

DB PAIRS AND VANISHING THEOREMS

SÁNDOR J KOVÁCS

In memoriam Professor Masayoshi Nagata

ABSTRACT. The main purpose of this article is to define the notion of Du Bois singularities for pairs and proving a vanishing theorem using this new notion. The main vanishing theorem specializes to a new vanishing theorem for resolutions of log canonical singularities.

1. INTRODUCTION

The class of rational singularities is one of the most important classes of singularities. Their essence lies in the fact that their cohomological behavior is very similar to that of smooth points. For instance, vanishing theorems can be easily extended to varieties with rational singularities. Establishing that a certain class of singularities is rational opens the door to using very powerful tools on varieties with those singularities.

Du Bois (or DB) singularities are probably somewhat harder to appreciate at first, but they are equally important. Their main importance comes from two facts: They are not too far from rational singularities, that is, they share many of their properties, but the class of Du Bois singularities is more inclusive than that of rational singularities. For instance, log canonical singularities are Du Bois, but not necessarily rational. The class of Du Bois singularities is also more stable under degeneration.

Recently there has been an effort to extend the notion of rational singularities to pairs. There are at least two approaches; Schwede and Takagi [ST08] are dealing with pairs (X, Δ) where $[\Delta] = 0$ while Kollár and Kovács [KK09] are studying pairs (X, Δ) where Δ is reduced.

The main goal of this article is to extend the definition of Du Bois singularities to pairs in the spirit of the latter approach.

Here is a brief overview of the paper.

In section 2 some basic properties of rational and DB singularities are reviewed, a few new ones are introduced, and the DB defect is defined. In

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section 3 I recall the definition and some basic properties of pairs, generalized pairs, and rational pairs. I define the notion of a DB pair and the DB defect of a generalized pair and prove a few basic properties. In section 4 I recall a relevant theorem from Deligne’s Hodge theory and derive a corollary that will be needed later. In section 5 one of the main results is proven. A somewhat weaker version is the following. See Theorem 5.4 for the stronger statement.

Theorem 1.1. *Rational pairs are DB pairs.*

This generalizes [Kov99, Theorem S] and [Sai00, 5.4] to pairs. In section 6 I prove a rather general vanishing theorem for DB pairs and use it to derive the following vanishing theorem for log canonical pairs.

Theorem 1.2. *Let (X, Δ) be a log canonical pair, $\pi : \tilde{X} \rightarrow X$ a log resolution of (X, Δ) . Let $\tilde{\Delta} = (\pi_*^{-1}[\Delta] + \text{Exc}_{\text{nkt}}(\pi))_{\text{red}}$. Then*

$$\mathcal{R}^i \pi_* \mathcal{O}_{\tilde{X}}(-\tilde{\Delta}) = 0 \quad \text{for } i > 0.$$

A philosophical consequence one might draw from this theorem is that log canonical pairs are not too far from being rational. One may even view this a vanishing theorem similar to the one in the definition of rational singularities cf. (2.1), (3.4) with a correction term as in vanishing theorems with multiplier ideals. Notice however, that this is in a dual form compared to Nadel’s vanishing, and hence does not follow from that, especially since the target is not necessarily Cohen-Macaulay.

Theorem 1.2 is also closely related to Steenbrink’s characterization of normal isolated Du Bois singularities [Ste83, 3.6] (cf. [DB81, 4.13], [KS09, 6.1]).

A weaker version of this theorem was the corner stone of a recent result on extending differential forms to a log resolution [GKKP10]. For details on how this theorem may be applied, see the original article. It is possible that the current theorem will lead to a strengthening of that result.

Definitions and Notation 1.3. Unless otherwise stated, all objects are assumed to be defined over \mathbb{C} , all schemes are assumed to be of finite type over \mathbb{C} and a morphism means a morphism between schemes of finite type over \mathbb{C} .

If $\phi : Y \rightarrow Z$ is a birational morphism, then $\text{Exc}(\phi)$ will denote the *exceptional set* of ϕ . For a closed subscheme $W \subseteq X$, the ideal sheaf of W is denoted by $\mathcal{I}_{W \subseteq X}$ or if no confusion is likely, then simply by \mathcal{I}_W . For a point $x \in X$, $\kappa(x)$ denotes the residue field of $\mathcal{O}_{X,x}$.

For morphisms $\phi : X \rightarrow B$ and $\vartheta : T \rightarrow B$, the symbol X_T will denote $X \times_B T$ and $\phi_T : X_T \rightarrow T$ the induced morphism. In particular, for $b \in B$ we write $X_b = \phi^{-1}(b)$. Of course, by symmetry, we also have the notation

$\vartheta_X : T_X \simeq X_T \rightarrow X$ and if \mathcal{F} is an \mathcal{O}_X -module, then \mathcal{F}_T will denote the \mathcal{O}_{X_T} -module $\vartheta_X^* \mathcal{F}$.

Let X be a complex scheme (i.e., a scheme of finite type over \mathbb{C}) of dimension n . Let $D_{\text{filt}}(X)$ denote the derived category of filtered complexes of \mathcal{O}_X -modules with differentials of order ≤ 1 and $D_{\text{filt,coh}}(X)$ the subcategory of $D_{\text{filt}}(X)$ of complexes K^\bullet , such that for all i , the cohomology sheaves of $Gr_{\text{filt}}^i K^\bullet$ are coherent cf. [DB81], [GNPP88]. Let $D(X)$ and $D_{\text{coh}}(X)$ denote the derived categories with the same definition except that the complexes are assumed to have the trivial filtration. The superscripts $+$, $-$, b carry the usual meaning (bounded below, bounded above, bounded). Isomorphism in these categories is denoted by \simeq_{qis} . A sheaf \mathcal{F} is also considered as a complex \mathcal{F}^\bullet with $\mathcal{F}^0 = \mathcal{F}$ and $\mathcal{F}^i = 0$ for $i \neq 0$. If K^\bullet is a complex in any of the above categories, then $h^i(K^\bullet)$ denotes the i -th cohomology sheaf of K^\bullet .

The right derived functor of an additive functor F , if it exists, is denoted by $\mathcal{R}F$ and $\mathcal{R}^i F$ is short for $h^i \circ \mathcal{R}F$. Furthermore, \mathbb{H}^i , \mathbb{H}_c^i , \mathbb{H}_Z^i , and \mathcal{H}_Z^i will denote $\mathcal{R}^i \Gamma$, $\mathcal{R}^i \Gamma_c$, $\mathcal{R}^i \Gamma_Z$, and $\mathcal{R}^i \mathcal{H}_Z$ respectively, where Γ is the functor of global sections, Γ_c is the functor of global sections with proper support, Γ_Z is the functor of global sections with support in the closed subset Z , and \mathcal{H}_Z is the functor of the sheaf of local sections with support in the closed subset Z . Note that according to this terminology, if $\phi : Y \rightarrow X$ is a morphism and \mathcal{F} is a coherent sheaf on Y , then $\mathcal{R}\phi_* \mathcal{F}$ is the complex whose cohomology sheaves give rise to the usual higher direct images of \mathcal{F} .

We will often use the notion that a morphism $f : A \rightarrow B$ in a derived category *has a left inverse*. This means that there exists a morphism $f^\ell : B \rightarrow A$ in the same derived category such that $f^\ell \circ f : A \rightarrow A$ is the identity morphism of A . I.e., f^ℓ is a *left inverse* of f .

Finally, we will also make the following simplification in notation. First observe that if $\iota : \Sigma \hookrightarrow X$ is a closed embedding of schemes then ι_* is exact and hence $\mathcal{R}\iota_* = \iota_*$. This allows one to make the following harmless abuse of notation: If $A \in \text{Ob } D(\Sigma)$, then, as usual for sheaves, we will drop ι_* from the notation of the object $\iota_* A$. In other words, we will, without further warning, consider A an object in $D(X)$.

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2. RATIONAL AND DU BOIS SINGULARITIES

Definition 2.1. Let X be a normal variety and $\phi : Y \rightarrow X$ a resolution of singularities. X is said to have *rational singularities* if $\mathcal{R}^i \phi_* \mathcal{O}_Y = 0$

for all $i > 0$, or equivalently if the natural map $\mathcal{O}_X \rightarrow \mathcal{R}\phi_*\mathcal{O}_Y$ is a quasi-isomorphism.

Du Bois singularities are defined via Deligne’s Hodge theory. We will need a little preparation before we can define them.

The starting point is Du Bois’s construction, following Deligne’s ideas, of the generalized de Rham complex, which we call the *Deligne-Du Bois complex*. Recall, that if X is a smooth complex algebraic variety of dimension n , then the sheaves of differential p -forms with the usual exterior differentiation give a resolution of the constant sheaf \mathbb{C}_X . I.e., one has a filtered complex of sheaves,

$$\mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \Omega_X^3 \xrightarrow{d} \dots \xrightarrow{d} \Omega_X^n \simeq \omega_X,$$

which is quasi-isomorphic to the constant sheaf \mathbb{C}_X via the natural map $\mathbb{C}_X \rightarrow \mathcal{O}_X$ given by considering constants as holomorphic functions on X . Recall that this complex *is not* a complex of quasi-coherent sheaves. The sheaves in the complex are quasi-coherent, but the maps between them are not \mathcal{O}_X -module morphisms. Notice however that this is actually not a shortcoming; as \mathbb{C}_X is not a quasi-coherent sheaf, one cannot expect a resolution of it in the category of quasi-coherent sheaves.

The Deligne-Du Bois complex is a generalization of the de Rham complex to singular varieties. It is a complex of sheaves on X that is quasi-isomorphic to the constant sheaf \mathbb{C}_X . The terms of this complex are harder to describe but its properties, especially cohomological properties are very similar to the de Rham complex of smooth varieties. In fact, for a smooth variety the Deligne-Du Bois complex is quasi-isomorphic to the de Rham complex, so it is indeed a direct generalization.

The construction of this complex, $\underline{\Omega}_X^\bullet$, is based on simplicial resolutions. The reader interested in the details is referred to the original article [DB81]. Note also that a simplified construction was later obtained in [Car85] and [GNPP88] via the general theory of polyhedral and cubic resolutions. An easily accessible introduction can be found in [Ste85]. Other useful references are the recent book [PS08] and the survey [KS09]. We will actually not use these resolutions here. They are needed for the construction, but if one is willing to believe the listed properties (which follow in a rather straightforward way from the construction) then one should be able to follow the material presented here. The interested reader should note that recently Schwede found a simpler alternative construction of (part of) the Deligne-Du Bois complex that does not need a simplicial resolution [Sch07]. For applications of the Deligne-Du Bois complex and Du Bois singularities other than the ones listed here see [Ste83], [Kol95, Chapter 12], [Kov99, Kov00b, KSS10, KK10].

The word ‘‘hyperresolution’’ will refer to either a simplicial, polyhedral, or cubic resolution. Formally, the construction of $\underline{\Omega}_X^\bullet$ is the same regardless the type of resolution used and no specific aspects of either types will be used.

The next theorem lists the basic properties of the Deligne-Du Bois complex:

Theorem 2.2 [DB81]. *Let X be a complex scheme of finite type. Then there exists a functorially defined object $\underline{\Omega}_X^\bullet \in \text{Ob } D_{\text{filt}}(X)$ such that using the notation*

$$\underline{\Omega}_X^p := Gr_{\text{filt}}^p \underline{\Omega}_X^\bullet[p],$$

it satisfies the following properties

(2.2.1)

$$\underline{\Omega}_X^\bullet \simeq_{qis} \mathbb{C}_X.$$

(2.2.2) $\underline{\Omega}_{(_)}^\bullet$ *is functorial, i.e., if $\phi: Y \rightarrow X$ is a morphism of complex schemes of finite type, then there exists a natural map ϕ^* of filtered complexes*

$$\phi^*: \underline{\Omega}_X^\bullet \rightarrow \mathcal{R}\phi_* \underline{\Omega}_Y^\bullet$$

Furthermore, $\underline{\Omega}_X^\bullet \in \text{Ob } (D_{\text{filt,coh}}^b(X))$ and if ϕ is proper, then ϕ^ is a morphism in $D_{\text{filt,coh}}^b(X)$.*

(2.2.3) *Let $U \subseteq X$ be an open subscheme of X . Then*

$$\underline{\Omega}_X^\bullet|_U \simeq_{qis} \underline{\Omega}_U^\bullet.$$

(2.2.4) *If X is proper, then there exists a spectral sequence degenerating at E_2 and abutting to the singular cohomology of X :*

$$E_2^{pq} = \mathbb{H}^q(X, \underline{\Omega}_X^p) \Rightarrow H^{p+q}(X, \mathbb{C}).$$

(2.2.5) *If $\varepsilon_\bullet: X_\bullet \rightarrow X$ is a hyperresolution, then*

$$\underline{\Omega}_X^\bullet \simeq_{qis} \mathcal{R}\varepsilon_{\bullet*} \underline{\Omega}_{X_\bullet}^\bullet.$$

In particular, $h^i(\underline{\Omega}_X^p) = 0$ for $i < 0$.

(2.2.6) *There exists a natural map, $\mathcal{O}_X \rightarrow \underline{\Omega}_X^0$, compatible with (2.2.2).*

(2.2.7) *If X is a normal crossing divisor in a smooth variety, then*

$$\underline{\Omega}_X^\bullet \simeq_{qis} \Omega_X^\bullet.$$

In particular,

$$\underline{\Omega}_X^p \simeq_{qis} \Omega_X^p.$$

(2.2.8) *If $\phi: Y \rightarrow X$ is a resolution of singularities, then*

$$\underline{\Omega}_X^{\dim X} \simeq_{qis} \mathcal{R}\phi_* \omega_Y.$$

(2.2.9) Let $\pi : \tilde{X} \rightarrow X$ be a projective morphism and $\Sigma \subseteq X$ a reduced closed subscheme such that π is an isomorphism outside of Σ . Let E denote the reduced subscheme of \tilde{X} with support equal to $\pi^{-1}(\Sigma)$. Then for each p one has an exact triangle of objects in the derived category,

$$\underline{\Omega}_X^p \longrightarrow \underline{\Omega}_\Sigma^p \oplus \mathcal{R}\pi_* \underline{\Omega}_{\tilde{X}}^p \xrightarrow{-} \mathcal{R}\pi_* \underline{\Omega}_E^p \xrightarrow{+1} \cdot$$

(2.2.10) Suppose $X = Y \cup Z$ is the union of two closed subschemes and denote their intersection by $W := Y \cap Z$. Then for each p one has an exact triangle of objects in the derived category,

$$\underline{\Omega}_X^p \longrightarrow \underline{\Omega}_Y^p \oplus \underline{\Omega}_Z^p \xrightarrow{-} \underline{\Omega}_W^p \xrightarrow{+1} \cdot$$

It turns out that the Deligne-Du Bois complex behaves very much like the de Rham complex for smooth varieties. Observe that (2.2.4) says that the Hodge-to-de Rham (a.k.a. Frölicher) spectral sequence works for singular varieties if one uses the Deligne-Du Bois complex in place of the de Rham complex. This has far reaching consequences and if the associated graded pieces $\underline{\Omega}_X^p$ turn out to be computable, then this single property leads to many applications.

Notice that (2.2.6) gives a natural map $\mathcal{O}_X \rightarrow \underline{\Omega}_X^0$, and we will be interested in situations when this map is a quasi-isomorphism. When X is proper over \mathbb{C} , such a quasi-isomorphism implies that the natural map

$$H^i(X^{\text{an}}, \mathbb{C}) \rightarrow H^i(X, \mathcal{O}_X) = \mathbb{H}^i(X, \underline{\Omega}_X^0)$$

is surjective because of the degeneration at E_1 of the spectral sequence in (2.2.4). Notice that this is the condition that is crucial for Kodaira-type vanishing theorems cf. [Kol95, §9].

Following Du Bois, Steenbrink was the first to study this condition and he christened this property after Du Bois. It should be noted that many of the ideas that play important roles in this theory originated from Deligne. Unfortunately the now standard terminology does not reflect this.

Definition 2.3. A scheme X is said to have *Du Bois singularities* (or *DB singularities* for short) if the natural map $\mathcal{O}_X \rightarrow \underline{\Omega}_X^0$ from (2.2.6) is a quasi-isomorphism.

REMARK 2.4. If $\varepsilon : X_\bullet \rightarrow X$ is a hyperresolution of X then X has DB singularities if and only if the natural map $\mathcal{O}_X \rightarrow \mathcal{R}\varepsilon_* \mathcal{O}_{X_\bullet}$ is a quasi-isomorphism.

EXAMPLE 2.5. It is easy to see that smooth points are DB and Deligne proved that normal crossing singularities are DB as well cf. (2.2.7), [DJ74, Lemme 2(b)].

In applications it is very useful to be able to take general hyperplane sections. The next statement helps with that.

Proposition 2.6. *Let X be a quasi-projective variety and $H \subset X$ a general member of a very ample linear system. Then $\underline{\Omega}_H \simeq_{qis} \underline{\Omega}_X \otimes_L \mathcal{O}_H$.*

Proof. Let $\varepsilon : X \rightarrow X$ be a hyperresolution. Since H is general, the fiber product $X \times_X H \rightarrow H$ provides a hyperresolution of H . Then the statement follows from (2.2.5) applied to both X and H . \square

We saw in (2.2.5) that $h^i(\underline{\Omega}_X^0) = 0$ for $i < 0$. In fact, there is a corresponding upper bound by [GNPP88, III.1.17], namely that $h^i(\underline{\Omega}_X^0) = 0$ for $i > \dim X$. It turns out that one can make a slightly better estimate.

Proposition 2.7 cf. [GKKP10, 13.7], [KSS10, 4.9]. *Let X be a positive dimensional variety (i.e., reduced). Then the i^{th} cohomology sheaf of $\underline{\Omega}_X^p$ vanishes for all $i \geq \dim X$, i.e., $h^i(\underline{\Omega}_X^p) = 0$ for all p and for all $i \geq \dim X$.*

Proof. For $i > \dim X$ or $p > 0$, the statement follows from [GNPP88, III.1.17]. The case $p = 0$ and $i = n := \dim X$ follows from either [GKKP10, 13.7] or [KSS10, 4.9]. \square

Another, much simpler fact that will be used later is the following:

Corollary 2.8. *If $\dim X = 1$, then $h^i(\underline{\Omega}_X^p) = 0$ for $i \neq 0$. In particular X is DB if and only if it is semi-normal.*

Proof. The first statement is a direct consequence of (2.7). For the last statement recall that the seminormalization of \mathcal{O}_X is exactly $h^0(\underline{\Omega}_X^0)$, and so X is seminormal if and only if $\mathcal{O}_X \simeq h^0(\underline{\Omega}_X^0)$ [Sai00, 5.2] (cf. [Sch06, 5.4.17], [Sch07, 4.8], and [Sch09, 5.6]). \square

Definition 2.9. The *DB defect* of X is the mapping cone of the morphism $\mathcal{O}_X \rightarrow \underline{\Omega}_X^0$. It is denoted by $\underline{\Omega}_X^\times$. As a simple consequence of the definition, one has an exact triangle,

$$\mathcal{O}_X \longrightarrow \underline{\Omega}_X^0 \longrightarrow \underline{\Omega}_X^\times \xrightarrow{+1} .$$

Notice that $h^0(\underline{\Omega}_X^\times) \simeq h^0(\underline{\Omega}_X^0)/\mathcal{O}_X$ and $h^i(\underline{\Omega}_X^\times) \simeq h^i(\underline{\Omega}_X^0)$ for $i > 0$.

Proposition 2.10. *Let X be a quasi-projective variety and $H \subset X$ a general member of a very ample linear system. Then $\underline{\Omega}_H^\times \simeq_{qis} \underline{\Omega}_X^\times \otimes_L \mathcal{O}_H$.*

Proof. This follows easily from the definition and 2.6. \square

The next simple observation explains the name of the DB defect.

Lemma 2.11. *A variety X is DB if and only if the DB defect of X is acyclic, that is, $\underline{\Omega}_X^\times \simeq_{qis} 0$.*

Proof. This follows directly from the definition. \square

Proposition 2.12. *Let $X = Y \cup Z$ be a union of closed subschemes with intersection $W = Y \cap Z$. Then one has an exact triangle of the DB defects of X, Y, Z , and W :*

$$\underline{\Omega}_X^\times \longrightarrow \underline{\Omega}_Y^\times \oplus \underline{\Omega}_Z^\times \xrightarrow{-} \underline{\Omega}_W^\times \xrightarrow{+1} .$$

Proof. Recall that there is an analogous exact triangle (a.k.a. a short exact sequence) for the structure sheaves of X, Y, Z , and W , which forms a commutative diagram with the exact triangle of (2.2.10),

$$\begin{array}{ccccccc} \mathcal{O}_X & \longrightarrow & \mathcal{O}_Y \oplus \mathcal{O}_Z & \xrightarrow{-} & \mathcal{O}_W & \xrightarrow{+1} & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & & \\ \underline{\Omega}_X^0 & \longrightarrow & \underline{\Omega}_Y^0 \oplus \underline{\Omega}_Z^0 & \xrightarrow{-} & \underline{\Omega}_W^0 & \xrightarrow{+1} & \longrightarrow \end{array} .$$

Then the statement follows by the (derived category version of the) 9-lemma. \square

3. PAIRS AND GENERALIZED PAIRS

3.A. Basic definitions

For an arbitrary proper birational morphism, $\phi : Y \rightarrow X$, $\text{Exc}(\phi)$ stands for the exceptional locus of ϕ . A \mathbb{Q} -divisor is a \mathbb{Q} -linear combination of integral Weil divisors; $\Delta = \sum a_i \Delta_i$, $a_i \in \mathbb{Q}$, Δ_i (integral) Weil divisor. For a \mathbb{Q} -divisor Δ , its *round-down* is defined by the formula: $\lfloor \Delta \rfloor = \sum \lfloor a_i \rfloor \Delta_i$, where $\lfloor a_i \rfloor$ is the largest integer not larger than a_i .

A *log variety* or *pair* (X, Δ) consists of an equidimensional variety (i.e., a reduced scheme of finite type over a field k) X and an effective \mathbb{Q} -divisor $\Delta \subseteq X$. A morphism of pairs $\phi : (Y, B) \rightarrow (X, \Delta)$ is a morphism $\phi : Y \rightarrow X$ such that $\phi(\text{supp } B) \subseteq \text{supp } \Delta$.

Let (X, Δ) be a pair with Δ a reduced integral divisor. Then (X, Δ) is said to have *simple normal crossings* or to be an *snc pair* at $p \in X$ if X is smooth at p , and there are local coordinates x_1, \dots, x_n on X in a neighbourhood of p such that $\text{supp } \Delta \subseteq (x_1 \cdots x_n = 0)$ near p . (X, Δ) is *snc* if it is snc at every $p \in X$.

A morphism of pairs $\phi : (Y, \Delta_Y) \rightarrow (X, \Delta)$ is a *log resolution* of (X, Δ) if $\phi : Y \rightarrow X$ is proper and birational, $\Delta_Y = \phi_*^{-1} \Delta$, and $(\Delta_Y)_{\text{red}} + \text{Exc}(\phi)$ is an snc divisor on Y .

Note that we allow (X, Δ) to be snc and still call a morphism with these properties a log resolution. Also note that the notion of a log resolution is not used consistently in the literature.

If (X, Δ) is a pair, then Δ is called a *boundary* if $\lfloor (1 - \varepsilon) \Delta \rfloor = 0$ for all $0 < \varepsilon < 1$, i.e., the coefficients of all irreducible components of Δ are in the interval $[0, 1]$. For the definition of *kl*, *dlt*, and *lc* pairs see [KM98].

Let (X, Δ) be a pair and $\mu : X^m \rightarrow X$ a proper birational morphism such that $\text{Exc}(\mu)$ is a divisor. Let $E = \sum a_i E_i$ be the discrepancy divisor, i.e., a linear combination of exceptional divisors such that

$$K_{X^m} + \mu_*^{-1} \Delta \sim_{\mathbb{Q}} \mu^*(K_X + \Delta) + E$$

and let $\Delta^m := \mu_*^{-1} \Delta + \sum_{a_i \leq -1} E_i$. For an irreducible divisor F on a birational model of X we define its discrepancy as its coefficient in E . Notice that as divisors correspond to valuations, this discrepancy is independent of the model chosen, it only depends on the divisor. A *non-klt place* of a pair (X, Δ) is an irreducible divisor F over X with discrepancy at most -1 and a *non-klt center* is the image of any non-klt place. $\text{Exc}_{\text{nklt}}(\mu)$ denotes the union of the loci of all non-klt places of ϕ .

Note that in the literature, non-klt places and centers are often called log canonical places and centers. For a more detailed and precise definition see [HK10, p.37].

Now if (X^m, Δ^m) is as above, then it is a *minimal dlt model* of (X, Δ) if it is a dlt pair and the discrepancy of every μ -exceptional divisor is at most -1 cf. [KK10]. Note that if (X, Δ) is lc with a minimal dlt model (X^m, Δ^m) , then $K_{X^m} + \Delta^m \sim_{\mathbb{Q}} \mu^*(K_X + \Delta)$.

3.B. Rational pairs

Recall the definition of a *log resolution* from (3.A): A morphism of pairs $\phi : (Y, \Delta_Y) \rightarrow (X, \Delta)$ is a *log resolution of (X, Δ)* if $\phi : Y \rightarrow X$ is proper and birational, $\Delta_Y = \phi_*^{-1} \Delta$, and $(\Delta_Y)_{\text{red}} + \text{Exc}(\phi)$ is an snc divisor on Y .

Definition 3.1. Let (X, Δ) be a pair and Δ an integral divisor. Then (X, Δ) is called a *normal pair* if there exists a log resolution $\phi : (Y, \Delta_Y) \rightarrow (X, \Delta)$ such that the natural morphism $\phi^\# : \mathcal{O}_X(-\Delta) \rightarrow \phi_* \mathcal{O}_Y(-\Delta_Y)$ is an isomorphism.

Definition 3.2. A pair (X, Δ) with Δ an integral divisor is called a *weakly rational pair* if there is a log resolution $\phi : (Y, \Delta_Y) \rightarrow (X, \Delta)$ such that the natural morphism $\mathcal{O}_X(-\Delta) \rightarrow \mathcal{R}\phi_* \mathcal{O}_Y(-\Delta_Y)$ has a left inverse.

Lemma 3.3. *Let (X, Δ) be a weakly rational pair. Then it is a normal pair.*

Proof. The 0th cohomology of the left inverse of $\mathcal{O}_X(-\Delta) \rightarrow \mathcal{R}\phi_* \mathcal{O}_Y(-\Delta_Y)$ gives a left inverse of $\phi^\# : \mathcal{O}_X(-\Delta) \rightarrow \phi_* \mathcal{O}_Y(-\Delta_Y)$. As the morphism ϕ is birational, the kernel of the left inverse of $\phi^\#$ is a torsion sheaf. However, since $\phi_* \mathcal{O}_Y(-\Delta_Y)$ is torsion-free, this implies that $\phi^\#$ is an isomorphism. \square

Definition 3.4. [KK09] Let (X, Δ) be a pair where Δ is an integral divisor. Then (X, Δ) is called a *rational pair* if there exists a log resolution $\phi : (Y, \Delta_Y) \rightarrow (X, \Delta)$ such that

(3.4.1) $\mathcal{O}_X(-\Delta) \simeq \phi_* \mathcal{O}_Y(-\Delta_Y)$, i.e., (X, Δ) is normal,

(3.4.2) $\mathcal{R}^i \phi_* \mathcal{O}_Y(-\Delta_Y) = 0$ for $i > 0$, and

(3.4.3) $\mathcal{R}^i \phi_* \omega_Y(\Delta_Y) = 0$ for $i > 0$.

Lemma 3.5. *Let (X, Δ) be a pair where Δ is an integral divisor. Then it is a rational pair if and only if it is a weakly rational pair and $\mathcal{R}^i \phi_* \omega_Y(\Delta_Y) = 0$ for $i > 0$.*

Proof. This follows directly from [KK09, 105]. □

REMARK 3.6. Note that the notion of a *rational pair* describes the “singularity” of the relationship between X and Δ . From the definition it is not clear for instance whether (X, Δ) being rational implies that X has rational singularities. I expect that this is not true, but at the time of writing this article I do not know an example of an irreducible X and an appropriate $\Delta \subset X$ such that (X, Δ) is a rational pair, but X does not have rational singularities.

REMARK 3.7. If $\Delta = \emptyset$, then (3.4.3) follows from Grauert-Riemenschneider vanishing and X is weakly rational if and only if it is rational by [Kov00a]. I do not know whether the same holds with $\Delta \neq \emptyset$.

3.C. Generalized pairs

Definition 3.8. A *generalized pair* (X, Σ) consists of an equidimensional variety (i.e., a reduced scheme of finite type over a field k) X and a subscheme $\Sigma \subseteq X$. A morphism of generalized pairs $\phi : (Y, \Gamma) \rightarrow (X, \Sigma)$ is a morphism $\phi : Y \rightarrow X$ such that $\phi(\Gamma) \subseteq \Sigma$. A *reduced generalized pair* is a generalized pair (X, Σ) such that Σ is reduced.

The *log resolution* of a generalized pair (X, W) is a proper birational morphism $\pi : \tilde{X} \rightarrow X$ such that $\text{Exc}(\pi)$ is a divisor and $\pi^{-1}W + \text{Exc}(\pi)$ is an snc divisor.

Let X be a complex scheme and Σ a closed subscheme whose complement in X is dense. Then $(X_\bullet, \Sigma_\bullet) \rightarrow (X, \Sigma)$ is a *good hyperresolution* if $X_\bullet \rightarrow X$ is a hyperresolution, and if $U_\bullet = X_\bullet \times_X (X \setminus \Sigma)$ and $\Sigma_\bullet = X_\bullet \setminus U_\bullet$, then, for all α , either Σ_α is a divisor with normal crossings on X_α or $\Sigma_\alpha = X_\alpha$. Notice that it is possible that X_\bullet has components that map into Σ . These component are contained in Σ_\bullet . For more details and the existence of such hyperresolutions see [DB81, 6.2] and [GNPP88, IV.1.21, IV.1.25, IV.2.1]. For a primer on hyperresolutions see the appendix of [KS09].

Let (X, Σ) be a reduced generalized pair. Consider the Deligne-Du Bois complex of (X, Σ) defined by Steenbrink [Ste85, §3]:

Definition 3.9. The *Deligne-Du Bois complex* of the reduced generalized pair (X, Σ) is the mapping cone of the natural morphism $\varrho : \underline{\Omega}_X^\bullet \rightarrow \underline{\Omega}_\Sigma^\bullet$ twisted by (-1) . In other words, it is an object $\underline{\Omega}_{X,\Sigma}^\bullet$ in $D_{\text{filt}}(X)$ such that it completes ϱ to an exact triangle:

$$(3.9.1) \quad \underline{\Omega}_{X,\Sigma}^\bullet \longrightarrow \underline{\Omega}_X^\bullet \longrightarrow \underline{\Omega}_\Sigma^\bullet \xrightarrow{+1} \cdot$$

The associated graded quotients of $\underline{\Omega}_{X,\Sigma}^\bullet$ will be denoted as usual:

$$\underline{\Omega}_{X,\Sigma}^p := \text{Gr}_{\text{filt}}^p \underline{\Omega}_{X,\Sigma}^\bullet[p].$$

Notice that the above triangle is in $D_{\text{filt}}(X)$ and hence for all $p \in \mathbb{N}$ we obtain another exact triangle:

$$(3.9.2) \quad \underline{\Omega}_{X,\Sigma}^p \longrightarrow \underline{\Omega}_X^p \longrightarrow \underline{\Omega}_\Sigma^p \xrightarrow{+1} \cdot$$

EXAMPLE 3.10. Let (X, Σ) be an snc pair. Then $\underline{\Omega}_{X,\Sigma}^\bullet \simeq_{\text{qis}} \Omega_X^\bullet(\log \Sigma)(-\Sigma)$.

The Deligne-Du Bois complex of a pair is functorial in the following sense:

Proposition 3.11. *Let $\phi : (Y, \Gamma) \rightarrow (X, \Delta)$ be a morphism of generalized pairs. Then there exists a filtered natural morphism $\underline{\Omega}_{X,\Sigma}^\bullet \rightarrow \mathcal{R}\phi_* \underline{\Omega}_{Y,\Gamma}^\bullet$.*

Proof. There exist compatible filtered natural morphisms $\underline{\Omega}_X^\bullet \rightarrow \mathcal{R}\phi_* \underline{\Omega}_Y^\bullet$ and $\underline{\Omega}_\Sigma^\bullet \rightarrow \mathcal{R}\phi_* \underline{\Omega}_\Gamma^\bullet$ by (2.2.2). They induce the following morphism between exact triangles,

$$\begin{array}{ccccccc} \underline{\Omega}_{X,\Sigma}^0 & \longrightarrow & \underline{\Omega}_X^0 & \longrightarrow & \underline{\Omega}_\Sigma^0 & \xrightarrow{+1} & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{R}\phi_* \underline{\Omega}_{Y,\Gamma}^0 & \longrightarrow & \mathcal{R}\phi_* \underline{\Omega}_Y^0 & \longrightarrow & \mathcal{R}\phi_* \underline{\Omega}_\Gamma^0 & \xrightarrow{+1} & \longrightarrow \end{array}$$

and thus one obtains the desired natural morphism. \square

It follows easily from the definition and (2.7) that we have the following bounds on the non-zero cohomology sheaves of $\underline{\Omega}_{X,\Sigma}^p$.

Proposition 3.12. *Let X be a positive dimensional variety. Then the i^{th} cohomology sheaf of $\underline{\Omega}_{X,\Sigma}^p$ vanishes for all $i \geq \dim X$, i.e., $h^i(\underline{\Omega}_{X,\Sigma}^p) = 0$ for all p and for all $i \geq \dim X$.*

Proof. This follows directly from (2.7) using the long exact cohomology sequence associated to (3.9.2). \square

3.D. *DB pairs and the DB defect*

Definition 3.13. Recall the short exact sequence for the restriction of regular functions from X to Σ :

$$0 \longrightarrow \mathcal{I}_{\Sigma \subseteq X} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_\Sigma \longrightarrow 0.$$

By (2.2.6) there exist compatible natural maps $\mathcal{O}_X \rightarrow \underline{\Omega}_X^0$ and $\mathcal{O}_\Sigma \rightarrow \underline{\Omega}_\Sigma^0$, and they induce a morphism between exact triangles,

$$(3.13.1) \quad \begin{array}{ccccccc} \mathcal{I}_{\Sigma \subseteq X} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_\Sigma & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow & & \\ \underline{\Omega}_{X,\Sigma}^0 & \longrightarrow & \underline{\Omega}_X^0 & \longrightarrow & \underline{\Omega}_\Sigma^0 & \xrightarrow{+1} & , \end{array}$$

A reduced generalized pair (X, Σ) will be called a *DB pair* if the natural morphism $\mathcal{I}_{\Sigma \subseteq X} \rightarrow \underline{\Omega}_{X,\Sigma}^0$ from (3.13.1) is a quasi-isomorphism.

REMARK 3.14. Note that just like the notion of a rational pair, the notion of a *DB pair* describes the “singularity” of the relationship between X and Σ . From the definition it is not clear for instance whether (X, Σ) being DB implies that X has DB singularities. I expect that this is not true, but at the time of writing this article I do not know an example of an irreducible X and an appropriate $\Sigma \subset X$ such that (X, Σ) is a DB pair, but X does not have DB singularities.

Proposition 3.15. *Let $\phi : (Y, \Gamma) \rightarrow (X, \Sigma)$ be a morphism of generalized pairs. Then there exists a natural morphism $\underline{\Omega}_{X,\Sigma}^0 \rightarrow \mathcal{R}\phi_* \underline{\Omega}_{Y,\Gamma}^0$ and a commutative diagram,*

$$\begin{array}{ccc} \mathcal{I}_{\Sigma \subseteq X} & \longrightarrow & \underline{\Omega}_{X,\Sigma}^0 \\ \downarrow & & \downarrow \\ \mathcal{R}\phi_* \mathcal{I}_{\Gamma \subseteq Y} & \longrightarrow & \mathcal{R}\phi_* \underline{\Omega}_{Y,\Gamma}^0 \end{array}$$

Proof. Similarly to (3.13.1) and one obtains a commutative diagram for (Y, Γ) :

$$\begin{array}{ccccccc} \mathcal{I}_{\Gamma \subseteq Y} & \longrightarrow & \mathcal{O}_Y & \longrightarrow & \mathcal{O}_\Gamma & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow & & \\ \underline{\Omega}_{Y,\Gamma}^0 & \longrightarrow & \underline{\Omega}_Y^0 & \longrightarrow & \underline{\Omega}_\Gamma^0 & \xrightarrow{+1} & . \end{array}$$

Then ϕ induces a morphism between these diagrams:

$$\begin{array}{ccccccc}
\mathcal{I}_{\Sigma \subseteq X} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_\Sigma & \xrightarrow{+1} & \\
\downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
& & \underline{\Omega}_{X,\Sigma}^0 & \longrightarrow & \underline{\Omega}_X^0 & \longrightarrow & \underline{\Omega}_\Sigma^0 \xrightarrow{+1} \\
& & \downarrow & & \downarrow & & \downarrow \\
\mathcal{R}\phi_* \mathcal{I}_{\Gamma \subseteq Y} & \longrightarrow & \mathcal{R}\phi_* \mathcal{O}_\Gamma & \longrightarrow & \mathcal{R}\phi_* \mathcal{O}_Y & \xrightarrow{+1} & \\
& \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
& & \mathcal{R}\phi_* \underline{\Omega}_{Y,\Gamma}^0 & \longrightarrow & \mathcal{R}\phi_* \underline{\Omega}_Y^0 & \longrightarrow & \mathcal{R}\phi_* \underline{\Omega}_\Gamma^0 \xrightarrow{+1} .
\end{array}$$

The front face of this diagram provides the one claimed in the statement. \square

Similarly to (2.9) we introduce the DB defect of the pair (X, Σ) :

Definition 3.16. The *DB defect* of the pair (X, Σ) is the mapping cone of the morphism $\mathcal{I}_{\Sigma \subseteq X} \rightarrow \underline{\Omega}_{X,\Sigma}^0$. It is denoted by $\underline{\Omega}_{X,\Sigma}^\times$. Again, one has the exact triangles,

$$(3.16.1) \quad \mathcal{I}_{\Sigma \subseteq X} \longrightarrow \underline{\Omega}_{X,\Sigma}^0 \longrightarrow \underline{\Omega}_{X,\Sigma}^\times \xrightarrow{+1} .$$

and

$$(3.16.2) \quad \underline{\Omega}_{X,\Sigma}^\times \longrightarrow \underline{\Omega}_X^\times \longrightarrow \underline{\Omega}_\Sigma^\times \xrightarrow{+1} .$$

And, again, one has that

$$(3.16.3) \quad h^0(\underline{\Omega}_{X,\Sigma}^\times) \simeq h^0(\underline{\Omega}_{X,\Sigma}^0) / \mathcal{I}_{\Sigma \subseteq X} \quad \text{and} \quad h^i(\underline{\Omega}_{X,\Sigma}^\times) \simeq h^i(\underline{\Omega}_{X,\Sigma}^0) \quad \text{for } i > 0.$$

Lemma 3.17. *Let (X, Σ) be a reduced generalized pair. Then the following are equivalent:*

- (3.17.1) *The pair (X, Σ) is DB.*
- (3.17.2) *The DB defect of (X, Σ) is acyclic, that is, $\underline{\Omega}_{X,\Sigma}^\times \simeq_{qis} 0$.*
- (3.17.3) *The induced natural morphism $\underline{\Omega}_X^\times \rightarrow \underline{\Omega}_\Sigma^\times$ is a quasi-isomorphism.*
- (3.17.4) *The induced natural morphism $h^i(\underline{\Omega}_X^\times) \rightarrow h^i(\underline{\Omega}_\Sigma^\times)$ is an isomorphism for all $i \in \mathbb{Z}$.*
- (3.17.5) *The induced natural morphism $h^i(\underline{\Omega}_X^0) \rightarrow h^i(\underline{\Omega}_\Sigma^0)$ is an isomorphism for all $i \neq 0$ and a surjection with kernel isomorphic to $\mathcal{I}_{\Sigma \subseteq X}$ for $i = 0$.*

REMARK 3.17.1. This statement also applies in the case when $\Sigma = \emptyset$, so it implies (2.11).

Proof. The equivalence of (3.17.1) and (3.17.2) follows from (3.16.1), the equivalence of (3.17.2) and (3.17.3) follows from (3.16.2), the equivalence of (3.17.3) and (3.17.4) follows from the definition of quasi-isomorphism, and the equivalence of (3.17.4) and (3.17.5) follows from the definition of the DB defect $\underline{\Omega}_{X,\Sigma}^\times$ (3.16) and (3.16.3). \square

Cutting by hyperplanes works the same way as in the absolute case:

Proposition 3.18. *Let (X, Σ) be a reduced general pair where X is a quasi-projective variety and $H \subset X$ a general member of a very ample linear system. Then $\underline{\Omega}_{H, H \cap \Sigma}^\times \simeq_{\text{qis}} \underline{\Omega}_{X, \Sigma}^\times \otimes_L \mathcal{O}_H$ and $\underline{\Omega}_{H, H \cap \Sigma}^\times \simeq_{\text{qis}} \underline{\Omega}_{X, \Sigma}^\times \otimes_L \mathcal{O}_H$.*

Proof. This follows directly from (2.6), (3.9.1), and (2.10). \square

We also have the following adjunction type statement.

Proposition 3.19. *Let $X = (Y \cup Z)_{\text{red}}$ be a union of closed reduced subschemes with intersection $W = (Y \cap Z)_{\text{red}}$. Then the DB defects of the pairs (X, Y) and (Z, W) are quasi-isomorphic. I.e.,*

$$\underline{\Omega}_{X, Y}^\times \simeq_{\text{qis}} \underline{\Omega}_{Z, W}^\times.$$

Proof. Consider the following diagram of exact triangles,

$$\begin{array}{ccccc} \underline{\Omega}_{X, Y}^\times & \longrightarrow & \underline{\Omega}_X^\times & \longrightarrow & \underline{\Omega}_Y^\times \xrightarrow{+1} \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ \underline{\Omega}_{Z, W}^\times & \longrightarrow & \underline{\Omega}_Z^\times & \longrightarrow & \underline{\Omega}_W^\times \xrightarrow{+1} \end{array},$$

where β and γ are the natural restriction morphisms and α is the morphism induced by β and γ on the mapping cones. Then by [KK10, 2.1] there exists an exact triangle

$$\mathbb{Q} \longrightarrow \underline{\Omega}_Y^\times \oplus \underline{\Omega}_Z^\times \longrightarrow \underline{\Omega}_W^\times \xrightarrow{+1}.$$

and a map $\sigma : \underline{\Omega}_X^\times \rightarrow \mathbb{Q}$ compatible with the above diagram such that α is an isomorphism if and only if σ is one. On the other hand, σ is indeed an isomorphism by (2.12) and so the statement follows. \square

4. COHOMOLOGY WITH COMPACT SUPPORT

Let X be a complex scheme of finite type and $\iota : \Sigma \hookrightarrow X$ a closed subscheme. Deligne's Hodge theory applied in this situation gives the following theorem:

Theorem 4.1. [Del74] *Let X be a complex scheme of finite type, $\iota : \Sigma \hookrightarrow X$ a closed subscheme and $j : U := X \setminus \Sigma \hookrightarrow X$. Then*

(4.1.1) The natural composition map $j_! \mathbb{C}_U \rightarrow \mathcal{I}_{\Sigma \subseteq X} \rightarrow \underline{\Omega}_{X, \Sigma}^\bullet$ is a quasi-isomorphism, i.e., $\underline{\Omega}_{X, \Sigma}^\bullet$ is a resolution of the sheaf $j_! \mathbb{C}_U$.

(4.1.2) The natural map $H_c^\bullet(U, \mathbb{C}) \rightarrow \mathbb{H}^\bullet(X, \underline{\Omega}_{X, \Sigma}^\bullet)$ is an isomorphism.

(4.1.3) If in addition X is proper, then the spectral sequence,

$$E_2^{p,q} = \mathbb{H}^q(X, \underline{\Omega}_{X, \Sigma}^p) \Rightarrow H_c^{p+q}(U, \mathbb{C})$$

degenerates at E_2 and abuts to the Hodge filtration of Deligne's mixed Hodge structure.

Proof. Consider an embedded hyperresolution of $\Sigma \subseteq X$:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varrho^\bullet} & X \\ \varepsilon^\bullet \downarrow & & \downarrow \varepsilon^\bullet \\ \Sigma & \xrightarrow{\varrho} & X \end{array}$$

Then by (2.2.5) and by definition $\underline{\Omega}_{X, \Sigma}^\bullet \simeq_{\text{qis}} \mathcal{R}\varepsilon_* \Omega_{X, \Sigma}^\bullet$. The statements then follow from [Del74, 8.1, 8.2, 9.3]. See also [GNPP88, IV.4]. \square

Corollary 4.2. *Let X be a proper complex scheme of finite type, $\iota : \Sigma \hookrightarrow X$ a closed subscheme and $j : U := X \setminus \Sigma \hookrightarrow X$. Then the natural map*

$$H^i(X, \mathcal{I}_{\Sigma \subseteq X}) \rightarrow \mathbb{H}^i(X, \underline{\Omega}_{X, \Sigma}^0)$$

is surjective for all $i \in \mathbb{N}$.

Proof. By (4.1.3) the natural composition map

$$H_c^i(U, \mathbb{C}) \rightarrow H^i(X, \mathcal{I}_{\Sigma \subseteq X}) \rightarrow \mathbb{H}^i(X, \underline{\Omega}_{X, \Sigma}^0)$$

is surjective. This clearly implies the statement. \square

5. DB PAIRS IN NATURE

Proposition 5.1. *Let (X, Σ) be a reduced generalized pair. If either X or Σ is DB, then the other one is DB if and only if (X, Σ) is a DB pair.*

Proof. Consider the exact triangle (3.16.2)

$$\underline{\Omega}_{X, \Sigma}^\times \longrightarrow \underline{\Omega}_X^\times \longrightarrow \underline{\Omega}_\Sigma^\times \xrightarrow{+1} .$$

Clearly, if one of the objects in this triangle is acyclic, then it is equivalent that the other two are acyclic. Then the statement follows by (2.11) and (3.17). \square

As one expects it from a good notion of singularity, smooth points are DB. For pairs, being smooth is replaced by being snc.

Corollary 5.2. *Let (X, Δ) be an snc pair. Then it is also a DB pair.*

Proof. This follows directly from (5.1) cf. (2.2.7) [Ste85, 3.2]. It also follows from (5.3). \square

Corollary 5.3. *Let (X, Δ) be a log canonical pair and $\Lambda \subset X$ an effective integral Weil divisor such that $\text{supp } \Lambda \subseteq \text{supp } \lfloor \Delta \rfloor$. Then (X, Λ) is a DB pair.*

Proof. By choice Λ is a union of non-klt centers of the pair (X, Δ) and hence by [KK10, Theorem 1.4] both X and Λ are DB. Then (X, Λ) is a DB pair by (5.1). \square

Theorem 5.4. *Let (X, Σ) be a reduced generalized pair. Assume that the natural morphism $\mathcal{I}_{\Sigma \subseteq X} \rightarrow \underline{\Omega}_{X, \Sigma}^0$ has a left inverse. Then (X, Σ) is a DB pair.*

Proof. We will mimic the proof of [Kov00b, 1.5]. The statement is local so we may assume that X is affine and hence quasi-projective

Lemma 5.5. *Assume that there exists a finite subset $P \subseteq X$ such that $(X \setminus P, \Sigma \setminus P)$ is a DB pair. Then the induced morphism*

$$H_P^i(X, \mathcal{I}_{\Sigma \subseteq X}) \rightarrow \mathbb{H}_P^i(X, \underline{\Omega}_{X, \Sigma}^0)$$

is surjective for all $i \in \mathbb{N}$.

Proof. Let \overline{X} be a projective closure of X and let $\overline{\Sigma}$ be the closure of Σ in \overline{X} . Let $Q = \overline{X} \setminus X$, $Z = P \dot{\cup} Q$, and $U = \overline{X} \setminus Z = X \setminus P$. Consider the exact triangle of functors,

$$(5.5.1) \quad \mathbb{H}_Z^0(\overline{X}, _) \longrightarrow \mathbb{H}^0(\overline{X}, _) \longrightarrow \mathbb{H}^0(U, _) \xrightarrow{+1}$$

and apply it to the morphism $\mathcal{I}_{\overline{\Sigma} \subseteq \overline{X}} \rightarrow \underline{\Omega}_{\overline{X}, \overline{\Sigma}}^0$. One obtains a morphism of two long exact sequences:

$$\begin{array}{ccccccc} \mathbb{H}^{i-1}(U, \mathcal{I}_{\overline{\Sigma} \subseteq \overline{X}}) & \longrightarrow & \mathbb{H}_Z^i(\overline{X}, \mathcal{I}_{\overline{\Sigma} \subseteq \overline{X}}) & \longrightarrow & \mathbb{H}^i(\overline{X}, \mathcal{I}_{\overline{\Sigma} \subseteq \overline{X}}) & \longrightarrow & \mathbb{H}^i(U, \mathcal{I}_{\overline{\Sigma} \subseteq \overline{X}}) \\ \downarrow \alpha_{i-1} & & \downarrow \beta_i & & \downarrow \gamma_i & & \downarrow \alpha_i \\ \mathbb{H}^{i-1}(U, \underline{\Omega}_{\overline{X}, \overline{\Sigma}}^0) & \longrightarrow & \mathbb{H}_Z^i(\overline{X}, \underline{\Omega}_{\overline{X}, \overline{\Sigma}}^0) & \longrightarrow & \mathbb{H}^i(\overline{X}, \underline{\Omega}_{\overline{X}, \overline{\Sigma}}^0) & \longrightarrow & \mathbb{H}^i(U, \underline{\Omega}_{\overline{X}, \overline{\Sigma}}^0). \end{array}$$

By assumption, α_i is an isomorphism for all i . By (4.2), γ_i is surjective for all i . Then by the 5-lemma, β_i is also surjective for all i .

By construction $P \cap Q = \emptyset$ and hence

$$\begin{aligned} H_Z^i(\overline{X}, \mathcal{I}_{\overline{\Sigma} \subseteq \overline{X}}) &\simeq H_P^i(\overline{X}, \mathcal{I}_{\overline{\Sigma} \subseteq \overline{X}}) \oplus H_Q^i(\overline{X}, \mathcal{I}_{\overline{\Sigma} \subseteq \overline{X}}) \\ \mathbb{H}_Z^i(\overline{X}, \underline{\Omega}_{\overline{X}, \overline{\Sigma}}^0) &\simeq \mathbb{H}_P^i(\overline{X}, \underline{\Omega}_{\overline{X}, \overline{\Sigma}}^0) \oplus \mathbb{H}_Q^i(\overline{X}, \underline{\Omega}_{\overline{X}, \overline{\Sigma}}^0) \end{aligned}$$

It follows that the natural map (which is also the restriction of β_i),

$$H_P^i(\overline{X}, \mathcal{I}_{\overline{\Sigma} \subseteq \overline{X}}) \rightarrow \mathbb{H}_P^i(\overline{X}, \underline{\Omega}_{\overline{X}, \overline{\Sigma}}^0)$$

is surjective for all i

Now, by excision on local cohomology one has that

$$H_P^i(\overline{X}, \mathcal{I}_{\overline{\Sigma} \subseteq \overline{X}}) \simeq H_P^i(X, \mathcal{I}_{\Sigma \subseteq X}) \quad \text{and} \quad \mathbb{H}_P^i(\overline{X}, \underline{\Omega}_{\overline{X}, \overline{\Sigma}}^0) \simeq \mathbb{H}_P^i(X, \underline{\Omega}_{X, \Sigma}^0).$$

and so (5.5) follows. \square

It is now relatively straightforward to finish the proof of 5.4:

By taking repeated hyperplane sections and using (3.18) we may assume that there exists a finite subset $P \subseteq X$ such that $(X \setminus P, \Sigma \setminus P)$ is a DB pair. Therefore we may apply (5.5).

By assumption, the natural morphism $\mathcal{I}_{\Sigma \subseteq X} \rightarrow \underline{\Omega}_{X, \Sigma}^0$ has a left inverse. This implies that applying any cohomology operator on this map induces an injective map on cohomology. In particular, this implies that the natural morphism

$$H_P^i(X, \mathcal{I}_{\Sigma \subseteq X}) \rightarrow \mathbb{H}_P^i(X, \underline{\Omega}_{X, \Sigma}^0)$$

is injective for all $i \in \mathbb{N}$. By (5.5) they are also surjective and hence an isomorphism. Thus the DB defect $\underline{\Omega}_{X, \Sigma}^\times$ is such that all of its local cohomology groups are zero:

$$\mathbb{H}_P^i(X, \underline{\Omega}_{X, \Sigma}^\times) = 0 \quad \text{for all } i.$$

On the other hand, by assumption $\underline{\Omega}_{X, \Sigma}^\times$ is supported entirely on P , so $\mathbb{H}^i(X \setminus P, \underline{\Omega}_{X, \Sigma}^\times) = 0$ as well. However, then $\mathbb{H}^i(X, \underline{\Omega}_{X, \Sigma}^\times) = 0$ by the long exact sequence induced by (5.5.1). Now $\dim P \leq 0$ so the spectral sequence that computes hypercohomology from the sheaf cohomology of the cohomology of the complex $\underline{\Omega}_{X, \Sigma}^\times$ degenerates and gives that for any $i \in \mathbb{N}$, $\mathbb{H}^i(X, \underline{\Omega}_{X, \Sigma}^\times) = H^0(X, h^i(\underline{\Omega}_{X, \Sigma}^\times))$, so, since we assumed that X is affine, it follows that $h^i(\underline{\Omega}_{X, \Sigma}^\times) = 0$ for all i . Therefore $\underline{\Omega}_{X, \Sigma}^\times \simeq_{\text{qis}} 0$ and thus the statement is proven. \square

Corollary 5.6. *Let (X, Δ) be a weakly rational pair. Then it is a DB pair.*

Proof. Let $\phi : (Y, \Delta_Y) \rightarrow (X, \Delta)$ be a log resolution. Then by (3.15) one has the commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_X(-\Delta) & \longrightarrow & \underline{\Omega}_{X, \Delta}^0 \\ \gamma^\ell \left(\begin{array}{c} \uparrow \\ \downarrow \\ \gamma \end{array} \right) & & \downarrow \alpha \\ \mathcal{R}\phi_* \mathcal{O}_Y(-\Delta_Y) & \xrightarrow[\simeq_{\text{qis}}]{\delta} & \mathcal{R}\phi_* \underline{\Omega}_{Y, \Delta_Y}^0 \end{array}$$

Recall that as (Y, Δ_Y) is an snc pair, it is also DB by (5.2) and hence δ is a quasi-isomorphism. By assumption (Y, Δ_Y) is a weakly rational pair so γ admits a left inverse γ^ℓ . Then $\gamma^\ell \circ \delta^{-1} \circ \alpha$ is a left inverse to $\mathcal{O}_X(-\Delta) \rightarrow \underline{\Omega}_{X, \Delta}^0$, so the statement follows from (5.4). \square

Corollary 5.7. *A rational pair is a DB pair.*

Proof. As a rational pair is also a weakly rational pair, this is straightforward from (5.6). \square

Corollary 5.8. *Let (X, Δ) be a dlt pair and $\Lambda \subset X$ an effective integral Weil divisor such that $\text{supp } \Lambda \subseteq \text{supp } [\Delta]$. Then (X, Λ) is a DB pair.*

Proof #1. A dlt pair is also an lc pair, so this follows from (5.3). \square

Proof #2. If (X, Δ) is a dlt pair, then (X, Λ) is a rational pair by [KK09, 111], so this also follows from (5.7). \square

6. VANISHING THEOREMS

The following is the main vanishing result of this paper. Note that a weaker version of it appeared in [GKKP10, 13.4].

Theorem 6.1. *Let (X, Σ) be a DB pair and $\pi : \tilde{X} \rightarrow X$ a proper birational morphism with $E := \text{Exc}(\pi)$. Let $\tilde{\Sigma} = E \cup \pi^{-1}\Sigma$ and $\Upsilon := \overline{\pi(E)} \setminus \Sigma$, both considered with their induced reduced subscheme structure. Further let $s \in \mathbb{N}$, $s > 0$ such that $h^i(\underline{\Omega}_{\Upsilon, \Upsilon \cap \Sigma}^{\circ}) = 0$ for $i \geq s$. Then*

$$\mathcal{R}^i \pi_* \underline{\Omega}_{\tilde{X}, \tilde{\Sigma}}^0 = 0 \quad \text{for all } i \geq s.$$

Proof. Let $\Gamma = \Sigma \cup \Upsilon$ and consider the exact triangle (2.2.9),

$$(6.1.1) \quad \underline{\Omega}_X^0 \longrightarrow \underline{\Omega}_\Gamma^0 \oplus \mathcal{R}^i \pi_* \underline{\Omega}_{\tilde{X}}^0 \longrightarrow \mathcal{R}^i \pi_* \underline{\Omega}_{\tilde{\Sigma}}^0 \xrightarrow{+1},$$

which induces the long exact sequence of sheaves:

$$\longrightarrow h^i(\underline{\Omega}_X^0) \xrightarrow{(\alpha^i, \sigma^i)} h^i(\underline{\Omega}_\Gamma^0) \oplus \mathcal{R}^i \pi_* \underline{\Omega}_{\tilde{X}}^0 \longrightarrow \mathcal{R}^i \pi_* \underline{\Omega}_{\tilde{\Sigma}}^0 \longrightarrow h^{i+1}(\underline{\Omega}_X^0) \longrightarrow$$

By (3.17) the natural morphism $\gamma^i : h^i(\underline{\Omega}_X^0) \rightarrow h^i(\underline{\Omega}_\Sigma^0)$ is an isomorphism for $i > 0$. By (3.19) and the assumption we obtain that $h^i(\underline{\Omega}_{\Gamma, \Sigma}^{\circ}) = 0$ for $i \geq s > 0$ and hence the natural morphism $\beta^i : h^i(\underline{\Omega}_\Gamma^0) \rightarrow h^i(\underline{\Omega}_\Sigma^0)$ is an isomorphism for $i \geq s$. Using the fact that $\gamma^i = \beta^i \circ \alpha^i$ we obtain that the morphism $\alpha^i : h^i(\underline{\Omega}_X^0) \rightarrow h^i(\underline{\Omega}_\Gamma^0)$ is an isomorphism for $i \geq s > 0$ and hence the natural restriction map

$$\varrho^i : \mathcal{R}^i \pi_* \underline{\Omega}_{\tilde{X}}^0 \rightarrow \mathcal{R}^i \pi_* \underline{\Omega}_{\tilde{\Sigma}}^0$$

is an isomorphism for $i \geq s$. This in turn implies that $\mathcal{R}^i \pi_* \underline{\Omega}_{\tilde{X}, \tilde{\Sigma}}^0 = 0$ for $i \geq s$ as desired. \square

As a corollary, a slight generalization of [GKKP10, 13.4] follows.

Corollary 6.2. *Let (X, Σ) be a DB pair and $\pi : \tilde{X} \rightarrow X$ a log resolution of (X, Σ) with $E := \text{Exc}(\pi)$. Let $\tilde{\Sigma} = E \cup \pi^{-1}\Sigma$ and $\Upsilon := \overline{\pi(E)} \setminus \tilde{\Sigma}$, both considered with their induced reduced subscheme structure. Then*

$$\mathcal{R}^i \pi_* \mathcal{I}_{\tilde{\Sigma} \subseteq \tilde{X}} = 0 \quad \text{for all } i \geq \max(\dim \Upsilon, 1).$$

In particular, if X is normal of dimension $n \geq 2$, then $\mathcal{R}^{n-1} \pi_ \mathcal{I}_{\tilde{\Sigma} \subseteq \tilde{X}} = 0$.*

Proof. Let $s = \max(\dim \Upsilon, 1)$. Then $h^i(\underline{\Omega}_{\Upsilon, \Upsilon \cap \Sigma}^0) = 0$ for $i \geq s$ by (3.12). As the pair $(\tilde{X}, \tilde{\Sigma})$ is snc, it is also DB and hence $\underline{\Omega}_{\tilde{X}, \tilde{\Sigma}}^0 \simeq_{\text{qis}} \mathcal{I}_{\tilde{\Sigma} \subseteq \tilde{X}}$. Therefore the statement follows from (6.1). \square

We have a stronger result for log canonical pairs.

Corollary 6.3. *Let (X, Δ) be a log canonical pair, $\pi : \tilde{X} \rightarrow X$ a log resolution of (X, Δ) . Let $\tilde{\Delta} = (\pi_*^{-1} \lfloor \Delta \rfloor + \text{Exc}_{\text{nkl}}(\pi))_{\text{red}}$. Then*

$$\mathcal{R}^i \pi_* \mathcal{O}_{\tilde{X}}(-\tilde{\Delta}) = 0 \quad \text{for } i > 0.$$

Proof. Consider the minimal dlt model $\mu : (X^{\text{m}}, \Delta^{\text{m}}) \rightarrow (X, \Delta)$ [KK10, 3.1]. Let $\Sigma := \lfloor \Delta \rfloor \cup \mu(\text{Exc}(\mu))$ considered with the induced reduced subscheme structure. From the definition of a minimal dlt model it follows that Σ is a union of non-klt centers of (X, Δ) . Then by [KK10, Theorem 1.4] both X and Σ are DB, and hence (X, Σ) is a DB pair by (5.1).

Since $(X^{\text{m}}, \Delta^{\text{m}})$ is dlt, $(X^{\text{m}}, \lfloor \Delta^{\text{m}} \rfloor)$ is a DB pair by (5.3). Therefore,

$$\underline{\Omega}_{X^{\text{m}}, \lfloor \Delta^{\text{m}} \rfloor}^0 \simeq_{\text{qis}} \mathcal{O}_{X^{\text{m}}}(-\lfloor \Delta^{\text{m}} \rfloor).$$

By the definition of a minimal dlt model $\lfloor \Delta^{\text{m}} \rfloor = (\pi^{-1}\Sigma)_{\text{red}} \supseteq \text{Exc}(\mu)$ and then it follows from (6.1) that $\mathcal{R}^i \mu_* \mathcal{O}_{X^{\text{m}}}(-\lfloor \Delta^{\text{m}} \rfloor) = 0$ for $i > 0$ and hence

$$(6.3.1) \quad \mathcal{R} \mu_* \mathcal{O}_{X^{\text{m}}}(-\lfloor \Delta^{\text{m}} \rfloor) \simeq_{\text{qis}} \mathcal{O}_X(-\lfloor \Delta \rfloor).$$

Next let $\sigma : \hat{X} \rightarrow X$ be a log resolution of (X, Δ) that factors through both π and μ . Then one has the following commutative diagram:

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\tau} & X^{\text{m}} \\ \lambda \downarrow & \searrow \sigma & \downarrow \mu \\ \tilde{X} & \xrightarrow{\pi} & X. \end{array}$$

Let $\hat{\Delta} \subseteq \hat{X}$ denote the strict transform of $\tilde{\Delta}$ on \hat{X} . It follows from the definition of a minimal dlt model that $\hat{\Delta}$ is also the strict transform of $\lfloor \Delta^{\text{m}} \rfloor$ from X^{m} .

As both (X^m, Δ^m) and $(\tilde{X}, \tilde{\Delta})$ are dlt, $(X^m, \lfloor \Delta^m \rfloor)$ and $(\tilde{X}, \tilde{\Delta})$ are also rational by [KK09, 111], so we have that

$$\begin{aligned} \mathcal{R}\tau_* \mathcal{O}_{\tilde{X}}(-\hat{\Delta}) &\simeq_{\text{qis}} \mathcal{O}_{X^m}(-\lfloor \Delta^m \rfloor), \text{ and} \\ \mathcal{R}\lambda_* \mathcal{O}_{\tilde{X}}(-\hat{\Delta}) &\simeq_{\text{qis}} \mathcal{O}_{\tilde{X}}(-\tilde{\Delta}) \end{aligned}$$

Therefore, by (6.3.1),

$$\begin{aligned} \mathcal{R}\pi_* \mathcal{O}_{\tilde{X}}(-\tilde{\Delta}) &\simeq_{\text{qis}} \mathcal{R}\pi_* \mathcal{R}\lambda_* \mathcal{O}_{\tilde{X}}(-\hat{\Delta}) \simeq_{\text{qis}} \mathcal{R}\sigma_* \mathcal{O}_{\tilde{X}}(-\hat{\Delta}) \simeq_{\text{qis}} \\ &\mathcal{R}\mu_* \mathcal{R}\tau_* \mathcal{O}_{\tilde{X}}(-\hat{\Delta}) \simeq_{\text{qis}} \mathcal{R}\mu_* \mathcal{O}_{X^m}(-\lfloor \Delta^m \rfloor) \simeq_{\text{qis}} \mathcal{O}_X(-\Delta). \end{aligned}$$

□

Finally, observe that (6.3) implies that log canonical singularities are not too far from being rational:

Corollary 6.4. *Let X be a variety with log canonical singularities and $\pi : \tilde{X} \rightarrow X$ a resolution of X with $E_{\text{lc}} := \text{ExC}_{\text{nkl}}(\pi)$. Then*

$$\mathcal{R}^i \pi_* \mathcal{O}_{\tilde{X}}(-E_{\text{lc}}) = 0 \quad \text{for } i > 0.$$

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UNIVERSITY OF WASHINGTON, DEPARTMENT OF MATHEMATICS, BOX 354350,
SEATTLE, WA 98195-4350, USA

E-mail address: skovacs@uw.edu

URL: <http://www.math.washington.edu/~kovacs>