Mass generation mechanism for spin 1/2 fermions in Dirac–Yang–Mills model equations with a symplectic gauge symmetry

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Abstract

In the Standard Model of electroweak interactions the fundamental fermions acquire masses by the Yukawa interaction with the (spin 0) Higgs field. In our model spin 1/2 fermions acquire masses by an interaction with (spin 1) gauge field with symplectic symmetry.

In [2, 3, 4, 5, 8] we develop a new approach to field theory, which based on so called *model equations of field theory*. In this paper we introduce Dirac–Yang–Mills model equations for spin 1/2 fermions interacting with two gauge fields simultaneously. One field has a unitary gauge symmetry and another has a symplectic gauge symmetry. There is no fermion's mass (m) term in the model Dirac equation. But there is the term $3m^3/16$ in the right hand part of the Yang–Mills equations with symplectic gauge symmetry. Hence, the constant $3m^3/16$ can be considered as a constant (charge) describing the interaction of a fermion with the symplectic gauge field.

Clifford algebra. Let $\mathcal{C}\ell(1,3)$ be the complex Clifford algebra [1] with the unity element e and with generators e^a, $a=0,1,2,3$, which satisfy the relations

$$e^a e^b + e^b e^a = 2\eta^{ab} e, \quad a,b=0,1,2,3,$$

where $\eta = \|\eta^{ab}\| = \mathrm{diag}(1,-1,-1,-1)$ is the diagonal matrix.

Let $\mathcal{C}\ell_k(1,3)$, $(k=0,1,2,3,4)$ be subspaces of rank k Clifford algebra elements and $\mathcal{C}\ell_{\mathrm{Even}}(1,3)$, $\mathcal{C}\ell_{\mathrm{Odd}}(1,3)$ be the subspaces of even and odd Clifford algebra elements respectively. By $\mathcal{C}\ell^{\mathbb{R}}(1,3)$ denote the real Clifford algebra.

Denote $\beta = e^0 \in \mathcal{C}\!\ell(1,3)$. Consider *an operation of pseudo-Hermitian conjugation* $* : \mathcal{C}\!\ell(1,3) \to \mathcal{C}\!\ell(1,3)$ such that $(e^a)^* = e^a$, $a = 0,1,2,3$ and

$$(\lambda U)^* = \bar{\lambda} U^*, \quad (UV)^* = V^* U^*, \quad (U+V)^* = U^* + V^*,$$

where U, V are arbitrary elements of $\mathcal{C}\!\ell(1,3)$ and $\lambda \in \mathbb{C}$. Now we can define *an operation of Hermitian conjugation* of Clifford algebra elements by the formula [6]

$$U^\dagger = \beta U^* \beta.$$

Symplectic Lie group and its real Lie algebra. Consider *the real symplectic Lie group* of matrices of even order $n = 2m$ and its Lie algebra

$$\begin{aligned} \mathrm{Sp}(m,\mathbb{R}) &= \{U \in \mathrm{Mat}(n,\mathbb{R}) : U^T S U = S\}, \\ \mathrm{sp}(m,\mathbb{R}) &= \{u \in \mathrm{Mat}(n,\mathbb{R}) : u^T S = -Su\}, \end{aligned}$$

where U^T is the transposed matrix, S is the block matrix

$$S = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix},$$

and I_m is the identity matrix of order m. Note that $S^2 = -\mathbf{1}$ ($\mathbf{1}$ is the identity matrix of order $2m$).

Symplectic Lie group of the Clifford algebra and its Lie algebra. Let us define two sets of Clifford algebra elements [7]

$$\begin{aligned} \mathrm{Sp}(\mathcal{C}\!\ell(1,3)) &= \{V \in \mathcal{C}\!\ell^{\mathbb{R}}_{\mathrm{Even}}(1,3) \oplus i\mathcal{C}\!\ell^{\mathbb{R}}_{\mathrm{Odd}}(1,3) : V^* V = e\}, \\ \mathrm{sp}(\mathcal{C}\!\ell(1,3)) &= \{v \in i\mathcal{C}\!\ell^{\mathbb{R}}_1(1,3) \oplus \mathcal{C}\!\ell^{\mathbb{R}}_2(1,3)\}. \end{aligned}$$

The set $\mathrm{Sp}(\mathcal{C}\!\ell(1,3))$ is closed with respect to the multiplication of Clifford algebra elements and forms a (Lie) group. This group is called *the symplectic group of Clifford algebra* $\mathcal{C}\!\ell(1,3)$. The set $\mathrm{sp}(\mathcal{C}\!\ell(1,3))$ is closed w.r.t. the commutator $[u,v] = uv - vu$ and forms the Lie algebra.

The following proposition is proved in [7]: *The group* $\mathrm{Sp}(\mathcal{C}\!\ell(1,3))$ *is isomorphic to the group* $\mathrm{Sp}(2,\mathbb{R})$ *and the Lie algebra* $\mathrm{sp}(\mathcal{C}\!\ell(1,3))$ *is isomorphic to the Lie algebra* $\mathrm{sp}(2,\mathbb{R})$*, i.e.*

$$\begin{aligned} \mathrm{Sp}(\mathcal{C}\!\ell(1,3)) &\simeq \mathrm{Sp}(2,\mathbb{R}), \qquad (1) \\ \mathrm{sp}(\mathcal{C}\!\ell(1,3)) &\simeq \mathrm{sp}(2,\mathbb{R}). \end{aligned}$$

Hermitian idempotents. Let $t \in \mathcal{C}\!\ell(1,3)$ be a nonzero element such that

$$t^2 = t, \quad t^\dagger = t, \quad \bar{t}J = Jt, \tag{2}$$

where $J = -e^1 e^3$. Such an element is called *a Hermitian idempotent*. In particular, we may take the Hermitian idempotents

$$\begin{aligned}
t_{(1)} &= \frac{1}{4}(e+e^0)(e+ie^{12}),\\
t_{(2)} &= \frac{1}{2}(e+e^0),\\
t_{(3)} &= \frac{1}{4}(3e+e^0+ie^{12}-ie^{012}),\\
t_{(4)} &= e.
\end{aligned}$$

A Hermitian idempotent t generates the left ideal $I(t)$, the two sided ideal $K(t)$, the Lie algebra $L(t)$, and the Lie group $G(t)$

$$\begin{aligned}
I(t) &= \{U \in \mathcal{C}\!\ell(1,3) : U = Ut\},\\
K(t) &= \{U \in I(t) : U = tU\},\\
L(t) &= \{U \in K(t) : U^\dagger = -U\},\\
G(t) &= \{U \in \mathcal{C}\!\ell(1,3) : U^\dagger U = e,\, U - e \in K(t)\}.
\end{aligned} \tag{3}$$

The Minkowski space. Let $\mathbb{R}^{1,3}$ be the Minkowski space with cartesian coordinates x^μ, where $\mu = 0,1,2,3$ and $\partial_\mu = \partial/\partial x^\mu$ are partial derivatives. We use Greek indices $\mu, \nu, \alpha, \beta, \ldots$ (run from 0 to 3) as tensor indices relative to coordinates x^μ. The Minkowski metric is given by the diagonal matrix η. By T^r_s denote the set of tensor fields of type (r,s) (of rank $r+s$) in Minkowski space. By $\mathrm{T}_{[s]}$ denote the set of rank s antisymmetric covariant tensor fields. In the sequel we consider tensors with values in Lie algebras. For example, if u_μ is a covector with values in a Lie algebra $L(t)$, then we write $u_\mu \in L(t)\mathrm{T}_1$.

In what follows the generators e^0, e^1, e^2, e^3 of Clifford algebra $\mathcal{C}\!\ell(1,3)$ and the fixed Hermitian idempotent t do not depend on x and they are scalars of the Minkowski space, i.e. they do not transform under Lorentzian changes of coordinates.

Model Dirac–Yang–Mills equations. Consider *the model Dirac–Yang–*

Mills equations [4, 8]

$$
\begin{aligned}
&ih^\mu(\partial_\mu\phi+\phi A_\mu-C_\mu\phi)-m\phi=0,\\
&\partial_\mu A_\nu-\partial_\nu A_\mu-[A_\mu,A_\nu]=F_{\mu\nu},\\
&\partial_\mu F^{\mu\nu}-[A_\mu,F^{\mu\nu}]=\phi^\dagger\beta ih^\nu\phi,\\
&\partial_\mu h^\nu-[C_\mu,h^\nu]=0,
\end{aligned}
\tag{4}
$$

where

1. The vector $ih^\mu=ih^\mu(x)\in\mathrm{sp}(\mathcal{C}\!\ell(1,3))\mathrm{T}^1$ is such that

$$
h^\mu h^\nu+h^\nu h^\mu=2\eta^{\mu\nu}e,\quad \mu,\nu=0,1,2,3. \tag{5}
$$

2. The element $\phi=\phi(x)\in I(t)$ is a scalar of Minkowski space (it does not transform under Lorentzian changes of coordinates) $(\phi(x)\to\phi(x(\acute{x})))$.

3. $A_\mu=A_\mu(x)\in L(t)\mathrm{T}_1$.

4. $F_{\mu\nu}=F_{\mu\nu}(x)\in L(t)\mathrm{T}_{[2]}$.

5. The mass m is a real constant.

6. $C_\mu=C_\mu(x)\in\mathrm{sp}(\mathcal{C}\!\ell(1,3))\mathrm{T}_1$.

We suppose that the idempotent t, the constant m, and the generators of Clifford algebra e^a are known and the variables $h^\mu,\phi,A_\mu,F_{\mu\nu},C_\mu$ are unknown. In this case equations (4) are called *model Dirac–Yang–Mills equations* (with local symplectic symmetry).

From the first equation in (4), using the identity $\frac{1}{4}h^\mu h_\mu=e$, we get the equation

$$
ih^\mu(\partial_\mu\phi+\phi A_\mu-B_\mu\phi)=0,
$$

where

$$
B_\mu=C_\mu-\frac{m}{4}ih_\mu\in\mathrm{sp}(\mathcal{C}\!\ell(1,3))\mathrm{T}_1.
$$

If we substitute the expression

$$
C_\mu=B_\mu+\frac{m}{4}ih_\mu\in\mathrm{sp}(\mathcal{C}\!\ell(1,3))\mathrm{T}_1
$$

into the equalities (the second equality is a consequence of the first one)

$$
\begin{aligned}
&\partial_\mu h^\nu-[C_\mu,h^\nu]=0,\\
&\partial_\mu C_\nu-\partial_\nu C_\mu-[C_\mu,C_\nu]=0,
\end{aligned}
$$

then we get

$$\partial_\mu ih^\nu - [B_\mu, ih^\nu] = \frac{m}{4}[ih_\mu, ih^\nu],$$
$$\partial_\mu B_\nu - \partial_\nu B_\mu - [B_\mu, B_\nu] = -\left(\frac{m}{4}\right)^2 [ih_\mu, ih_\nu]. \tag{6}$$

From the first equality if follows that

$$\partial_\mu h^\mu - [B_\mu, h^\mu] = 0.$$

We denote

$$G_{\mu\nu} = -\left(\frac{m}{4}\right)^2 [ih_\mu, ih_\nu].$$

Using the relations (6) and the relations

$$\frac{1}{4}h^\mu h_\mu = e, \quad h^\mu h^\nu h_\mu = h_\mu h^\nu h^\mu = -2h^\nu,$$

we see that

$$\partial_\mu G^{\mu\nu} - [B_\mu, G^{\mu\nu}] = \frac{3}{16}m^3 ih^\nu.$$

Therefore we have proved that if the variables $\phi, h^\mu,\ A_\mu, C_\mu, F_{\mu\nu}$ satisfy conditions (5) and equations (4), then the variables

$$\phi,\ h^\mu,\ A_\mu,\ B_\mu = C_\mu - \frac{m}{4}ih_\mu,\ F_{\mu\nu},\ G_{\mu\nu} = -\left(\frac{m}{4}\right)^2 [ih_\mu, ih_\nu]$$

satisfy the equations

$$\begin{aligned}
&ih^\mu(\partial_\mu\phi + \phi A_\mu - B_\mu\phi) = 0,\\
&\partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] = F_{\mu\nu},\\
&\partial_\mu F^{\mu\nu} - [A_\mu, F^{\mu\nu}] = \phi^\dagger\beta ih^\nu\phi,\\
&\partial_\mu B_\nu - \partial_\nu B_\mu - [B_\mu, B_\nu] = G_{\mu\nu},\\
&\partial_\mu G^{\mu\nu} - [B_\mu, G^{\mu\nu}] = \frac{3}{16}m^3 ih^\nu.
\end{aligned} \tag{7}$$

This system of equations contains two pairs of Yang–Mills equations for the fields $(A_\mu, F_{\mu\nu})$ and $(B_\mu, G_{\mu\nu})$ respectively.

Consider the system of equations (7), where the idempotent t, the real constant m, and the generators of Clifford algebra e^a are known and the variables $h^\mu, \phi, A_\mu, F_{\mu\nu}, B_\mu, G_{\mu\nu}$ are unknown and such that

1a. The vector $ih^\mu = ih^\mu(x) \in \mathrm{sp}(\mathcal{C}\ell(1,3))\mathrm{T}^1$ satisfies conditions (5).

2a. The element $\phi = \phi(x) \in I(t)$ is a scalar of the Minkowski space.

3a. $A_\mu = A_\mu(x) \in L(t)\mathrm{T}_1$.

4a. $F_{\mu\nu} = F_{\mu\nu}(x) \in L(t)\mathrm{T}_{[2]}$.

5a. $B_\mu = B_\mu(x) \in \mathrm{sp}(\mathcal{C}\ell(1,3))\mathrm{T}_1$.

6a. $G_{\mu\nu} = G_{\mu\nu}(x) \in \mathrm{sp}(\mathcal{C}\ell(1,3))\mathrm{T}_{[2]}$.

This system of equation is called *the model Dirac–Yang–Mills system of equations with two Yang–Mills fields.*

Suppose that the variables

$$\phi,\ h^\mu,\ A_\mu,\ B_\mu,\ F_{\mu\nu},\ G_{\mu\nu}$$

satisfy the conditions **1a-6a** and satisfy equations (7). We see the vector h^μ at the right hand part of the Yang–Mills equations

$$\begin{aligned} &\partial_\mu B_\nu - \partial_\nu B_\mu - [B_\mu, B_\nu] = G_{\mu\nu}, \\ &\partial_\mu G^{\mu\nu} - [B_\mu, G^{\mu\nu}] = \frac{3}{16} m^3 i h^\nu. \end{aligned}$$

Therefore the vector field h^μ satisfies the non-abelian conservation law

$$\partial_\mu h^\mu - [B_\mu, h^\mu] = 0. \tag{8}$$

However the identities (6) can't be fulfilled. Hence we may consider the system of equations (7) as a generalization of the system of equations (4).

Properties of the model Dirac–Yang–Mills equations. A transformation of variables in the system of equation (7) is called *equivalent transformation* if this system of equation written for transformed variables has the same form as the system of equation in initial variables. In this case we say that system (7) is *covariant* w.r.t. this transformation of variables.

An equivalent transformation of variables in the system of equation (7) is called *symmetry* if the generators e^a and the Hermitian idempotent t do not transform (see [4, 8] for details).

Let us discuss the properties of the model equations (7) that related to equivalent transformations and symmetries. Let $\Theta = \{h^\mu, \phi, B_\mu, G_{\mu\nu}, A_\mu, F_{\mu\nu}\}$ satisfy the equations (7).

1. (Symmetry). All the variables in the system of equation (7) are tensors (scalars are rank 0 tensors). Therefore this system of equations is covariant under Lorentzian changes of coordinates.

2. Consider bilinear forms of the model Dirac–Yang–Mills equations (7)

$$iJ^{\mu_1\ldots\mu_k} = i^{\frac{k(k-1)}{2}+1}\phi^\dagger\beta h^{[\mu_1}\ldots h^{\mu_k]}\phi \in L(t)\mathrm{T}^{[k]}.$$

Bilinear forms $J^{\mu_1\ldots\mu_k}$ are the components of contravariant antisymmetric tensors of rank k with values in Hermitian elements of the Clifford Algebra $\mathcal{C}\!\ell(1,3)$. Eigenvalues of these bilinear forms are real.

3. The vector

$$iJ^\mu = \phi^\dagger\beta i h^\mu\phi \in L(t)\mathrm{T}^1$$

satisfy non-abelian conservative law

$$\partial_\mu J^\mu - [A_\mu, J^\mu] = 0.$$

4. The equations (7) are covariant under the following global transformation defined by a unitary element $U\in\mathcal{C}\!\ell(1,3)$, $U^\dagger = U^{-1}$, $\partial_\mu U = 0$,

$$\phi\to\phi U,\quad A_\mu\to U^{-1}A_\mu U,\quad F_{\mu\nu}\to U^{-1}F_{\mu\nu}U,\quad t\to U^{-1}tU. \qquad (9)$$

5. (Symmetry). The equations (7) are covariant under the local (gauge) transformation with $U=U(x)\in G(t)$

$$\phi\to\phi U,\quad A_\mu\to U^{-1}A_\mu U - U^{-1}\partial_\mu U,\quad F_{\mu\nu}\to U^{-1}F_{\mu\nu}U. \qquad (10)$$

Note that for $U\in G(t)$ we have $[U,t]=0$ and under the considered transformation the Hermitian idempotent t does not transform.

6. (Symmetry). System of equation (7) is covariant w.r.t. the local (gauge) transformation of variables $\Theta\to\hat\Theta$ induced by an element $W=W(x)\in \mathrm{Sp}(\mathcal{C}\!\ell(1,3))$:

$$\hat\phi = W^{-1}\phi,\quad \hat h^\mu = W^{-1}h^\mu W,\quad \hat B_\mu = W^{-1}B_\mu W - W^{-1}\partial_\mu W.$$

7. System of equation (7) is covariant w.r.t. the discreet transformation (complex conjugation) of variables

$$ih^\mu\to\overline{ih^\mu},\quad (h^\mu\to-\bar h^\mu),\quad t\to\bar t,$$

$$\phi \to \bar{\phi}, \quad A_\mu \to \bar{A}_\mu, \quad F_{\mu\nu} \to \bar{F}_{\mu\nu}, \quad B_\mu \to \bar{B}_\mu,$$

here we suppose that $\bar{e}^{a_1 \ldots a_k} = e^{a_1 \ldots a_k}$.

8. (Symmetry). System of equation (7) is covariant w.r.t. the discreet transformation of variables

$$\begin{aligned} &ih^\mu \to \overline{ih^\mu}, \quad (h^\mu \to -\bar{h}^\mu), \quad \phi \to \bar{\phi}J, \quad B_\mu \to \bar{B}_\mu, \\ &A_\mu \to J^{-1}\bar{A}_\mu J, \quad F_{\mu\nu} \to J^{-1}\bar{F}_{\mu\nu}J, \end{aligned}$$

where $J = -e^1 e^3$.

Discussion of the model. We have introduced system of equation (7), which consists of three parts

- The model Dirac equation

  $$ih^\mu(\partial_\mu \phi + \phi A_\mu - B_\mu \phi) = 0$$

  for the wave function (ih^μ, ϕ) of spin 1/2 particle.

- The first pair of Yang–Mills equations

  $$\begin{aligned} &\partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] = F_{\mu\nu}, \\ &\partial_\mu F^{\mu\nu} - [A_\mu, F^{\mu\nu}] = \phi^\dagger \beta i h^\nu \phi, \end{aligned} \qquad (11)$$

  describes the Yang–Mills field $(A_\mu, F_{\mu\nu})$ with the gauge group $G(t)$ that is isomorphic to a subgroup of the unitary group U(4). According to the Standard Model, if the gauge group $G(t)$ is isomorphic to one of three groups – U(1), U(1) × SU(2), SU(3), then system of equation (11) can be used for the description of the electromagnetic (QED) interaction, the electroweak (EW) interaction, and the strong (QCD) interaction respectively.

- The second pair of Yang–Mills equations

  $$\begin{aligned} &\partial_\mu B_\nu - \partial_\nu B_\mu - [B_\mu, B_\nu] = G_{\mu\nu}, \\ &\partial_\mu G^{\mu\nu} - [B_\mu, G^{\mu\nu}] = \frac{3}{16} m^3 i h^\nu \end{aligned} \qquad (12)$$

  describes the Yang–Mills field $(B_\mu, G_{\mu\nu})$ with the symplectic group $\mathrm{Sp}(\mathcal{C}\ell(1,3))$ of gauge symmetry. The dimension of the Lie algebra sp($\mathcal{C}\ell(1,3)$) is equal to 10. Hence the Yang–Mills field $(B_\mu, G_{\mu\nu})$ describes 10 types of spin 1 elementary particles (mediators), which interact with the initial spin 1/2 particle (wave function (ih^μ, ϕ)). The model Dirac equation does not contain the mass m of spin 1/2 particle. We see mass m only at the right hand part of Yang–Mills equations (12) in the term $3m^3/16$. Therefore the constant $3m^3/16$ can be considered as a charge of spin 1/2 particle relevant to the gauge field $(B_\mu, G_{\mu\nu})$.

Conclusion. In considered model, which based on equations (7), spin 1/2 particles acquire masses by interaction between these fermions and the (spin 1) gauge field with symplectic symmetry.

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