

## OPTIMALLY SPARSE FRAMES

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*Dedicated to the memory of Nigel J. Kalton, who was a great person, friend, and mathematician.*

ABSTRACT. Frames have established themselves as a means to derive redundant, yet stable decompositions of a signal for analysis or transmission, while also promoting sparse expansions. However, when the signal dimension is large, the computation of the frame measurements of a signal typically requires a large number of additions and multiplications, and this makes a frame decomposition intractable in applications with limited computing budget. To address this problem, in this paper, we introduce sparsity of a frame as a new paradigm. In our terminology, a *sparse frame* is a frame whose elements have a sparse representation in an orthonormal basis, thereby enabling low-complexity frame decompositions. To introduce a precise meaning of *optimality*, we take the sum of the numbers of vectors needed of this orthonormal basis when expanding each frame vector as sparsity measure. We then analyze the recently introduced algorithm *Spectral Tetris* for construction of unit norm tight frames and prove that the tight frames generated by this algorithm are in fact optimally sparse with respect to the standard unit vector basis. Finally, we show that even the generalization of Spectral Tetris for the construction of unit norm frames associated with a given frame operator produces optimally sparse frames.

## 1. INTRODUCTION

Frames are nowadays a standard methodology in applied mathematics, computer science, and engineering when redundant, yet stable expansions are required. Examples include sampling theory [19], data quantization [5, 3], quantum measurements [20], coding [2, 26], image processing [8, 25], wireless communication [21, 22, 27], time-frequency analysis [17, 28], speech recognition [1], and bioimaging [15]; see also [23, 24] for a beautiful survey and further references. The typical application exploits the decomposition of a signal  $x \in \mathbb{R}^n$  into its frame components, which requires computation of the frame measurements, i.e., the inner products between the signal  $x$  and the frame vectors  $(\varphi_i)_{i=1}^N$ , say. However, if the dimension  $n$  of the ambient space is large and the frame vectors have ‘many’ non-zero entries, the computational complexity of the computation of the frame measurements might be high; in fact, for applications with constraints on the available computing power and bandwidth

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for data processing, computing the frame measurements and hence the frame decomposition might be intractable.

In this paper we tackle this problem by constructing frames which have very few non-zero entries, thereby reducing the number of required additions and multiplications when computing frame measurements significantly. This viewpoint can be also slightly generalized by assuming that there exists a unitary transformation mapping the frame into one having this ‘sparsity property’. Sparsity of fusion frames, which were introduced in [13] as a mathematical framework for distributed processing thereby going beyond frame theory, was already defined in [7] as a concept. However, the paradigm we aim for in this paper differs from the one introduced in [7] for fusion frames when restricting to the case of frames, since we here aim for an overall sparsity of the frame.

Frame constructions have a long history; browsing through the literature, however, it becomes evident that all constructions for unit norm tight frames – those frames most advantageous for applications – only produce such frames for very special cases such as harmonic frames, see also [10, 14]. Very recently, a real breakthrough for the construction of unit norm tight frames was achieved through the introduction of the so-called *Spectral Tetris* algorithm in [12]. For most combinations of the number of frame vectors and the dimension of the ambient space, this procedure indeed generates a unit norm tight frame. A few months ago, an extension of Spectral Tetris to construct unit norm frames with prescribed frame operator was introduced in [7] to allow additional flexibility in the design process.

Surprisingly, in this paper we show that all unit norm frames which this extended Spectral Tetris algorithm generates are *optimally sparse* in the sense of the total number of non-zero entries in the frame vectors. We also explicitly determine the exact minimum value of the non-zero entries. This result then immediately provides an explicit construction for optimally sparse unit norm frames with a prescribed frame operator. Along the way, we introduce *block decompositions* as a novel structural property of unit norm frames, which we anticipate to be useful also in other settings.

**1.1. Main Contribution.** Our main contribution is hence two-fold: Firstly, we introduce sparsity of a frame as a novel paradigm in frame theory. More precisely, we introduce the notion of a sparse frame as well as a sparsity measure for such frames, thereby allowing for optimality results. Secondly, we analyze an extended version of Spectral Tetris and prove that this algorithm indeed constructs optimally sparse frames. Thus, Spectral Tetris can serve as an algorithm for computing frames with this desirable property, and our results show that it is not possible to derive sparser frames through a different procedure.

**1.2. Impact on Applications.** This paper proclaims the consideration of the computational complexity of a frame in the sense of efficient computation of the frame measurements as a new direction in frame theory. The particular result derived in this paper provides for the first time an algorithmic construction of such computationally efficient frames. Since the method is applicable for almost all combinations of the number of frame vectors and the dimension of the ambient space, we expect this construction to be beneficial for all those applications which require frame measurements if there is some flexibility in choosing the utilized frame. Here we, in particular, think of sensing applications, where the sensors can be designed, or coding strategies, which allow the freedom to choose the codes appropriately.

Even more, applications previously being prevented from using frame decompositions due to constraints such as computing power and bandwidth, or even limited space to store the synthesis matrix, might now be able to implement resilience to robustness by exploiting sparse frames constructed by the Spectral Tetris algorithm.

**1.3. Outline.** This paper is organized as follows. In Section 2, we first fix the terminology we require from frame theory and then review the extended version of the Spectral Tetris algorithm. A novel sparsity measure for a frame will then be introduced in Section 3 together with a notion of optimality. In Section 4, a structural property of frames suitable for our analysis is first introduced, and finally we state and prove our main result Theorem 4.5. We finish with some conclusions and discussions in Section 5.

## 2. FRAME CONSTRUCTION

We first review the classical as well as the extended version of the Spectral Tetris algorithm from [7]. To stand on common ground, we start by fixing our terminology while briefly reviewing the basic definitions and notations related to frames.

**2.1. Frames.** A sequence  $\Phi = (\varphi_i)_{i=1}^N$  in  $\mathbb{R}^n$  is called a *frame* for  $\mathbb{R}^n$ , if it is a – typically, but not necessarily linearly dependent – spanning set. This definition is equivalent to asking for the existence of constants  $0 < A \leq B < \infty$  such that

$$A\|x\|^2 \leq \sum_{i=1}^N |\langle x, \varphi_i \rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathbb{R}^n.$$

When  $A$  is chosen as the largest possible value and  $B$  as the smallest for these inequalities to hold, then we call them the (*optimal*) *frame bounds*. If  $A$  and  $B$  can be chosen as  $A = B$ , then the frame  $\Phi$  is called *A-tight*, and if  $A = B = 1$  is possible,  $\Phi$  is a *Parseval frame*.  $\Phi$  is called *equal-norm*, if there exists some  $c > 0$  such that  $\|\varphi_i\| = c$  for all  $i = 1, \dots, N$ , and it is *unit-norm* if  $c = 1$ .

Frames allow the analysis of data by studying the associated *frame coefficients*  $(\langle x, \varphi_i \rangle)_{i=1}^N$ , where the operator  $T$  defined by  $T : \mathbb{R}^n \rightarrow \ell_2(\{1, 2, \dots, N\})$ ,  $x \mapsto (\langle x, \varphi_i \rangle)_{i=1}^N$  is called the *analysis operator*. The adjoint  $T^*$  of the analysis operator is typically referred to as the *synthesis operator* and satisfies  $T^*((c_i)_{i=1}^N) = \sum_{i=1}^N c_i \varphi_i$ . Later, the synthesis operator will play an essential role, and we will write it in the matrix form  $[\varphi_1 | \dots | \varphi_N]$  with the frame vectors as columns. In the sequel we refer to this matrix as the *synthesis matrix*. The main operator associated with a frame, which provides a stable reconstruction process, is the *frame operator*

$$S = T^*T : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad x \mapsto \sum_{i=1}^N \langle x, \varphi_i \rangle \varphi_i,$$

a positive, self-adjoint, invertible operator on  $\mathbb{R}^n$ . In the case of an  $A$ -tight frame, we have  $S = A \cdot \text{Id}_{\mathbb{R}^n}$ , and in case of a Parseval frame,  $S = \text{Id}_{\mathbb{R}^n}$ . In general,  $S$  allows for the reconstruction of a signal  $x \in \mathbb{R}^n$  through the reconstruction formula

$$x = \sum_{i=1}^N \langle x, S^{-1} \varphi_i \rangle \varphi_i. \tag{1}$$

Redundancy is obviously the crucial property of a frame ensuring resilience to noise and erasures while simultaneously enabling us to choose the expansion coefficients appropriately. The particular choice of coefficients displayed in (1) is the smallest in  $\ell_2$  norm [16], hence it contains the least energy. Recently, a different view point has received rapidly increasing attention, namely to choose the coefficient sequence to be sparse in the sense of having only few non-zero entries, thereby allowing data compression while preserving perfect recoverability (see, e.g., [6] and the references therein). In this context, for later use, we will denote the support of a vector  $x \in \mathbb{R}^n$ , i.e., the number of non-zero entries, by  $\text{supp } x$ .

Finally, we should mention that, customarily, redundancy of a frame  $(\varphi_i)_{i=1}^N$  for  $\mathbb{R}^n$  is measured by  $\frac{N}{n}$ , i.e., the number of frame vectors divided by the dimension of the ambient space. Since this measure is exceptionally crude and not sensitive to local behavior of the frame vectors, the notions of *upper* and *lower redundancy* have been suggested in [4] as a finer redundancy measure.

**2.2. The Spectral Tetris Algorithm.** Spectral Tetris was first introduced in [12] as an algorithm to generate unit norm tight frames for any number of frame vectors  $N$ , say, and for any ambient dimension  $n$  provided that  $\frac{N}{n} \geq 2$ . This algorithm can indeed be considered a breakthrough in frame construction, since it is the first systematic construction of unit norm tight frames. Before, only a number of very special classes of unit norm tight frames such as harmonic frames have been known.

An extension to the construction of unit norm frames having a desired frame operator associated with eigenvalues  $\lambda_1, \dots, \lambda_n \geq 2$  satisfying  $\sum_{j=1}^n \lambda_j = N$  was then introduced and analyzed in [7] – in fact, an even more general algorithm for the construction of fusion frames was stated therein. The frame-version of this algorithm is what we intend to analyze in this paper. Figure 1 states the steps of this version of the algorithm, which we coin *Spectral Tetris for Frames*; in short, STF. We wish to remark that the original form of the algorithm in [12] requires the sequence of eigenvalues to be in decreasing order, i.e.  $\lambda_1 \geq \dots \geq \lambda_n$ . This assumption, however, was made only for classification reasons, and it is easily seen that it can be dropped. Since in the sequel, we will consider carefully chosen, presumably non-decreasing, sequences of eigenvalues, the gained freedom is essential for our analysis.

### 3. NEW PARADIGM FOR FRAME CONSTRUCTIONS: SPARSITY

**3.1. Classical Sparsity.** Over the past few years, sparsity has become a key concept in various areas of applied mathematics, computer science, and electrical engineering. Sparse signal processing methodologies explore the fundamental fact that many types of signals can be represented by only a few non-zero coefficients when choosing a suitable basis or, more generally, a frame. A signal representable by only  $k$ , say, basis or frame elements is called *k-sparse*. If signals possess such a sparse representation, they can in general be recovered from few measurements using  $\ell_1$  minimization techniques (see, e.g., [6, 9, 18] and the references therein).

**3.2. Sparse Frames.** In this paper, however, we pose a different question concerning sparsity, viewing sparsity from a very different standpoint. Typically, data processing applications face low on-board computing power and/or a small bandwidth budget. When the signal

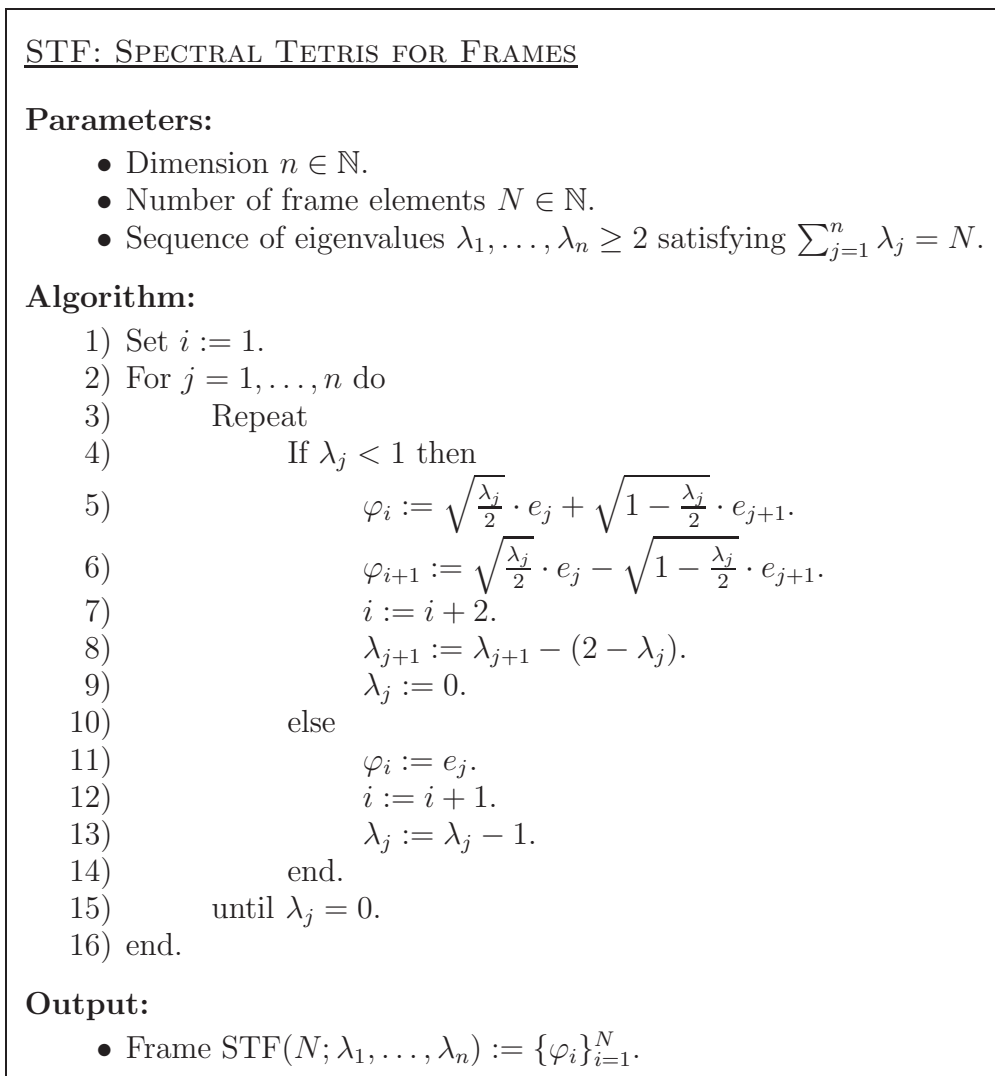


FIGURE 1. The Spectral Tetris algorithm for constructing an  $N$ -element unit norm frame  $\text{STF}(N; \lambda_1, \dots, \lambda_n)$  for  $\mathbb{R}^n$  with an associated frame operator having eigenvalues  $\lambda_1, \dots, \lambda_n$ .

dimension is large, the decomposition of the signal into its frame measurements requires a large number of additions and multiplications, which may be infeasible for on-board data processing. Also the space required for storing the synthesis matrix of the frame might be huge. It would hence be a significant improvement, if each frame vector would contain very few non-zero entries, hence – phrasing it differently – be sparse in the standard unit vector basis, which ensures low-complexity processing. Since we are interested in the performance of the whole frame, the total number of non-zero entries in the frame vectors seems to be a suitable sparsity measure. This viewpoint can also be slightly generalized by assuming that there exists a unitary transformation mapping the frame into one having this ‘sparsity’ property.

**3.3. Sparseness Measure.** Taking these considerations into account, we are led to proclaim the following definition for a sparse frame:

**Definition 3.1.** *Let  $(e_j)_{j=1}^n$  be an orthonormal basis for  $\mathbb{R}^n$ . Then a frame  $(\varphi_i)_{i=1}^N$  for  $\mathbb{R}^n$  is called  $k$ -sparse with respect to  $(e_j)_{j=1}^n$ , if, for each  $i \in \{1, \dots, N\}$ , there exists  $J_i \subseteq \{1, \dots, n\}$  such that*

$$\varphi_i \in \text{span}\{e_j : j \in J_i\}$$

and

$$\sum_{i=1}^n |J_i| = k. \quad (2)$$

The attentive reader will have realized that this definition differs from the definition stated in [7] for fusion frames (see [13]) when restricting to the special case of frames. The exact relation is the following: Let  $(e_j)_{j=1}^n$  be an orthonormal basis for  $\mathbb{R}^n$ , let  $(\varphi_i)_{i=1}^N$  be a frame for  $\mathbb{R}^n$ , and, for each  $i \in \{1, \dots, N\}$ , let  $J_i \subseteq \{1, \dots, n\}$  such that  $\varphi_i \in \text{span}\{e_j : j \in J_i\}$ . Then, in the sense of [7], the frame is  $\max\{|J_i| : i = 1, \dots, n\}$ -sparse, whereas in our Definition 3.1, the frame is  $\sum_{i=1}^n |J_i|$  sparse. Thus our definition encodes the true overall sparsity which is the sparsity required for frame processing in contrast to the more local version of [7].

One can certainly imagine other sparsity measures dependent on the requirements and constraints of the application at hand. Instead of (2), a weighted version could be considered with the weights chosen depending on the computational constraints of the application. Also, (2) could be regarded as the  $\ell_1$  norm of the sequence  $\{|J_i| : i = 1, \dots, n\}$ , and a different viewpoint might lead us to considering a different norm instead – as it was done in [7] for the  $\ell_\infty$  norm.

**3.4. Notion of Optimality.** We now have the necessary machinery at hand to introduce a notion of an *optimally* sparse frame. Optimality will typically – as also in this paper – be considered within a particular class of frames, for instance, in the class of unit norm tight frames.

**Definition 3.2.** *Let  $\mathcal{F}$  be a class of frames for  $\mathbb{R}^n$ , let  $(\varphi_i)_{i=1}^N \in \mathcal{F}$ , and let  $(e_j)_{j=1}^n$  be an orthonormal basis for  $\mathbb{R}^n$ . Then  $(\varphi_i)_{i=1}^N$  is called optimally sparse in  $\mathcal{F}$  with respect to  $(e_j)_{j=1}^n$ , if  $(\varphi_i)_{i=1}^N$  is  $k_1$ -sparse with respect to  $(e_j)_{j=1}^n$  and there does not exist a frame  $(\psi_i)_{i=1}^N \in \mathcal{F}$  which is  $k_2$ -sparse with respect to  $(e_j)_{j=1}^n$  with  $k_2 < k_1$ .*

We wish to emphasize the strong dependence of sparsity on the chosen basis. Also an optimally sparse frame is in general not uniquely determined; we present an example for this observation in Subsection 4.2.

#### 4. AN OPTIMALITY RESULT FOR SPARSE FRAMES

We now seek a construction for an optimally sparse unit norm frame with prescribed properties. As already elaborated upon before, the condition we impose is having a given frame operator, which, in particular, also includes operators with equal eigenvalues corresponding to tight frames. This frame operator will in the following be always determined by its eigenvalues. Hence we are interested in optimal sparsity within the following class:

Let  $n, N > 0$  and let the real values  $\lambda_1, \dots, \lambda_n \geq 2$  satisfy  $\sum_{j=1}^n \lambda_j = N$ . Then the class of unit norm frames  $(\varphi_i)_{i=1}^N$  in  $\mathbb{R}^n$  whose frame operator has eigenvalues  $\lambda_1, \dots, \lambda_n$  will be denoted by

$$\mathcal{F}(N, \{\lambda_i\}_{i=1}^n).$$

It is important to mention that by writing  $\{\lambda_i\}_{i=1}^n$ , we wish to indicate that the ordering does not play a role here, however, multiplicities are counted. Also the reader should note that  $\mathcal{F}(N, \{\lambda_i\}_{i=1}^n)$  does contain infinitely many elements for any set of real values  $\lambda_1, \dots, \lambda_n \geq 2$ .

It might be beneficial for the reader to mention at this point that we will discuss the analysis presented in Subsections 4.1 to 4.3 in the important special case of tight frames in Subsection 4.4 for illustrative purposes.

**4.1. Novel Structural Property of Synthesis Matrices.** Aiming for determining the maximally achievable sparsity for a class  $\mathcal{F}(N, \{\lambda_i\}_{i=1}^n)$ , we first need to introduce a particular measure associated with the set of eigenvalues  $\{\lambda_i\}_{i=1}^n$ . This measure indicates the maximal number of partial sums which are an integer; here one maximizes over all reorderings of the eigenvalues. Later, this will allow us to – roughly speaking – reduce the analysis to the blocks between two such integer partial sums, and hence, in particular, the maximally achievable sparsity will depend on this number.

**Definition 4.1.** *A finite sequence of real values  $\lambda_1, \dots, \lambda_n$  is ordered blockwise, if for any permutation  $\pi$  of  $\{1, \dots, n\}$  the set of partial sums  $\{\sum_{j=1}^s \lambda_j : s = 1, \dots, n\}$  contains at least as many integers as the set  $\{\sum_{j=1}^s \lambda_{\pi(j)} : s = 1, \dots, n\}$ . The maximal block number of a finite sequence of real values  $\lambda_1, \dots, \lambda_n$ , denoted by  $\mu(\lambda_1, \dots, \lambda_n)$ , is the number of integers in  $\{\sum_{j=1}^s \lambda_{\sigma(j)} : s = 1, \dots, n\}$ , where  $\sigma$  is a permutation of  $\{1, \dots, n\}$  such that  $\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)}$  is ordered blockwise.*

Surprisingly, the notion of maximal block number can illuminatingly be transferred to a particular decomposition property of the synthesis matrix of a frame. Let us first define the decomposition property we are interested in:

**Definition 4.2.** *Let  $n, N > 0$ , and let  $(\varphi_i)_{i=1}^N$  be a frame for  $\mathbb{R}^n$ . Then we say that the synthesis matrix of  $(\varphi_i)_{i=1}^N$  has block decomposition of order  $m$ , if there exists a partition  $\{1, \dots, N\} = I_1 \cup \dots \cup I_m$  such that, for any  $k_1 \in I_{i_1}$  and  $k_2 \in I_{i_2}$  with  $i_1 \neq i_2$ , we have  $\text{supp } \varphi_{k_1} \cap \text{supp } \varphi_{k_2} = \emptyset$  and  $m$  is maximal.*

The following result now connects the maximal block number of the sequence of eigenvalues of a frame operator with the block decomposition order of an associated frame.

**Proposition 4.3.** *Let  $n, N > 0$  and let the real values  $\lambda_1, \dots, \lambda_n \geq 2$  satisfy  $\sum_{j=1}^n \lambda_j = N$ . Then the synthesis matrix of any frame in the class  $\mathcal{F}(N, \{\lambda_i\}_{i=1}^n)$  has block decomposition of order at most  $\mu(\lambda_1, \dots, \lambda_n)$ .*

*Proof.* Suppose  $(\varphi_i)_{i=1}^N \in \mathcal{F}(N, \{\lambda_i\}_{i=1}^n)$  has block decomposition of order  $\nu > \mu(\lambda_1, \dots, \lambda_n)$ , and let  $\{1, \dots, N\} = I_1 \cup \dots \cup I_\nu$  be a corresponding partition. For  $j = 1, \dots, \nu$ , let  $S_j$  be the common support set of the vectors  $(\varphi_i)_{i \in I_j}$ , i.e.,  $k \in S_j$  if and only if  $k \in \text{supp } \varphi_i$  for some

$i \in I_j$ . Now let  $r_k$  denote the  $k$ -th row of the synthesis matrix of  $(\varphi_i)_{i=1}^N$ . Then  $S_1, \dots, S_\nu$  is a partition of  $\{1, \dots, n\}$  and, for every  $j = 1, \dots, \nu$ ,

$$|I_j| = \sum_{k \in I_j} \|\varphi_k\|^2 = \sum_{k \in S_j} \|r_k\|^2 = \sum_{k \in S_j} \lambda_k.$$

This implies that the maximal block number of  $\lambda_1, \dots, \lambda_n$  is at least  $\nu$ , which is a contradiction. Thus the synthesis matrix of the arbitrarily chosen frame  $(\varphi_i)_{i=1}^N$  in the class  $\mathcal{F}(N, \{\lambda_i\}_{i=1}^n)$  has block decomposition of order at most  $\mu(\lambda_1, \dots, \lambda_n)$ .  $\square$

**4.2. Maximally Achievable Sparsity.** Having introduced the required new notions, we are now in a position to state the exact value for the maximally achievable sparsity for a class  $\mathcal{F}(N, \{\lambda_i\}_{i=1}^n)$ . It is not initially clear that this optimal sparsity can always be attained. With Theorem 4.5 we will prove that this is indeed the case; in fact, Theorem 4.5 also provides an explicit construction of those frames.

**Theorem 4.4.** *Let  $n, N > 0$ , and let the real values  $\lambda_1, \dots, \lambda_n \geq 2$  satisfy  $\sum_{j=1}^n \lambda_j = N$ . Then any frame in  $\mathcal{F}(N, \{\lambda_i\}_{i=1}^n)$  has sparsity at least*

$$N + 2(n - \mu(\lambda_1, \dots, \lambda_n))$$

*with respect to any orthonormal basis.*

*Proof.* We first study the case that  $\mu(\lambda_1, \dots, \lambda_n) = 1$ . For this, let  $T^*$  denote the synthesis matrix of a frame in  $\mathcal{F}(N, \{\lambda_i\}_{i=1}^n)$  with respect to a fixed orthonormal basis. For the sake of brevity, in the sequel we will use the phrase that two rows of  $T^*$  *have overlap of size  $k$* , if the intersection of their supports is a set of size  $k$ . Note that, since the rows of  $T^*$  are orthogonal, it is not possible that two rows of  $T^*$  have overlap 1.

Fix now an arbitrary row  $r_1$  of  $T^*$ . Since, by Proposition 4.3,  $T^*$  has block decomposition of order 1, there exists a row  $r_2$  whose overlap with  $r_1$  is of size  $\geq 2$ . Similarly, there has to exist a row different from  $r_1$  and  $r_2$  which has overlap of size  $\geq 2$  with  $r_1$  or  $r_2$ . Iterating this procedure will provide an order  $r_1, r_2, \dots, r_n$  such that, for each row  $r_j$ , there exists some  $k < j$  such that  $r_j$  has overlap of size  $\geq 2$  with  $r_k$ . Since all columns in  $T^*$  are unit norm, for each column  $c$ , there exists a minimal  $j$  for which the entry  $c_{r_j}$  is non-zero. This yields  $N$  non-zero entries in  $T^*$ . In addition, each row  $r_2$  through  $r_n$  has at least 2 non-zero entries coming from the overlap, which are different from the just accounted for  $N$  entries. This sums up to a total of at least  $2(n - 1)$  non-zero coefficients. Consequently, the synthesis matrix has at least  $N + 2(n - 1)$  non-zero entries, as desired.

Finally, suppose  $\mu := \mu(\lambda_1, \dots, \lambda_n) > 1$ . By Proposition 4.3,  $T^*$  has block decomposition of order at most  $\mu$ . Performing the same construction as above, there exist at most  $\mu$  rows  $r_j$  (including the first one) which do not have overlap with a row  $r_k$ ,  $k < j$ . Thus the synthesis matrix  $T^*$  must at least contain  $N + 2(n - \mu)$  non-zero entries.  $\square$

It should be mentioned that an optimally sparse frame from  $\mathcal{F}(N, \{\lambda_i\}_{i=1}^n)$  is in general not uniquely determined. Various examples can be constructed along the following line: For simplicity, we choose  $n = 4$  and  $N = 9$  and construct a tight frame, i.e.,  $\lambda_1 = \dots = \lambda_4 = \frac{9}{4}$ . Then, by Theorem 4.4, the maximal achievable sparsity is  $9 + 2(4 - 1) = 15$ . The following



matrices are synthesis matrices with respect to the standard unit vector basis of two different frames in  $\mathcal{F}(9, \{\frac{9}{4}\}_{i=1}^9)$ , the first in fact being generated by Spectral Tetris:

$$\begin{bmatrix} 1 & 1 & \sqrt{\frac{1}{8}} & \sqrt{\frac{1}{8}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{7}{8}} & -\sqrt{\frac{7}{8}} & \sqrt{\frac{1}{4}} & \sqrt{\frac{1}{4}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{3}{4}} & -\sqrt{\frac{3}{4}} & \sqrt{\frac{3}{8}} & \sqrt{\frac{3}{8}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{5}{8}} & -\sqrt{\frac{5}{8}} & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & \sqrt{\frac{5}{8}} & \sqrt{\frac{5}{8}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{3}{8}} & -\sqrt{\frac{3}{8}} & \sqrt{\frac{3}{8}} & \sqrt{\frac{3}{8}} & \sqrt{\frac{3}{8}} & \sqrt{\frac{3}{8}} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{1}{8}} & -\sqrt{\frac{1}{8}} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{5}{8}} & -\sqrt{\frac{5}{8}} & 0 & 1 \end{bmatrix}.$$

**4.3. Main Result.** Having set the benchmark, we now prove that frames constructed by Spectral Tetris in fact achieve the optimal sparsity rate. For this, we would like to remind the reader that the frame constructed by Spectral Tetris (see Figure 1) was denoted by  $\text{STF}(N; \lambda_1, \dots, \lambda_n)$ .

**Theorem 4.5.** *Let  $n, N > 0$ , and let the real values  $\lambda_1, \dots, \lambda_n \geq 2$  be ordered block-wise and satisfy  $\sum_{j=1}^n \lambda_j = N$ . Then the frame  $\text{STF}(N; \lambda_1, \dots, \lambda_n)$  is optimally sparse in  $\mathcal{F}(N, \{\lambda_i\}_{i=1}^n)$  with respect to the standard unit vector basis. That is, this frame is  $N + 2(n - \mu(\lambda_1, \dots, \lambda_n))$ -sparse with respect to the standard unit vector basis.*

*Proof.* Let  $(\varphi_i)_{i=1}^N$  be the frame  $\text{STF}(N; \lambda_1, \dots, \lambda_n)$ . We will first show that its synthesis matrix has block decomposition of order  $\mu := \mu(\lambda_1, \dots, \lambda_n)$ . For this, let  $k_0 = 0$ , and let  $k_1, \dots, k_\mu \in \mathbb{N}$  be chosen such that  $m_i := \sum_{j=k_{i-1}+1}^{k_i} \lambda_j$  is an integer for every  $i = 1, \dots, \mu$ . Moreover, let  $m_0 = 0$ . Further, note that  $k_\mu = n$  and  $m_\mu = N$ , since  $\sum_{j=1}^n \lambda_j$  is an integer by hypothesis. The steps of Spectral Tetris (STF) for computing  $\text{STF}(m_1; \lambda_1, \dots, \lambda_{k_1})$  and  $\text{STF}(N; \lambda_1, \dots, \lambda_n)$  coincide until the dimension for computing  $\text{STF}(N; \lambda_1, \dots, \lambda_n)$  exceeds  $m_1$  and  $k_1$ . Therefore, the first  $k_1$  entries of the first  $m_1$  vectors of both constructions coincide. Continuing the computation of  $\text{STF}(N; \lambda_1, \dots, \lambda_n)$  will set the remaining entries of the first  $m_1$  vectors and also the first  $k_1$  entries of the remaining vectors to zero. Thus, any of the first  $k_1$  vectors has disjoint support from any of the vectors constructed later on. Repeating this argument for  $k_2$  until  $k_\mu$ , we obtain that the synthesis matrix has a block decomposition of order  $\mu$ ; the corresponding partition of the frame vectors being

$$\bigcup_{i=1}^{\mu} \{\varphi_{m_{i-1}+1}, \dots, \varphi_{m_{i-1}+m_i}\}.$$

To compute the number of non-zero entries in the synthesis matrix generated by Spectral Tetris, we let  $i \in \{1, \dots, \mu\}$  be arbitrarily fixed and compute the number of non-zero entries of the vectors  $\varphi_{m_{i-1}+1}, \dots, \varphi_{m_{i-1}+m_i}$ . Spectral Tetris ensures that each of the rows  $k_{i-1} + 1$

up to  $k_i - 1$  intersects the support of the subsequent row on a set of size 2. Thus, there exist  $2(k_i - k_{i-1} - 1)$  frame vectors with two non-zero entries. The remaining  $m_i - 2(k_i - k_{i-1} - 1)$  frame vectors will have only one entry, yielding a total number of  $m_i + 2(k_i - k_{i-1} - 1)$  non-zero entries in the vectors  $\varphi_{m_{i-1}+1}, \dots, \varphi_{m_{i-1}+m_i}$ .

Summarizing, the total number of non-zero entries in the frame vectors of  $(\varphi_i)_{i=1}^N$  is

$$\sum_{i=1}^{\mu} m_i + 2(k_i - k_{i-1} - 1) = \left( \sum_{i=1}^{\mu} m_i \right) + 2 \left( k_{\mu} - \left( \sum_{i=1}^{\mu} 1 \right) \right) = N + 2(n - \mu),$$

which is by Theorem 4.4 the maximal achievable sparsity.  $\square$

The reader will have realized that Spectral Tetris generates frames which are ‘only’ optimally sparse with respect to the standard unit vector basis. This seems at first sight like a drawback. However, if sparsity with respect to a different orthonormal basis is required, Spectral Tetris can easily be modified to accommodate this request by using vectors of this orthonormal basis instead of the standard unit vector basis when filling in the frame vectors in Steps 5, 6 and 11 in STF (cf. Figure 1). It is a straightforward exercise to show that this modified Spectral Tetris algorithm then generates a frame which is optimally sparse with respect to this new orthonormal basis.

**4.4. Special Case: Constructing Optimally Sparse Tight Frames.** In the special case of equal eigenvalues, i.e., of tight frames, with  $N$  elements in  $\mathbb{R}^n$ , all eigenvalues need to equal  $\frac{N}{n}$  for the equality  $\sum_{j=1}^n \lambda_j = N$  to be satisfied. The maximal block number can be easily computed to be  $\gcd(N, n)$ . Theorem 4.5 then takes the following form:

**Corollary 4.6.** *For all strictly positive  $n$  and  $N$ , the frame  $STF(N; \frac{N}{n}, \dots, \frac{N}{n})$  is optimally sparse in  $\mathcal{F}(N, \{\frac{N}{n}\}_{i=1}^N)$  with respect to the standard unit vector basis. That is, this frame is  $N + 2(n - \gcd(N, n))$ -sparse with respect to the standard unit vector basis.*

## 5. CONCLUSIONS AND DISCUSSION

In this paper we considered the design of frames which enable efficient computations of the associated frame measurements. This led to the introduction of the notion of a sparse frame as well as a sparsity measure for such frames, thereby introducing optimal sparsity as a new paradigm into the construction of frames. We then analyzed an extended version of Spectral Tetris for frames and proved that the frames constructed by this algorithm are indeed optimally sparse. This shows that Spectral Tetris can serve as an algorithm for computing frames with this desirable property, and our results prove that it is not possible to derive sparser frames through a different procedure.

We would finally like to point out that the analysis in this paper leads to several intriguing open problems for future research; a few examples are stated in the sequel.

- *Eigenvalues also smaller than 2.* It is still an open problem whether and how Spectral Tetris extends to sets of eigenvalues, if some eigenvalues are smaller than 2. Some partial results by extending Spectral Tetris by inserting larger DFT matrices than the previously exploited  $2 \times 2$ -matrices were derived in [11]. However, from the results in [11] it can be deduced that this procedure does not always lead to optimally sparse

frames even in the case when all eigenvalues are equal, i.e., the tight frame case. Hence, extensive research will be necessary to introduce an appropriate – in the sense of optimal sparsity – extension of Spectral Tetris.

- *Extension to other classes of frames.* Depending on the application, other desiderata might be requested from a frame such as, for instance, equi-angularity. For such a class of frames, the question of an optimally sparse frame can and should similarly be posed.
- *Relative sparsity/Compressibility.* Taking numerical considerations and perturbations into account, it will be necessary to extend the notion of sparsity to relative sparsity/compressibility for frames and analyze optimality for such.

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