EQUIVARIANT MAPS BETWEEN CALOGERO-MOSER SPACES

GEORGE WILSON

ABSTRACT. We add a last refinement to the results of [BW1] and [BW2] relating ideal classes of the Weyl algebra to the Calogero-Moser varieties: we show that the bijection constructed in those papers is *uniquely determined* by its equivariance with respect to the automorphism group of the Weyl algebra.

1. Introduction and statement of results

Let A be the Weyl algebra $\mathbb{C}\langle x,y\rangle/(xy-yx-1)$, and let $\mathcal{R}$ be the space of noncyclic right ideal classes of A (that is, isomorphism classes of noncyclic finitely generated rank 1 torsion-free right A-modules). Let $\mathcal{C}$ be the disjoint union of the *Calogero-Moser spaces* $\mathcal{C}_n$ $(n \geq 1)$: we recall that $\mathcal{C}_n$ is the space of all simultaneous conjugacy classes of pairs of $n\times n$ matrices (X,Y) such that $[X,Y]+I$ has rank 1. It is a smooth irreducible affine variety of dimension $2n$ (see [W]). For simplicity, in what follows we shall use the same notation (X,Y) for a pair of matrices and for the corresponding point of $\mathcal{C}_n$. Let G be the group of $\mathbb{C}$-automorphisms of A, and let Γ and Γ' be the isotropy groups of the generators y and x of A. Thus Γ consists of all automorphisms of the form

$$\Phi_p(x) = x - p(y)\,, \quad \Phi_p(y) = y$$

where p is a polynomial; and similarly Γ' consists of all automorphisms of the form

$$\Psi_q(x) = x\,, \quad \Psi_q(y) = y - q(x)$$

where q is a polynomial. According to Dixmier (see [D]), G is generated by the subgroups Γ and Γ'. There is an obvious action of G on $\mathcal{R}$; we let G act on $\mathcal{C}$ by the formulae

$$\Phi_p(X,Y) = (X+p(Y),\,Y)\,, \quad \Psi_q(X,Y) = (X,\,Y+q(X))\,. \tag{1.1}$$

According to [BW1] this G-action is *transitive* on each space $\mathcal{C}_n$. The main result of [BW1] was the following.

Theorem 1.1. *There is a bijection between the spaces $\mathcal{R}$ and $\mathcal{C}$ which is equivariant with respect to the above actions of G.*

This bijection constructed in [BW1] was obtained in a quite different way in [BW2]. The proof in [BW2] that the two constructions agree used the fact that equivariance was known in both cases; thus to prove that the bijections coincide, it was enough to check one point in each G-orbit, that is, in each space $\mathcal{C}_n$. The result to be proved in the present note is that even this (not difficult) check was unnecessary.

Theorem 1.2. *There is only one G-equivariant bijection between the spaces $\mathcal{R}$ and $\mathcal{C}$.*

Clearly, it is equivalent to show that there is no nontrivial G-equivariant bijection from $\mathcal{C}$ to itself. We shall show a little more, namely, that (apart from the identity) there is no G-equivariant map (for short: G-map) at all from $\mathcal{C}$ to itself. Since a G-map must take each orbit onto another orbit, that amounts to the following assertion.

Theorem 1.3. (i) *For any $n \geq 1$, let $f : \mathcal{C}_n \to \mathcal{C}_n$ be a G-map. Then f is the identity.*
(ii) *For $n \neq m$ there is no G-map from $\mathcal{C}_n$ to $\mathcal{C}_m$.*

Since $\mathcal{C}_n$ and the action of G on it are defined by simple formulae involving matrices, the proof of Theorem 1.3 is just an exercise in linear algebra. Quite possibly there is a simpler solution to the exercise than the one given below.

The first part of Theorem 1.3 is equivalent to the statement that the isotropy group of any point of $\mathcal{C}$ (or $\mathcal{R}$) coincides with its normalizer in G (see section 6 below); in particular, these isotropy groups are not normal in G, confirming a suspicion of Stafford (see [St], p. 636). Stafford's conjecture seems to have been the motivation for Kouakou's work [K], which contains a result equivalent to ours. The proof in [K] looks quite different from the present one, because Kouakou does not use the spaces $\mathcal{C}_n$, but rather the alternative description of $\mathcal{R}$ (due to Cannings and Holland, see [CH]) as the adelic Grassmannian of [W]. I have not entirely succeeded in following the details of [K]; in any case, it seems worthwhile to make available the independent verification of the result offered here.

Remark. We have excluded from $\mathcal{R}$ the cyclic ideal class, corresponding to the Calogero-Moser space $\mathcal{C}_0$ (which is a point). The reason is very trivial: since there is always a map from any space to a point, part (ii) of Theorem 1.3 would be false if we included $\mathcal{C}_0$. However, Theorem 1.2 would still be true.

2. Proof of Theorem 1.3 in the case $n < m$

If we accept (cf. [BW1], section 11) that the $\mathcal{C}_n$ are homogeneous spaces for the (infinite-dimensional) *algebraic* group G, then Theorem 1.3 becomes obvious in the case $n < m$. Indeed, any G-map from $\mathcal{C}_n$ to $\mathcal{C}_m$ would have to be a surjective map of *algebraic varieties*, which is clearly impossible if $n < m$, because then $\mathcal{C}_m$ has greater dimension ($2m$) than $\mathcal{C}_n$. For readers who are not convinced by this argument, we offer a more elementary one, based on the following lemma.

Lemma 2.1. *Let $f : \mathcal{C}_n \to \mathcal{C}_m$ be a G-map. Suppose that $f(X,Y) = (P,Q)$, and that P is diagonalizable. Then every eigenvalue of P is an eigenvalue of X.*

Proof. Let χ be the minimum polynomial of X: then in $\mathcal{C}_m$ we have

$$(P,Q) = f(X,Y) = f(X, Y + \chi(X)) = (P, Q + \chi(P))$$

(where the last step used the fact that f has to commute with the action of $\Psi_\chi \in G$). That means that there is an invertible matrix A such that

$$APA^{-1} = P \text{ and } AQA^{-1} = Q + \chi(P) .$$

We may assume that $P = \operatorname{diag}(p_1, \ldots, p_m)$ is diagonal. Then since the p_i are distinct (see [W], Proposition 1.10), A is diagonal too, so taking the diagonal entries in the last equation gives $q_{ii} = q_{ii} + \chi(p_i)$, whence $\chi(p_i) = 0$ for all i. Thus $\chi(P) = 0$, so the minimum polynomial of P divides χ. The lemma follows. □

Corollary 2.2. *If $n < m$ there is no G-map $f : \mathcal{C}_n \to \mathcal{C}_m$.*

Proof. Choose $(P, Q) \in \mathcal{C}_m$ with P diagonalizable. Since $\mathcal{C}_m$ is just one G-orbit, f is surjective, so we can choose $(X, Y) \in \mathcal{C}_n$ with $f(X, Y) = (P, Q)$. But then Lemma 2.1 says that X is an $n \times n$ matrix with more than n distinct eigenvalues, which is impossible. □

3. The base-point

A useful subgroup of G is the group R of *scaling transformations*, defined by

$$R_\lambda(x) = \lambda x\,,\ R_\lambda(y) = \lambda^{-1} y \quad (\lambda \in \mathbb{C}^\times)\,.$$

It acts on $\mathcal{C}_n$ in a similar way:

$$R_\lambda(X, Y) = (\lambda^{-1} X,\, \lambda Y)\,. \tag{3.1}$$

Lemma 3.1. *Suppose that the conjugacy class $(X, Y) \in \mathcal{C}_n$ is fixed by the group R . Then X and Y are both nilpotent.*

Proof. Let μ be an eigenvalue of (say) Y . Then for any $\lambda \in \mathbb{C}^\times$, $\lambda\mu$ is an eigenvalue of λY , which is (by hypothesis) conjugate to Y . Thus $\lambda\mu$ is an eigenvalue of Y for every $\lambda \in \mathbb{C}^\times$, which is impossible unless $\mu = 0$. Hence all eigenvalues of Y must be 0 , that is, Y must be nilpotent. The same argument applies to X . □

The converse to Lemma 3.1 is also true, but we shall use that fact only for the pair (X_0, Y_0) given by

$$X_0 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & n-1 & 0 \end{pmatrix}, \quad Y_0 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{3.2}$$

We shall regard (X_0, Y_0) as the *base-point* in $\mathcal{C}_n$. In the rather trivial case $n = 1$, we have $\mathcal{C}_1 = \mathbb{C}^2$, and we interpret (X_0, Y_0) as $(0, 0)$.

Lemma 3.2. *The (conjugacy class of) the pair $(X_0, Y_0) \in \mathcal{C}_n$ is fixed by the group R .*

Proof. For $\lambda \in \mathbb{C}^\times$, let $d(\lambda)$ be the diagonal matrix

$$d(\lambda) := \mathrm{diag}(\lambda, \lambda^2, \dots, \lambda^n)\,.$$

Then $d(\lambda)^{-1} X d(\lambda) = \lambda^{-1} X$ and $d(\lambda)^{-1} Y d(\lambda) = \lambda Y$. □

Corollary 3.3. *Let $f : \mathcal{C}_n \to \mathcal{C}_m$ be a G-map, and let $f(X_0, Y_0) = (P, Q)$. Then P and Q are nilpotent.*

Proof. This follows at once from Lemmas 3.1 and 3.2, since a G-map must respect the fixed point set of any subgroup of G . □

4. Proof of Theorem 1.3 in the case $n > m$

The remaining parts of the proof use the following trivial fact.

Lemma 4.1. *Let* $(X, Y) \in \mathcal{C}_n$ *, let* p *be any polynomial, and let* χ *be divisible by the minimum polynomial of* $X + p(Y)$ *. Then the automorphism* $\Phi_{-p}\Psi_\chi\Phi_p$ *fixes* (X, Y) *.*

Proof. Since $\chi(X + p(Y)) = 0$ we have

$$\begin{aligned}\Phi_{-p}\Psi_\chi\Phi_p(X,\,Y) &= \Phi_{-p}\Psi_\chi(X + p(Y),\,Y)\\ &= \Phi_{-p}(X + p(Y),\,Y)\\ &= (X,\,Y)\ ,\end{aligned}$$

as claimed. □

Proposition 4.2. *If* $n > m > 0$ *there is no* G*-map* $f : \mathcal{C}_n \to \mathcal{C}_m$ *.*

Proof. We apply Lemma 4.1 to the base-point $(X_0, Y_0) \in \mathcal{C}_n$, with $p(t) = t^{n-1}$. The minimum (= characteristic) polynomial of $X_0 + Y_0^{n-1}$ is

$$\chi(t) := \det(tI - X_0 - Y_0^{n-1}) = t^n - (n-1)!\ . \tag{4.1}$$

Now suppose that $f : \mathcal{C}_n \to \mathcal{C}_m$ is a G-map, and let $f(X_0, Y_0) = (P, Q)$: according to Corollary 3.3, P and Q are nilpotent. They are of size less than n , so we have $P^{n-1} = Q^{n-1} = 0$. Thus

$$\begin{aligned}\Phi_{-p}\Psi_\chi\Phi_p(P,\,Q) &= \Phi_{-p}\Psi_\chi(P + Q^{n-1},\,Q)\\ &= \Phi_{-p}\Psi_\chi(P,\,Q)\\ &= \Phi_{-p}(P,\,Q + P^n - (n-1)!I)\\ &= \Phi_{-p}(P,\,Q - (n-1)!I)\\ &= (\text{something},\,Q - (n-1)!I)\ .\end{aligned}$$

Now, $Q - (n-1)!I$ is not conjugate to Q (because their eigenvalues are different), hence $\Phi_{-p}\Psi_\chi\Phi_p$ does not fix (P, Q) . So by Lemma 4.1, the isotropy group of (X_0, Y_0) is not contained in the isotropy group of $f(X_0, Y_0)$. This contradiction shows that f does not exist. □

5. Proof of Theorem 1.3 in the case $n = m$

It remains to show that there is no nontrivial G-map from $\mathcal{C}_n$ to itself. Note that because $\mathcal{C}_n$ is a single orbit, any such map must be bijective, and must map each point of $\mathcal{C}_n$ to a point with *the same* isotropy group. In the case $n = 1$ the result follows (for example) from Lemma 2.1, so from now on we shall assume that $n \geq 2$. Let $f : \mathcal{C}_n \to \mathcal{C}_n$ be a G-map, and let $f(X_0,\,Y_0) = (P, Q)$. Again, Corollary 3.3 says that P and Q are nilpotent. We aim to show that (P, Q) can only be $(X_0,\,Y_0)$, whence f is the identity. We remark first that if $Q^{n-1} = 0$, then the calculation in the proof of Proposition 4.2 still gives a contradiction; thus the Jordan form of Q consists of just one block, so we may assume that $Q = Y_0$. Now, it is not hard to classify all the points $(X,\,Y_0) \in \mathcal{C}_n$ with X nilpotent (see [W], p.26 for the elementary argument): there are exactly n of them, and they all have the form $(X(\boldsymbol{a}), Y_0)$, where $\boldsymbol{a} := (a_1, \ldots, a_{n-1})$ and $X(\boldsymbol{a})$ denotes the subdiagonal matrix

$$X(\boldsymbol{a}) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ a_1 & 0 & 0 & \ldots & 0 \\ 0 & a_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & a_{n-1} & 0 \end{pmatrix} . \tag{5.1}$$

The possible vectors $\boldsymbol{a}$ that give points of $\mathcal{C}_n$ are

$$\boldsymbol{a} = (1, 2, \ldots, r-1; -(n-r), \ldots, -2, -1) \quad \text{for} \quad 1 \le r \le n \tag{5.2}$$

(so $r = n$ gives X_0). Thus so far we have shown that $f(X_0, Y_0)$ must be one of these points $(X(\boldsymbol{a}), Y_0)$. To finish the argument, we need the following easy calculations of characteristic polynomials (the first generalizes (4.1)):

$$\det(tI - X(\boldsymbol{a}) - Y_0^{n-1}) \;=\; t^n - \prod_1^{n-1} a_i \;; \tag{5.3}$$

$$\det(tI - X(\boldsymbol{a}) - Y_0^{n-2}) \;=\; t^n - (\prod_1^{n-2} a_i + \prod_2^{n-1} a_i)\, t \;, \tag{5.4}$$

where the last formula holds only for $n \ge 3$. If $\boldsymbol{a}$ is one of the vectors (5.2) with $1 < r < n$, then the right hand side of (5.4) is just t^n; that is, $X(\boldsymbol{a}) + Y_0^{n-2}$ is nilpotent. In fact it is easy to check that the pair $(X(\boldsymbol{a}) + Y_0^{n-2}, Y_0)$ is conjugate to $(X(\boldsymbol{a}), Y_0)$; that is, the map $(X, Y) \mapsto (X + Y^{n-2}, Y)$ fixes $(X(\boldsymbol{a}), Y_0)$. It does not fix (X_0, Y_0), so $f(X_0, Y_0)$ cannot be any of these points $(X(\boldsymbol{a}), Y_0)$. It remains only to see that f cannot map (X_0, Y_0) to the pair corresponding to $r = 1$ in (5.2): let us call it (X_1, Y_0).

If n is *even* we use (5.3): the characteristic polynomial of $X_0 + Y_0^{n-1}$ is $\chi(t) := t^n - (n-1)!$ while the characteristic polynomial of $X_1 + Y_0^{n-1}$ is $t^n + (n-1)!$, so that $\chi(X_1 + Y_0^{n-1}) = -2(n-1)!I$. We now apply Lemma 4.1 with $p(t) = t^{n-1}$. According to that lemma, the map $\Phi_{-p}\Psi_\chi\Phi_p$ fixes (X_0, Y_0); on the other hand

$$\begin{aligned} \Phi_{-p}\Psi_\chi\Phi_p(X_1, Y_0) &= \Phi_{-p}\Psi_\chi(X_1 + Y_0^{n-1}, Y_0) \\ &= \Phi_{-p}(X_1 + Y_0^{n-1}, Y_0 - 2(n-1)!I) \\ &= (\text{something}, Y_0 - 2(n-1)!I) \;. \end{aligned}$$

Since $Y_0 - 2(n-1)!I$ is not conjugate to Y_0, this shows that $\Phi_{-p}\Psi_\chi\Phi_p$ does not fix (X_1, Y_0). Thus in this case $f(X_0, Y_0)$ cannot be equal to (X_1, Y_0)

Finally, if n is *odd*, we have a similar calculation using (5.4). Setting $\alpha := (n-1)! + (n-2)!$, the characteristic polynomial of $X_0 + Y_0^{n-2}$ is $\chi(t) := t^n - \alpha t$ while the characteristic polynomial of $X_1 + Y_0^{n-2}$ is $t^n + \alpha t$, so that $\chi(X_1 + Y_0^{n-2}) = -2\alpha(X_1 + Y_0^{n-2})$. We now apply Lemma 4.1 with $p(t) = t^{n-2}$. The map $\Phi_{-p}\Psi_\chi\Phi_p$ fixes (X_0, Y_0); on the other hand

$$\begin{aligned} \Phi_{-p}\Psi_\chi\Phi_p(X_1, Y_0) &= \Phi_{-p}\Psi_\chi(X_1 + Y_0^{n-2}, Y_0) \\ &= \Phi_{-p}(X_1 + Y_0^{n-2}, Y_0 - 2\alpha(X_1 + Y_0^{n-2})) \\ &= (\text{something}, Y_0 - 2\alpha(X_1 + Y_0^{n-2})) \;. \end{aligned}$$

The matrix $Y_0 - 2\alpha(X_1 + Y_0^{n-2})$ is not nilpotent, for example because its square does not have trace zero. Hence $\Phi_{-p}\Psi_\chi\Phi_p$ does not fix (X_1, Y_0), and the proof is finished.

6. Other formulations of Theorem 1.3

The remarks in this section are at the level of "groups acting on sets": that is, we may as well suppose that $\mathcal{R}$ denotes any set acted on by a group G. We are interested in the condition

(6.1) there is no nontrivial G-map $f : \mathcal{R} \to \mathcal{R}$

("nontrivial" means "not the identity map"). As we observed above, that is equivalent to the two conditions

(6.2a) each G-orbit in $\mathcal{R}$ satisfies (6.1);

(6.2b) if O_1 and O_2 are distinct orbits, there is no G-map from O_1 to O_2.

Let us reformulate these conditions in terms of the isotropy groups G_M of the points $M \in \mathcal{R}$. If H and K are subgroups of G, then any G-map from G/H to G/K to must have the form $\varphi(gH) = g(xK)$ for some $x \in G$. This is well-defined if and only if we have

$$x^{-1}Hx \subseteq K \ .$$

In the case $H = K$, that says that $x \in N_G(H)$, where N_G denotes the normalizer in G: it follows that the G-maps from G/H to itself correspond 1–1 to the points of $N_G(H)/H$. Thus the conditions (6.2) are equivalent to

(6.3a) for any $M \in \mathcal{R}$, we have $G_M = N_G(G_M)$;

(6.3b) if M and N are on different orbits, no conjugate of G_M is in G_N .

Finally, we note that the conditions (6.3) are equivalent to the single assertion

(6.4) if $G_M \subseteq G_N$, then $M = N$.

Indeed, suppose (6.4) holds, and let $x \in N_G(G_M)$, that is, $xG_Mx^{-1} \subseteq G_M$, or $G_{xM} \subseteq G_M$. By (6.4), we then have $xM = M$, that is, $x \in G_M$. Thus (6.4) $\Rightarrow$ (6.3a). Now, if (6.3b) is false, we have $xG_Mx^{-1} \subseteq G_N$, that is, $G_{xM} \subseteq G_N$, for some $x \in G$ and some M, N on different orbits. But since they are on different orbits, $xM \neq N$, so (6.4) is false. Thus (6.4) $\Rightarrow$ (6.3b).

Conversely, suppose (6.3) holds, and let M, N be such that $G_M \subseteq G_N$. By (6.3b), M and N are on the same orbit, so $M = xN$ for some $x \in G$; hence $G_M = xG_Nx^{-1} \subseteq G_N$. Thus $x \in N_G(G_N)$, so by (6.3a), $x \in G_N$: hence $M = N$, as desired.

It is in the form (6.4) that our result is stated in [K].

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Mathematical Institute, 24–29 St Giles, Oxford OX1 3LB, UK
E-mail address: wilsong@maths.ox.ac.uk