\rho}, P_{x,j,\theta,-}^{\epsilon,\rho}\}$. Moreover in the neighborhood of $P_{x,j,\theta,\pm}^{\epsilon,\rho}$ the curve $\psi_{L_j}^X(\partial H_{j,\theta}^{\epsilon,\rho}(x) \setminus \Gamma_{x,j,\theta}^{\epsilon,\rho})$ is to the right of $\psi_{L_j}^X(\Gamma_{x,j,\theta}^{\epsilon,\rho})$. Analogously as in subsection 5.4.2 we obtain that $\Re(e^{i\theta_1}X)$ is almost transversal to $\partial H_{j,\theta}^{\epsilon,\rho}(x)$ at both $P_{x,j,\theta,-}^{\epsilon,\rho}$ and $P_{x,j,\theta,+}^{\epsilon,\rho}$ for any $\theta_1 \in (-\theta,\theta)$. Lemma 5.5 implies that $H_{j,\theta}^{\epsilon,\rho}(x)$ is $\Re(e^{i\theta_1}X)$-convex for any $\theta_1 \in (-\theta,\theta)$.

Let us consider the flow $\Re(e^{i\theta}X)$. The proof for $\Re(e^{-i\theta}X)$ is analogous. We can proceed as in the previous paragraphs to obtain that $\Re(e^{i\theta}X)$ is almost transversal to $\partial H_{j,\theta}^{\epsilon,\rho}(x)$ at any point outside of $\Gamma_{x,j,\theta,-}^{\epsilon,\rho}$. The analysis of the properties of $P_{x,j,\theta,-}^{\epsilon,\rho}$ also implies $\exp(se^{i\theta}X)(P_{x,j,\theta,-}^{\epsilon,\rho}) \notin \overline{H_{j,\theta}^{\epsilon,\rho}(x)}$ for any $s < 0$ in a neighborhood of 0.

Given $Q \in H_{j,\theta}^{\epsilon,\rho}(x)$ we denote $\Gamma_Q = \Gamma(e^{i\theta}X, Q, H_{j,\theta}^{\epsilon,\rho})$ and $(s_-,s_+) = \mathcal{I}(\Gamma_Q)$. Clearly $\Gamma_Q(s_+) \in \partial H_{j,\theta}^{\epsilon,\rho}(x) \setminus \Gamma_{x,j,\theta,-}^{\epsilon,\rho}$, thus $\Re(e^{i\theta}X)$ is almost transversal to $\partial H_{j,\theta}^{\epsilon,\rho}(x)$ at $\Gamma_Q(s_+)$. Analogously if $\Gamma_Q(s_-) \notin \Gamma_{x,j,\theta,-}^{\epsilon,\rho}$ the vector field $\Re(e^{i\theta}X)$ is almost transversal to $\partial H_{j,\theta}^{\epsilon,\rho}(x)$ at $\Gamma_Q(s_-)$. Otherwise we obtain $\Gamma_Q(s_-) = \Gamma_{x,j,0}^{\epsilon,\rho}(-M)$. We deduce

$$\mathcal{I}(\Gamma(e^{i\theta}X, Q, \overline{H_{j,\theta}^{\epsilon,\rho}})) = [s_- - \sup \mathcal{I}(\Gamma_{x,j,\theta,-}^{\epsilon,\rho}), s_+]$$

and then

$$(s_-,s_+) = \{s \in \mathcal{I}(\Gamma(e^{i\theta}X, Q, \overline{H_{j,\theta}^{\epsilon,\rho}})) : \Gamma(e^{i\theta}X, Q, \overline{H_{j,\theta}^{\epsilon,\rho}})(s) \in H_{j,\theta}^{\epsilon,\rho}\}.$$

Lemma 5.5 implies that $H_{j,\theta}^{\epsilon,\rho}(x)$ is $Re(e^{i\theta}X)$-convex.

There exists $s_0 \in \mathbb{R}^+$ such that $\exp(se^{i\theta_1}X)(Q) \in H_{j,\theta}^{\epsilon,\rho}(x)$ for all $s \in (0, s_0)$, $\theta_1 \in [-\theta, \theta]$ and $Q \in \Gamma_{x,j,0}^{\epsilon,\rho}[-M, M]$. Since the set $H_{j,\theta}^{\epsilon,\rho}(x)$ is $Re(e^{i\theta_1}X)$-convex for $\theta_1 \in [-\theta, \theta]$ we deduce that the sets $\psi_{L_j}^X(H_{j,\theta}^{\epsilon,\rho}(x))$ and $\psi_{L_j}^X(Q) + e^{i[-\theta,\theta]}(\mathbb{R}^- \cup \{0\})$ are disjoint for any $Q \in \Gamma_{x,j,0}^{\epsilon,\rho}[-M, M]$. Therefore $\psi_{L_j}^X(H_{j,\theta}^{\epsilon,\rho}(x))$ is contained in the set $\psi_{L_j}^X(T_{iX}^{\epsilon,j}(x)) + W_{\theta,M}$ (see def. 5.3). □

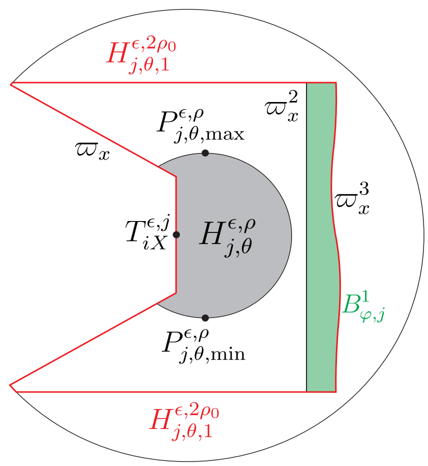


FIGURE 11. Picture of $H_{j,\theta,1}^{\epsilon,2\rho_0}$ and $B_{\varphi,j}^1$ in the $\psi_{L_j}^X$ coordinate

Step 3. In this step we define a subset $H_{j,\theta,1}^{\epsilon,2\rho_0}$ of $H_{j,\theta}^{\epsilon,2\rho_0}$ and sort of a fundamental domain $B_{\varphi,j}^1$ of $\varphi_{|H_{j,\theta,1}^{\epsilon,2\rho_0}}$ such that $B_{\varphi,j}^1(x) \subset H_{\Lambda,j}^{\lambda}(x)$ for $x \in (0,\delta)I_{\Lambda}^{\lambda}$ and $B_{\varphi,j}^1(x) \subset H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}(x)$ for $x \in (0,\delta)I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. Moreover we have $H_{j,\theta}^{\epsilon,\rho} \subset H_{j,\theta,1}^{\epsilon,2\rho_0}$ for $\rho >> 0$. As a consequence a Fatou coordinate of φ defined in $B_{\varphi,j}^1$ can be extended to $H_{j,\theta}^{\epsilon,\rho}$ (see Step 4). In other words to define a Fatou coordinate of φ in $H_{j,\theta}^{\epsilon,\rho}$ we can restrict ourselves to $B_{\varphi,j}^1$. This is a key property in next subsection's results.

The fundamental domain $B_{\varphi,j}^1(x)$ has different properties than $B_{\varphi}^2(T_{iX}^{\epsilon,j}(x))$ (see Step 3 of subsection 5.4). The former one tends to the origin when $x \to 0$ whereas the latter one tends to $B_{\varphi}^2(T_{iX}^{\epsilon,j}(0))$. This discrepancy is justified by the methods that we use in subsection 5.7 to study the asymptotic developments of Lavaurs vector fields.

More precisely, let $\exp(Y_k)$ be a k-convergent normal form of φ. In subsection 5.7 we want to find Fatou coordinates $\psi_{j,\lambda,k}^{\varphi}$ of φ in $H_{j,\theta}^{\epsilon,\rho}$ such that $\psi_{j,\lambda,k}^{\varphi}$ satisfies

$$\psi_{j,\lambda,k}^{\varphi} - \psi_{L_j,k} = O((y \circ \varphi - y)^{c_k})$$

where $\psi_{L_j,k}$ is a Fatou coordinate of Y_k and $\lim_{k\to\infty} c_k = \infty$. We obtain $\psi_{j,\lambda,k}^{\varphi}$ in a fundamental domain by using synthesis and solving a $\overline{\partial}$ equation. Then we extend the result by iteration. If the fundamental domain is $B_{\varphi}^2(T_{iX}^{\epsilon,j}(x))$ then $O(y \circ \varphi - y)$ is a $O(1)$ and such property does not improve by iteration. On the other hand the choice of $B^1_{\varphi,j}$ turns out to be good since we have

$$\sup_{(x,y')\in B^1_{\varphi,j}(x)} |y\circ\varphi - y|(x,y') = O((y\circ\varphi - y)(x,y))$$

when $x\to 0$ in $H^{\epsilon,\rho}_{j,\theta}$.

Let us explicit the construction. We define $arc^{\epsilon,\rho}_{j,\theta}(x_0) = \partial H^{\epsilon,\rho}_{j,\theta}(x_0)\cap\{|y|=\rho x_0\}$. By proposition 4.2 the sections $P^{\epsilon,\rho}_{x,j,\theta,+}$ and $P^{\epsilon,\rho}_{x,j,\theta,-}$ are asymptotically continuous in $B(0,\delta)\setminus\{0\}$. Hence the arc $arc^{\epsilon,\rho}_{j,\theta}(r,\tilde\lambda)$ converges in adapted coordinates $(x,w)=(x,y/x)$ to an arc $arc^{\epsilon,\rho}_{j,\theta}(0,\lambda_0)$ when $(r,\tilde\lambda)\to(0,\lambda_0)$. The set $H^{\epsilon,\rho}_{j,\theta}(r,\tilde\lambda)$ tends to some set $\tilde H^{\epsilon,\rho}_{j,\theta}(0,\lambda_0)$ in adapted coordinates when $(r,\tilde\lambda)\to(0,\lambda_0)$. More precisely $\tilde H^{\epsilon,\rho}_{j,\theta}(0,\lambda_0)$ is contained in $\{|w|>\rho\}$ and its boundary is contained in the union of $arc^{\epsilon,\rho}_{j,\theta}(0,\lambda_0)$, a trajectory in $Tr_{\leftarrow\infty}(-e^{-i\theta}X_0(\lambda_0))$ and a trajectory in $Tr_{\leftarrow\infty}(-e^{i\theta}X_0(\lambda_0))$ for $\lambda_0\in\mathbb{S}^1$. We denote $\psi^{\mathcal{C}_0}_{\lambda_0}$ the Fatou coordinate of $X_0(\lambda_0)$ defined in the neighborhood of $\overline{\tilde H^{\epsilon,\rho}_{j,\theta}(0,\lambda_0)}$ such that $\psi^{\mathcal{C}_0}_{\lambda_0}(\infty)=0$. We obtain

$$\lim_{(r,\tilde\lambda)\to(0,\lambda_0)} (r^{\nu(\mathcal{E}_0)}\psi^X_{L_j})(\overline{H^{\epsilon,\rho}_{j,\theta}(r,\tilde\lambda)}) = \psi^{\mathcal{C}_0}_{\lambda_0}(\overline{\tilde H^{\epsilon,\rho}_{j,\theta}(0,\lambda_0)})$$

in the Hausdorff topology for any $\lambda_0\in\mathbb{S}^1$.

We proceed as in subsection 5.4.2. The maximum (resp. the minimum) of the function $Im(r^{\nu(\mathcal{E}_0)}\psi^X_{L_j})$ in $\overline{\tilde H^{\epsilon,\rho}_{j,\theta}(r,\tilde\lambda)}$ is attained at a unique point $P^{\epsilon,\rho}_{j,\theta,\max}(r,\tilde\lambda)$ (resp. $P^{\epsilon,\rho}_{j,\theta,\min}(r,\tilde\lambda)$) contained in $arc^{\epsilon,\rho}_{j,\theta}(r,\tilde\lambda)$ for any $(r,\tilde\lambda)\in[0,\delta)\times\mathbb{S}^1$. Consider $a_0,a_1\in\mathbb{R}^+$ such that $a_0+4i[-a_1,a_1]$ is contained in $(r^{\nu(\mathcal{E}_0)}\psi^X_{L_j})(\tilde H^{\epsilon,2\rho_0}_{j,\pi/2}(r,\tilde\lambda))$ for any $(r,\tilde\lambda))\in[0,\delta)\times\mathbb{S}^1$. We choose $\rho\geq 2\rho_0$ such that $C_6/\rho^{\nu(\mathcal{E}_0)}\leq\min(a_0,a_1)/2$. The equation (20) implies that $\psi^X_{L_j}(H^{\epsilon,\rho}_{j,\theta}(x))$ is contained in

$$\left[\frac{-a_0}{2|x|^{\nu(\mathcal{E}_0)}},\frac{a_0}{2|x|^{\nu(\mathcal{E}_0)}}\right]+i\left[\frac{-a_1}{2|x|^{\nu(\mathcal{E}_0)}},\frac{a_1}{2|x|^{\nu(\mathcal{E}_0)}}\right]$$

for any $x\in B(0,\delta)\setminus\{0\}$. We have

$$\overline{\tilde H^{\epsilon,2\rho_0}_{j,\theta}(r,\tilde\lambda)}\cap T(\mathcal{C}_0)^{2\rho_0}_X(r,\tilde\lambda)\subset\{P^{\epsilon,2\rho_0}_{j,\theta,\min}(r,\tilde\lambda),P^{\epsilon,2\rho_0}_{j,\theta,\min}(r,\tilde\lambda)\}\ \forall(r,\tilde\lambda)\in[0,\delta)\times\mathbb{S}^1.$$

As a consequence there exists $\theta'\in(0,\theta]$ such that the angle between $Re(X)$ and $arc^{\epsilon,2\rho_0}_{j,\theta}(x)$ is greater or equal than θ' at any point in

$$\cup_{x\in B(0,\delta)\setminus\{0\}}\left(arc^{\epsilon,2\rho_0}_{j,\theta}(x)\cap\left\{Im(\psi^X_{L_j})\in\left[\frac{-(3+1/2)a_1}{|x|^{\nu(\mathcal{E}_0)}},\frac{(3+1/2)a_1}{|x|^{\nu(\mathcal{E}_0)}}\right]\right\}\right).$$

The angle between $Re(X)$ and $\partial H^{\epsilon,2\rho_0}_{j,\theta}(x)$ is greater or equal than $\theta'=\min(\theta',\theta)$ at any point in

$$\partial H^{\epsilon,2\rho_0}_{j,\theta}(x)\cap\left\{Im(\psi^X_{L_j})\in\left[\frac{-(3+1/2)a_1}{|x|^{\nu(\mathcal{E}_0)}},\frac{(3+1/2)a_1}{|x|^{\nu(\mathcal{E}_0)}}\right]\right\}\ \forall x\in B(0,\delta).$$

The previous set is a union of two curves, namely a curve ϖ_x containing $T^{\epsilon,j}_{iX}(x)$ and a curve ϖ'_x contained in $\{|y|=2\rho_0|x|\}$. The curve ϖ_x is parametrized by $Im(\psi^X_{L_j})$. Indeed given $x\in B(0,\delta)\setminus\{0\}$ and $s\in(7/2)a_1[-1/|x|^{\nu(\mathcal{E}_0)},1/|x|^{\nu(\mathcal{E}_0)}]$ there exists a unique $d(\varpi_x,s)\in\mathbb{R}$ such that $d(\varpi_x,s)+is\in\psi^X_{L_j}(\varpi_x)$. We define

$$\varpi^1_x=\varphi(\varpi_x)\cap\left\{Im(\psi^X_{L_j})\in\left[\frac{-3a_1}{|x|^{\nu(\mathcal{E}_0)}},\frac{3a_1}{|x|^{\nu(\mathcal{E}_0)}}\right]\right\}\ \forall x\in B(0,\delta)\setminus\{0\}.$$

We can use the equation (22) and the techniques in Step 3 of subsection 5.2 to obtain that ϖ_x^1 is transversal to $\Re(X)$ for any $x \in B(0,\delta) \setminus \{0\}$; moreover we obtain $d(\varpi_x, s) \leq d(\varpi_x^1, s)$ for all $x \in B(0,\delta) \setminus \{0\}$ and $s \in [-3a_1/|x|^{\nu(\mathcal{E}_0)}, 3a_1/|x|^{\nu(\mathcal{E}_0)}]$. The curve $\psi_{L_j}^X(\varpi_x^1)$ is to the right of $\psi_{L_j}^X(\varpi_x)$. We define the curves ϖ_x^2 and ϖ_x^3 contained in $H_{j,\theta}^{\epsilon,2\rho_0}(x)$ such that

$$(|x|^{\nu(\mathcal{E}_0)}\psi_{L_j}^X)(\varpi_x^2) = a_0 + \frac{7}{2}i[-a_1,a_1], \ \varpi_x^3 = \varphi(\varpi_x^2) \cap \left\{ |Im(\psi_{L_j}^X)| \leq \frac{3a_1}{|x|^{\nu(\mathcal{E}_0)}} \right\}.$$

We proceed as in Step 3 of subsection 5.2 to prove that ϖ_x^3 is transversal to $\Re(X)$ for any $x \in B(0,\delta) \setminus \{0\}$ and $d(\varpi_x^2, s) \leq d(\varpi_x^3, s)$ for all $x \in B(0,\delta) \setminus \{0\}$ and $s \in [-3a_1/|x|^{\nu(\mathcal{E}_0)}, 3a_1/|x|^{\nu(\mathcal{E}_0)}]$. The curve $\psi_{L_j}^X(\varpi_x^3)$ is to the right of $\psi_{L_j}^X(\varpi_x^2)$.

Definition 5.22. *We define $H_{j,\theta,1}^{\epsilon,2\rho_0}(x)$ as the subset of $H_{j,\theta}^{\epsilon,2\rho_0}(x)$ such that*

$$\psi_{L_j}^X(H_{j,\theta,1}^{\epsilon,2\rho_0}(x)) = \cup_{s \in [-a_1/|x|^{\nu(\mathcal{E}_0)}, a_1/|x|^{\nu(\mathcal{E}_0)}]}((d(\varpi_x,s), d(\varpi_x^3,s)) + is).$$

Definition 5.23. *Given $l \in \{1,2,3\}$ we denote $B_{\varphi,j}^l(x)$ the subset of $H_{j,\pi/2}^{\epsilon,2\rho_0}(x)$ such that*

$$\psi_{L_j}^X(B_{\varphi,j}^l(x)) = \cup_{s \in [-la_1/|x|^{\nu(\mathcal{E}_0)}, la_1/|x|^{\nu(\mathcal{E}_0)}]}([d(\varpi_x^2,s), d(\varpi_x^3,s)) + is).$$

Clearly $B_{\varphi,j}^1(x)$ is contained in $H_{j,\theta,1}^{\epsilon,2\rho_0}(x)$.

Step 4. Next we prove that a Fatou coordinate of φ in $B_{\varphi,j}^1$ extends by iteration to a Fatou coordinate in $H_{j,\theta}^{\epsilon,\rho}$.

Lemma 5.9. *Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$. Let $\Upsilon = \exp(X)$ be a k-convergent normal form. Let $j \in \mathcal{D}(\varphi)$, $\theta \in (0,\pi/2]$. Consider $x \in B(0,\delta)\setminus\{0\}$ and $P \in H_{j,\theta}^{\epsilon,\rho}(x)$. Then there exists $l_0(P) \in \mathbb{Z}$ such that $\{P,\ldots,\varphi^{l_0}(P)\} \subset H_{j,\theta,1}^{\epsilon,2\rho_0}(x)$ and $\varphi^{l_0}(P) \in B_{\varphi,j}^1(x)$.*

Proof. Suppose that $j \in \mathcal{D}_1(\varphi)$ without lack of generality. Suppose that $\{P,\ldots,\varphi^{l_1}(P)\}$ is contained in $H_{j,\theta,1}^{\epsilon,2\rho_0}(x)$ for some $l_1 \in \mathbb{N} \cup \{0\}$. We have

$$\psi_{L_j}^X(\varphi^{l_1+1}(P)) = \psi_{L_j}^X(P) + (l_1+1) + \sum_{l=0}^{l_1} \Delta_\varphi(\varphi^l(P)).$$

The set $\psi_{L_j}^X(H_{j,\theta}^{\epsilon,\rho}(x))$ is contained in the set $\psi_{L_j}^X(T_{iX}^{\epsilon,j}(x)) + W_{\theta,M}$ (lemma 5.8). Thus equations (21), (22) and lemma 5.2 imply that there exists a constant $K_3 \in \mathbb{R}^+$ independent of P, x and l_1 such that

$$|\psi_{L_j}^X(\varphi^{l_1+1}(P)) - \psi_{L_j}^X(P) - (l_1+1)| \leq \frac{K_3}{(1+|\psi_{L_j}^X(P)|)^{k-1}}. \tag{23}$$

Suppose that the lemma is false. Then Step 3 implies that there exists $l_2 \in \mathbb{N} \cup \{0\}$ such that

$$\{P,\ldots,\varphi^{l_2}(P)\} \subset H_{j,\theta,1}^{\epsilon,2\rho_0}(x) \text{ and } |Im(\psi_{L_j}^X(\varphi^{l_2+1}(P)))| > a_1/|x|^{\nu(\mathcal{E}_0)}.$$

The choice of ρ implies

$$\frac{a_1}{2|x|^{\nu(\mathcal{E}_0)}} \leq |Im(\psi_{L_j}^X(\varphi^{l_2+1}(P))) - Im(\psi_{L_j}^X(P))| \leq K_3.$$

We obtain a contradiction. □

We can extend $\ddot{\psi}_{j,\lambda}^{\varphi}$ and $\ddot{\psi}_{j,\lambda}^{\varphi} - \ddot{\psi}_{j,\lambda'}^{\varphi}$ to $H_{j,\theta}^{\epsilon,\rho}$ by defining

$$\ddot{\psi}_{j,\lambda}^{\varphi}(P) = \ddot{\psi}_{j,\lambda}^{\varphi}(\varphi^{l_0}(P)) - l_0 \text{ and } (\ddot{\psi}_{j,\lambda}^{\varphi} - \ddot{\psi}_{j,\lambda'}^{\varphi})(P) = (\ddot{\psi}_{j,\lambda}^{\varphi} - \ddot{\psi}_{j,\lambda'}^{\varphi})(\varphi^{l_0}(P)). \tag{24}$$

The lemma 5.9 and property (23) imply

$$|(\ddot{\psi}_{j,\lambda}^{\varphi}(P) - \psi_{L_j}^{X}(P)) - (\ddot{\psi}_{j,\lambda}^{\varphi}(\varphi^{l_0}(P)) - \psi_{L_j}^{X}(\varphi^{l_0}(P)))| \leq \frac{K_3}{(1+|\psi_{L_j}^{X}(P)|)^{k-1}}. \tag{25}$$

Since $Re(\psi_{L_j}^{X}(Q)) \geq a_0/|x|^{\nu(\mathcal{E}_0)}$ for any $Q \in B_{\varphi,j}^{1}(x)$ we can use equation (20) to obtain $B_{\varphi,j}^{1}(x) \subset \{x\} \times \overline{B}(0, \sqrt[\nu(\mathcal{E}_0)]{C_6/a_0}|x|)$. We deduce

$$\lim_{x \in (0,\delta) I_{\Lambda}^{\lambda},\ Q \in B_{\varphi,j}^{1}(x),\ Q \to (0,0)} \ddot{\psi}_{j,\lambda}^{\varphi}(Q) - \psi_{L_j}^{X}(Q) = (\ddot{\psi}_{j,\lambda}^{\varphi} - \psi_{L_j}^{X})(0,0)$$

by proposition 5.6 since $B_{\varphi,j}^{1}(x) \subset H_{\Lambda,j}^{\lambda}(x)$ for any $x \in (0,\delta) I_{\Lambda}^{\lambda}$. This implies

$$\lim_{Q \in H_{j,\theta}^{\epsilon,\rho,\lambda},\ Q \to (0,0)} \ddot{\psi}_{j,\lambda}^{\varphi}(Q) - \psi_{L_j}^{X}(Q) = (\ddot{\psi}_{j,\lambda}^{\varphi} - \psi_{L_j}^{X})(0,0).$$

as a consequence of equation (25). The previous discussion leads us to

Proposition 5.10. *Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$. Let Υ be a 2-convergent normal form. Consider $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}}) \in \mathcal{M}$, $\lambda \in \mathbb{S}^1$, $j \in \mathcal{D}(\varphi)$ and $\theta \in (0,\pi/2]$. Then there exists $\rho \geq 2\rho_0$ such that the function $\ddot{\psi}_{j,\Lambda,\lambda}^{\varphi} - \psi_{L_j}^{X}$ is continuous in $\overline{H_{j,\theta}^{\epsilon,\rho,\lambda}}$ and holomorphic in the interior of $H_{j,\theta}^{\epsilon,\rho,\lambda}$. Moreover ρ depends only on X, φ and Λ.*

The definition of $\ddot{\psi}_{j,\lambda}^{\varphi} - \ddot{\psi}_{j,\lambda'}^{\varphi}$ in equation (24) and proposition 5.9 immediately imply

Proposition 5.11. *Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ with 2-convergent normal form $\exp(X)$. Let $\Lambda, \Lambda' \in \mathcal{M}$. Consider $\lambda, \lambda' \in \mathbb{S}^1$, $j \in \mathcal{D}(\varphi)$ and $\theta \in (0,\pi/2]$. Then there exist $K \in \mathbb{R}^+$ and $\rho \geq 2\rho_0$ such that*

$$|\ddot{\psi}_{j,\Lambda,\lambda}^{\varphi} - \ddot{\psi}_{j,\Lambda',\lambda'}^{\varphi}|(x,y) \leq \frac{e^{-K/|x|^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',-}}}}{2} + \frac{e^{-K/|x|^{e_{\Lambda,\Lambda',j}^{\lambda,\lambda',+}}}}{2} \leq e^{-K/|x|^{\tilde{e}_{d_{\Lambda,\Lambda'}^{\lambda,\lambda'}+1}}}$$

for any $(x,y) \in H_{j,\theta}^{\epsilon,\rho,\lambda,\lambda'}$. Moreover ρ depends only on X, φ and Λ.

5.7. **Asymptotics of Fatou coordinates.** Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$. Consider a 2-convergent normal form $\Upsilon = \exp(X)$. Consider $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}}) \in \mathcal{M}$ and the dynamical splitting $\digamma_{\Lambda}$ in remark 4.11. Let $j \in \mathbb{Z}/(2\nu(\mathcal{E}_0)\mathbb{Z})$. We use the notations in the previous subsection.

We denote $f = y \circ \varphi - y$, $\log \varphi = \hat{u} f \partial/\partial y$ and $X = u f \partial/\partial y$ for some units $\hat{u} \in \mathbb{C}[[x,y]]$ and $u \in \mathbb{C}\{x,y\}$. We obtain $\hat{u} - u \in (f^2)$. Fix $k \geq \max(5, 4\nu(\mathcal{E}_0))$. Let $\exp(Y_k)$ be a k-convergent normal form. It is of the form $\exp(u_k f \partial/\partial y)$ for some unit $u_k \in \mathbb{C}\{x,y\}$. We choose $\epsilon_k \in (0,\epsilon]$ such that u_k does not vanish at any point of $B(0,\delta) \times B(0,\epsilon_k)$. By construction we have $\hat{u} - u \in (f^2)$ and $\hat{u} - u_k \in (f^k)$. We obtain $u - u_k \in (f^2)$.

The set $H_{j,0}^{\epsilon_k,2\rho_0} \cup H_{j,+}^{\epsilon_k,2\rho_0} \cup H_{j,-}^{\epsilon_k,2\rho_0}$ is contained in $H_{j,0}^{\epsilon,2\rho_0} \cup H_{j,+}^{\epsilon,2\rho_0} \cup H_{j,-}^{\epsilon,2\rho_0}$. A Fatou coordinate ψ_k of Y_k is a solution of the differential equation $u_k f \partial \psi_k/\partial y = 1$.

Thus we can obtain a Fatou coordinate ψ_k of Y_k in $H_{j,0}^{\epsilon_k,2\rho_0} \cup H_{j,+}^{\epsilon_k,2\rho_0} \cup H_{j,-}^{\epsilon_k,2\rho_0}$ of the form $\psi_{L_j,k} = \psi_{L_j}^X + h$ where $h \in \mathcal{O}(B(0,\delta) \times B(0,\epsilon_k))$ is a solution of

$$\frac{\partial h}{\partial y} = \frac{1}{u_k f} - \frac{1}{uf} = \frac{1}{uu_k}\frac{u-u_k}{f}.$$

Such a solution exists since $u - u_k \in (f^2)$. Moreover we can suppose $h(0,0) = 0$. Denote $\psi_k = \psi_{L_j,k}$.

Denote $I_\lambda = (0,\delta)\lambda e^{i(-\pi/(4\nu(\mathcal{E}_0)),\pi/(4\nu(\mathcal{E}_0)))}$. In this subsection we build Fatou coordinates $\psi_{j,\lambda,k}^{\varphi}$ of φ in $\cup_{x \in I_\lambda}(H_{j,\theta}^{\epsilon,\rho}(x) \cap H_{j,\theta/2}^{\epsilon_k,\rho}(x))$ such that

$$\psi_{j,\lambda,k}^{\varphi} - \psi_{L_j,k} = O(f^{k-5\nu(\mathcal{E}_0)/(\nu(\mathcal{E}_0)+1)}).$$

Let $X_{j,\lambda,k}^{\varphi} = v_k f \partial/\partial y$ be the holomorphic vector field such that $X_{j,\lambda,k}^{\varphi}(\psi_{j,\lambda,k}^{\varphi}) \equiv 1$. The previous equation implies $v_k - u_k = O(f^{k-5\nu(\mathcal{E}_0)/(\nu(\mathcal{E}_0)+1)})$. Thus the asymptotic development of $X_{j,\lambda,k}^{\varphi}$ is $\log\varphi$ up to order $f^{k-5\nu(\mathcal{E}_0)/(\nu(\mathcal{E}_0)+1)}$. The flatness properties in subsection 5.5 imply that the difference $X_{j,\lambda,k}^{\varphi} - X_{H_{\Lambda,j}^{\lambda}}^{\varphi}$ is exponentially small (see corollary 5.1 and def. 5.11 for the definition of $X_{H_{\Lambda,j}^{\lambda}}^{\varphi}$). In this way we deduce in section 6 that $\log\varphi$ is an asymptotic development of $X_{H_{\Lambda,j}^{\lambda}}^{\varphi}$. The construction of $\psi_{j,\lambda,k}^{\varphi}$ is based on finding a well-behaved C^∞ Fatou coordinate and then deducing the existence of a holomorphic one by solving a $\overline{\partial}$ equation.

Definition 5.24. *We define*

$$\Delta_{\varphi^l,k} = \psi_k \circ \varphi^l - (\psi_k + l) \in \mathbb{C}\{x,y\} \cap (f^k) \text{ for } l \in \{-1,1\}.$$

By considering a smaller $\epsilon_k > 0$ we can suppose that $\Delta_{\varphi^{-1},k}$ is defined in a neighborhood of $\overline{B}(0,\delta) \times \overline{B}(0,\epsilon_k)$. We define the coordinates

$$\begin{cases} z = \psi_k(x,y) \\ \xi = \frac{1}{x^{\nu(\mathcal{E}_0)}} \end{cases} \quad \text{in } H_{j,0}^{\epsilon_k,2\rho_0} \cup H_{j,+}^{\epsilon_k,2\rho_0} \cup H_{j,-}^{\epsilon_k,2\rho_0}.$$

We define $U' = \cup_{x \in B(0,\delta)\setminus\{0\},\ s \in B(0,1/4)} \exp(sX)(B_{\varphi,j}^2(x))$. Analogously as for equation (21) there exists $K_4 \in \mathbb{R}^+$ such that

$$|\Delta_{\varphi^{-1},k}| \leq \frac{K_4}{(1+|\psi_{L_j}^X|)^{k(1+1/\nu(\mathcal{E}_0))}} \text{ in } H_{j,0}^{\epsilon_k,2\rho_0} \cup H_{j,+}^{\epsilon_k,2\rho_0} \cup H_{j,-}^{\epsilon_k,2\rho_0}. \tag{26}$$

By construction we have $|\psi_{L_j}^X|(x,y) \geq a_0/(2|x|^{\nu(\mathcal{E}_0)})$ in U'. We obtain

$$|\Delta_{\varphi^{-1},k}|(x,y) \leq K_5|x|^{k(\nu(\mathcal{E}_0)+1)} \ \forall (x,y) \in U'. \tag{27}$$

where $K_5 = K_4(2/a_0)^{k(1+1/\nu(\mathcal{E}_0))}$.

Definition 5.25. *We define the mapping*

$$\sigma(\xi,z) = (\xi, z + \varrho(Re(z) - a_0|\xi|)\Delta_{\varphi^{-1},k}(\xi,z))$$

where ϱ is the function defined in Step 1 of subsection 5.2.

The mapping σ is defined in U'. In fact it conjugates the diffeomorphisms $\varphi(\xi,z)$ and $(\xi, z+1)$. Proceeding as in Step 3 of subsection 5.2 we obtain a constant $K_6 \in \mathbb{R}^+$ such that

$$\left|\frac{\partial \Delta_{\varphi^{-1},k}}{\partial z}\right|(\xi,z) \leq \frac{K_6}{|\xi|^{k(1+1/\nu(\mathcal{E}_0))}} \ \forall (\xi,z) \in U'. \tag{28}$$

We have

$$|z \circ \sigma(\xi, z) - z| \leq \frac{K_5}{|\xi|^{k(1+1/\nu(\mathcal{E}_0))}} \text{ and } ||(\mathcal{J}\sigma)(\xi, z) - Id|| < \frac{K_5 \sup_{\mathbb{R}} |\partial \varrho / \partial t| + 2K_6}{|\xi|^{k(1+1/\nu(\mathcal{E}_0))}}$$

for any $(\xi, z) \in U'$. Indeed σ is a C^∞ diffeomorphism from U' to $\sigma(U')$.

Fix $\lambda_0 \in \mathbb{S}^1$. Consider the sector $\tilde{I}_{\lambda_0} = (0, \delta)\lambda_0 e^{i(-7\pi/(24\nu(\mathcal{E}_0)), 7\pi/(24\nu(\mathcal{E}_0)))}$ in the x coordinate. It corresponds to $(\delta^{-\nu(\mathcal{E}_0)}, \infty)\lambda_0^{-\nu(\mathcal{E}_0)} e^{i[-7\pi/24, 7\pi/24]}$ in the ξ coordinate. We define $U'_{\lambda_0} = \cup_{x \in \tilde{I}_{\lambda_0}} U'(x)$.

Definition 5.26. *We define the functions*

$$h_1(\xi, z) = Re(2a_1\xi\lambda_0^{\nu(\mathcal{E}_0)}) - Im(z) \text{ and } h_2(\xi, z) = Re(2a_1\xi\lambda_0^{\nu(\mathcal{E}_0)}) + Im(z)$$

in $H_{j,0}^{\epsilon_k, 2\rho_0} \cup H_{j,+}^{\epsilon_k, 2\rho_0} \cup H_{j,-}^{\epsilon_k, 2\rho_0}$. *We define the functions* $\tau_1 = h_1 \circ \sigma$ *and* $\tau_2 = h_2 \circ \sigma$ *in* U'_{λ_0}.

The functions τ_1, τ_2 satisfy $\tau_1 \circ \varphi = \tau_1$ and $\tau_2 \circ \varphi = \tau_2$ in $U' \cap \varphi^{-1}(U')$. We have

$$a_1|\xi| < 2\cos(7\pi/24)a_1|\xi| \leq Re(2a_1\xi\lambda_0^{\nu(\mathcal{E}_0)}) \leq 2a_1|\xi| \ \forall \xi \in \tilde{I}_{\lambda_0}.$$

The set $\{(\xi, z) \in B_{\varphi,j}^3 : \xi \in \tilde{I}_{\lambda_0} \text{ and } h_1(\xi, z) > 0 < h_2(\xi, z)\}$ is contained in $B_{\varphi,j}^2$ and contains $\cup_{\xi \in \tilde{I}_{\lambda_0}} B_{\varphi,j}^1(\xi)$. The set

$$\{(\xi, z) \in U'_{\lambda_0} \cap B_{\varphi,j}^3 : \xi \in \tilde{I}_{\lambda_0} \text{ and } \tau_1(\xi, z) > 0 < \tau_2(\xi, z)\}$$

contains $\cup_{\xi \in \tilde{I}_{\lambda_0}} B_{\varphi,j}^1(\xi)$. We define

$$\tilde{U}'_{\lambda_0} = \{(\xi, z) \in U'_{\lambda_0} : \tau_1(\xi, z)\tau_2(\xi, z) > 1\}$$

and

$$\tilde{B}_{\varphi,j}^2 = \{(\xi, z) \in U'_{\lambda_0} \cap B_{\varphi,j}^3 : \xi \in \tilde{I}_{\lambda_0} \text{ and } \tau_1(\xi, z)\tau_2(\xi, z) > 1\}.$$

The set $\tilde{B}_{\varphi,j}^2$ contains both $\cup_{\xi \in \tilde{I}_{\lambda_0}} B_{\varphi,j}^1(\xi)$ and a fundamental domain of $\varphi_{|\tilde{U}'_{\lambda_0}}$. We denote $U_{\lambda_0}^*$ the space of orbits of $\varphi_{|\tilde{U}'_{\lambda_0}}$.

5.7.1. *The $\overline{\partial}$ equation.* Our goal is defining a Fatou coordinate $\psi_{j,\lambda_0,k}^{\varphi}$ of φ in $\tilde{U}'_{\lambda_0}$. We can define $\psi_{j,\lambda_0,k}^{\varphi} = \sigma \circ \psi_k$ if σ is holomorphic. In general we look for a function v such that

$$\sigma_v(\xi, z) = (\xi, z + \varrho(Re(z) - a_0|\xi|)\Delta_{\varphi^{-1},k}(\xi, z) + v(\xi, z))$$

is a holomorphic mapping conjugating $\varphi(\xi, z)$ and $(\xi, z+1)$. The latter condition is equivalent to $v \circ \varphi \equiv v$ whereas the former condition is equivalent to the equation $\overline{\partial} v = \Omega$ where $\Omega = -\overline{\partial}(z \circ \sigma(\xi, z))$ is a $(0,1)$ form. Since $z \circ \sigma \circ \varphi(\xi, z) = z \circ \sigma + 1$ we obtain $\varphi^*\Omega \equiv \Omega$. The form Ω represents an element of $H^{0,1}(U_{\lambda_0}^*)$. It suffices to find a function v defined in $U_{\lambda_0}^*$ such that $\overline{\partial} v = \Omega$.

We have

$$\Omega = -\Delta_{\varphi^{-1},k}(\xi, z)\overline{\partial}(\varrho(Re(z) - a_0|\xi|)).$$

Moreover Ω is of the form $\Omega = A(\xi, z)d\overline{\xi} + B(\xi, z)d\overline{z}$ where A, B satisfy

$$|A(\xi, z)| \leq \frac{a_0 K_5 \sup_{\mathbb{R}} |\partial \varrho / \partial t|}{2|\xi|^{k(1+1/\nu(\mathcal{E}_0))}}, \ |B(\xi, z)| \leq \frac{K_5 \sup_{\mathbb{R}} |\partial \varrho / \partial t|}{2|\xi|^{k(1+1/\nu(\mathcal{E}_0))}} \quad \forall (\xi, z) \in U'_{\lambda_0}.$$

Definition 5.27. *Since τ_l is defined in $U^*_{\lambda_0}$ we define $\omega_l = \partial\tau_l \in H^{1,0}(U^*_{\lambda_0})$ for $l \in \{1,2\}$. We denote $\overline{\partial}\omega_l = \sum c^l_{bd}\overline{\omega_b} \wedge \omega_d$ for $l \in \{1,2\}$. We consider the decomposition*

$$dc^l_{bd} = \sum_{g=1}^{2} \partial_g(c^l_{bd})w_g + \sum_{g=1}^{2} \overline{\partial_g}(c^l_{bd})\overline{w_g}. \tag{29}$$

We define the volume elements

$$dV_0 = (i/2)^2 dz \wedge d\overline{z} \wedge d\zeta \wedge d\overline{\zeta} \text{ and } dV = (i/2)^2 \omega_1 \wedge \overline{\omega}_1 \wedge \omega_2 \wedge \overline{\omega}_2.$$

The forms dV_0 and dV are defined in U'_{λ_0}. The form dV is also defined in $U^*_{\lambda_0}$.

Lemma 5.10. *Denote $e = k(1 + 1/\nu(\mathcal{E}_0))$. We have*

- *The forms ω_1 and ω_2 compose a base of the cotangent space in every point of U'_{λ_0} and $U^*_{\lambda_0}$.*
- *There exists $K_7 \geq 1$ such that*
$$\frac{1}{K_7}dV_0 \leq dV \leq K_7 dV_0 \quad \text{in } U'_{\lambda_0}.$$
- *We have $\partial\omega_1 = \partial\omega_2 = 0$, $\overline{\partial}(\omega_1 + \omega_2) = 0$ and*
$$|c^l_{bd}| \leq K_8/|\xi^e| \ , \ |\partial_g c^l_{bd}| \leq K_8/|\xi^e| \text{ and } |\overline{\partial}_g c^l_{bd}| \leq K_8/|\xi^e|$$
for some $K_8 > 0$ and any $(l, b, d, g) \in \{1,2\}^4$.

Proof. Denote $\rho(\xi, z) = \varrho(Re(z) - a_0|\xi|)$ and $\Delta = \Delta_{\varphi^{-1},k}$. We have

$$\frac{\partial\tau_1}{\partial\xi} = a_1\lambda_0^{\nu(\mathcal{E}_0)} - \frac{\partial\rho}{\partial\xi}Im(\Delta) + \frac{i}{2}\frac{\partial\Delta}{\partial\xi}\rho.$$

We have $\Delta = O(1/\xi^e)$ by equation (27). We use Cauchy's integral formula as in Step 3 of subsection 5.2 to obtain a constant $K_9 \in \mathbb{R}^+$ such that

$$|\Delta|, \left|\frac{\partial\Delta}{\partial\xi}\right|, \left|\frac{\partial\Delta}{\partial z}\right|, \left|\frac{\partial^2\Delta}{\partial\xi\partial z}\right|, \left|\frac{\partial^2\Delta}{\partial z^2}\right|, \left|\frac{\partial^2\Delta}{\partial\xi^2}\right| \leq \frac{K_9}{|\xi|^e}$$

in U'. Therefore we obtain

$$\left|\frac{\partial\tau_l}{\partial\xi} - a_1\lambda_0^{\nu(\mathcal{E}_0)}\right|(\xi, z) \leq \frac{1}{2}\frac{K_9}{|\xi|^e}(a_0 \sup_{\mathbb{R}} |\partial\varrho/\partial t| + 1) \ \forall (\xi, z) \in U'_{\lambda_0} \ \forall l \in \{1,2\}.$$

The proof for τ_2 is analogous. We also have

$$\left|\frac{\partial\tau_1}{\partial z} - \frac{i}{2}\right|(\xi, z) \leq \frac{K_9}{2|\xi|^e}\left(\sup_{\mathbb{R}} |\varrho'| + 1\right), \ \left|\frac{\partial\tau_2}{\partial z} + \frac{i}{2}\right|(\xi, z) \leq \frac{K_9}{2|\xi|^e}\left(\sup_{\mathbb{R}} |\varrho'| + 1\right)$$

for any $(\xi, z) \in U'_{\lambda_0}$. Thus the two first points of the lemma hold true.

Consider a composition $\partial_l \circ ... \circ \partial_1$ of operators where $l \leq 3$ and ∂_b is either $\partial/\partial\xi$, $\partial/\partial\overline{\xi}$, $\partial/\partial z$ or $\partial/\partial\overline{z}$ for $b \leq 3$. Consider an operator $\partial' = \partial_{b_d} \circ \ldots \circ \partial_{b_1}$ where $d \leq l$ and $1 \leq b_1 < \ldots < b_d \leq l$. Let l'_ξ be the number of times that we apply the operators $\partial/\partial\xi$ and $\partial/\partial\overline{\xi}$ in ∂'. Suppose $l'_\xi \geq 1$. We have that $\partial'(-a_0|\xi|) = O(\xi^{-l'_\xi+1})$. Since $(\partial_l \circ ... \circ \partial_1)(\rho)$ is a polynomial with rational coefficients in ϱ', $\varrho^{2)}$ and $\varrho^{3)}$ and functions of the form $(\partial_{b_d} \circ \ldots \circ \partial_{b_1})(-a_0|\xi|)$ we deduce that $(\partial_l \circ ... \circ \partial_1)(\rho)$ is bounded in U'_{λ_0}.

We have that the form $\overline{\partial}\omega_1$ is equal to

$$-Im\left(\frac{\partial^2\rho}{\partial\xi\partial\overline{\xi}}\Delta+\frac{\partial\Delta}{\partial\xi}\frac{\partial\rho}{\partial\overline{\xi}}\right)d\overline{\xi}\wedge d\xi+\left(-\frac{\partial^2\rho}{\partial\xi\partial\overline{z}}Im(\Delta)-\frac{i}{2}\frac{\overline{\partial\Delta}}{\partial z}\frac{\partial\rho}{\partial\xi}+\frac{i}{2}\frac{\partial\Delta}{\partial\xi}\frac{\partial\rho}{\partial\overline{z}}\right)d\overline{z}\wedge d\xi+$$
$$\left(-\frac{\partial^2\rho}{\partial\overline{\xi}\partial z}Im(\Delta)-\frac{i}{2}\frac{\overline{\partial\Delta}}{\partial\xi}\frac{\partial\rho}{\partial z}+\frac{i}{2}\frac{\partial\Delta}{\partial z}\frac{\partial\rho}{\partial\overline{\xi}}\right)d\overline{\xi}\wedge dz-Im\left(\frac{\partial^2\rho}{\partial z\partial\overline{z}}\Delta+\frac{\partial\Delta}{\partial z}\frac{\partial\rho}{\partial\overline{z}}\right)d\overline{z}\wedge dz.$$

Since

$$\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}=\begin{pmatrix}\frac{\partial\tau_1}{\partial\xi} & \frac{\partial\tau_1}{\partial z}\\ \frac{\partial\tau_2}{\partial\xi} & \frac{\partial\tau_2}{\partial z}\end{pmatrix}\begin{pmatrix}d\xi\\ dz\end{pmatrix}\implies\begin{pmatrix}d\xi\\ dz\end{pmatrix}=\frac{\begin{pmatrix}\frac{\partial\tau_2}{\partial z} & -\frac{\partial\tau_1}{\partial z}\\ -\frac{\partial\tau_2}{\partial\xi} & \frac{\partial\tau_1}{\partial\xi}\end{pmatrix}}{2a_1\lambda_0^{-\nu(\mathcal{E}_0)}\frac{\partial\tau_2}{\partial z}}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$$

the coefficients c_{bd}^l are of the form $\tilde{c}_{bd}^l/(2a_1|\partial\tau_2/\partial z|)^2$ where numerator and denominator are polynomials with complex coefficients in Δ, $\partial\Delta/\partial\xi$, $\partial\Delta/\partial z$, their complex conjugates and functions of the form $(\partial_g\circ...\circ\partial_1)(\rho)$ with $g\leq 2$. There is no independent term in the Δ variables for $\tilde{c}_{bd}^l$ and $(2a_1|\partial\tau_2/\partial z|)^2-a_1^2$. Therefore dc_{bd}^l is a $O(1/|x|^e)$ in the base $\{d\xi,d\overline{\xi},dz,d\overline{z}\}$ and then in the base $\{\omega_1,\omega_2,\overline{\omega}_1,\overline{\omega}_2\}$ for any $(l,b,d)\in\{1,2\}^3$. □

5.7.2. *Stein character of the space of orbits.* Next step is proving that $U_{\lambda_0}^*$ is pseudoconvex. We define the function

$$u_2=u_2^0+u_2^1+u_2^2+u_2^3$$

where $u_2^0=-\ln(h_1h_2-1)$, $u_2^1=5\ln|\xi|$, $u_2^2=-\ln(|\xi|^2-1/\delta^{2\nu(\mathcal{E}_0)})$ and

$$u_2^3=-\ln\left([Im(\ln(\xi\lambda_0^{\nu(\mathcal{E}_0)}))+7\pi/24][7\pi/24-Im(\ln(\xi\lambda_0^{\nu(\mathcal{E}_0)}))]\right).$$

Since none of the functions u_2^l depends on $Re(z)$ then u_2 is a C^∞ function of $\sigma(U_{\lambda_0}^*)$. The function u_2 is a good candidate to be a C^∞ p.s.h. exhaustion of $\sigma(U_{\lambda_0}^*)$. We define $u_1=u_2\circ\sigma$.

Lemma 5.11. *The function u_1 is a p.s.h. C^∞ exhaustion of $U_{\lambda_0}^*$ for $\delta>0$ small enough. In particular $U_{\lambda_0}^*$ is a Stein manifold.*

Proof. The function u_1 is C^∞ and in order to prove that it is an exhaustion it suffices to prove that u_2 is an exhaustion of $\sigma(U_{\lambda_0}^*)$. The set $\sigma(U_{\lambda_0}^*)$ is biholomorphic to the subset $S=(\xi,e^{2\pi iz})(\sigma(\tilde{U}_{\lambda_0}'))$ of $\mathbb{C}^2$ since we have to identify (ξ,z) and $(\xi,z+1)$. The functions $\ln|\xi|$ and u_2^3 are bounded by below. The function $2\ln|\xi|+u_2^2$ is also bounded by below. The functions h_1 and h_2 are smaller than $4a_1|\xi|$, we have

$$u_2^0+2\ln|\xi|\geq\ln(|\xi|^2/(16a_1^2|\xi|^2))=-2\ln(4a_1)$$

and then $u_2^0+2\ln|\xi|$ is also bounded by below. Since

$$u_2=(u_2^0+2\ln|\xi|)+\ln|\xi|+(u_2^2+2\ln|\xi|)+u_2^3$$

the function u_2 extends continuously to $\overline{S}$ by defining $(u_2)_{|\partial S}\equiv\infty$. It suffices to prove the compactness of $S_K=\{(\xi,w)\in S:u_2(\xi,w)\leq K\}$ for any $K\in\mathbb{R}^+$. The set S_K is closed in $\mathbb{C}^2$ since $(u_2)_{|\partial S}\equiv\infty$. There exists $K_1'\in\mathbb{R}$ such that $\ln|\xi|\leq K_1'$ for any $(\xi,w)\in S_K$. The set S_K is contained in $|\xi|\leq e^{K_1'}$. Moreover since $-2a_1|\xi|<Im(z)<2a_1|\xi|$ in $\sigma(\tilde{U}_{\lambda_0}')$ then

$$|\xi|\leq e^{K_1'}\text{ and }|e^{2\pi iz}|\leq e^{4\pi a_1\exp(K_1')}\quad\forall(\xi,z)\in\sigma(\tilde{U}_{\lambda_0}')\cap\{u_2\leq K\}.$$

The set S_K is closed and bounded and then compact. Thus $\sigma(U^*_{\lambda_0}) \cap \{u_2 \leq K\}$ is a compact set for any $K \in \mathbb{R}^+$.

The remainder of the proof is devoted to show that u_1 is a p.s.h. funtion in $U^*_{\lambda_0}$. It suffices to prove that $u = -\ln(\tau_1\tau_2 - 1)$ is strictly p.s.h in $U^*_{\lambda_0}$ since it is straightforward to prove that $u_2^l \circ \sigma$ is p.s.h. for $l \in \{1, 2, 3\}$. We have

$$\overline{\partial} u = \frac{\partial u}{\partial \tau_1}\overline{\partial}\tau_1 + \frac{\partial u}{\partial \tau_2}\overline{\partial}\tau_2$$

and then

$$\partial\overline{\partial} u = \sum_{bl} \frac{\partial^2 u}{\partial\tau_b \partial\tau_l}\omega_b \wedge \overline{\omega}_l + \frac{\partial u}{\partial \tau_1}\partial\overline{\partial}\tau_1 + \frac{\partial u}{\partial \tau_2}\partial\overline{\partial}\tau_2.$$

We denote $\overline{\partial}\omega_l = \sum_{bd} c^l_{bd}\overline{\omega_b} \wedge \omega_d$ for $l \in \{1, 2\}$. We can express $\partial\overline{\partial} u$ as

$$\sum_{bl} \omega_b \wedge \overline{\omega}_l \left(\frac{\partial^2 u}{\partial\tau_b \partial\tau_l} + c^1_{lb}\frac{\partial u}{\partial \tau_1} + c^2_{lb}\frac{\partial u}{\partial \tau_2} \right).$$

We define the coefficients u_{bl} to satisfy $\partial\overline{\partial} u = \sum_{bl} u_{bl}\omega_b \wedge \overline{\omega}_l$. Then we have

$$u_{bl} = \frac{\partial^2 u}{\partial\tau_b \partial\tau_l} + c^1_{lb}\left(\frac{\partial u}{\partial \tau_1} - \frac{\partial u}{\partial \tau_2} \right).$$

We define now $\rho = \tau_1\tau_2 - 1$. We can calculate the first derivatives of u to obtain

$$\frac{\partial u}{\partial \tau_1} = -\frac{\tau_2}{\rho}\ ,\ \frac{\partial u}{\partial \tau_2} = -\frac{\tau_1}{\rho} \ \text{ and } \ \frac{\partial u}{\partial \tau_1} - \frac{\partial u}{\partial \tau_2} = \frac{\tau_1 - \tau_2}{\rho}$$

and then the second derivatives

$$\frac{\partial^2 u}{\partial {\tau_1}^2} = \frac{{\tau_2}^2}{\rho^2}\ ,\ \frac{\partial^2 u}{\partial {\tau_2}^2} = \frac{{\tau_1}^2}{\rho^2} \ \text{ and } \ \frac{\partial^2 u}{\partial \tau_1 \partial \tau_2} = \frac{1}{\rho^2}.$$

We can apply these calculations to show that $\sum_{bl} u_{bl}\zeta_b\overline{\zeta}_l$ is greater or equal than

$$\frac{(\tau_2^2|\zeta_1|^2 + \tau_1^2|\zeta_2|^2 + (\zeta_1\overline{\zeta}_2 + \overline{\zeta}_1\zeta_2))}{\rho^2} - \frac{2K_8|\tau_1 - \tau_2||\xi|^{-k(1+1/\nu(\mathcal{E}_0))}}{\rho}(|\zeta_1|^2 + |\zeta_2|^2).$$

where $K_8 \in \mathbb{R}^+$ is the constant in lemma 5.10. We have

$$\frac{(\tau_2^2|\zeta_1|^2 + \tau_1^2|\zeta_2|^2 + (\zeta_1\overline{\zeta}_2 + \overline{\zeta}_1\zeta_2))}{\rho^2} \geq \frac{1}{\rho^2}\frac{{\tau_1}^2{\tau_2}^2 - 1}{{\tau_1}^2 + \tau_2^2}(|\zeta_1|^2 + |\zeta_2|^2).$$

The right-hand side is equal to

$$\frac{\tau_1\tau_2 + 1}{\rho({\tau_1}^2 + {\tau_2}^2)}(|\zeta_1|^2 + |\zeta_2|^2)$$

and since $(\tau_1\tau_2 + 1)/({\tau_1}^2 + {\tau_2}^2) \geq 2/(\tau_1 + \tau_2)^2$ if $\tau_1\tau_2 > 1$ then

$$\frac{(\tau_2^2|\zeta_1|^2 + \tau_1^2|\zeta_2|^2 + (\zeta_1\overline{\zeta}_2 + \overline{\zeta}_1\zeta_2))}{\rho^2} \geq \frac{1}{\rho}\frac{2}{(\tau_1 + \tau_2)^2}(|\zeta_1|^2 + |\zeta_2|^2).$$

We remark that $\tau_1 + \tau_2 = 4a_1 Re(\xi\lambda_0^{\nu(\mathcal{E}_0)})$ and $|\tau_2 - \tau_1| < 4a_1|\xi|$, we deduce

$$\sum_{bl} u_{bl}\zeta_b\overline{\zeta}_l \geq \frac{1}{\rho}\left(\frac{1}{8a_1^2 Re(\xi\lambda_0^{\nu(\mathcal{E}_0)})^2} - 8a_1K_8|\xi|^{-k(1+1/\nu(\mathcal{E}_0))+1} \right)(|\zeta_1|^2 + |\zeta_2|^2).$$

Since $k(1+1/\nu(\mathcal{E}_0)) > 3$ we get

$$\sum_{bl} u_{bl}(\xi,z)\zeta_b\overline{\zeta}_l \geq \frac{1}{\rho}\frac{1}{9a_1^2 Re(\xi\lambda_0^{\nu(\mathcal{E}_0)})^2}(|\zeta_1|^2+|\zeta_2|^2) \ \forall(\xi,z)\in\tilde{U}'_{\lambda_0} \ \forall(\zeta_1,\zeta_2)\in\mathbb{C}^2$$

for $\delta > 0$ small enough. The function u is strictly p.s.h. in $U^*_{\lambda_0}$. □

5.7.3. *Estimates for the solution of the $\overline{\partial}$ equation.* The equation $\overline{\partial}v = \Omega$ has a solution in $U^*_{\lambda_0}$ since $U^*_{\lambda_0}$ is a Stein manifold. We could define

$$\psi^{\varphi}_{j,\lambda_0,k} = \psi_{L_j,k} + \varrho(Re(z) - a_0|\xi|)\Delta_{\varphi^{-1},k} + v$$

as a Fatou coordinate of φ in $\tilde{U}'_{\lambda_0}$. Unfortunately such a solution does not necessarily satisfy good estimates as the one we will prove later on (see proposition 5.13). Thus we need a well-behaved solution of the $\overline{\partial}$ equation. We will use the following theorem due to Hörmander (see [13]):

Theorem 5.1. *Let $\mathcal{U}$ be a Stein manifold of dimension n with an hermitian metric. Let $\omega_1,\ldots,\omega_n \in \Omega^{1,0}(\mathcal{U})$ be C^∞ forms such that $\{\omega_1,\ldots,\omega_n\}$ is an orthonormal base of $T^*\mathcal{U}$ at any point. Denote $dV = (i/2)^n\omega_1\wedge\overline{\omega}_1\wedge\ldots\wedge\omega_n\wedge\overline{\omega}_n$ the volume element. Denote $\partial\omega_d = \frac{1}{2}\sum_{bl} a^d_{bl}\omega_b\wedge\omega_l$ and $\overline{\partial}\omega_d = \sum_{bl} c^d_{bl}\overline{\omega_b}\wedge\omega_l$ for $1\leq d\leq n$. Suppose that there exist $\theta_0,\theta_1 : \mathcal{U}\to\mathbb{R}$ with $|c^d_{bl}| \leq \theta_0 \geq |a^d_{bl}|$, $|\partial_g c^d_{bl}| \leq \theta_1 \geq |\partial_g a^d_{bl}|$ and $|\overline{\partial}_g c^d_{bl}| \leq \theta_1 \geq |\overline{\partial}_g a^d_{bl}|$ in $\mathcal{U}$ for any $(g,b,l,d)\in\{1,\ldots,n\}^4$. Let $\phi:\mathcal{U}\to\mathbb{R}$ be a C^2 strictly p.s.h. function. Suppose that there exists a continuous $\theta:\mathcal{U}\to\mathbb{R}^+$ such that*

$$\sum_{bl}\phi_{bl}\zeta_b\overline{\zeta}_l \geq (\theta + A(\theta_0^2+\theta_1))\sum_{d=1}^{n}|\zeta_d|^2$$

in $\mathcal{U}$ where $\partial\overline{\partial}\phi = \sum_{bl}\phi_{bl}\omega_b\wedge\overline{\omega}_l$ and $A\in\mathbb{R}^+$ is a universal constant depending only on the dimension of $\mathcal{U}$. Consider a $\overline{\partial}$-closed $(0,1)$ form $\Omega\in C^\infty(\mathcal{U})$ such that

$$\int_{\mathcal{U}}\theta^{-1}|\Omega|^2 e^{-\phi}dV < \infty.$$

Then there exists a complex function $v\in C^\infty(\mathcal{U})$ such that $\overline{\partial}v = \Omega$ and

$$\int_{\mathcal{U}}|v|^2 e^{-\phi}dV \leq \int_{\mathcal{U}}\theta^{-1}|\Omega|^2 e^{-\phi}dV.$$

Proposition 5.12. *There exists a solution $v : U^*_{\lambda_0}\to\mathbb{C}$ of $\overline{\partial}v = \Omega$ such that*

$$\int_{U^*_{\lambda_0}}|v|^2|\xi|^{2k(1+1/\nu(\mathcal{E}_0))-10}(\tau_1\tau_2-1)dV < K_{13}$$

for some $K_{13}\in\mathbb{R}^+$.

Proof. Let us apply theorem 5.1. In our situation we define $\mathcal{U} = U^*_{\lambda_0}$ and

$$\phi = -2(k(1+1/\nu(\mathcal{E}_0)) - 5)\ln|\xi| - \ln(\tau_1\tau_2 - 1).$$

The function ϕ is p.s.h. in $U^*_{\lambda_0}$. We can choose $\theta_0 = \theta_1 = K_8/|\xi|^{k(1+1/\nu(\mathcal{E}_0))}$ by lemma 5.10. Denote $\rho = \tau_1\tau_2 - 1$. We define $\theta = 1/(10a_1^2\rho Re(\xi\lambda_0^{\nu(\mathcal{E}_0)})^2)$. We have

$$\sum_{bl}\phi_{bl}\zeta_b\overline{\zeta}_l \geq \sum_{bl}u_{bl}\zeta_b\overline{\zeta}_l \geq \left(\frac{1}{9a_1^2\rho Re(\xi\lambda_0^{\nu(\mathcal{E}_0)})^2}\right)(|\zeta_1|^2+|\zeta_2|^2).$$

Now we use $\rho Re(\xi\lambda_0^{\nu(\mathcal{E}_0)})^2 A(\theta_0^2+\theta_1) = O(\xi^{4-k(1+1/\nu(\mathcal{E}_0))})$ to prove

$$\frac{1}{9a_1^2\rho Re(\xi\lambda_0^{\nu(\mathcal{E}_0)})^2} \geq \theta + A(\theta_0^2+\theta_1).$$

Next step is proving

$$\int_{U^*_{\lambda_0}} \theta^{-1}|\Omega|^2 e^{-\phi} dV < \infty.$$

Since $\Omega = -\overline{\partial}(\varrho(Re(z)-a_0|\xi|))\Delta_{\varphi^{-1},k}$ and $\overline{\partial}(\varrho(Re(z)-a_0|\xi|))$ is bounded (see proof of lemma 5.10) then we obtain $\Omega = O(|\xi|^{-e})$. The integral is smaller or equal than

$$K_{10}\int_{U^*_{\lambda_0}} |\xi|^4|\xi|^{-2e}|\xi|^{2(e-5)}|\xi|^2 dV$$

for $e = k(1+1/\nu(\mathcal{E}_0))$ and some positive constant K_{10}. We deduce

$$\int_{U^*_{\lambda_0}} \theta^{-1}|\Omega|^2 e^{-\phi} dV \leq K_{11}\int_{U'_{\lambda_0}} \frac{1}{|\xi|^4} dV_0$$

for some $K_{11} > 0$. The area of $U'_{\lambda_0}(\xi)$ is a $O(\xi)$ when $\xi \to \infty$. Thus we have

$$\int_{U^*_{\lambda_0}} \theta^{-1}|\Omega|^2 e^{-\phi} dV \leq K_{12}\int_{\tilde{I}_{\lambda_0}} \frac{1}{|\xi|^3}(i/2) d\xi \wedge d\overline{\xi} < K_{13}$$

for some positive constants K_{12} and K_{13}. We obtain the existence of a function v defined in $U^*_{\lambda_0}$ such that $\overline{\partial}v = \Omega$ and $\int_{U^*_{\lambda_0}} |v|^2 e^{-\phi} dV < K_{13}$. Hence we obtain

$$\int_{U^*_{\lambda_0}} |v|^2|\xi|^{2e-10}(\tau_1\tau_2 - 1) dV < K_{13}$$

as intended. □

Definition 5.28. *We define* $U = \cup_{x\in B(0,\delta)\setminus\{0\},\ s\in B(0,1/8)} \exp(sX)(B^2_{\varphi,j}(x))$,

$$I_{\lambda_0} = (0,\delta)\lambda_0 e^{i(-\pi/(4\nu(\mathcal{E}_0)),\pi/(4\nu(\mathcal{E}_0)))} \text{ and } U_{\lambda_0} = \cup_{x\in I_{\lambda_0}} U(x).$$

We denote $\tilde{U}_{\lambda_0} = \{(\xi,z)\in U_{\lambda_0} : \tau_1(\xi,z) > 3 < \tau_2(\xi,z)\}$.

Lemma 5.12. *Let* $L = \varrho(Re(z)-a_0|\xi|)\Delta_{\varphi^{-1},k} + v$. *There exists* $K_{15}\in\mathbb{R}^+$ *such that* $|L| \leq K_{15}|\psi_k|^{5-k(1+1/\nu(\mathcal{E}_0))}$ *in* $\tilde{U}_{\lambda_0}$.

Proof. The function L is holomorphic in $\tilde{U}'_{\lambda_0}$. It is not defined in general in $U^*_{\lambda_0}$. The next step is estimating the modulus of the function L. We have

$$\int_{U^*_{\lambda_0}\cap\{\tau_1\geq 2\}\cap\{\tau_2\geq 2\}} |v|^2|\xi|^{2e-10} dV \leq \frac{K_{13}}{3}.$$

Since $|L|^2 \leq 2|v|^2 + 2|\Delta_{\varphi^{-1},k}|^2$ we deduce that

$$\int_{\tilde{U}'_{\lambda_0}\cap\{\tau_1\geq 2\}\cap\{\tau_2\geq 2\}} |L|^2|\xi|^{2e-10} dV_0 \leq K_{14}$$

for some $K_{14} > 0$.

Let $(\xi_0, z_0) \in \tilde{U}_{\lambda_0}$. Consider $\kappa \in \mathbb{R}^+$ such that $a_0\kappa \leq 1/32$ and $a_1\kappa \leq 1/4$. The polydisk D_{ξ_0,z_0} of center (ξ_0, z_0) and poli-radius $(\kappa, 1/16)$ is contained in the set $\tilde{U}'_{\lambda_0} \cap \{\tau_1 \geq 2\} \cap \{\tau_2 \geq 2\}$. We have

$$L(\xi_0, z_0)\xi_0^{e-5} = \frac{16^2}{\pi^2\kappa^2} \int_{D_{\xi_0,z_0}} L(\xi, z)\xi^{e-5} dV_0.$$

This implies

$$|L(\xi_0, z_0)\xi_0^{e-5}| \leq \frac{16^2}{\pi^2\kappa^2} \sqrt{\int_{D_{\xi_0,z_0}} |L(\xi, z)\xi^{e-5}|^2 dV_0} \sqrt{\int_{D_{\xi_0,z_0}} dV_0} \leq \frac{16}{\pi\kappa}\sqrt{K_{14}}$$

for any $(\xi_0, z_0) \in \tilde{U}_{\lambda_0}$. As a consequence L is a $O(\xi^{-e+5}) = O(\psi_k^{5-e})$ in $\tilde{U}_{\lambda_0}$. □

We define

$$\psi^{\varphi}_{j,\lambda_0,k} = \psi_{L_j,k} + L(x, y).$$

It is a holomorphic Fatou coordinate of φ in $\tilde{U}_{\lambda_0}$. We obtain

Lemma 5.13. *Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$. Let Υ be a 2-convergent normal form. Consider $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}}) \in \mathcal{M}$, $\lambda \in \mathbb{S}^1$, $j \in \mathcal{D}(\varphi)$ and $k \geq \max(5, 4\nu(\mathcal{E}_0))$. Then there exists $K_{16} \in \mathbb{R}^+$ such that*

$$|\psi^{\varphi}_{j,\lambda,k} - \psi_{L_j,k}|(x, y) \leq \frac{K_{16}}{(1 + |\psi^X_{L_j}(x, y)|)^{k(1+1/\nu(\mathcal{E}_0))-5}}$$

for any $(x, y) \in \tilde{U}_\lambda$.

5.7.4. *Well-behaved Fatou coordinates.*

Proposition 5.13. *Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$. Let Υ be a 2-convergent normal form. Consider $\Lambda \in \mathcal{M}$, $\lambda \in \mathbb{S}^1$, $j \in \mathcal{D}(\varphi)$, $\theta \in (0, \pi/2]$ and $k \geq \max(5, 4\nu(\mathcal{E}_0))$. Then there exists $\rho \geq 2\rho_0$ such that the function $\psi^{\varphi}_{j,\lambda,k}$ satisfies*

$$|\psi^{\varphi}_{j,\lambda,k} - \psi_{L_j,k}|(x, y) \leq \frac{K_{17}}{(1 + |\psi^X_{L_j}(x, y)|)^{k-1}} \tag{30}$$

for some $K_{17} \in \mathbb{R}^+$ and any $(x, y) \in \cup_{x' \in I_\lambda}(H^{\epsilon,\rho}_{j,\theta}(x') \cap H^{\epsilon_k,\rho}_{j,\theta/2}(x'))$. Moreover ρ depends only on X, φ and Λ.

Proof. Let ρ be the same constant defined in proposition 5.10.

We claim that $\tilde{U}_\lambda(x)$ contains a neighborhood of $\overline{B^1_{\varphi,j}(x)}$ for any $x \in I_\lambda$. In fact we have

$$|Im(z \circ \sigma)| \leq |Im(z)| + 1 = |Im(\psi_k)| + 1 \leq |Im(\psi^X_{L_j})| + 2 \leq a_1|\xi| + 2$$

in $B^1_{\varphi,j}$. Since $Re(\xi\lambda^{\nu(\mathcal{E}_0)}) > |\xi|\cos(\pi/4)$ in I_λ we obtain

$$a_1|\xi| + 2 < 2a_1 Re(\xi\lambda^{\nu(\mathcal{E}_0)}) - 3 \ \forall \xi \in I_\lambda.$$

As a consequence $\overline{B^1_{\varphi,j}(x)}$ is contained in $\tilde{U}_\lambda(x)$ for any $x \in I_\lambda$. Therefore $\psi^{\varphi}_{j,\lambda,k}$ is defined in $\cup_{x \in I_\lambda} B^1_{\varphi,j}(x)$. We can suppose that $j \in \mathcal{D}_1(\varphi)$ without lack of generality.

Given $x \in B(0, \delta) \setminus \{0\}$ and $P \in H^{\epsilon,\rho}_{j,\theta}(x) \cap H^{\epsilon_k,\rho}_{j,\theta/2}(x)$ there exists $l_0(P) \in \mathbb{N} \cup \{0\}$ such that $\{P, \ldots, \varphi^{l_0}(P)\} \subset H^{\epsilon,2\rho_0}_{j,\theta,1}(x) \cap H^{\epsilon_k,2\rho_0}_{j,\theta/2}(x)$ and $\varphi^{l_0}(P) \in B^1_{\varphi,j}(x)$ by lemma

5.9. By defining $\psi_{j,\lambda,k}^{\varphi}(P)=\psi_{j,\lambda,k}^{\varphi}(\varphi^{l_0}(P))-l_0$ we extend the function $\psi_{j,\lambda,k}^{\varphi}$ to $H_{j,\theta}^{\epsilon,\rho}(x)\cap H_{j,\theta/2}^{\epsilon_k,\rho}(x)$. We have

$$(\psi_{j,\lambda,k}^{\varphi}-\psi_{L_j,k})(P)=(\psi_{j,\lambda,k}^{\varphi}-\psi_{L_j,k})(\varphi^{l_0}(P))+(\psi_{L_j,k}(\varphi^{l_0}(P))-\psi_{L_j,k}(P)-l_0)$$

and then we obtain (see def. 5.24)

$$(\psi_{j,\lambda,k}^{\varphi}-\psi_{L_j,k})(P)=O(\xi^{1-k})+\sum_{l=0}^{l_0-1}\Delta_{\varphi,k}(\varphi^{l}(P))$$

by lemma 5.13 and $k\geq 4\nu(\mathcal{E}_0)$. Since $\Delta_{\varphi,k}=O(1/(1+|\psi_{L_j}^{X}|)^k)$ in $H_{j,\theta,1}^{\epsilon,2\rho_0}$ lemma 5.2 implies

$$\psi_{j,\lambda,k}^{\varphi}-\psi_{L_j,k}=O(\xi^{-(k-1)})+O\left(\frac{1}{(1+|\psi_{L_j}^{X}|)^{k-1}}\right)\ \text{in}\ \cup_{x\in I_\lambda}(H_{j,\theta}^{\epsilon,\rho}(x)\cap H_{j,\theta/2}^{\epsilon_k,\rho}(x)).$$

We have $\psi_{L_j}^{X}\sim 1/y^{\nu(\mathcal{E}_0)}$ by equation (20). Thus $\psi_{L_j}^{X}$ is a $O(1/x^{\nu(\mathcal{E}_0)})=O(\xi)$ in $H_{j,\theta}^{\epsilon_k,\rho}$. We deduce $\xi^{-1}=O(|\psi_{L_j}^{X}|^{-1})$ in $H_{j,\theta}^{\epsilon,\rho}(x)\cap H_{j,\theta/2}^{\epsilon_k,\rho}(x)$. Hence equation (30) holds true for some $K_{17}\in\mathbb{R}^+$ and any $(x,y)\in\cup_{x'\in I_\lambda}(H_{j,\theta}^{\epsilon,\rho}(x')\cap H_{j,\theta/2}^{\epsilon_k,\rho}(x'))$. □

6. Multi-summability of the infinitesimal generator

The goal of this section is proving the multi-summable nature (with respect to the parameter x) of the Fatou coordinates and Lavaurs vector fields of an element φ of $\text{Diff}_{tp1}(\mathbb{C}^2,0)$. In the latter case we explain how the infinitesimal generator of φ is summable.

The subsection 6.1 is a fast review of the results of summability theory that we are going to use. In subsection 6.2 we study the extensions of the Ecalle-Voronin invariants

$$\ddot{\xi}_{j,\Lambda,\lambda}^{\varphi}(x,z)=\ddot{\psi}_{j+1,\Lambda,\lambda}^{\varphi}\circ(x,\ddot{\psi}_{j,\Lambda,\lambda}^{\varphi})^{-1}(x,z).$$

for $j\in\mathcal{D}(\varphi)$, $\Lambda\in\mathcal{M}$ and $\lambda\in\mathbb{S}^1$. At first sight the definition does not make sense since $H_{\Lambda,j}^{\lambda}\cap H_{\Lambda,j+1}^{\lambda}=\emptyset$ but this problem can be solved by extending slightly the domains of definition of $\ddot{\psi}_{j,\Lambda,\lambda}^{\varphi}$ and $\ddot{\psi}_{j+1,\Lambda,\lambda}^{\varphi}$. We prove in theorem 6.1 that the family $\{\ddot{\psi}_{j,\Lambda,\lambda}^{\varphi}\}_{(j,\Lambda,\lambda)\in\mathcal{D}(\varphi)\times\mathcal{M}\times\mathbb{S}^1}$ represents a multi-summable object. The proof is based on the estimates provided in prop. 5.9. The summable nature is concentrated in the x variable since all our estimates are exponentially small functions of x. We study the properties of the Lavaurs vector fields in subsection 6.3. Given $j\in\mathcal{D}(\varphi)$ the Lavaurs vector fields $\{X_{j,\Lambda,\lambda}^{\varphi}\}_{(\Lambda,\lambda)\in\mathcal{M}\times\mathbb{S}^1}$ (def. 6.7) represent a multi-summable object whose asymptotic development is of the form $\hat{X}_j^{\varphi}=\left(\sum_{k=0}^{\infty}g_{j,k}^{\varphi}(y)x^k\right)\partial/\partial y$ where the coefficients $g_{j,k}^{\varphi}$ are defined in the petal of order j of $\varphi_{|x=0}$ or more precisely in $\cup_{\theta\in(0,\pi/2]}H_{j,\theta}^{\epsilon,\rho,\lambda}(0)$. The proof is based on the estimates of prop. 5.11. The infinitesimal generator $\log\varphi$ is of the form $\left(\sum_{k=0}^{\infty}\hat{g}_k^{\varphi}(y)x^k\right)\partial/\partial y$. Then given $k\in\mathbb{N}\cup\{0\}$ it is natural to ask whether the power series $\hat{g}_k^{\varphi}$ is a $\nu(\mathcal{E}_0)$-summable function whose sums are the functions $g_{j,k}^{\varphi}$ for $j\in\mathcal{D}(\varphi)$. The answer is affirmative (theorem 6.6 of subsection 6.4).

6.1. **Multi-summability of formal power series.** For the sake of completeness we introduce the notations in [8] and [14].

We consider $\mathcal{V}_\lambda$ the set of open subsets of $\mathbb{C}^*$ containing a set of the form $(0,\zeta)\lambda e^{i(-\zeta,\zeta)}$ for some $\zeta \in \mathbb{R}^+$.

Let us introduce some sheafs defined in $\mathbb{S}^1$. Let $\mathcal{A}$ be the sheaf of rings such that $\mathcal{A}_\lambda$ is the set of holomorphic functions f defined in some $W \in \mathcal{V}_\lambda$ and admitting an asymptotyc development $\hat{f} = \sum_{l \geq 0} a_l x^l$ at the origin, i.e. we have

$$|f(x) - \sum_{l=0}^{b-1} a_l x^l| \leq c_b |x|^b \text{ in } W \text{ for some } c_b \in \mathbb{R}^+.$$

We denote $\mathcal{A}^{<0}$ the subsheaf of $\mathcal{A}$ whose elements f satisfy $\hat{f} \equiv 0$.

Given $s \in \mathbb{R}^+$ we define $\mathcal{A}_{(s)}$ the subsheaf of $\mathcal{A}$ such that an element f of $\mathcal{A}_{(s),\lambda}$ defined in some $W \in \mathcal{V}_\lambda$ satisfies

$$|f(x) - \sum_{l=0}^{b-1} a_l x^l| \leq c^b (b!)^s |x|^b$$

for any $x \in W$ and some $c \in \mathbb{R}^+$ independent of b. The sheaf $\mathcal{A}_{(s)}$ is the sheaf of functions admitting a Gevrey asymptotic expansion of order s. If $\hat{f} = \sum_{l=0}^{\infty} a_l x^l$ is a formal power series such that $|a_b| \leq c^b (b!)^s$ for any $b \in \mathbb{N}$ and some $c \in \mathbb{R}^+$ we say that $\hat{f}$ is a formal power series of Gevrey order s. We denote $\mathbb{C}[[x]]_s$ the set of formal power series of Gevrey order s.

Given $k \geq 0$ we define $\mathcal{A}^{\leq -k}$ the subsheaf of $\mathcal{A}^{<0}$ such that an element f of $\mathcal{A}_\lambda^{\leq -k}$ defined in some $W \in \mathcal{V}_\lambda$ satisfies $|f(x)| \leq A e^{-B/|x|^k}$ for any $x \in W$ and some $A, B \in \mathbb{R}^+$. By convention $\mathcal{A}^{\leq -\infty}$ only contains the zero function.

Definition 6.1. *Let $\tilde{e} = (\tilde{e}_1, \ldots, \tilde{e}_{\tilde{q}}) \in (\mathbb{R}^+)^{\tilde{q}}$ and $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}}) \in (\mathbb{S}^1)^{\tilde{q}}$. We define $I_l(\lambda, \upsilon) = \lambda e^{i[-\frac{\pi}{2\tilde{e}_l} - \upsilon, \frac{\pi}{2\tilde{e}_l} + \upsilon]}$ for $\lambda \in \mathbb{S}^1$, $\upsilon \in \mathbb{R}^+ \cup \{0\}$ and $1 \leq l \leq \tilde{e}_{\tilde{q}}$. We say that the pair $(\tilde{e}, \Lambda)$ is admissible if*

- *We have $0 < \tilde{e}_1 < \ldots < \tilde{e}_{\tilde{q}}$*
- *$I_{l+1}(\lambda_{l+1}, 0) \subset I_l(\lambda_l, 0)$ for any $1 \leq l < \tilde{q}$.*

Definition 6.2. [14] *Let $\tilde{e} = (\tilde{e}_1, \ldots, \tilde{e}_{\tilde{q}}) \in (\mathbb{R}^+)^{\tilde{q}}$ and $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}}) \in (\mathbb{S}^1)^{\tilde{q}}$. We set $I_0(\lambda_0, 0) = \mathbb{S}^1$ and $\tilde{e}_{\tilde{q}+1} = \infty$. Assume that $(\tilde{e}, \Lambda)$ is admissible. Let $\hat{\phi} \in \mathbb{C}[[x]]$ be a formal power series expansion. We will say that $\hat{\phi}$ is $(\tilde{e}_1, \ldots, \tilde{e}_{\tilde{q}})$-summable in the multi-direction Λ, with sum $\phi_{\tilde{q}}$, if:*

(i) $\hat{\phi} \in \mathbb{C}[[x]]_{\frac{1}{\tilde{e}_1}}$,

(ii) *there exists a sequence $(\phi_0, \ldots, \phi_{\tilde{q}})$ where:*

   a) *$\phi_0 \in \Gamma(\mathbb{S}^1; \mathcal{A}/\mathcal{A}^{\leq -\tilde{e}_1})$ and ϕ_0 corresponds to $\hat{\phi}$ by the natural isomorphism*

$$\Gamma(\mathbb{S}^1; \mathcal{A}/\mathcal{A}^{\leq -\tilde{e}_1}) \to \mathbb{C}[[x]]_{\frac{1}{\tilde{e}_1}},$$

   b) *$\phi_j \in \Gamma(I_j(\lambda_j, 0); \mathcal{A}/\mathcal{A}^{-\tilde{e}_{j+1}})$ $(j = 0, \ldots, \tilde{q})$, and $\phi_{j|I_{j+1}} = \phi_{j+1}$ modulo $\mathcal{A}^{\leq -\tilde{e}_{j+1}}$, for $j = 0, \ldots, \tilde{q} - 1$.*

The next proposition (see [1], page 69) is a criterium to identify multi-summable functions.

Proposition 6.1. *Let $\tilde{e} = (\tilde{e}_1, \ldots, \tilde{e}_{\tilde{q}}) \in (\mathbb{R}^+)^{\tilde{q}}$ and $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}}) \in (\mathbb{S}^1)^{\tilde{q}}$. We set $I_0(\lambda_0, 0) = \mathbb{S}^1$ and $\tilde{e}_{\tilde{q}+1} = \infty$. Assume that $\tilde{e}_1 > 1/2$ and $(\tilde{e}, \Lambda)$ is admissible. For $\zeta, \tau \in \mathbb{R}^+$, assume existence of $f(z;\lambda)$ (for every $\lambda \in \mathbb{S}^1$), analytic in the sector $I^\lambda = (0,\tau)\lambda e^{i(-\zeta/2,\zeta/2)}$ and bounded at the origin, such that for every λ_1, λ_2 with $I^{\lambda_1} \cap I^{\lambda_2} \neq \emptyset$ we have: If $\lambda_1, \lambda_2 \in I_l(\lambda_l, 0)$ for some l, $0 \leq l \leq \tilde{q}$, then*

$$f(z;\lambda_1) - f(z;\lambda_2) \in \mathcal{A}^{\leq -\tilde{e}_{l+1}}(I^{\lambda_1} \cap I^{\lambda_2}).$$

Then there exists a (unique) $(\tilde{e}_1, \ldots, \tilde{e}_{\tilde{q}})$-summable power series $\hat{\phi}$ in Λ with sum $f(z;\lambda_{\tilde{q}})$.

Let $e \in \mathbb{R}^+$. A power series $\hat{\phi} \in \mathbb{C}[[x]]$ is e-summable if it is e-summable in any direction outside of a finite set. A power series $\hat{\phi} \in \mathbb{C}[[x]]$ is $(\tilde{e}_1, \ldots, \tilde{e}_{\tilde{q}})$-summable if it has at most finitely many singular directions of each level $\tilde{e}_l$, $1 \leq l \leq \tilde{q}$ (see [1]). We will use the following result:

Lemma 6.1. *Let $\hat{\phi} \in \mathbb{C}[[x]]$ and $\tilde{e} = (\tilde{e}_1, \ldots, \tilde{e}_{\tilde{q}}) \in (\mathbb{R}^+)^{\tilde{q}}$. Fix $\lambda_j \in \mathbb{S}^1$. Suppose that there exists a sequence $(\lambda_{1,n}, \ldots, \lambda_{j-1,n}, \lambda_{j+1,n}, \ldots, \lambda_{\tilde{q},n}) \in (\mathbb{S}^1)^{\tilde{q}-1}$ such that*

- *Given $\Lambda_n = (\lambda_{1,n}, \ldots, \lambda_{j-1,n}, \lambda_j, \lambda_{j+1,n}, \ldots, \lambda_{\tilde{q},n})$ the pair $(\tilde{e}, \Lambda_n)$ is admissible for any $n \in \mathbb{N}$*
- *$\lim_{n\to\infty} \lambda_{k,n} = \lambda_k$ for any $k < j$*
- *$\hat{\phi}$ is $\tilde{e}$-summable in Λ_n for any $n \in \mathbb{N}$*

for any admissible pair $(\tilde{e}, \Lambda)$ with $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}})$. Then λ_j is a regular direction of level $\tilde{e}_j$.

Next lemma is of technical type. It will be used to identify the asymptotic development of the Lavaurs vector fields associated to an element of $\mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$.

Lemma 6.2. *Fix $\nu \geq 2$, $\lambda_0 \in \mathbb{S}^1$, $a, n \in \mathbb{N}$, $c_1, c_2 \in \mathbb{R}^+$ and $b \in \mathbb{R}^+$ with $b > \pi/a$. Denote $\lambda_k = \lambda_0 e^{2\pi i k/a}$, $\lambda_k' = \lambda_k e^{-i\pi/a}$ and $I_k = \lambda_k e^{i(-b,b)}$ for $k \in \mathbb{Z}/a\mathbb{Z}$. Consider $c \in \mathbb{R}^+$ such that the function $t \mapsto e^{-c_2 t^{-\nu}} t^{-(n+1)}$ is increasing in $(0,c)$. Let $\{h_k\}_{k \in \mathbb{Z}/a\mathbb{Z}}$ be a family of holomorphic functions satisfying*

- *h_k is holomorphic in $(0,c)I_k$ for any $k \in \mathbb{Z}/a\mathbb{Z}$.*
- *$|h_k - h_{k-1}| \leq c_1 e^{-c_2/|x|^\nu}$ in $(0,c)\lambda_k' e^{i(-(b-\pi/a), b-\pi/a)}$ for any $k \in \mathbb{Z}/a\mathbb{Z}$.*

Suppose that there exists $\tau \in \mathbb{R}^+$ such that $|h_k| \leq \tau$ in $(0,c)I_k$ for any $k \in \mathbb{Z}/a\mathbb{Z}$. The function h_k has a $1/\nu$ Gevrey asymptotic development $\sum_{l=0}^{\infty} \hat{h}_l x^l$ independent of k. Then we obtain

$$|\hat{h}_n| \leq \frac{2^n \tau}{c^n} + e^{-c_2/(2c^\nu)} \tag{31}$$

if $c \in \mathbb{R}^+$ is small enough.

Proof. The functions h_k share a $1/\nu$ Gevrey asymptotic development since they define a section h of $(\mathcal{A}/\mathcal{A}^{\leq -\nu})(\mathbb{S}^1)$ and $\Gamma(\mathbb{S}^1; \mathcal{A}/\mathcal{A}^{\leq -\nu}) \to \mathbb{C}[[x]]_{\frac{1}{\nu}}$ is an isomorphism.

Denote $\theta = (b - \pi/a)/2$ and $c_0 = \min(\sin(\theta), 1/2)$. By replacing b with $b - \theta$ we can suppose that $h_k - h_{k-1}$ is defined in $(0,c)\lambda_k' e^{i(-(b-\pi/a+\theta), b-\pi/a+\theta)}$ for any $k \in \mathbb{Z}/a\mathbb{Z}$.

Let us construct, for $j \in \mathbb{Z}/a\mathbb{Z}$, holomorphic functions $\tilde{h}_k$ defined in $(0, c/2)I_k$, defining the same element $h \in \Gamma(\mathbb{S}^1; \mathcal{A}/\mathcal{A}^{\leq -\nu})$ and whose Gevrey development is easier to estimate. We use the Cauchy-Heine transform (see [1], chapter 4) to define the function

$$h_k^\sharp(x) = \frac{1}{2\pi i}\int_0^{c\lambda_k'} (h_k - h_{k-1})(w)(w-x)^{-1}dw. \tag{32}$$

It is defined in $(0,c/2)(\mathbb{S}^1 \setminus \lambda_k' e^{i[-\theta,\theta]})$. Moreover we obtain $|w-x|/|w| \geq \sin(\theta)$ for all $w \in (0,c)\lambda_k'$ and $x \in (0,c/2)(\mathbb{S}^1 \setminus \lambda_k' e^{i[-\theta,\theta]})$. Given $\lambda \in \lambda_k' e^{i(-(b-\pi/a+\theta), b-\pi/a+\theta)}$ we can extend $h_k^\sharp$ to $(0,c/2)(\mathbb{S}^1 \setminus \lambda e^{i[-\theta,\theta]})$ by replacing the path of integration $[0,c]\lambda_k'$ in equation (32) with the union γ_λ of $[0,c]\lambda$ and an arc in $\partial B(0,c)$ joining $c\lambda$ and $c\lambda_k'$. We obtain $|w-x|/|w| \geq c_0$ for all $w \in \gamma_\lambda$ and $x \in (0,c/2)(\mathbb{S}^1 \setminus \lambda e^{i[-\theta,\theta]})$. The function $h_k^\sharp$ is holomorphic in $(0,c/2)(-\lambda_k')e^{i(-(b-\pi/a+\pi), b-\pi/a+\pi)}$. It is multi-valuated and satisfies $h_k^\sharp(x) - h_k^\sharp(e^{2\pi i}x) = (h_k - h_{k-1})(x)$. We define

$$\tilde{h}_k(x) = \sum_{l=1}^{k} h_l^\sharp(x) + \sum_{l=k+1}^{a} h_l^\sharp(e^{2\pi i}x)$$

in $(0,c/2)I_k$ for $k \in \mathbb{Z}/a\mathbb{Z}$. By construction we obtain $\tilde{h}_k - \tilde{h}_{k-1} = h_k - h_{k-1}$ and

$$|\tilde{h}_k(x)| \leq \frac{1}{2\pi}\frac{a}{c_0} c(b-\pi/a+\theta+1) \sup_{t\in(0,c)} (c_1 e^{-c_2 t^{-\nu}} t^{-1})$$

for all $k \in \mathbb{Z}/a\mathbb{Z}$ and $x \in (0,c/2)I_k$. By simple calculus we can see that if $e^{-c_2 t^{-\nu}} t^{-(n+1)}$ is increasing in $(0,c)$ then so is $e^{-c_2 t^{-\nu}} t^{-1}$. As a consequence we obtain

$$|\tilde{h}_k(x)| \leq \frac{1}{2\pi}\frac{ac_1}{c_0}(b-\pi/a+\theta+1)e^{-c_2 c^{-\nu}}$$

for all $k \in \mathbb{Z}/a\mathbb{Z}$ and $x \in (0,c/2)I_k$.

Let $\sum_{l=0}^{\infty} h_l^\flat x^l$ be the $1/\nu$ Gevrey development associated to the element of $\Gamma(\mathbb{S}^1; \mathcal{A}/\mathcal{A}^{\leq -\nu})$ defined by the functions $\tilde{h}_k$ for $k \in \mathbb{Z}/a\mathbb{Z}$. We have

$$h_n^\flat = \frac{1}{2\pi i}\sum_{k=1}^{a}\int_0^{c\lambda_k'} (h_k - h_{k-1})(w) w^{-(n+1)} dw$$

by the properties of the Cauchy-Heine transform. Since $e^{-c_2 t^{-\nu}} t^{-(n+1)}$ is increasing in $(0,c)$ we deduce $|h_n^\flat| \leq (2\pi)^{-1} acc_1 e^{-c_2 c^{-\nu}} c^{-(n+1)}$. Let $h - \tilde{h}$ be the function defined in $B(0,c/2)$ such that $(h-\tilde{h})_{|(0,c/2)I_k} = h_k - \tilde{h}_k$ for $k \in \mathbb{Z}/a\mathbb{Z}$. We have

$$|(h-\tilde{h})(x)| \leq \tau + \frac{ac_1(b-\pi/a+\theta+1)}{2c_0\pi} e^{-c_2 c^{-\nu}} \ \forall x \in B(0,c/2).$$

Cauchy's integral formula implies

$$|\hat{h}_n| \leq \frac{2^n}{c^n}\left(\tau + \frac{ac_1(b-\pi/a+\theta+1)}{2c_0\pi} e^{-c_2 c^{-\nu}}\right) + \frac{ac_1}{2c^n\pi} e^{-c_2 c^{-\nu}}.$$

A straightforward argument provides the estimate (31). □

6.2. **Multi-summability of Fatou coordinates.** Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ with 2-convergent normal form $\Upsilon = \exp(X)$. Consider $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}}) \in \mathcal{M}$ and the dynamical splitting $\digamma_\Lambda$ in remark 4.11.

Let $j \in \mathcal{D}_s(\varphi)$ and $\lambda \in \mathbb{S}^1$. Our aim is to define

$$\ddot{\xi}^\varphi_{j,\Lambda,\lambda}(x,z) = \ddot{\psi}^\varphi_{j+1,\Lambda,\lambda} \circ (x, \ddot{\psi}^\varphi_{j,\Lambda,\lambda})^{-1}(x,z).$$

We can interpret the set $(x, \ddot{\psi}_{j,\lambda}^{\varphi})(H_{\Lambda,j}^{\lambda})$ as a chart coordinate system in which φ is of the form $(x, z+1)$. The map $\ddot{\xi}_{j,\lambda}^{\varphi}$ is then a transition function commuting with $(x, z+1)$. The main problem in order to define $\ddot{\xi}_{j,\lambda}^{\varphi}$ is that $H_{\Lambda,j}^{\lambda} \cap H_{\Lambda,j+1}^{\lambda} = \emptyset$. Anyway in next lemma we extend the Fatou coordinates by iteration in order to obtain a common domain of definition for both $\ddot{\psi}_{j,\lambda}^{\varphi}$ and $\ddot{\psi}_{j+1,\lambda}^{\varphi}$.

Remark 6.1. *The family $\{\ddot{\xi}_{j,\Lambda,\lambda}^{\varphi}\}_{j \in \mathcal{D}(\varphi)}$ for $\Lambda \in \mathcal{M}$ and $\lambda \in \mathbb{S}^1$ is an extension of the Ecalle-Voronin invariants of $\varphi_{|x=0}$ in I_{Λ}^{λ}.*

Lemma 6.3. *Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ with 2-convergent normal form $\Upsilon = \exp(X)$. Consider $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}}) \in \mathcal{M}$, $\lambda \in \mathbb{S}^1$, $j \in \mathcal{D}_s(\varphi)$ and $s \in \{-1, 1\}$. There exists $M \in \mathbb{R}^+$ such that the function $\ddot{\xi}_{j,\Lambda,\lambda}^{\varphi}(x,z)$ commutes with $z \mapsto z+1$ and is defined in $[0,\delta)I_{\Lambda}^{\lambda} \times s(\mathbb{R} + i(-\infty, -M))$. Moreover there exists*

$$\lim_{(x, Im(z)) \to (x_0, -s\infty)} \ddot{\xi}_{j,\Lambda,\lambda}^{\varphi}(x,z) - z$$

for any $x_0 \in [0,\delta)I_{\Lambda}^{\lambda}$. The function $e^{2\pi i z} \circ \ddot{\xi}_{j,\Lambda,\lambda}^{\varphi}(x, (\ln z)/(2\pi i))$ is holomorphic in $(0,\delta)\lambda e^{i(-\upsilon_{\Lambda}, \upsilon_{\Lambda})} \times \{z \in \mathbb{P}^1(\mathbb{C}) : |z|^s > e^{2\pi M}\}$. The constant M does not depend on Λ, λ or j.

Proof. Consider $\Gamma_{x,k,\lambda} = \Gamma(\aleph_{\Lambda,\lambda} X, T_{iX}^{\epsilon,k}(x), T_0)$ for $k \in \mathcal{D}(\varphi)$ and $x \in [0,\delta)I_{\Lambda}^{\lambda}$. Suppose $s = 1$ without lack of generality. We denote

$$B_{X,j,\lambda}(x) = \cup_{t \in [0,2]} \exp(tX)(\Gamma_{x,j,\lambda}) \text{ and } B_{X,j+1,\lambda}(x) = \cup_{t \in [-2,0]} \exp(tX)(\Gamma_{x,j+1,\lambda})$$

for $x \in [0,\delta)I_{\Lambda}^{\lambda}$. Let $H_{j,j+1}^{\lambda}$ be the element of the set $Reg^*(\epsilon, \aleph_{\Lambda,\lambda} X, I_{\Lambda}^{\lambda})$ such that $T_{iX}^{\epsilon,j}(0) \in \overline{H_{j,j+1}^{\lambda}(0)} \ni T_{iX}^{\epsilon,j+1}(0)$. We define

$$E_{X,j,\lambda}(x) = B_{X,j,\lambda}(x) \cup B_{X,j+1,\lambda}(x) \cup H_{j,j+1}^{\lambda}(x) \text{ for } x \in [0,\delta)I_{\Lambda}^{\lambda}.$$

Since $E_{X,j,\lambda}(x)$ is simply connected for $x \in [0,\delta)I_{\Lambda}^{\lambda}$ we can extend $\psi_{L_j}^{X}$ to $E_{X,j,\lambda}$.

There exists $M \in \mathbb{R}^+$ such that any trajectory $\Gamma(X, P, E_{X,j,\lambda})$ for $P \in B_{X,j,\lambda}(x)$, $x \in [0,\delta)I_{\Lambda}^{\lambda}$ and $Im(\psi_{L_j}^{X}(P)) < -M$ intersects $B_{X,j+1,\lambda}(x)$. More precisely, there exists $t_0(P) \in \mathbb{R}^+$ such that $\exp(tX)(P) \in E_{X,j,\lambda}$ for any $t \in [-t_0(P), 0]$ and $\exp(-t_0(P)X)(P) \in B_{X,j+1,\lambda}(x)$. The constant M does not depend on Λ, λ or j. We have

$$|\Delta_{\varphi}| \leq \frac{K_{18}}{(1 + |\psi_{L_j}^{X}|)^2} \text{ in } E_{X,j,\lambda}$$

for some $K_{18} \in \mathbb{R}^+$ by proposition 5.2. We have $\psi_{L_j}^{X} \circ \varphi^{-1} = \psi_{L_j}^{X} - 1 - \Delta_{\varphi} \circ \varphi^{-1}$. Lemma 5.2 implies

$$|\psi_{L_j}^{X} \circ \varphi^{-l} - (\psi_{L_j}^{X} - l)|(P) \leq \frac{K_{19}}{1 + |\psi_{L_j}^{X}(P)|}$$

for all orbit $\{P, \varphi^{-1}(P), \ldots, \varphi^{-l}(P)\}$ contained in $E_{X,j,\lambda}(x)$ and $x \in [0,\delta)I_{\Lambda}^{\lambda}$. We can use the previous inequality to show that there exists $t_1(P) \in \mathbb{N}$ such that

$$\varphi^{-l}(P) \in E_{X,j,\lambda} \ \forall 1 \leq l \leq t_1(P) \text{ and } \varphi^{-t_1(P)}(P) \in B_{X,j+1,\lambda}$$

for any $P \in \cup_{x \in [0,\delta)I_{\Lambda}^{\lambda}} B_{X,j,\lambda}(x)$ with $Im(\psi_{L_j}^{X}(P)) < -M$. We consider a greater $M \in \mathbb{R}^+$ if necessary. Thus we can extend $\ddot{\psi}_{j+1,\lambda}^{\varphi}$ to $B_{X,j,\lambda}(x) \cap \{Im(\psi_{L_j}^{X}) < -M\}$

for any $x \in [0,\delta)I_{\Lambda}^{\lambda}$. Since we have

$$|(\ddot{\psi}_{j+1,\lambda}^{\varphi} - \psi_{L_j}^{X})(P) - (\ddot{\psi}_{j+1,\lambda}^{\varphi} - \psi_{L_j}^{X})(\varphi^{-t_1(P)}(P))| \leq \frac{K_{19}}{1+|\psi_{L_j}^{X}(P)|}$$

for any $P \in \cup_{x \in [0,\delta)I_{\Lambda}^{\lambda}} B_{X,j,\lambda}(x)$ with $Im(\psi_{L_j}^{X}(P)) < -M$ then $\ddot{\psi}_{j+1,\lambda}^{\varphi} - \psi_{L_j}^{X}$ is continuous in $\cup_{x \in [0,\delta)I_{\Lambda}^{\lambda}} \overline{B_{X,j,\lambda}(x)} \cap \{Im(\psi_{L_j}^{X}) < -M\}$ and so is the mapping $\ddot{\psi}_{j+1,\lambda}^{\varphi} - \ddot{\psi}_{j,\lambda}^{\varphi}$. The mapping $\ddot{\xi}_{j,\lambda}^{\varphi}(x,z)$ commutes with $(x, z+1)$. Therefore it is defined in $[0,\delta)I_{\Lambda}^{\lambda} \times (\mathbb{R} + i(-\infty,-M))$ up to consider a greater $M \in \mathbb{R}^{+}$. Moreover we obtain

$$\lim_{(x,Im(z)) \to (x_0,-\infty)} \ddot{\xi}_{j,\lambda}^{\varphi}(x,z) - z = (\ddot{\psi}_{j+1,\lambda}^{\varphi} - \ddot{\psi}_{j,\lambda}^{\varphi})(\alpha^{\aleph_{\Lambda,\lambda}X}(H_{\Lambda,j}^{\lambda}(x))).$$

Thus $e^{2\pi i z} \circ \ddot{\xi}_{j,\lambda}^{\varphi}(x, (\ln z)/(2\pi i))$ is holomorphic in $[0,\delta)I_{\Lambda}^{\lambda} \times \{|z| > e^{2\pi M}\}$. □

Next we study the dependence of $\ddot{\xi}_{j,\Lambda,\lambda}^{\varphi}$ on $\Lambda \in \mathcal{M}$ and $\lambda \in \mathbb{S}^1$. We provide the estimates implying the multi-summability of the extensions of the Ecalle-Voronin invariants.

Theorem 6.1. *Let $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2,0)$ with 2-convergent normal form $\Upsilon = \exp(X)$. Consider $\Lambda, \Lambda' \in \mathcal{M}$, $\lambda, \lambda' \in \mathbb{S}^1$, $j \in \mathcal{D}_s(\varphi)$ and $s \in \{-1,1\}$. Then*

$$|\ddot{\xi}_{j,\Lambda,\lambda}^{\varphi}(x,z) - \ddot{\xi}_{j,\Lambda',\lambda'}^{\varphi}(x,z)| \leq K_{20} e^{-K/|x|^{\tilde{e}_{d_{\Lambda,\Lambda'}^{\lambda,\lambda'}+1}}} \tag{33}$$

in $[0,\delta)I_{\Lambda,\Lambda'}^{\lambda,\lambda'} \times s(\mathbb{R} + i(-\infty,-M))$ for some $K_{20} \in \mathbb{R}^{+}$.

Proof. Consider the notations in the proof of lemma 6.3. Denote $d = d_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. We have

$$|\ddot{\psi}_{j,\Lambda,\lambda}^{\varphi} - \ddot{\psi}_{j,\Lambda',\lambda'}^{\varphi}| \leq e^{-K/|x|^{\tilde{e}_{d+1}}} \text{ and } |\ddot{\psi}_{j+1,\Lambda,\lambda}^{\varphi} - \ddot{\psi}_{j+1,\Lambda',\lambda'}^{\varphi}| \leq e^{-K/|x|^{\tilde{e}_{d+1}}}$$

in $H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}$ and $H_{\Lambda,\Lambda',j+1}^{\lambda,\lambda'}$ respectively by prop. 5.9.

Given $x \in [0,\delta)I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$ consider a connected path γ_x contained in $B_{X,j,\lambda}(x)$ such that $\ddot{\psi}_{j,\lambda}^{\varphi}(\gamma_x)$ is of the form $[a(x), a(x)+1] - isM$. Given $x_0 \in [0,\delta)I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$ and $z_0 \in \ddot{\psi}_{j,\lambda}^{\varphi}(\gamma_{x_0})$ we consider the point $(x_0,y_0) \in \gamma_{x_0}$ such that $\ddot{\psi}_{j,\lambda}^{\varphi}(x_0,y_0) = z_0$. We consider the point (x_0,y_1) such that $\ddot{\psi}_{j,\lambda'}^{\varphi}(x_0,y_1) = z_0$. Since

$$|\ddot{\psi}_{j,\lambda}^{\varphi}(x_0,y_1) - \ddot{\psi}_{j,\lambda'}^{\varphi}(x_0,y_1)| \leq e^{-K/|x_0|^{\tilde{e}_{d+1}}}$$

by proposition 5.9 we obtain $|\ddot{\psi}_{j,\lambda}^{\varphi}(x_0,y_1) - \ddot{\psi}_{j,\lambda}^{\varphi}(x_0,y_0)| \leq e^{-K/|x_0|^{\tilde{e}_{d+1}}}$. There exists $K_1' \in \mathbb{R}^{+}$ such that $|\ddot{\psi}_{j,\lambda}^{\varphi} \circ (x, \psi_{L_j}^{X})^{-1}(x,z) - z| \leq K_1'$ in a neighborhood of $\cup_{x \in [0,\delta)I_{\Lambda,\Lambda'}^{\lambda,\lambda'}} \{x\} \times \psi_{L_j}^{X}(B_{X,j,\lambda}(x))$ since $\ddot{\psi}_{j,\lambda}^{\varphi} - \psi_{L_j}^{X}$ is bounded. By using Cauchy's integral formula we obtain $K_2' > 0$ such that

$$\left| \frac{\partial(\ddot{\psi}_{j,\lambda}^{\varphi} \circ (x, \psi_{L_j}^{X})^{-1})}{\partial z} - 1 \right| \leq K_2'$$

in a neighborhood of $\cup_{x \in [0,\delta)I_{\Lambda,\Lambda'}^{\lambda,\lambda'}} \{x\} \times \psi_{L_j}^{X}(B_{X,j,\lambda}(x))$. We can suppose $K_2' \leq 1/2$ by considering a greater $M \in \mathbb{R}^{+}$. Thus we obtain $|\partial(\psi_{L_j}^{X} \circ (x, \ddot{\psi}_{j,\lambda}^{\varphi})^{-1})/\partial z| \leq 2$ in a neighborhood of $\cup_{x \in [0,\delta) I_{\Lambda,\Lambda'}^{\lambda,\lambda'}} \{x\} \times \ddot{\psi}_{j,\lambda}^{\varphi}(B_{X,j,\lambda}(x))$. We deduce

$$|\psi_{L_j}^X(x_0, y_1) - \psi_{L_j}^X(x_0, y_0)| \leq 2e^{-K/|x_0|^{\tilde{e}_{d+1}}}.$$

We have

$$|\ddot{\psi}_{j+1,\lambda}^{\varphi}(x_0, y_0) - \ddot{\psi}_{j+1,\lambda'}^{\varphi}(x_0, y_1)| \leq |\ddot{\psi}_{j+1,\lambda}^{\varphi}(x_0, y_0) - \ddot{\psi}_{j+1,\lambda}^{\varphi}(x_0, y_1)| + e^{-K/|x_0|^{\tilde{e}_{d+1}}}.$$

There exists $K_3' \in \mathbb{R}^+$ such that $|\partial(\ddot{\psi}_{j+1,\lambda}^{\varphi} \circ (x, \psi_{L_j}^X)^{-1})/\partial z - 1| \leq K_3'$ in a neighborhood of $\cup_{x \in [0,\delta) I_{\Lambda,\Lambda'}^{\lambda,\lambda'}} \{x\} \times [\psi_{L_j}^X(B_{X,j,\lambda}(x)) \cap s(\mathbb{R} + i(-\infty, -M))]$. The proof is analogous to the one for $\ddot{\psi}_{j,\lambda}^{\varphi}$. This property implies

$$|\ddot{\psi}_{j+1,\lambda}^{\varphi}(x_0, y_0) - \ddot{\psi}_{j+1,\lambda'}^{\varphi}(x_0, y_1)| \leq 2(1 + K_3')e^{-K/|x_0|^{\tilde{e}_{d+1}}} + e^{-K/|x_0|^{\tilde{e}_{d+1}}}.$$

Denote $K_{20} = 2(1 + K_3') + 1$. The function $(\ddot{\xi}_{j,\Lambda,\lambda}^{\varphi} - \ddot{\xi}_{j,\Lambda',\lambda'}^{\varphi})(x_0, (\ln z)/(2\pi i))$ is defined in $\{z \in \mathbb{P}^1(\mathbb{C}) : |z|^s > e^{2\pi M}\}$ and is bounded by above by $K_{20} e^{-K/|x_0|^{\tilde{e}_{d+1}}}$ in $\partial B(0, e^{2\pi s M})$. The modulus maximum principle implies equation (33). □

Definition 6.3. *Let $j \in \mathcal{D}_s(\varphi)$. Since $\ddot{\xi}_{j,\Lambda,\lambda}^{\varphi}$ commutes with $(x, z + 1)$, it is of the form*

$$\ddot{\xi}_{j,\Lambda,\lambda}^{\varphi} = z + \ddot{a}_{j,\Lambda,\lambda,0}^{\varphi}(x) + \sum_{l=1}^{\infty} \ddot{a}_{j,\Lambda,\lambda,l}^{\varphi}(x) e^{-2\pi i s l z}$$

where $\ddot{a}_{j,\Lambda,\lambda,l}^{\varphi}$ is continuous in $[0,\delta) I_{\Lambda}^{\lambda}$ and holomorphic in $(0,\delta) \dot{I}_{\Lambda}^{\lambda}$ for any $l \geq 0$.

The properties of the families $\{\ddot{a}_{j,\Lambda,\lambda,l}^{\varphi}\}_{(\Lambda,\lambda) \in \mathcal{M} \times \mathbb{S}^1}$ and $\{\ddot{\xi}_{j,\Lambda,\lambda}^{\varphi}\}_{(\Lambda,\lambda) \in \mathcal{M} \times \mathbb{S}^1}$ are analogous.

Theorem 6.2. *Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ with 2-convergent normal form $\Upsilon = \exp(X)$. Consider $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}}) \in \mathcal{M}$, $j \in \mathcal{D}_s(\varphi)$ and $s \in \{-1, 1\}$. Then the function $\ddot{a}_{j,\Lambda,\lambda,l}^{\varphi}$ is $(\tilde{e}_1, \ldots, \tilde{e}_{\tilde{q}})$-summable in the multi-direction Λ for any $l \in \mathbb{N} \cup \{0\}$. Moreover its asymptotic development $\ddot{a}_{j,l}^{\varphi}$ does not depend on Λ. In particular $\ddot{a}_{j,l}^{\varphi}$ is a $(\tilde{e}_1, \ldots, \tilde{e}_{\tilde{q}})$-summable power series for any $l \in \mathbb{N} \cup \{0\}$.*

Proof. Fix $l \in \mathbb{N} \cup \{0\}$. Let $\Lambda' \in \mathcal{M}$ and $\lambda, \lambda' \in \mathbb{S}^1$. Consider the curve γ_x defined in the proof of theorem 6.1 for $x \in [0,\delta) I_{\Lambda,\Lambda'}^{\lambda,\lambda'}$. We have

$$|\ddot{a}_{j,\Lambda,\lambda,l}^{\varphi} - \ddot{a}_{j,\Lambda',\lambda',l}^{\varphi}|(x) \leq \left| \int_{\ddot{\psi}_{j,\lambda}^{\varphi}(\gamma_x)} e^{2\pi i s l z} (\ddot{\xi}_{j,\Lambda,\lambda}^{\varphi}(x,z) - \ddot{\xi}_{j,\Lambda',\lambda'}^{\varphi}(x,z)) dz \right|$$

and then

$$|\ddot{a}_{j,\Lambda,\lambda,l}^{\varphi} - \ddot{a}_{j,\Lambda',\lambda',l}^{\varphi}|(x) \leq e^{2\pi l M} K_{20} e^{-K/|x|^{\tilde{e}_{d_{\Lambda,\Lambda'}^{\lambda,\lambda'}+1}}}.$$

We obtain

$$\ddot{a}_{j,\Lambda,\lambda,l}^{\varphi} - \ddot{a}_{j,\Lambda',\lambda',l}^{\varphi} \in \mathcal{A}^{\leq -\tilde{e}_1}(\dot{I}_{\Lambda,\Lambda'}^{\lambda,\lambda'}). \tag{34}$$

Suppose $\Lambda' = \Lambda$. If $\lambda, \lambda' \in I_k(\lambda_k, 0)$ for some $k \in \{0, \ldots, \tilde{q}\}$ we obtain $d_{\Lambda}^{\lambda} \geq k \leq d_{\Lambda}^{\lambda'}$ and $d_{\Lambda,\Lambda}^{\lambda,\lambda'} \geq k$. We obtain

$$\ddot{a}_{j,\Lambda,\lambda,l}^{\varphi} - \ddot{a}_{j,\Lambda,\lambda',l}^{\varphi} \in \mathcal{A}^{\leq -\tilde{e}_{k+1}}(\dot{I}_{\Lambda,\Lambda}^{\lambda,\lambda'}).$$

Proposition 6.1 implies that $\ddot{a}^{\varphi}_{j,\Lambda,\lambda,l}$ is $(\tilde{e}_1, \dots, \tilde{e}_{\tilde{q}})$-summable in Λ. Denote $\ddot{a}^{\varphi}_{j,\Lambda,l}$ its asymptotic development. Property (34) for $\lambda, \lambda' \in \mathbb{S}^1$ implies $\ddot{a}^{\varphi}_{j,\Lambda,l} = \ddot{a}^{\varphi}_{j,\Lambda',l}$ for all $\Lambda, \Lambda' \in \mathcal{M}$ (see condition (ii) a) in def. 6.2).

Consider $(\lambda_1, \dots, \lambda_{\tilde{q}}) \in (\mathbb{S}^1)^{\tilde{q}}$ where $\lambda_k \notin \tilde{\Xi}^k_X$ (see def. 4.10). There exist $s_1, s_2 \in \{-1, 1\}$ such that

$$(\lambda_1 e^{is_1\zeta}, \dots, \lambda_{k-1} e^{is_1\zeta}, \lambda_k, \lambda_{k+1} e^{is_2\zeta}, \dots, \lambda_{\tilde{q}} e^{is_2\zeta}) \in \mathcal{M}$$

for any ζ in a neighborhood of 0 in $\mathbb{R}^+$. The singular directions of order $\tilde{e}_k$ of $\ddot{a}^{\varphi}_{j,l}$ are contained in the finite set $\tilde{\Xi}^k_X$ for any $1 \leq k \leq \tilde{q}$ by lemma 6.1. Therefore $\ddot{a}^{\varphi}_{j,l} \in \mathbb{C}[[x]]$ is $(\tilde{e}_1, \dots, \tilde{e}_{\tilde{q}})$-summable. □

Definition 6.4. *We define* $\zeta_\varphi(x) = -\pi i/\nu(\mathcal{E}_0)^{-1} \sum_{P \in (SingX)(x)} Res(X, P)$ *(see def. 4.4). The function* $\zeta_\varphi(x)$ *is holomorphic in the neighborhood of* $x = 0$*. It does not depend on the choice of* X*.*

Let us review the normalizing conditions for Fatou coordinates introduced in subsection 8.1 of [16]. By making analytic extension along the arc going from $T^{\epsilon,j}_{iX}(0)$ to $T^{\epsilon,j+1}_{iX}(0)$ in counter clock wise sense we can define $\psi^X_{L_{j+1}} - \psi^X_{L_j}$. It depends only on x and it is holomorphic in a neighborhood of $x = 0$. By considering Fatou coordinates of the form $\dot{\psi}^X_j = \psi^X_{L_j} + c_j(x)$ for convenient holomorphic functions $c_1, \dots, c_{2\nu(\mathcal{E}_0)}$ defined in $B(0, \delta)$ we can obtain $(\dot{\psi}^X_{j+1} - \dot{\psi}^X_j)(x, y) = \zeta_\varphi(x)$ for any $j \in \mathcal{D}(\varphi)$. Let $y = \gamma_1(x)$, ..., $y = \gamma_p(x)$ the irreducible components of $Fix(\varphi)$. There exist continuous functions

$$b_j : [0, \delta)I^\lambda_\Lambda \to \mathbb{C} \text{ for } 1 \leq j \leq 2\nu(\mathcal{E}_0) \text{ and } c^{\varphi}_{\Lambda,\lambda,k} : [0, \delta)I^\lambda_\Lambda \to \mathbb{C} \text{ for } 1 \leq k \leq p,$$

holomorphic in $(0, \delta)\dot{I}^\lambda_\Lambda$, such that

$$(\ddot{\psi}^{\varphi}_{j,\Lambda,\lambda} + b_j(x) - \dot{\psi}^X_j)(x, \gamma_k(x)) = c^{\varphi}_{\Lambda,\lambda,k}(x) \tag{35}$$

for all $x \in [0, \delta)I^\lambda_\Lambda$, $j \in \mathcal{D}(\varphi)$ and $1 \leq k \leq p$ such that $(x, \gamma_k(x)) \in \overline{H^\lambda_{\Lambda,j}}$ (see [16]). The relevant property is that a function $c^{\varphi}_{\Lambda,\lambda,k}$ does not depend on the choice of $j \in \mathcal{D}(\varphi)$. We define

$$\dot{\psi}^{\varphi}_{j,\Lambda,\lambda}(x, y) = \ddot{\psi}^{\varphi}_{j,\Lambda,\lambda}(x, y) + b_j(x) - c^{\varphi}_{\Lambda,\lambda,1}(x) \ \text{ for } j \in \mathcal{D}(\varphi).$$

We obtain $(\dot{\psi}^{\varphi}_{j,\Lambda,\lambda} - \dot{\psi}^X_j)(x, \gamma_1(x)) = 0$ for all $x \in [0, \delta)I^\lambda_\Lambda$ and $j \in \mathcal{D}(\varphi)$ such that $(x, \gamma_1(x)) \in \overline{H^\lambda_{\Lambda,j}}$.

Definition 6.5. *(section 8.1, prop. 8.1* [16]*) The family* $\{\dot{\psi}^{\varphi}_{j,\Lambda,\lambda}\}_{(j,\Lambda,\lambda) \in \mathcal{D}(\varphi) \times \mathcal{M} \times \mathbb{S}^1}$ *is called a homogeneous privileged (with respect to* $y = \gamma_1(x)$*) system of Fatou coordinates of* φ*. We define the extension of the Ecalle-Voronin invariants*

$$\dot{\xi}^{\varphi}_{j,\Lambda,\lambda}(x, z) = \dot{\psi}^{\varphi}_{j+1,\Lambda,\lambda} \circ (x, \dot{\psi}^{\varphi}_{j,\Lambda,\lambda})^{-1}(x, z)$$

for $(j, \Lambda, \lambda) \in \mathcal{D}(\varphi) \times \mathcal{M} \times \mathbb{S}^1$*. We have*

$$\dot{\xi}^{\varphi}_{j,\Lambda,\lambda}(x, z) = z + \zeta_\varphi(x) + \sum_{l=1}^{\infty} \dot{a}^{\varphi}_{j,\Lambda,\lambda,l}(x) e^{-2\pi i s l z}$$

for $j \in \mathcal{D}_s(\varphi)$ *and* $s \in \{-1,1\}$. *Since the system* $\{\dot{\psi}_j^X\}_{j\in\mathcal{D}(\varphi)}$ *is unique up to a holomorphic additive function then so is* $\{\dot{\psi}_{j,\Lambda,\lambda}^{\varphi}\}_{(j,\Lambda,\lambda)\in\mathcal{D}(\varphi)\times\mathcal{M}\times\mathbb{S}^1}$. *More precisely, any other homogeneous privileged system of Fatou coordinates of* φ *is of the form* $\{\dot{\psi}_{j,\Lambda,\lambda}^{\varphi}+c\}_{(j,\Lambda,\lambda)\in\mathcal{D}(\varphi)\times\mathcal{M}\times\mathbb{S}^1}$ *where* $c(x)$ *is a holomorphic function defined in* $B(0,\delta)$.

Let us remark that given $\Lambda \in \mathcal{M}$ and $\lambda \in \mathbb{S}^1$ we have

$$\sum_{k=1}^{2\nu(\mathcal{E}_0)} \ddot{a}_{k,\Lambda,\lambda,0}^{\varphi} \equiv 2\nu(\mathcal{E}_0)\zeta_\varphi.$$

The analogous result holds true for $\{\dot{\psi}_{j,\Lambda,\lambda}^{\varphi}\}_{j\in\mathcal{D}(\varphi)}$ and the property is preserved if we replace $\breve{\psi}_{j_0,\Lambda,\lambda}^{\varphi}$ with $\breve{\psi}_{j_0,\Lambda,\lambda}^{\varphi} + c_{j_0}(x)$ in a a system of Fatou coordinates $\{\breve{\psi}_{j,\Lambda,\lambda}^{\varphi}\}_{j\in\mathcal{D}(\varphi)}$. Since $\psi_{j,\Lambda,\lambda}^{\varphi} - \dot{\psi}_{j,\Lambda,\lambda}^{\varphi}$ is a function of x for any $j \in \mathcal{D}(\varphi)$ the result follows. This discussion motivates the next choice of normalizing condition for multi-summable Fatou coordinates.

Definition 6.6. *We define* $\psi_{1,\Lambda,\lambda}^{\varphi} \equiv \ddot{\psi}_{1,\Lambda,\lambda}^{\varphi}$. *We define*

$$\psi_{j,\Lambda,\lambda}^{\varphi}(x,y) = \ddot{\psi}_{j,\Lambda,\lambda}^{\varphi}(x,y) - \sum_{k=1}^{j-1} \ddot{a}_{k,\Lambda,\lambda,0}^{\varphi}(x) + (j-1)\zeta_\varphi(x)$$

for $j = 2,\dots,2\nu(\mathcal{E}_0)$. *We define the extension of the Ecalle-Voronin invariants*

$$\xi_{j,\Lambda,\lambda}^{\varphi}(x,z) = \psi_{j+1,\Lambda,\lambda}^{\varphi} \circ (x, \psi_{j,\Lambda,\lambda}^{\varphi})^{-1}(x,z)$$

for $j \in \mathcal{D}(\varphi)$. *The family* $\{\psi_{j,\Lambda,\lambda}^{\varphi}\}_{(j,\Lambda,\lambda)\in\mathcal{D}(\varphi)\times\mathcal{M}\times\mathbb{S}^1}$ *is called a homogeneous* $(\tilde{e}_1,\dots,\tilde{e}_{\tilde{q}})$*-summable system of Fatou coordinates of* φ. *It is defined up to a* $(\tilde{e}_1,\dots,\tilde{e}_{\tilde{q}})$*-summable function, i.e. any other homogeneous* $(\tilde{e}_1,\dots,\tilde{e}_{\tilde{q}})$*-summable system of Fatou coordinates of* φ *is of the form* $\{\psi_{j,\Lambda,\lambda}^{\varphi}+c_{\Lambda,\lambda}\}_{(j,\Lambda,\lambda)\in\mathcal{D}(\varphi)\times\mathcal{M}\times\mathbb{S}^1}$ *where* $c_{\Lambda,\lambda}(x)$ *is a* $(\tilde{e}_1,\dots,\tilde{e}_{\tilde{q}})$*-summable function.*

Remark 6.2. *Different choices of* 2*-convergent normal forms can provide different homogeneous* $(\tilde{e}_1,\dots,\tilde{e}_{\tilde{q}})$*-summable systems of Fatou coordinates of* φ.

Remark 6.3. *The definition implies*

$$\xi_{j,\Lambda,\lambda}^{\varphi} = z + \zeta_\varphi(x) + \sum_{l=1}^{\infty} a_{j,\Lambda,\lambda,l}^{\varphi}(x) e^{-2\pi i s l z}$$

for $j = 1,\dots,2\nu(\mathcal{E}_0)$. *The function* $\psi_{j,\Lambda,\lambda}^{\varphi} - \dot{\psi}_{j,\Lambda,\lambda}^{\varphi}$ *depends only on* x *for any* $j \in \{1,2\}$. *It is continuous in* $[0,\delta)I_\Lambda^\lambda$ *and holomorphic in* $(0,\delta)\dot{I}_\Lambda^\lambda$. *Moreover it does not depend on* $j \in \mathcal{D}(\varphi_1)$ *since both systems of Fatou coordinates are homogeneous.*

Theorem 6.3. *Let* $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ *with* 2*-convergent normal form* $\exp(X)$. *Let* $\Lambda,\Lambda' \in \mathcal{M}$, $\lambda,\lambda' \in \mathbb{S}^1$ *and* $j \in \mathcal{D}(\varphi)$. *Then there exists* $K \in \mathbb{R}^+$ *such that*

$$|\psi_{j,\Lambda,\lambda}^{\varphi} - \psi_{j,\Lambda',\lambda'}^{\varphi}|(x,y) \leq e^{-K/|x|^{\tilde{e}_{d_{\Lambda,\Lambda'}^{\lambda,\lambda'}+1}}}$$

for any $(x,y) \in H_{\Lambda,\Lambda',j}^{\lambda,\lambda'}$.

Theorem 6.4. *Let* $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$ *with* 2*-convergent normal form* $\exp(X)$*. Let* $\Lambda, \Lambda' \in \mathcal{M}$*,* $\lambda, \lambda' \in \mathbb{S}^1$*,* $j \in \mathcal{D}(\varphi)$ *and* $\theta \in (0, \pi/2]$*. Then there exist* $K \in \mathbb{R}^+$ *and* $\rho \geq 2\rho_0$ *such that*

$$|\psi^{\varphi}_{j,\Lambda,\lambda} - \psi^{\varphi}_{j,\Lambda',\lambda'}|(x,y) \leq e^{-K/|x|^{\tilde{e}_{d^{\lambda,\lambda'}_{\Lambda,\Lambda'}+1}}}$$

for any $(x, y) \in H^{\epsilon,\rho,\lambda,\lambda'}_{j,\theta}$*. Moreover* ρ *depends only on* X*,* φ *and* Λ*.*

Remark 6.4. *As in theorem 6.2 the combinatorics of sectors in the* x *variable of the family* $\{\psi^{\varphi}_{j,\Lambda,\lambda}\}_{(\Lambda,\lambda)\in\mathcal{M}\times\mathbb{S}^1}$ *corresponds to a multi-summable function. It is justified to say that* $\{\psi^{\varphi}_{j,\Lambda,\lambda}\}_{(\Lambda,\lambda)\in\mathcal{M}\times\mathbb{S}^1}$ *is* $(\tilde{e}_1, \ldots, \tilde{e}_{\tilde{q}})$*-summable in the* x*-variable. It would be more rigorous to say that the family* $\{\psi^{\varphi}_{j,\Lambda,\lambda} - \psi^{X}_{L_j}\}_{(\Lambda,\lambda)\in\mathcal{M}\times\mathbb{S}^1}$ *is multi-summable since its elements are bounded.*

Remark 6.5. *The previous theorems are deduced from prop. 5.9, 5.11 and the* $(\tilde{e}_1, \ldots, \tilde{e}_{\tilde{q}})$*-summability of the power series* $-\sum_{k=1}^{j-1} \ddot{a}^{\varphi}_{k,0}(x) + (j-1)\zeta_{\varphi}(x)$ *for any* $j \leq 2\nu(\mathcal{E}_0)$*. Thus the lemma 6.3 and the theorems 6.1 and 6.2 hold true for* $\{\xi^{\varphi}_{j,\Lambda,\lambda}\}_{(j,\Lambda,\lambda)\in\mathcal{D}(\varphi)\times\mathcal{M}\times\mathbb{S}^1}$ *and* $\{a^{\varphi}_{j,\Lambda,\lambda,k}\}_{(j,\Lambda,\lambda,k)\in\mathcal{D}(\varphi)\times\mathcal{M}\times\mathbb{S}^1\times\mathbb{N}}$*. Then Ecalle-Voronin invariants are* $(\tilde{e}_1, \ldots, \tilde{e}_{\tilde{q}})$*-summable in the* x*-variable.*

Remark 6.6. *Let* $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$ *with convergent normal form* $\exp(X)$*. Suppose that* $Fix(\varphi)$ *is a curve* $y = \gamma(x)$*. The sets of unstable directions* $\mathcal{U}^j_X$ *(see subsection 4.1) are empty for any* $1 \leq j \leq q$*. Therefore the Fatou coordinates* $\{\psi^{\varphi}_j\}_{j\in\mathcal{D}(\varphi)}$ *and Ecalle-Voronin invariants* $\{\xi^{\varphi}_j\}_{j\in\mathcal{D}(\varphi)}$ *are analytic. Indeed we have*

$$\xi^{\varphi}_j = z + \zeta_{\varphi}(x) + \sum_{l=1}^{\infty} a^{\varphi}_{j,l}(x) e^{-2\pi i s l z}$$

where $a^{\varphi}_{j,l} \in \mathcal{O}(B(0,\delta))$ *for all* $j \in \mathcal{D}_s(\varphi)$*,* $s \in \{-1, 1\}$ *and* $l \in \mathbb{N}$*.*

6.3. **Multi-summability of Lavaurs vector fields.** Let $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$ with 2-convergent normal form $\Upsilon = \exp(X)$. Consider $\Lambda \in \mathcal{M}$, $\lambda \in \mathbb{S}^1$, $j \in \mathcal{D}(\varphi)$.

Definition 6.7. *We define the Lavaurs vector field (see cor. 5.1)*

$$X^{\varphi}_{j,\Lambda,\lambda} = \frac{1}{\partial\psi^{\varphi}_{j,\Lambda,\lambda}/\partial y} \frac{\partial}{\partial y}.$$

It is defined in $H^{\lambda}_{\Lambda,j}$ *and in* $H^{\epsilon,\rho,\lambda}_{j,\theta}$ *(see Step 1 of subsection 5.6) for any* $\theta \in (0, \pi/2]$*. We denote* $g^{\varphi}_{j,\Lambda,\lambda} = 1/(\partial\psi^{\varphi}_{j,\Lambda,\lambda}/\partial y)$*.*

Theorem 6.5. *Let* $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$ *with* 2*-convergent normal form* $\Upsilon = \exp(X)$*. Consider* $j \in \mathcal{D}_s(\varphi)$ *and* $s \in \{-1, 1\}$*. There exists a development*

$$\hat{X}^{\varphi}_j = \left(\sum_{k=0}^{\infty} g^{\varphi}_{j,k}(y) x^k\right) \frac{\partial}{\partial y}$$

such that

- $g^{\varphi}_{j,k}$ *is holomorphic in* $\cup_{\theta\in(0,\pi/2]} H^{\epsilon,\rho,\lambda}_{j,\theta}(0)$ *for any* $k \in \mathbb{N} \cup \{0\}$*.*
- *The series* $\sum_{k=0}^{\infty} g^{\varphi}_{j,k}(y_0) x^k$ *is the asymptotic development of the* $(\tilde{e}_1, \ldots, \tilde{e}_{\tilde{q}})$*-summable function* $g^{\varphi}_{j,\Lambda,\lambda}(x, y_0)$ *for all* $(0, y_0) \in \cup_{\theta\in(0,\pi/2]} H^{\epsilon,\rho,\lambda}_{j,\theta}(0)$ *and* $\Lambda \in \mathcal{M}$*.*

Let us remark that the set $\cup_{\theta \in (0,\pi/2]} H_{j,\theta}^{\epsilon,\rho,\lambda}(0)$ does not depend on ρ or λ. It only depends on j and ϵ.

Proof. Let us prove that $g_{j,\Lambda,\lambda}^{\varphi}$ is $(\tilde{e}_1, \dots, \tilde{e}_{\tilde{q}})$-summable in the x variable.

Fix $\theta \in (0, \pi/2]$. Denote $X = g(x,y)\partial/\partial y$. Let $\Lambda, \Lambda' \in \mathcal{M}$ and $\lambda, \lambda' \in \mathbb{S}^1$. We have

$$|\psi_{j,\Lambda,\lambda}^{\varphi} - \psi_{j,\Lambda',\lambda'}^{\varphi}|(x,y) \leq e^{-K/|x|^{\tilde{e}_{d_{\Lambda,\Lambda'}^{\lambda,\lambda'}+1}}}$$

for any $(x,y) \in H_{j,\theta}^{\epsilon,\rho,\lambda,\lambda'}$ by theorem 6.4. The definition of $X_{j,\Lambda,\lambda}^{\varphi}$ implies

$$g_{j,\Lambda,\lambda}^{\varphi} - g_{j,\Lambda',\lambda'}^{\varphi} = \frac{1}{\partial \psi_{j,\Lambda,\lambda}^{\varphi}/\partial y} - \frac{1}{\partial \psi_{j,\Lambda',\lambda'}^{\varphi}/\partial y} = \frac{\partial \psi_{j,\Lambda',\lambda'}^{\varphi}/\partial y - \partial \psi_{j,\Lambda,\lambda}^{\varphi}/\partial y}{(\partial \psi_{j,\Lambda,\lambda}^{\varphi}/\partial y)(\partial \psi_{j,\Lambda',\lambda'}^{\varphi}/\partial y)}.$$

Let us consider the function $h_{j,\Lambda,\lambda} = \partial(\psi_{j,\Lambda,\lambda}^{\varphi} - \psi_{L_j}^{X})/\partial y$. It satisfies

$$h_{j,\Lambda,\lambda} = \frac{\partial(\psi_{j,\Lambda,\lambda}^{\varphi} - \psi_{L_j}^{X})}{\partial \psi_{L_j}^{X}} \frac{\partial \psi_{L_j}^{X}}{\partial y} = \frac{\partial(\psi_{j,\Lambda,\lambda}^{\varphi} - \psi_{L_j}^{X})}{\partial \psi_{L_j}^{X}} \frac{1}{g}.$$

By using proposition 5.10 and Cauchy's integral formula we obtain that $gh_{j,\Lambda,\lambda}$ is a continuous function defined in in $\overline{H_{j,\theta}^{\epsilon,\rho,\lambda}}$ whose value at $(0,0)$ is equal to 0. Moreover the restriction $(gh_{j,\Lambda,\lambda})(0,y)$ to $x=0$ does not depend on Λ or λ. Thus there exists $K_1' \geq 1$ such that

$$\frac{1}{K_1'} \leq \frac{g_{j,\Lambda,\lambda}^{\varphi}}{g} \leq K_1' \text{ and } \frac{1}{K_1'} \leq \frac{g_{j,\Lambda',\lambda'}^{\varphi}}{g} \leq K_1'$$

in $H_{j,\theta}^{\epsilon,\rho,\lambda}$ and $H_{j,\theta}^{\epsilon,\rho,\lambda'}$ respectively. Since

$$\frac{\partial \psi_{j,\Lambda',\lambda'}^{\varphi}}{\partial y} - \frac{\partial \psi_{j,\Lambda,\lambda}^{\varphi}}{\partial y} = \frac{\partial(\psi_{j,\Lambda',\lambda'}^{\varphi} - \psi_{j,\Lambda,\lambda}^{\varphi})}{\partial \psi_{L_j}^{X}} \frac{\partial \psi_{L_j}^{X}}{\partial y} = \frac{\partial(\psi_{j,\Lambda',\lambda'}^{\varphi} - \psi_{j,\Lambda,\lambda}^{\varphi})}{\partial \psi_{L_j}^{X}} \frac{1}{g}$$

we deduce analogously that

$$\left| \frac{\partial \psi_{j,\Lambda',\lambda'}^{\varphi}}{\partial y} - \frac{\partial \psi_{j,\Lambda,\lambda}^{\varphi}}{\partial y} \right| \leq K_2' e^{-K/|x|^{\tilde{e}_{d_{\Lambda,\Lambda'}^{\lambda,\lambda'}+1}}} \frac{1}{|g|}$$

in $H_{j,\theta}^{\epsilon,\rho,\lambda,\lambda'}$ for some $K_2' \in \mathbb{R}^+$. Therefore we obtain

$$|g_{j,\Lambda,\lambda}^{\varphi} - g_{j,\Lambda',\lambda'}^{\varphi}| \leq |g| K_2' (K_1')^2 e^{-K/|x|^{\tilde{e}_{d_{\Lambda,\Lambda'}^{\lambda,\lambda'}+1}}}$$

in $H_{j,\theta}^{\epsilon,\rho,\lambda,\lambda'}$. Notice that $H_{j,\theta}^{\epsilon,\rho,\lambda}(0) = H_{j,\theta}^{\epsilon,\rho,\lambda,\lambda'}(0)$. The result is obtained by proceeding as in the proof of theorem 6.2. □

6.4. **Analyzing the infinitesimal generator.** Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2, 0)$ with 2-convergent normal form $\Upsilon = \exp(X)$.

Definition 6.8. *The infinitesimal generator* $\log \varphi$ *of* φ *is of the form*

$$\log \varphi = \left(\sum_{k=0}^{\infty} \hat{g}_k^{\varphi}(y) x^k \right) \frac{\partial}{\partial y}$$

where $\hat{g}_k^{\varphi} \in \mathbb{C}[[x]]$ *for any* $k \in \mathbb{N} \cup \{0\}$.

Analogously as in th. 6.5 we prove that the family $\{g_{j,k}^{\varphi}\}_{j\in\mathcal{D}(\varphi)}$ represents a $\nu(\mathcal{E}_0)$-summable function for any $k\in\mathbb{N}\cup\{0\}$ (th. 6.6). We claim that this function coincides with $\hat{g}_k^{\varphi}$. Roughly speaking the Lavaurs vector fields $(1/\partial\psi_{j,\lambda,k}^{\varphi})\partial/\partial y$ are very tangent to $\hat{u}f\partial/\partial y$ by prop. 5.13. The proof is completed by using that the functions of the form $\psi_{j,\Lambda,\lambda}^{\varphi}(x,y)-\psi_{j,\lambda,k}^{\varphi}(x,y)$ are (up to an additive function of x) exponentially small in the x variable (prop. 6.2).

Denote $f=y\circ\varphi-y$, $\log\varphi=\hat{u}f\partial/\partial y$ and $X=uf\partial/\partial y$ for some units $\hat{u}\in\mathbb{C}[[x,y]]$, $u\in\mathbb{C}\{x,y\}$. We obtain $\hat{u}-u\in(f^2)$. Fix $k\geq\max(5,4\nu(\mathcal{E}_0))$. Let $\exp(Y_k)$ be a k-convergent normal form and $B(0,\delta)\times B(0,\epsilon_k)$ as defined in subsection 5.7. The vector field Y_k is of the form $u_kf\partial/\partial y$ where $u_k\in\mathbb{C}\{x,y\}$ is a unit such that $\hat{u}-u\in(f^k)$.

Proposition 6.2. *Let $\varphi\in\mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ with 2-convergent normal form $\Upsilon=\exp(X)$. Then $\hat{g}_b^{\varphi}$ is an asymptotic development in $H_{j,\theta}^{\epsilon,\rho}(0)$ at $x=0$ of $g_{j,b}^{\varphi}$ for all $j\in\mathcal{D}(\varphi)$, $\theta\in(0,\pi/2]$ and $b\in\mathbb{N}\cup\{0\}$.*

Proof. Denote $\nu=\nu(\mathcal{E}_0)$. Fix $k\geq\max(5,4\nu)$. Consider $\lambda\in\mathbb{S}^1$. We have

$$\frac{1}{\partial\psi_{j,\lambda,k}^{\varphi}/\partial y}-\frac{1}{\partial\psi_{L_j,k}/\partial y}=\frac{(\partial(\psi_{L_j,k}-\psi_{j,\lambda,k}^{\varphi})/\partial\psi_{L_j,k})(\partial\psi_{L_j,k}/\partial y)}{(\partial\psi_{L_j,k}/\partial y)(\partial\psi_{j,\lambda,k}^{\varphi}/\partial y)},$$

see subsection 5.7 for the definition of $\psi_{L_j,k}$. We denote $g_{j,\lambda,k}^{\varphi}=1/(\partial\psi_{j,\lambda,k}^{\varphi}/\partial y)$. We obtain

$$\left|g_{j,\lambda,k}^{\varphi}-u_kf\right|=O\left(\frac{f}{(1+|\psi_{L_j}^X(x,y)|)^{k-1}}\right)=O(f^{\frac{\nu k+1}{\nu+1}})$$

in $\cup_{x\in I_\lambda}(H_{j,\theta}^{\epsilon,\rho}(x)\cap H_{j,\theta/2}^{\epsilon_k,\rho}(x))$ by proposition 5.13. Consider $\lambda'\in\mathbb{S}^1$ such that $I_\lambda\cap I_{\lambda'}\neq\emptyset$. The function $\psi_{j,\lambda,k}^{\varphi}-\psi_{j,\lambda',k}^{\varphi}$ is defined in $\cup_{x\in I_{\lambda'}\cap I_\lambda}B_{\varphi,j}^1(x)$ (see Step 3 of subsection 5.6). We argue as in the proof of prop. 5.9 to obtain $K_1'\in\mathbb{R}^+$ such that

$$|\psi_{j,\lambda,k}^{\varphi}-\psi_{j,\lambda',k}^{\varphi}|\leq e^{-K_1'/|x|^{\nu}}$$

in $\cup_{x\in I_{\lambda'}\cap I_\lambda}(H_{j,\theta}^{\epsilon,\rho}(x)\cap H_{j,\theta/2}^{\epsilon_k,\rho}(x))$. This implies

$$g_{j,\lambda,k}^{\varphi}-g_{j,\lambda',k}^{\varphi}=O\left(f\frac{\partial(\psi_{j,\lambda',k}^{\varphi}-\psi_{j,\lambda,k}^{\varphi})}{\partial\psi_{L_j}^X}\right)=O\left(e^{-K_1'/|x|^{\nu}}f\right)\tag{36}$$

in $\cup_{x\in I_{\lambda'}\cap I_\lambda}(H_{j,\theta}^{\epsilon,\rho}(x)\cap H_{j,\theta/2}^{\epsilon_k,\rho}(x))$.

We have $H_{j,\theta}^{\epsilon,\rho}\cap H_{j,\theta/2}^{\epsilon_k,\rho}\subset B(0,\delta)\times e^{i[-\zeta,\zeta]}(0,\epsilon)$ for some $\zeta\in\mathbb{R}^+$ (prop. 4.1). Lemma 4.3 and remark 4.5 imply $|\psi_{L_j}^X|\leq C_1'/|y|^{\nu}$ in $\mathcal{E}_0\cap(B(0,\delta)\times e^{i[-\zeta,\zeta]}(0,\epsilon))$ for some $C_1'\in\mathbb{R}^+$. Cauchy's integral formula provides

$$\left|\frac{\partial\psi_{L_j}^X}{\partial x}\right|(x_0,y)=\frac{1}{2\pi}\left|\int_{x\in\partial B(x_0,|y|/(2\rho_0))}\frac{\psi_{L_j}^X(x,y)}{(x-x_0)^2}dx\right|\leq\frac{C_1'2\rho_0}{|y|^{\nu+1}}$$

for any $(x_0,y)\in\tilde{\mathcal{E}}_0\cap(B(0,\delta)\times e^{i[-\zeta,\zeta]}(0,\epsilon))$ (see section 3). Consider points $(0,y)\in H_{j,\theta}^{\epsilon,\rho}\cap H_{j,\theta/2}^{\epsilon_k,\rho}$ and $x\in\overline{B}(0,|y|^{\nu+2})$. We have

$$|\psi_{L_j}^X(x,y)-\psi_{L_j}^X(0,y)|\leq C_1'2\rho_0\frac{|x|}{|y|^{\nu+1}}\leq C_1'2\rho_0|y|.$$

Therefore (x,y) is in the neighborhood of $H_{j,\theta}^{\epsilon,\rho} \cap H_{j,\theta/2}^{\epsilon_k,\rho}$ if y is in a neighborhood of 0. As a consequence the property (36) holds true for points $(x,y) \in T_0$ such that $(0,y)$ is in a neighborhood of $(0,0)$ in $H_{j,\theta}^{\epsilon,\rho} \cap H_{j,\theta/2}^{\epsilon_k,\rho}$ and $x \in \overline{B}(0,|y|^{\nu+2})$. The function f is a $O(y^{\nu+1}) + O(x)$ and then a $O(y^{\nu+1})$ in $\{(x,y) \in \mathbb{C}^2 : |x| \leq |y|^{\nu+2}\}$.

Fix $(0,y_0) \in H_{j,\theta}^{\epsilon,\rho} \cap H_{j,\theta/2}^{\epsilon_k,\rho}$ and the ball $B(0,|y_0|^{\nu+2})$. The equation (36) implies that the family of functions $\{g_{j,\lambda,k}^{\varphi}(x,y_0)\}_{\lambda \in \mathbb{S}^1}$ has a common $1/\nu$ Gevrey asymptotic development $\sum_{b=0}^{\infty} g_{j,b,k}^{\varphi}(y_0)x^b$. We apply lemma 6.2 to the functions $(g_{j,\lambda,k}^{\varphi} - u_k f)(x,y_0)$ to obtain that $g_{j,b,k}^{\varphi}(y_0)$ satisfies

$$\left| g_{j,b,k}^{\varphi}(y_0) - \frac{1}{b!}\frac{\partial^b (u_k f)}{\partial x^b}(0,y_0) \right| = O\left(\frac{(y_0^{\nu+1})^{\frac{\nu k+1}{\nu+1}}}{y_0^{b(\nu+2)}} \right) + e^{\frac{-K_1'}{2|y_0|^{\nu(\nu+2)}}} = O\left(y_0^{\nu k+1-b(\nu+2)} \right).$$

Consider $\Lambda \in \mathcal{M}$, $\lambda \in \mathbb{S}^1$. Then $\psi_{j,\Lambda,\lambda}^{\varphi} - \psi_{j,\lambda,k}^{\varphi}$ is defined in $\cup_{x \in I_\lambda \cap (0,\delta) I_\Lambda^\lambda} B_{\varphi,j}^1(x)$. The functions $\psi_{j,\lambda,k}^{\varphi} - \psi_{L_j}^X$ and $\psi_{j,\Lambda,\lambda}^{\varphi} - \psi_{L_j}^X$ are bounded in $\cup_{x \in I_\lambda \cap (0,\delta) I_\Lambda^\lambda} B_{\varphi,j}^1(x)$ by lemma 5.13 and proposition 5.6 respectively. Therefore $\psi_{j,\Lambda,\lambda}^{\varphi} - \psi_{j,\lambda,k}^{\varphi}$ is bounded. By using $(Im\, \psi_{L_j}^X)(B_{\varphi,j}^1(x)) = [-a_1/|x|^\nu, a_1/|x|^\nu]$ and proceeding as in prop. 5.9 we obtain that there exist $K_2' \in \mathbb{R}^+$ and a holomorphic function $a_0(x) \in \mathcal{O}(I_\lambda \cap (0,\delta) I_\Lambda)$ such that

$$|\psi_{j,\Lambda,\lambda}^{\varphi}(x,y) - \psi_{j,\lambda,k}^{\varphi}(x,y) - a_0(x)| \leq e^{-K_2'/|x|^{\nu(\mathcal{E}_0)}} \text{ in } \cup_{x \in I_\lambda \cap (0,\delta) I_\Lambda^\lambda} B_{\varphi,j}^1(x).$$

The estimate holds true in $\cup_{x \in I_{\lambda'} \cap (0,\delta) I_\Lambda^\lambda} (H_{j,\theta}^{\epsilon,\rho}(x) \cap H_{j,\theta/2}^{\epsilon_k,\rho}(x))$ too since the orbits by φ of points in this set intersect $\cup_{x \in I_\lambda \cap (0,\delta) I_\Lambda^\lambda} B_{\varphi,j}^1(x)$ (lemma 5.9) and both sides are invariant by φ. We obtain

$$g_{j,\lambda,k}^{\varphi} - g_{j,\Lambda,\lambda}^{\varphi} = O\left(e^{-K_2'/|x|^{\nu(\mathcal{E}_0)}} f \right)$$

in $\cup_{x \in I_{\lambda'} \cap (0,\delta) I_\Lambda^\lambda} (H_{j,\theta}^{\epsilon,\rho}(x) \cap H_{j,\theta/2}^{\epsilon_k,\rho}(x))$. This implies $g_{j,b}^{\varphi} = g_{j,b,k}^{\varphi}$ in $(H_{j,\theta}^{\epsilon,\rho} \cap H_{j,\theta/2}^{\epsilon_k,\rho})(0)$. Since $(\sum_{k=0}^{\infty} \hat{g}_k^{\varphi}(y)x^k) - u_k f \in (f^{k+1})$ we deduce that $\hat{g}_b^{\varphi}$ is an asymptotic development of $g_{j,b}^{\varphi}$ in $(H_{j,\theta}^{\epsilon,\rho} \cap H_{j,\theta/2}^{\epsilon_k,\rho})(0)$ and then in $H_{j,\theta}^{\epsilon,\rho}(0)$ since $H_{j,\theta}^{\epsilon,\rho}$ and $H_{j,\theta}^{\epsilon,\rho} \cap H_{j,\theta/2}^{\epsilon_k,\rho}$ coincide in a neighborhood of $(0,0)$. □

Theorem 6.6. *Let $\varphi \in \text{Diff}_{tp1}(\mathbb{C}^2,0)$ with 2-convergent normal form $\Upsilon = \exp(X)$. Then $\hat{g}_b^{\varphi}$ is a $\nu(\mathcal{E}_0)$-summable function whose sum in $\cup_{\theta \in (0,\pi/2]} H_{j,\theta}^{\epsilon,\rho}(0)$ is equal to $g_{j,b}^{\varphi}$ for all $j \in \mathcal{D}(\varphi)$ and $b \in \mathbb{N} \cup \{0\}$.*

Proof. Denote $\nu = \nu(\mathcal{E}_0)$. Fix $\theta \in (0,\pi/2]$. Fix $\Lambda \in \mathcal{M}$. Consider $\lambda \in \mathbb{S}^1$. The function $(g_{j+1,\Lambda,\lambda}^{\varphi} - g_{j,\Lambda,\lambda}^{\varphi})(x,y_0)$ represents a $(\tilde{e}_1,\ldots,\tilde{e}_{\tilde{q}})$-summable function for any $(0,y_0) \in H_{j,\theta}^{\epsilon,\rho}(0) \cap H_{j+1,\theta}^{\epsilon,\rho}(0)$ by th. 6.5. In fact we have

$$(g_{j+1,\Lambda,\lambda}^{\varphi} - g_{j,\Lambda,\lambda}^{\varphi})(x,y) - (g_{j+1,\Lambda',\lambda'}^{\varphi} - g_{j,\Lambda',\lambda'}^{\varphi})(x,y) = O(f e^{-K/|x|^{\tilde{e}_{d_{\Lambda,\Lambda'}^{\lambda,\lambda'}+1}}})$$

in $H_{j,\theta}^{\epsilon,\rho,\lambda,\lambda'} \cap H_{j+1,\theta}^{\epsilon,\rho,\lambda,\lambda'}$ for all $\Lambda' \in \mathcal{M}$ and $\lambda,\lambda' \in \mathbb{S}^1$ (see proof of theorem 6.5).

We have

$$\xi_{j,\Lambda,\lambda}^{\varphi} - z = \zeta_\varphi(x) + \sum_{l=1}^{\infty} a_{j,\Lambda,\lambda,l}^{\varphi}(x) e^{-2\pi i s l z}$$

where $\sum_{l=1}^{\infty} a_{j,\Lambda,\lambda,l}^{\varphi}(x)w^l$ is a bounded function in $\{(x,w) \in (0,\delta)I_{\Lambda}^{\lambda} \times B(0,e^{-2\pi M})\}$. We deduce that $\xi_{j,\Lambda,\lambda}^{\varphi} - z - \zeta_{\varphi}(x)$ is a $O(e^{-2\pi i s z})$. We obtain

$$\psi_{j+1,\Lambda,\lambda}^{\varphi} - \psi_{j,\Lambda,\lambda}^{\varphi} - \zeta_{\varphi}(x) = O(e^{-2\pi i s \psi_{j,\Lambda,\lambda}^{\varphi}}) = O(e^{-2\pi |Im(\psi_{L_j}^X)|})$$

in $H_{j,\theta}^{\epsilon,\rho} \cap H_{j+1,\theta}^{\epsilon,\rho} \cap \{sIm(\psi_{L_j}^X) < -M\}$. Since $\theta > 0$ we get that there exists $c_\theta \in \mathbb{R}^+$ such that $|Im(\psi_{L_j}^X)| \geq c_\theta |\psi_{L_j}^X|$ in $H_{j,\theta}^{\epsilon,\rho} \cap H_{j+1,\theta}^{\epsilon,\rho} \cap \{sIm(\psi_{L_j}^X) < -M\}$ and then $O(e^{-2\pi |Im(\psi_{L_j}^X)|}) = O(e^{-2\pi c_\theta |\psi_{L_j}^X|})$. We obtain

$$g_{j+1,\Lambda,\lambda}^{\varphi} - g_{j,\Lambda,\lambda}^{\varphi} = \frac{(\partial(\psi_{j,\Lambda,\lambda}^{\varphi} - \psi_{j+1,\Lambda,\lambda}^{\varphi})/\partial \psi_{L_j}^X)(\partial \psi_{L_j}^X/\partial y)}{(\partial \psi_{j,\Lambda,\lambda}^{\varphi}/\partial y)(\partial \psi_{j,\Lambda,\lambda}^{\varphi}/\partial y)} = O(fe^{-2\pi c_\theta |\psi_{L_j}^X|})$$

in $H_{j,\theta}^{\epsilon,\rho} \cap H_{j+1,\theta}^{\epsilon,\rho} \cap \{sIm(\psi_{L_j}^X) < -M\}$. Moreover $\psi_{L_j}^X$ satisfies $|\psi_{L_j}^X| \geq C_6^{-1}/|y|^{\nu}$ in $H_{j,\theta}^{\epsilon,\rho}$ by equation (20). The previous discussion implies

$$g_{j+1,\Lambda,\lambda}^{\varphi} - g_{j,\Lambda,\lambda}^{\varphi} = O\left(fe^{-\frac{2\pi c_\theta}{C_6}\frac{1}{|y|^{\nu}}}\right)$$

in $H_{j,\theta}^{\epsilon,\rho} \cap H_{j+1,\theta}^{\epsilon,\rho} \cap \{sIm(\psi_{L_j}^X) < -M\}$.

We proceed as in proposition 6.2 (using lemma 6.2) to obtain

$$(g_{j+1,b}^{\varphi} - g_{j,b}^{\varphi})(y) = O\left(\frac{fe^{-\frac{2\pi c_\theta}{C_6}\frac{1}{|y|^{\nu}}}}{y^{b(\nu+2)}}\right) + O\left(e^{-K/(2|y|^{\nu(\nu+2)})}\right) = O\left(e^{-\frac{\pi c_\theta}{C_6}\frac{1}{|y|^{\nu}}}\right) \tag{37}$$

in $H_{j,\theta}^{\epsilon,\rho}(0) \cap H_{j+1,\theta}^{\epsilon,\rho}(0) \cap \{sIm(\psi_{L_j}^X) < -M\}$.

The curve $\{y \in H_{j,\theta}^{\epsilon,\rho}(0) : \psi_{L_j}^X(0,y) \in \mathbb{R}\}$ adheres a direction $\lambda_j \in \mathbb{S}^1$ at $y = 0$. We have $\lambda_{k'} = \lambda_k e^{\pi i \nu^{-1}(k'-k)}$ for all $k,k' \in \mathbb{Z}/(2\nu\mathbb{Z})$. Since $\psi_{L_j}^X \sim 1/y^{\nu}$ in $H_{j,\theta}^{\epsilon,\rho}(0)$ we deduce that $H_{j,\theta}^{\epsilon,\rho}(0)$ contains a sector $\{y \in (0,\epsilon_\theta)\lambda_j e^{i[-\frac{\pi-2\theta}{\nu},\frac{\pi-2\theta}{\nu}]}\}$. Indeed it contains sectors of the form $(0,\epsilon')\lambda_j e^{i[-(\pi-\theta)\nu^{-1}+\eta,(\pi-\theta)\nu^{-1}-\eta]}$ for any $\eta \in \mathbb{R}^+$. Analogously $H_{j,\theta}^{\epsilon,\rho}(0) \cap H_{j+1,\theta}^{\epsilon,\rho}(0) \cap \{sIm(\psi_{L_j}^X) < -M\}$ contains a sector $\{y \in (0,\epsilon_\theta')\lambda_j e^{\frac{\pi i}{2\nu}} e^{i[-\frac{\pi-3\theta}{2\nu},\frac{\pi-3\theta}{2\nu}]}\}$ in which property (37) holds true. The function $g_{j,b}$ is bounded at the origin in $H_{j,\theta}^{\epsilon,\rho}(0)$ for $j \in \mathbb{Z}/(2\nu\mathbb{Z})$ since it has an asymptotic development (prop. 6.2). Therefore prop. 6.1 implies that there exists a unique ν-summable function $\hat{\phi}_b$ such that its sum in $H_{j,\theta}^{\epsilon,\rho}(0)$ is $g_{j,b}$ for all $j \in \mathbb{Z}/(2\nu\mathbb{Z})$ and $\theta \in (0,\pi/2]$. Moreover $\{\lambda_1 e^{\frac{\pi i}{\nu}}, \ldots, \lambda_{2\nu} e^{\frac{\pi i}{\nu}}\}$ is the set of singular directions of $\hat{\phi}_b$. Since asymptotic developments are unique we obtain $\hat{g}_b^{\varphi} = \hat{\phi}_b$. In particular $\hat{g}_b^{\varphi}$ is a ν-summable function for any $b \in \mathbb{N} \cup \{0\}$. □

Remark 6.7. *Let $\varphi \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ and $\Upsilon = \exp(g\partial/\partial y)$ a 2-convergent normal form of φ. We can describe the asymptotic behavior of the Lavaurs vector fields in the neighborhood of $Fix(\varphi) \cup \{x=0\}$.*

- *The vector field $X_{j,\Lambda,\lambda}^{\varphi} = g_{j,\Lambda,\lambda}^{\varphi}\partial/\partial y$ coincides with $\log\varphi$ until the first non-zero term in the neighborhood of $Fix(\varphi)$. More precisely, $g_{j,\Lambda,\lambda}^{\varphi}/g - 1$ is a continuous function in $H_{j,\theta}^{\epsilon,\rho,\lambda}$ vanishing at $\overline{H_{j,\theta}^{\epsilon,\rho,\lambda}} \cap Fix(\varphi)$. This result is corollary 7.3 in* [16] *whose proof is based in prop. 7.3* [16]. *Since prop. 5.10 is the analogue of prop. 7.3* [16] *for multi-transversal flows the same result holds true.*
- *Fixed $j \in \mathcal{D}(\varphi)$ the family $\{X_{j,\Lambda,\lambda}^{\varphi}\}_{(\Lambda,\lambda)\in\mathcal{M}\times\mathbb{S}^1}$ is multi-summable in the variable x.*

7. Applications

In this section we adapt the analytic conjugation theorem in [16] to the multisummable setting. We also express the analytic class of $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ as a function of the analytic classes of the one dimensional germs in the family $\{\varphi_{|x=x_n}\}_{n\in\mathbb{N}}$ where x_n is a sequence of elements of $B(0,\delta) \setminus \{0\}$ such that $\lim_{n\to\infty} x_n = 0$.

7.1. Actions of conjugations on Fatou coordinates.

Given $\varphi, \eta \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$, we are interested on studying whether the analytic conjugacy of $\varphi_{|x=x_n}$ and $\eta_{|x=x_n}$ by an analytic mapping κ_n defined in an open set in $\{x_n\} \times \mathbb{C}$ for a sequence $x_n \to 0$ implies that φ and η are analytically conjugated. No continuous dependence with respect to the parameter x of the mapping $(x_n, y) \mapsto \kappa_n(y)$ is required. In this subsection we prove that even if the mappings κ_n are unrelated some of their properties behave uniformly.

Definition 7.1. *(see [16]) We say that η is a s-mapping if η is a biholomorphism from $B(0, s)$ onto $\eta(B(0, s))$. If $\eta(B(0, s))$ is contained in $B(0, S)$ then we say that η is a (s,S)-mapping.*

Next we adapt the results in section 10.1 of [16] to the context of slow decaying functions (see def. 4.36).

Definition 7.2. *Let $X = x^a g(x, y)\partial/\partial y \in \mathcal{X}(\mathbb{C}^2, 0)$ with $g \in \mathbb{C}\{x, y\}$ and $g(0, y) \not\equiv 0$. The radical ideal $\sqrt{(g)}$ is generated by an element $h \in \mathbb{C}\{x, y\}$. We define $N(X)$ as the order of $h(0, y)$ at $y = 0$. In fact $N(X)$ is the cardinal of the set $SingX \cap \{x = x_0\}$ for $x_0 \neq 0$.*

Definition 7.3. *Let $X \in \mathcal{X}(\mathbb{C}^2, 0)$. We denote $\kappa_{|(SingX)(x_0)} \cong Id$ if $N(X) > 1$ and $\kappa_{|(SingX)(x_0)} \equiv Id$. Suppose $N(X) = 1$. The set $SingX$ has a unique component $y = \gamma(x)$ different than $x = 0$. We denote $\kappa_{|(SingX)(x_0)} \cong Id$ if $\kappa(\gamma(x_0)) = \gamma(x_0)$ and $\kappa'(\gamma(x_0)) = 1$.*

The previous definition implies that $\kappa - y$ vanishes in $(SingX)(x_0)$ and the sum of the vanishing multiplicities is greater or equal than 2.

Next we see that we can control the image of s-mappings.

Lemma 7.1. *Let $X \in \mathcal{X}(\mathbb{C}^2, 0)$ with $N(X) \geq 1$. Let $s : B(0, \delta) \setminus \{0\} \to \mathbb{R}^+$ be a slow decaying function and $\tau \in (0, 1/4]$. There exists a neighborhood V of 0 in the x-line such that for any $x_0 \in V \setminus \{0\}$ each $s(x_0)$-mapping satisfying $\kappa_{|(SingX)(x_0)} \cong Id$ is a $(s(x_0)\tau, 2s(x_0)\tau)$-mapping.*

Proof. Let $\gamma_1(x_0)$ and $\gamma_2(x_0)$ be two different points of $(SingX)(x_0)$ if $N(X) > 1$. Otherwise we consider $\gamma_1(x_0) = \gamma_2(x_0) \in (SingX)(x_0)$. We define

$$\kappa_1(y) = \frac{\kappa((s(x_0) - |\gamma_1(x_0)|)y + \gamma_1(x_0)) - \gamma_1(x_0)}{(s(x_0) - |\gamma_1(x_0)|)(\partial\kappa/\partial y)(\gamma_1(x_0))}.$$

Then κ_1 is a Schlicht function. Denote $\upsilon(x_0) = (\gamma_2(x_0)-\gamma_1(x_0))/(s(x_0)-|\gamma_1(x_0)|)$. We have $\kappa_1(\upsilon(x_0)) = \upsilon(x_0)/(\partial\kappa/\partial y)(\gamma_1(x_0))$. Koebe's distortion theorem (see [3], page 65) implies $|(\partial\kappa/\partial y)(\gamma_1(x_0))| \leq (1 + |\upsilon(x_0)|)^2$. We have

$$\sup_{y\in B(0,s(x_0)\tau)} |\kappa(y)| \leq (s(x_0) - |\gamma_1(x_0)|)(1 + |\upsilon(x_0)|)^2 \sup_{y\in B(0,A(x_0,\tau))} |\kappa_1(y)| + |\gamma_1(x_0)|$$

where $A(x_0,\tau) = (s(x_0)\tau + |\gamma_1(x_0)|)/(s(x_0) - |\gamma_1(x_0)|)$. By Koebe's distortion theorem we obtain $\sup_{y\in B(0,A(x_0,\tau))} |\kappa_1(y)| \leq A(x_0,\tau)/(1-A(x_0,\tau))^2$ and

$$\sup_{y\in B(0,s(x_0)\tau)} |\kappa(y)| \leq (s(x_0)-|\gamma_1(x_0)|)(1+|\upsilon(x_0)|)^2 \frac{A(x_0,\tau)}{(1-A(x_0,\tau))^2} + |\gamma_1(x_0)|.$$

Since

$$\lim_{x_0\to 0} A(x_0,\tau) = \lim_{x_0\to 0} \tau \frac{1+\frac{|\gamma_1(x_0)|}{s(x_0)\tau}}{1-\frac{|\gamma_1(x_0)|}{s(x_0)}} = \tau.$$

we get $\sup_{y\in B(0,s(x_0)\tau)} |\kappa(y)| \leq 2s(x_0)\tau$ for any x_0 in a neighborhood of 0. □

Lemma 7.2. *Let $\varphi_1,\varphi_2 \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ with convergent normal forms $\exp(X_1)$ and $\exp(X_2)$ respectively. Suppose $Fix(\varphi_1) = Fix(\varphi_2)$ is a curve $y=\gamma(x)$. Consider an analytic diffeomorphism $\sigma(x,y) = (x,\sigma_2(x,y)) \in \mathrm{Diff}_p(\mathbb{C}^2,0)$ conjugating X_1 and X_2. Let $s: B(0,\delta)\setminus\{0\} \to \mathbb{R}^+$ be a slow decaying function and $\tau \in (0,1/4]$. There exists a neighborhood V of 0 in the x-line such that for any $x_0 \in V\setminus\{0\}$ each $s(x_0)$-mapping κ conjugating $(\varphi_1)_{|x=x_0}$ and $(\varphi_2)_{|x=x_0}$ is a $(s(x_0)\tau, s(x_0)\tau')$-mapping where $\tau' = 2\sup_{x\in B(0,\delta)} |\partial(y\circ\sigma)/\partial y|(x,\gamma(x))$.*

Proof. Denote ζ_1, ζ_2 and υ the germs of diffeomorphisms induced by $(\varphi_1)_{|x=x_0}$, $(\sigma^{-1}\circ\varphi_2\circ\sigma)_{|x=x_0}$ and $\sigma^{-1}_{|x=x_0}\circ\kappa$ respectively in the neighborhood of $\gamma(x_0)$. The mapping υ conjugates diffeomorphisms ζ_1, ζ_2 with common convergent normal form $\exp(X_1)_{|x=x_0}$. Denote $\nu = \nu(\mathcal{E}_0)$. The diffeomorphism υ is of the form

$$\upsilon = \sigma_\lambda \circ \exp(t\log\zeta_2)\circ\hat{\sigma}(\zeta_1,\zeta_2)$$

where $\hat{\sigma}(\zeta_1,\zeta_2)$ is the unique element of $\widehat{\mathrm{Diff}}(\mathbb{C},\gamma(x_0))$ conjugating ζ_1, ζ_2 of the form $y + O((y-\gamma(x_0))^{\nu+2})$, $t\in\mathbb{C}$ and σ_λ is a periodic element of $\widehat{\mathrm{Diff}}(\mathbb{C},\gamma(x_0))$ commuting with ζ_2 such that $\upsilon'(\gamma(x_0)) = \sigma_\lambda'(\gamma(x_0)) = \lambda \in e^{2\pi i\mathbb{Q}}$. We obtain $|\upsilon'(\gamma(x_0))| = 1$ and then $|\kappa'(\gamma(x_0))| = |\partial(y\circ\sigma)/\partial y|(x_0,\gamma(x_0))$.

We define

$$\kappa_1(y) = \frac{\kappa((s(x_0)-|\gamma(x_0)|)y+\gamma(x_0)) - \gamma(x_0)}{(s(x_0)-|\gamma(x_0)|)(\partial\kappa/\partial y)(\gamma(x_0))}.$$

Then κ_1 is a Schlicht function. By the Koebe's distortion theorem (see [3], page 65) we get

$$\sup_{y\in B(0,s(x_0)\tau)} |\kappa(y)| \leq (s-|\gamma|)(x_0)\left|\frac{\partial(y\circ\sigma)}{\partial y}\right|(x_0,\gamma(x_0)) \sup_{y\in B(0,A(x_0,\tau))} |\kappa_1(y)| + |\gamma(x_0)|$$

where $A(x_0,\tau) = (s(x_0)\tau + |\gamma(x_0)|)/(s(x_0)-|\gamma(x_0)|)$. By arguing as in lemma 7.1 we obtain that κ is a $(s(x_0)\tau, s(x_0)\tau')$-mapping. □

Now we provide uniform quantitative estimates for s-mapping conjugacies.

Lemma 7.3. *Let $\varphi_1,\varphi_2 \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ with common convergent normal form $\exp(X)$ where $X = u(x,y)\prod_{j=1}^N (y-\gamma_j(x))^{n_j}\partial/\partial y$ and $u \in \mathbb{C}\{x,y\}$ is a unit. Denote $\nu = \nu(\mathcal{E}_0)$. Let $s: B(0,\delta)\setminus\{0\}\to\mathbb{R}^+$ be a slow decaying function. There exists an open set $0\in V\subset\mathbb{C}$ such that for any $x_0\in V\setminus\{0\}$ and any $s(x_0)$-mapping κ conjugating $(\varphi_1)_{|x=x_0}$, $(\varphi_2)_{|x=x_0}$ with $\kappa_{|(SingX)(x_0)} \cong Id$ then κ is of the form $y + J_\kappa(y)\prod_{j=1}^N (y-\gamma_j(x_0))^{n_j}$ where $\sup_{B(0,s(x_0)/4)} |J_\kappa| < (8^\nu 6)/s^\nu(x_0)$.*

Proof. We have $\kappa = y+(y-\gamma_1(x_0))\dots(y-\gamma_N(x_0))A(y)$ for some $A \in \mathcal{O}(B(0,s(x_0)))$ by hypothesis. By lemma 7.1 and the modulus maximum principle we obtain

$$\sup_{B(0,s(x_0)/4)} |A| = \sup_{y\in B(0,s(x_0)/4)} \frac{|\kappa(y)-y|}{|(y-\gamma_1(x_0))\dots(y-\gamma_N(x_0))|} \leq \frac{3s(x_0)/4}{((s(x_0)/4)/2)^N}$$

for any x_0 in a pointed neighborhood of 0. We have that

$$\left|\frac{\partial\kappa}{\partial y}(\gamma_j(x_0))-1\right| \leq \frac{8^{N-1}6}{s^{N-1}(x_0)} \prod_{k\in\{1,\dots,N\}\setminus\{j\}} |\gamma_j(x_0)-\gamma_k(x_0)|.$$

Fix $j \in \{1,\dots,N\}$. We claim that $(y-\gamma_j(x_0))^{n_j}$ divides $\kappa - y$. We can suppose $n_j > 1$. Denote ζ_1, ζ_2 and υ the germs of diffeomorphisms induced by $(\varphi_1)_{|x=x_0}$, $(\varphi_2)_{|x=x_0}$ and κ respectively in the neighborhood of $\gamma_j(x_0)$. The diffeomorphism υ is of the form

$$\upsilon = \sigma_\lambda \circ \exp(t\log\zeta_2)\circ\hat{\sigma}(\zeta_1,\zeta_2)$$

where $\hat{\sigma}(\zeta_1,\zeta_2)$ is the unique element of $\widehat{\mathrm{Diff}}(\mathbb{C},\gamma_j(x_0))$ conjugating ζ_1, ζ_2 of the form $y+O((y-\gamma_j(x_0))^{n_j+1})$, $t\in\mathbb{C}$ and σ_λ is a periodic element of $\widehat{\mathrm{Diff}}(\mathbb{C},\gamma_j(x_0))$ commuting with ζ_2 such that $\upsilon'(\gamma_j(x_0)) = \sigma_\lambda'(\gamma_j(x_0)) = \lambda \in < e^{2\pi i/(n_j-1)} >$. We obtain $\lambda = 1$ for $N(X)=1$ by hypothesis. We obtain $\lambda=1$ for $N(X)>1$ and x_0 in a neighborhood of 0 since $N\geq 2$ implies $\lim_{x_0\to 0}(\partial\kappa/\partial y)(\gamma_j(x_0)) = 1$. Thus $\sigma_\lambda \equiv Id$ and then $\kappa - y$ belongs to $(y-\gamma_j(x_0))^{n_j}$. The function J_κ belongs to $\mathcal{O}(B(0,s(x_0)))$. Analogously as for A we obtain $\sup_{B(0,s(x_0)/4)}|J_\kappa| \leq 8^\nu 6/s^\nu(x_0)$ for any $x_0 \neq 0$. □

We want to interpret the estimates in lemma 7.3 in terms of the Fatou coordinates of the common normal form. We define $\kappa_t(y) = y+t(\kappa(y)-y)$ for $y\in B(0,s(x_0))$ and $t\in\mathbb{C}$. Let ψ^X be a holomorphic integral of the time form of X. We can define the function $\psi^X\circ\kappa(x,y)-\psi^X(x,y)$ analogously as Δ_φ. The continuous path that we use to extend ψ^X is parametrized by $t\to\kappa_t(x,y)$ for $t\in[0,1]$. The function $\psi^X\circ\kappa-\psi^X$ is well-defined and holomorphic in $B(0,s(x_0))\setminus SingX$.

Lemma 7.4. *Let $\varphi_1,\varphi_2 \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ with common convergent normal form $\exp(X)$. Let s be a slow decaying function. Then there exists $\tau\in\mathbb{R}^+$ and $C'\in\mathbb{R}^+$ such that we have $\sup_{B(0,s(x_0)\tau)}|\psi^X\circ\kappa-\psi^X| \leq C'/s^\nu(x_0)$ for any $s(x_0)$-mapping κ conjugating $\varphi_1(x_0,y)$ and $\varphi_2(x_0,y)$ with $\kappa_{|(SingX)(x_0)} \cong Id$ and any x_0 in a pointed neighborhood of 0. In particular we obtain that $\psi^X\circ\kappa-\psi^X$ belongs to $\mathcal{O}(B(0,s(x_0)\tau))$.*

Proof. Denote $X(y) = u(x,y)(y-\gamma_1(x))^{n_1}\dots(y-\gamma_N(x))^{n_N}$ where $u\in\mathbb{C}\{x,y\}$ is a unit. Denote $\nu=\nu(\mathcal{E}_0)$. Consider $\tau\in(0,1/4)$. The function $\kappa_t(y) = y+t(\kappa(y)-y)$ satisfies $\kappa_t(B(0,s(x_0)\tau))\subset B(0,3s(x_0)\tau)$ for any $t\in[0,1]$ by lemma 7.1. By lemma 7.3 we have

$$\left|\frac{\partial\kappa_t}{\partial t}(y)\right| = |\kappa(y)-y| \leq \frac{(8^\nu 6)/s^\nu(x_0)}{|u\circ\kappa_t(y)|}|X(y)\circ\kappa_t(y)|\left|\frac{\prod_{j=1}^N(y-\gamma_j(x_0))^{n_j}}{\prod_{j=1}^N(y-\gamma_j(x_0))^{n_j}\circ\kappa_t(y)}\right|$$

for all $y\in B(0,s(x_0)\tau)$ and $t\in[0,1]$. We have

$$\frac{(y-\gamma_j(x_0))\circ\kappa_t(y)}{y-\gamma_j(x_0)} = 1+t\frac{\kappa(y)-y}{y-\gamma_j(x_0)}.$$

Lemma 7.3 implies

$$\left|\frac{\kappa(y)-y}{y-\gamma_j(x_0)}\right| \leq \frac{8^\nu 6}{s^\nu(x_0)}(2s(x_0)\tau)^\nu = 16^\nu 6\tau^\nu \quad \forall y \in B(0, s(x_0)\tau).$$

By considering a smaller $\tau \in \mathbb{R}^+$ we can suppose $16^\nu 6\tau^\nu < 1/2$. We obtain

$$\left|\frac{\partial \kappa_t}{\partial t}(y)\right| \leq \frac{1}{s^\nu(x_0)}\frac{(8^\nu 6)2}{|u(0,0)|}2^{\nu+1}|X(y)\circ\kappa_t(y)|$$

for all $y \in B(0, s(x_0)\tau)$ and $t \in [0,1]$. Denote $C' = (2^{4\nu+3}3)/|u(0,0)|$. We deduce that

$$|\psi^X \circ \kappa_t - \psi^X|(y) = \left|\int_0^t \frac{\partial(\psi^X\circ\kappa_\upsilon(y))}{\partial \upsilon}d\upsilon\right| = \left|\int_0^t \frac{\partial \psi^X}{\partial y}\circ\kappa_\upsilon(y)\frac{\partial \kappa_\upsilon(y)}{\partial \upsilon}d\upsilon\right| \leq \frac{C't}{s^\nu(x_0)}$$

for all $y \in B(0, s(x_0)\tau) \setminus (SingX)(x_0)$ and $t \in [0,1]$. By Riemann's theorem $\psi^X \circ \kappa - \psi^X$ belongs to $\mathcal{O}(B(0, s(x_0)\tau))$ and $|\psi^X \circ \kappa - \psi^X| \leq C'/s^\nu(x_0)$. □

Let $\varphi_1, \varphi_2 \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$ with common convergent normal form. Given a formal conjugation $\hat{\eta} \in \widehat{\text{Diff}}_p(\mathbb{C}^2, 0)$ we express the condition $\hat{\eta} \in \text{Diff}(\mathbb{C}^2, 0)$ in terms of the extensions of Ecalle-Voronin invariants $\{\xi_{j,\Lambda,\lambda}^{\varphi_1}\}_{(j,\Lambda,\lambda)\in\mathcal{D}(\varphi_1)\times\mathcal{M}\times\mathbb{S}^1}$ and $\{\xi_{j,\Lambda,\lambda}^{\varphi_2}\}_{(j,\Lambda,\lambda)\in\mathcal{D}(\varphi_2)\times\mathcal{M}\times\mathbb{S}^1}$.

Proposition 7.1. *Let $\varphi_1, \varphi_2 \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$ with common convergent normal form $\exp(X)$. Let s be a slow decaying function. Fix $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}}) \in \mathcal{M}$ and $\lambda \in \mathbb{S}^1$. Consider $x_0 \in (0,\delta)I_\Lambda^\lambda$ and a $s(x_0)$-mapping κ conjugating $(\varphi_1)_{|x=x_0}$ and $(\varphi_2)_{|x=x_0}$ with $\kappa_{|(SingX)(x_0)} \cong Id$. Then there exists $c(x_0) \in \mathbb{C}$ such that*

$$\xi_{j,\Lambda,\lambda}^{\varphi_2}(x_0, z) = (z + c(x_0)) \circ \xi_{j,\Lambda,\lambda}^{\varphi_1}(x_0, z) \circ (z - c(x_0)) \quad \forall j \in \mathcal{D}(\varphi_1)$$

and $|c(x_0)| \leq C'/s^{\nu(\mathcal{E}_0)}(x_0)$. The constant C' depends on Λ and λ but it does not depend on x_0.

Proof. Denote $\nu = \nu(\mathcal{E}_0)$. Since s is bounded we can suppose $s < \epsilon$ by replacing s with $s\tau_0$ for some $\tau_0 \in (0,1)$. By lemma 7.4 there exist $C_1' \in \mathbb{R}^+$ and $\tau_1 \in (0,1)$ such that $\sup_{B(0,s(x_0)\tau_1)} |\psi^X \circ \kappa - \psi^X| \leq C_1'/s^\nu(x_0)$.

Let $j \in \mathcal{D}(\varphi_1)$ and $\tau \in (0,1)$. We define $H_{s\tau,j}$ the element of $Reg(s\tau, \aleph_{\Lambda,\lambda}X, I_\Lambda^\lambda)$ contained in $H_{\Lambda,j}^\lambda$ (see def. 5.11). We obtain that $\kappa(H_{s\tau,j}(x_0)) \subset H_{\Lambda,j}^\lambda(x_0)$ for $\tau \in \mathbb{R}^+$ small enough and any $j \in \mathcal{D}(\varphi_1)$ by proposition 4.6.

Consider an irreducible component $y = \gamma(x)$ of $SingX$. We are in the situation of prop. 10.1 in [16]. By considering homogeneous privileged (with respect to $y = \gamma(x)$) systems of Fatou coordinates

$$\{\dot{\psi}_{j',\Lambda',\lambda'}^{\varphi_1}\}_{(j',\Lambda',\lambda')\in\mathcal{D}(\varphi_1)\times\mathcal{M}\times\mathbb{S}^1} \quad \text{and} \quad \{\dot{\psi}_{j',\Lambda',\lambda'}^{\varphi_2}\}_{(j',\Lambda',\lambda')\in\mathcal{D}(\varphi_2)\times\mathcal{M}\times\mathbb{S}^1}$$

of φ_1 and φ_2 respectively we obtain

$$\dot{\psi}_{j,\Lambda,\lambda}^{\varphi_2} \circ \kappa - \dot{\psi}_{j,\Lambda,\lambda}^{\varphi_1} = \tilde{c}(x_0) \text{ in } H_{s\tau,j}(x_0) \quad \forall j \in \mathcal{D}(\varphi_1) \tag{38}$$

where $\tilde{c}(x_0) \equiv (\psi^X \circ \kappa - \psi^X)(x_0, \gamma(x_0))$ (proposition 10.1 of [16]). We obtain the inequality $|\tilde{c}(x_0)| \leq C_1'/s^\nu(x_0)$. Let $c_{\Lambda,\lambda}^{\varphi_l} : [0,\delta)I_\Lambda^\lambda \to \mathbb{C}$ be the function defined by $c_{\Lambda,\lambda}^{\varphi_l} = \psi_{j,\Lambda,\lambda}^{\varphi_l} - \dot{\psi}_{j,\Lambda,\lambda}^{\varphi_l}$ for $l \in \{1,2\}$ (see remark 6.3). Equation (38) leads us to

$$\psi_{j,\Lambda,\lambda}^{\varphi_2} \circ \kappa - \psi_{j,\Lambda,\lambda}^{\varphi_1} = c(x_0) \text{ in } H_{s\tau,j}(x_0) \quad \forall j \in \mathcal{D}(\varphi_1)$$

by defining $c(x_0) = \tilde{c}(x_0) + c^{\varphi_2}_{\Lambda,\lambda}(x_0) - c^{\varphi_1}_{\Lambda,\lambda}(x_0)$. We obtain (see def. 6.6)

$$\xi^{\varphi_2}_{j,\Lambda,\lambda}(x_0, z) = (z + c(x_0)) \circ \xi^{\varphi_1}_{j,\Lambda,\lambda}(x_0, z) \circ (z - c(x_0)) \ \ \forall j \in \mathcal{D}(\varphi_1).$$

Thus there exists $C' \in \mathbb{R}^+$ independent of x_0 such that $|c(x_0)| \leq C'/s^{\nu}(x_0)$. □

Proposition 7.2. [16] *Let $\varphi_1, \varphi_2 \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$ with common convergent normal form $\exp(X)$. Fix $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}}) \in \mathcal{M}$ and $\lambda \in \mathbb{S}^1$. Fix a constant $M > 0$. Suppose that*

$$\xi^{\varphi_2}_{j,\Lambda,\lambda}(x_0, z) = (z + c(x_0)) \circ \xi^{\varphi_1}_{j,\Lambda,\lambda}(x_0, z) \circ (z - c(x_0)) \ \ \forall j \in \mathcal{D}(\varphi_1)$$

for some $x_0 \in [0, \delta) I^{\lambda}_{\Lambda}$ and $c(x_0) \in B(0, M)$. Then there exists a s-mapping κ such that $\kappa \circ (\varphi_1)_{|x=x_0} = (\varphi_2)_{|x=x_0} \circ \kappa$ and $\kappa_{|(SingX)(x_0)} \cong Id$. The constant $s \in \mathbb{R}^+$ does not depend on x_0.

7.2. **Theorem of analytic conjugation.** In this subsection we prove an analogue (theorem 7.2) of theorem 10.1 in [16] for the multi-summable setting. Roughly speaking two elements in $\text{Diff}_{tp1}(\mathbb{C}^2, 0)$ with common normal form are analytically conjugated if and only if their homogeneous multi-summable systems of extensions of Ecalle-Voronin invariants coincide up to the action of a multi-summable family of transformations of the form $(x, z + c(x))$.

Definition 7.4. *Let $\varphi_1, \varphi_2 \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$. We denote $\varphi_1 \sim \varphi_2$ if there exists $\sigma \in \text{Diff}\,(\mathbb{C}^2, 0)$ conjugating φ_1 and φ_2 such that $\sigma_{|Fix(\varphi_1)} \equiv Id$.*

Theorem 7.1. [16] *Let $\varphi_1, \varphi_2 \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$ with common convergent normal form $\exp(X)$. Suppose $N(X) = 1$. Then $\varphi_1 \sim \varphi_2$ if and only if there exist $c \in \mathbb{C}\{x\}$ and $k \in \mathbb{Z}/(\nu(\mathcal{E}_0)\mathbb{Z})$ such that*

$$\xi^{\varphi_2}_{j}(x, z) \equiv \xi^{\varphi_1}_{j+2k}(x, z - c(x)) + c(x)$$

for any $j \in \mathcal{D}(\varphi_1)$.

Theorem 7.2. *Let $\varphi_1, \varphi_2 \in \text{Diff}_{tp1}(\mathbb{C}^2, 0)$ with common convergent normal form $\exp(X)$. Suppose $N(X) > 1$. Assume $\varphi_1 \sim \varphi_2$. Then there exist a $(\tilde{e}_1, \ldots, \tilde{e}_{\tilde{q}})$-summable function $\{c_{\Lambda,\lambda}(x)\}_{(\Lambda,\lambda) \in \mathcal{M} \times \mathbb{S}^1}$, where $c_{\Lambda,\lambda}$ is defined in $(0, \delta)\dot{I}^{\lambda}_{\Lambda}$, such that*

$$\xi^{\varphi_2}_{j,\Lambda,\lambda}(x, z) \equiv \xi^{\varphi_1}_{j,\Lambda,\lambda}(x, z - c_{\Lambda,\lambda}(x)) + c_{\Lambda,\lambda}(x)$$

for any $j \in \mathcal{D}(\varphi_1)$.

The reciprocal of the theorem is also true. In fact it is a direct consequence of proposition 7.2 and the theorem 7.3.

Proof. Let $\sigma(x, y) = (x, \sigma_2(x, y) \in \text{Diff}_p(\mathbb{C}^2, 0)$ be the mapping conjugating φ_1 and φ_2 such that $\sigma_{|Fix(\varphi_1)} \equiv Id$. The mapping σ is a diffeomorphism defined in a neighborhood $B(0, \delta) \times B(0, s)$ of the origin for some $s \in \mathbb{R}^+$. By considering a smaller $s \in \mathbb{R}^+$ if necessary we obtain that the function $\psi^X \circ \sigma - \psi^X$ is bounded in $B(0, \delta) \times B(0, s)$ (lemma 7.4). Fix $j' \in \mathcal{D}(\varphi_1)$, we define $L^s_{j'}$ the unique element of $\mathcal{P}^s_X$ contained in $L_{j'}$. The set $\sigma(L^s_j)$ is contained in L_j for some $s \in \mathbb{R}^+$ and any $j \in \mathcal{D}(\varphi_1)$ by proposition 4.6. Consider a point $(0, y_0) \in L^s_{j'}$.

Proposition 7.1 implies that

$$c_{\Lambda,\lambda} = \psi^{\varphi_2}_{j,\Lambda,\lambda} \circ \sigma - \psi^{\varphi_1}_{j,\Lambda,\lambda}$$

defines a function $c_{\Lambda,\lambda} : [0,\delta) I_{\Lambda}^{\lambda} \to \mathbb{C}$ of x. The definition of $c_{\Lambda,\lambda}$ does not depend on $j \in \mathcal{D}(\varphi_1)$. Moreover $c_{\Lambda,\lambda}$ is continuous in $[0,\delta) I_{\Lambda}^{\lambda}$ and holomorphic in $(0,\delta)\dot{I}_{\Lambda}^{\lambda}$. We obtain

$$\xi_{j,\Lambda,\lambda}^{\varphi_2}(x,z) \equiv \xi_{j,\Lambda,\lambda}^{\varphi_1}(x, z - c_{\Lambda,\lambda}(x)) + c_{\Lambda,\lambda}(x) \quad \forall j \in \mathcal{D}(\varphi_1).$$

It suffices to prove that for all $\Lambda, \Lambda' \in \mathcal{M}$ and $\lambda, \lambda' \in \mathbb{S}^1$ we have

$$|c_{\Lambda,\lambda} - c_{\Lambda',\lambda'}|(x) = O\left(e^{-K/|x|^{\tilde{e}_{d_{\Lambda,\Lambda'}^{\lambda,\lambda'}+1}}} \right)$$

for some $K \in \mathbb{R}^+$. We have

$$(c_{\Lambda,\lambda} - c_{\Lambda',\lambda'})(x) = (\psi_{j',\Lambda,\lambda}^{\varphi_2} - \psi_{j',\Lambda',\lambda'}^{\varphi_2})(\sigma(x,y_0)) - (\psi_{j',\Lambda,\lambda}^{\varphi_1} - \psi_{j',\Lambda',\lambda'}^{\varphi_1})(x,y_0)$$

in a neighborhood of $x=0$ in $(0,\delta)(\dot{I}_{\Lambda}^{\lambda} \cap \dot{I}_{\Lambda'}^{\lambda'})$. The points (x,y_0) and $\sigma(x,y_0)$ belong to $H_{\Lambda,\Lambda',j'}^{\lambda,\lambda'}$ for any $x \in (0,\delta)(\dot{I}_{\Lambda}^{\lambda} \cap \dot{I}_{\Lambda'}^{\lambda'})$ in a neighborhood of 0. Theorem 6.3 implies that $c_{\Lambda,\lambda} - c_{\Lambda',\lambda'}$ is exponentially small of the proper order. □

7.3. **Isolated zeros theorem for analytic conjugacy.** This subsection is devoted to the proof of theorem 7.3. Let us notice that theorem 7.3 stands by itself; it makes no reference to Fatou coordinates or other multi-summable objects in the paper.

Definition 7.5. *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. Consider a convergent normal form* $\exp(X)$ *of φ. Denote $X = g(x,y)\partial/\partial y$. We define $\nu(\varphi) = \nu(g(0,y)) - 1$ where $\nu(g(0,y))$ is the order of the series $g(0,y)$ at $y=0$. The definition does not depend on the choice of X.*

Theorem 7.3. *Let $\varphi, \eta \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ with $Fix(\varphi) = Fix(\eta)$. Then $\varphi \sim \eta$ if and only if there exists a $\nu(\varphi)$ slow decaying function s and a sequence $x_n \to 0$ contained in $B(0,\delta)\setminus\{0\}$ such that for any $n \in \mathbb{N}$ the restrictions $\varphi_{|x=x_n}$ and $\eta_{|x=x_n}$ are conjugated by an injective holomorphic mapping κ_n defined in $B(0,s(x_n))$ and fixing the points in $Fix(\varphi) \cap \{x = x_n\}$.*

Proof. We start proving some normalizing conditions. We can suppose that $\varphi, \eta \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$ up to replace φ and η with $(x^{1/k},y)\circ\varphi\circ(x^k,y)$ and $(x^{1/k},y)\circ\eta\circ(x^k,y)$ respectively for some $k \in \mathbb{N}$. It suffices to prove the theorem in this context since given $k \in \mathbb{N}$ we have

$$\varphi \sim \eta \Leftrightarrow (x^{1/k},y) \circ \varphi \circ (x^k,y) \sim (x^{1/k},y) \circ \eta \circ (x^k,y)$$

by lemma 10.9 in [16]. Let us stress that the $\nu(\varphi)$ slow decaying character is invariant by ramification $x \to x^k$.

Let $\exp(X)$ and $\exp(Y)$ be convergent normal forms of φ and η respectively. Since $Fix(\varphi) = Fix(\eta)$ we obtain

$$X(y) = u(x,y)\prod_{j=1}^{p}(y - \gamma_j(x))^{n_j} \text{ and } Y(y) = v(x,y)\prod_{j=1}^{p}(y-\gamma_j(x))^{n_j'}$$

where $u, v \in \mathbb{C}\{x,y\}$ are units. The mapping κ_n conjugates the germs of $\varphi_{|x=x_n}$ and $\eta_{|x=x_n}$ in the neighborhood of $y = \gamma_j(x_n)$. Since the analytic conjugation κ_n preserves the order of tangency with the identity we get $n_j = n_j'$ for any $1 \le j \le p$.

Denote $w_{n,j} = (x_n, \gamma_j(x_n))$. An analytic conjugation preserves residues too (prop. 5.8 [15]), thus we obtain

$$Res(X, w_{n,j}) = Res(\varphi, w_{n,j}) = Res(\eta, w_{n,j}) = Res(Y, w_{n,j})$$

for all $1 \leq j \leq p$ and $n \in \mathbb{N}$ (see def. 2.6 and 5.2). Given $1 \leq j \leq p$ the functions

$$x \to Res(\varphi, (x, \gamma_j(x))) \text{ and } x \to Res(\eta, (x, \gamma_j(x)))$$

are meromorphic (prop. 5.2 [15]). We obtain $Res(\varphi, (x, \gamma_j(x))) \equiv Res(\eta, (x, \gamma_j(x)))$ for any $1 \leq j \leq p$. The equality $(X(y)) = (Y(y))$ of ideals of $\mathbb{C}\{x, y\}$ and

$$Res(X, (x, \gamma_j(x))) \equiv Res(Y, (x, \gamma_j(x))) \ \forall 1 \leq j \leq p$$

imply that the diffeomorphisms $\exp(X)$ and $\exp(Y)$ are analytically conjugated by a mapping $\sigma(x, y) = (x, \sigma_2(x, y))$ such that $\sigma_{|SingX} \equiv Id$ (prop. 5.10 [15]). Up to replace s with $s\tau$ for some $\tau \in \mathbb{R}^+$ we obtain that $\sigma^{-1} \circ \kappa_n$ is a $s(x_n)$-mapping for $n \in \mathbb{N}$ (lemmas 7.1 and 7.2). Up to replace η with $\sigma^{-1} \circ \eta \circ \sigma$ and κ_n with $\sigma^{-1} \circ \kappa_n$ for $n \in \mathbb{N}$ we can suppose that φ and η have common normal form $\exp(X)$.

Suppose $N(X) = 1$. We obtain that $\kappa_n'(\gamma_1(x_n)) \in < e^{2\pi i/\nu} >$ for any $n \in \mathbb{N}$ by arguing as in lemma 7.3. Up to taking a subsequence of $\{x_n\}_{n \in \mathbb{N}}$ we can suppose that there exists $\mu \in < e^{2\pi i/\nu} >$ such that $\kappa_n'(\gamma_1(x_n)) = \mu$ for any $n \in \mathbb{N}$. There exists $\sigma_0 \in \mathrm{Diff}_p(\mathbb{C}^2, 0)$ conjugating the vector fields X and

$$X_0 = \frac{(y - \gamma(x))^{\nu+1}}{1 + (y - \gamma(x))^{\nu} Res(X, (x, \gamma(x)))} \frac{\partial}{\partial y}.$$

Since $(x, \mu(y - \gamma(x)) + \gamma(x))$ preserves X_0 then $\sigma = \sigma_0^{-1} \circ (x, \mu(y - \gamma(x)) + \gamma(x)) \circ \sigma_0$ conjugates X with itself. Moreover σ satisfies $(\partial(y \circ \sigma)/\partial y)(x, \gamma(x)) \equiv \mu$. As in the previous paragraph up to replace η with $\sigma^{-1} \circ \eta \circ \sigma$ and κ_n with $\sigma^{-1} \circ \kappa_n$ for $n \in \mathbb{N}$ we can suppose $\kappa_n'(\gamma(x_n)) = 1$ for any $n \in \mathbb{N}$. In this way we obtain that $\kappa_n \cong Id$ for any $n \in \mathbb{N}$ independently on whether $N(X) = 1$ or $N(X) > 1$.

The goal is proving that there exists $s_0 \in \mathbb{R}^+$ such that $\varphi_{x=x_0}$ and $\eta_{|x=x_0}$ are conjugated by a s_0-mapping ϕ_{x_0} with $\phi_{x_0} \cong Id$ for any x_0 in a pointed neighborhood of 0. Then the Main Theorem of [16] implies $\varphi \sim \eta$.

By taking a subsequence of $\{x_n\}_{n \in \mathbb{N}}$ we can suppose that x_n adheres to a direction $\lambda \in \mathbb{S}^1$. Fix $\Lambda = (\lambda_1, \ldots, \lambda_{\tilde{q}}) \in \mathcal{M}$ with λ_q close to λ. Consider the set

$$E_\upsilon^\lambda(\varphi) = \{(j, l) \in \mathcal{D}_\upsilon(\varphi) \times \mathbb{N} : a_{j,\Lambda,\lambda,l}^\varphi \not\equiv 0\} \text{ for } \upsilon \in \{-1, 1\}$$

and $E^\lambda(\varphi) = E_{-1}^\lambda(\varphi) \cup E_1^\lambda(\varphi)$. Proposition 7.1 implies that there exists a sequence $\{c_n\}_{n \in \mathbb{N}}$ of complex numbers such that

$$\xi_{j,\Lambda,\lambda}^\eta(x_n, z) = (z + c_n) \circ \xi_{j,\Lambda,\lambda}^\varphi(x_n, z) \circ (z - c_n) \ \ \forall j \in \mathcal{D}(\varphi).$$

Moreover we have $|c_n| \leq C'/s^\nu(x_n)$ for some $C' \in \mathbb{R}^+$ and any $n \in \mathbb{N}$. We have

$$a_{j,\Lambda,\lambda,l}^\eta(x_n) = a_{j,\Lambda,\lambda,l}^\varphi(x_n) e^{2\pi i \upsilon l c_n} \tag{39}$$

for all $n \in \mathbb{N}$, $\upsilon \in \{-1, 1\}$ and $j \in \mathcal{D}_\upsilon(\varphi)$. The multi-summable character implies for both $a_{j,\Lambda,\lambda,l}^\varphi$ and $a_{j,\Lambda,\lambda,l}^\eta$ that either they are identically 0 or never vanishing in a neighborhood of 0 in $(0, \delta) I_\Lambda^\lambda$. We obtain $E^\lambda(\varphi) = E^\lambda(\eta)$.

Consider $(j, l) \in E_\upsilon^\lambda(\varphi)$. We obtain

$$e^{-\frac{2\pi l C'}{s^\nu(x_n)}} \leq \left| \frac{a_{j,\Lambda,\lambda,l}^\eta}{a_{j,\Lambda,\lambda,l}^\varphi} \right| (x_n) \leq e^{\frac{2\pi l C'}{s^\nu(x_n)}} \ \forall n \in \mathbb{N}.$$

Given $\iota \in \mathbb{R}^+$ we have $2\pi l C'/s^{\nu}(x_n) < \iota |\ln |x_n||$ for any $n \in \mathbb{N}$ big enough since s is a ν slow decaying function. This implies

$$|x_n|^{\iota} \leq \left| \frac{a^{\eta}_{j,\Lambda,\lambda,l}}{a^{\varphi}_{j,\Lambda,\lambda,l}} \right| (x_n) \leq \frac{1}{|x_n|^{\iota}} \ \forall n >> 1. \tag{40}$$

The multi-summability of $a^{\varphi}_{j,\Lambda,\lambda,l}$ and $a^{\eta}_{j,\Lambda,\lambda,l}$ implies that $a^{\eta}_{j,\Lambda,\lambda,l}/a^{\varphi}_{j,\Lambda,\lambda,l}$ is a function of the form $x^e h_{j,\Lambda,\lambda,\lambda}$ where e belongs to $\mathbb{Z}$ and $h_{j,l,\Lambda,\lambda}$ is a $(\tilde{e}_1, \ldots, \tilde{e}_{\tilde{q}})$-summable function such that $h_{j,l,\Lambda,\lambda}(0) \neq 0$. Equation (40) implies $e = 0$. We consider a continuous function $c^{j,l}_{\lambda}$ defined in a neighborhood of 0 in $[0,\delta)\dot{I}^{\lambda}_{\Lambda}$ such that $e^{2\pi i \upsilon l c^{j,l}_{\lambda}} \equiv a^{\eta}_{j,\Lambda,\lambda,l}/a^{\varphi}_{j,\Lambda,\lambda,l}$.

Consider $(j,l) \in E^{\lambda}_{\upsilon}(\varphi)$ and $(j',l') \in E^{\lambda}_{\upsilon'}(\varphi)$. The equation (39) implies

$$\left(\frac{a^{\eta}_{j,\Lambda,\lambda,l}}{a^{\varphi}_{j,\Lambda,\lambda,l}} \right)^{\upsilon l'} (x_n) = \left(\frac{a^{\eta}_{j',\Lambda,\lambda,l'}}{a^{\varphi}_{j',\Lambda,\lambda,l'}} \right)^{\upsilon' l} (x_n) \ \ \forall n \in \mathbb{N}.$$

Therefore we obtain

$$\left(\frac{a^{\eta}_{j,\Lambda,\lambda,l}}{a^{\varphi}_{j,\Lambda,\lambda,l}} \right)^{\upsilon l'} (x) = \left(\frac{a^{\eta}_{j',\Lambda,\lambda,l'}}{a^{\varphi}_{j',\Lambda,\lambda,l'}} \right)^{\upsilon' l} (x) \ \ \forall x \in (0,\delta)\dot{I}^{\lambda}_{\Lambda} \tag{41}$$

by the multi-summable character of the functions involved.

Suppose $E^{\lambda}(\varphi) \neq \emptyset$. Let us consider (j_1,l_1), $\ldots$, (j_b,l_b) such that $(j,l) \in E^{\lambda}(\varphi)$ implies that l belongs to the ideal $(l_1,\ldots,l_b)$ of $\mathbb{Z}$. Let x_{n_0} be a point such that $a^{\varphi}_{j_k,\Lambda,\lambda,l_k}(x_{n_0}) \neq 0$ for any $1 \leq k \leq b$. We can choose the function c^{j_k,l_k}_{λ} such that $c^{j_k,l_k}_{\lambda}(x_{n_0}) = c_{n_0}$ for any $1 \leq k \leq b$. Equation (41) and $c^{j_k,l_k}_{\lambda}(x_{n_0}) = c^{j_d,l_d}_{\lambda}(x_{n_0})$ imply

$$e^{2\pi i l_k l_d c^{j_k,l_k}_{\lambda}} \equiv e^{2\pi i l_k l_d c^{j_d,l_d}_{\lambda}} \implies c^{j_k,l_k}_{\lambda} - c^{j_d,l_d}_{\lambda} \in \mathbb{Z}/(l_k l_d) \implies c^{j_k,l_k}_{\lambda} \equiv c^{j_d,l_d}_{\lambda}$$

for all $1 \leq k,d \leq b$. We denote $c_{\lambda} = c^{j_k,l_k}_{\lambda}$ for $1 \leq k \leq b$. We define $c_{\lambda} \equiv 0$ for the case $E^{\lambda}(\varphi) = \emptyset$.

We have $(j_k,l_k) \in E^{\lambda}_{\upsilon_k}(\varphi)$ for $1 \leq k \leq b$. Consider $(j,l) \in E^{\lambda}_{\upsilon}(\varphi)$. There exist $m_1,\ldots,m_b \in \mathbb{Z}$ such that $l = m_1 l_1 + \ldots + m_b l_b$. We have

$$e^{2\pi i \upsilon_k l_k c_n} = \frac{a^{\eta}_{j_k,\Lambda,\lambda,l_k}}{a^{\varphi}_{j_k,\Lambda,\lambda,l_k}}(x_n) \ \forall k \in \{1,\ldots,b\} \ \forall n >> 1.$$

We obtain

$$\frac{a^{\eta}_{j,\Lambda,\lambda,l}}{a^{\varphi}_{j,\Lambda,\lambda,l}}(x_n) = e^{2\pi i \upsilon l c_n} = \prod_{k=1}^{b} \left(\frac{a^{\eta}_{j_k,\Lambda,\lambda,l_k}}{a^{\varphi}_{j_k,\Lambda,\lambda,l_k}} \right)^{\upsilon_k \upsilon m_k} (x_n) \ \forall n >> 1.$$

The multi-summability character of the functions involved in the previous equality implies

$$\frac{a^{\eta}_{j,\Lambda,\lambda,l}}{a^{\varphi}_{j,\Lambda,\lambda,l}} \equiv \prod_{k=1}^{b} \left(\frac{a^{\eta}_{j_k,\Lambda,\lambda,l_k}}{a^{\varphi}_{j_k,\Lambda,\lambda,l_k}} \right)^{\upsilon_k \upsilon m_k}.$$

We also have

$$e^{2\pi i \upsilon_k l_k c_{\lambda}} \equiv \frac{a^{\eta}_{j_k,\Lambda,\lambda,l_k}}{a^{\varphi}_{j_k,\Lambda,\lambda,l_k}} \ \forall k \in \{1,\ldots,b\}$$

by construction. Therefore we obtain

$$e^{2\pi i \upsilon l c_\lambda} \equiv \prod_{k=1}^{b} \left(\frac{a^{\eta}_{j_k,\Lambda,\lambda,l_k}}{a^{\varphi}_{j_k,\Lambda,\lambda,l_k}} \right)^{\upsilon_k \upsilon m_k} \equiv \frac{a^{\eta}_{j,\Lambda,\lambda,l}}{a^{\varphi}_{j,\Lambda,\lambda,l}}$$

for any $(j,l) \in E^{\lambda}(\varphi)$. We deduce

$$\xi^{\eta}_{j,\Lambda,\lambda}(x,z) \equiv \xi^{\varphi}_{j,\Lambda,\lambda}(x, z - c_\lambda(x)) + c_\lambda(x) \quad \forall j \in \mathcal{D}(\varphi).$$

By prop. 7.2 there exists $s_0 \in \mathbb{R}^+$ such that for any $x_0 \in [0,\delta)\dot{I}^{\lambda}_{\Lambda}$ there exists a s_0-mapping κ_{x_0} with $\kappa_{x_0} \cong Id$ conjugating $\varphi_{|x=x_0}$ and $\eta_{|x=x_0}$. Indeed it is possible to choose $\{\kappa_x\}_{x \in [0,\delta)\dot{I}^{\lambda}_{\Lambda}}$ such that there exists a continuous map κ defined in $[0,\delta)\dot{I}^{\lambda}_{\Lambda} \times B(0,s_0)$ and holomorphic in $(0,\delta)\dot{I}^{\lambda}_{\Lambda} \times B(0,s_0)$ satisfying $\kappa \circ \varphi = \eta \circ \kappa$ and $\kappa_{|x=x_0} = \kappa_{x_0}$ for any $x_0 \in [0,\delta)\dot{I}^{\lambda}_{\Lambda}$.

By considering $\lambda' \in \mathbb{S}^1$ such that $\dot{I}^{\lambda}_{\Lambda} \cap I^{\lambda'}_{\Lambda} \neq \emptyset$ we can repeat the previous argument to enlarge the family $\{\kappa_x\}_{x \in [0,\delta)\dot{I}^{\lambda}_{\Lambda}}$ to obtain a family $\{\kappa_x\}_{x \in [0,\delta)(\dot{I}^{\lambda}_{\Lambda} \cup \dot{I}^{\lambda'}_{\Lambda})}$ satisfying analogous properties for some smaller $s_0 \in \mathbb{R}^+$. By varying $\lambda' \in \mathbb{S}^1$ we obtain a family $\{\kappa_x\}_{x \in B(0,\delta)}$ of s_0-mappings such that κ_{x_0} conjugates $\varphi_{|x=x_0}$ and $\eta_{|x=x_0}$ and satisfies $\kappa_{x_0} \cong Id$ for any $x_0 \in B(0,\delta)$. The Main Theorem in [16] implies $\varphi \sim \eta$. □

Remark 7.1. *Theorem 7.3 can be generalized for codimension finite resonant diffeomorphisms. If the linear part of $\varphi_{|x=0}$ is p periodic we replace in the theorem the sets $Fix(\varphi)$ and $Fix(\eta)$ with $Fix(\varphi^p)$ and $Fix(\eta^p)$ respectively. Indeed we have $(\partial(y \circ \varphi)/\partial y)(0,0) = (\partial(y \circ \eta)/\partial y)(0,0)$ by continuity and $\varphi^p \sim \eta^p$ by theorem 7.3. Proposition 5.4 of* [16] *implies $\varphi \sim \eta$.*

Corollary 7.1. *Let $\varphi, \eta \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ with $Fix(\varphi) = Fix(\eta)$. Suppose there exist $s \in \mathbb{R}^+$ and a sequence $x_n \to 0$ contained in $B(0,\delta) \setminus \{0\}$ such that for any $n \in \mathbb{N}$ the restrictions $\varphi_{|x=x_n}$ and $\eta_{|x=x_n}$ are conjugated by an injective holomorphic mapping κ_n defined in $B(0,s)$ and fixing the points in $Fix(\varphi) \cap \{x = x_n\}$. Then we obtain $\varphi \sim \eta$.*

Remark 7.2. *The theorem 7.3 allows us to reduce the problem of analytic classification of elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ to well-behaved directions in the parameter space. For instance consider a diffeomorphism of the form*

$$\varphi(x,y) = (x, y + u(x,y)(y-x)(y+x))$$

for some unit $u \in \mathbb{C}\{x,y\}$ such that $u(0,0) = 1$. In any sector of the form

$$S = (0,\delta)e^{i(\pi/2+\upsilon, 3\pi/2-\upsilon)} \text{ or } S = (0,\delta)e^{i(-\pi/2+\upsilon, \pi/2-\upsilon)}$$

for $\upsilon \in \mathbb{R}^+$ the diffeomorphism $\varphi_{|x=x_0}$ has an attracting and a repelling fixed point for any $x_0 \in S$. Indeed an analytic system of invariants in S can be constructed by comparing the linearizing mappings in both fixed points. This is Glutsyuk's point of view [6]. *Roughly speaking, conjugation in $(B(0,\delta) \setminus i\mathbb{R}^*) \times B(0,\epsilon)$ implies conjugation in a neighborhood of the origin.*

Remark 7.3. *Consider $X \in \mathcal{X}(\mathbb{C}^2,0)$ such that $\exp(X) \in \mathrm{Diff}_{tp1}(\mathbb{C}^2,0)$. Prop. 11.1 in* [16] *shows that there exist $\varphi, \eta \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ with normal form $\exp(X)$ such that*

- $\varphi \not\sim \eta$

- *φ and η are conjugated by an analytic injective multi-valued, in the x variable, mapping σ.*
- *σ is defined in a domain $|y| < C_0 / \sqrt[\nu(X)]{|\ln x|}$ for some $C_0 \in \mathbb{R}^+$.*
- *σ satisfies $\sigma_{|Fix(\varphi)} \equiv Id$ and $\sigma(e^{2\pi i}x, y) = \eta \circ \sigma(x, y)$.*

The Main Theorem is optimal and it does not hold true for non-slow decaying functions. Equivalently a domain of the form $|y| < C_0 / \sqrt[\nu(X)]{|\ln x|}$ is maximal as a domain of definition of a mapping σ satisfying the previous properties. If σ is defined in a substantially bigger domain $|y| < s(x)$, i.e $\lim_{x \to 0} s(x)/(1/\sqrt[\nu(X)]{|\ln |x||}) = \infty$, then we obtain $\varphi \sim \eta$.

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