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Existence of solutions and separation from singularities for a class of fourth order degenerate parabolic equations

Giulio Schimperna
Dipartimento di Matematica, Università di Pavia,
Via Ferrata 1, I-27100 Pavia, Italy
E-mail: giusch04@unipv.it

Sergey Zelik
Department of Mathematics, University of Surrey,
Guildford, GU2 7XH, United Kingdom
E-mail: S.Zelik@surrey.ac.uk

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Abstract

A nonlinear parabolic equation of the fourth order is analyzed. The equation is characterized by a mobility coefficient that degenerates at 0. Existence of at least one weak solution is proved by using a regularization procedure and deducing suitable a-priori estimates. If a viscosity term is added and additional conditions on the nonlinear terms are assumed, then it is proved that any weak solution becomes instantaneously strictly positive. This in particular implies uniqueness for strictly positive times and further time-regularization properties. The long-time behavior of the problem is also investigated and the existence of trajectory attractors and, under more restrictive conditions, of strong global attractors is shown.

Key words: degenerate fourth-order parabolic equation, separation from singularities, long-time behavior.

AMS (MOS) subject classification: 35K35, 35K65, 37L30.

1 Introduction

This paper is devoted to the analysis of the following class of fourth order parabolic equations:

$$u_t - \operatorname{div}(b(u)\nabla w) = 0, \tag{1.1}$$
$$w = \delta u_t - \Delta u + f(u) + \gamma(u) - g, \tag{1.2}$$

on $\Omega \times (0, +\infty)$, Ω being a bounded smooth subset of $\mathbb{R}^d$, $d \in \{2, 3\}$, coupled with the initial and boundary conditions

$$u|_{t=0} = u_0, \quad \text{in } \Omega, \tag{1.3}$$
$$\partial_{\boldsymbol{n}} u = b(u)\nabla w \cdot \boldsymbol{n} = 0, \quad \text{on } \partial\Omega. \tag{1.4}$$

The function $b(u)$ represents a solution-dependent mobility coefficient that possibly *degenerates* at 0 as a power of u (cf. (2.1) below), while the sum $f + \gamma$ stands for the derivative of a configuration potential W. In particular, we assume that f is the (dominating) monotone part, with $f(u) \sim u^{-\kappa}$ for some $\kappa > 1$, and that γ is a bounded and globally summable perturbation that accounts for possible nonconvexity of W. The coefficient δ in (1.2) is assumed to be nonnegative, with $\delta > 0$ describing the presence of *viscosity* effects. Finally, g is a smooth external forcing term.

In the two-dimensional case, problem (1.1)-(1.4) can describe the evolution of some classes of thin liquid films, with u representing the height of the film. Then, the singular behavior of f near 0 accounts for the presence of short-range repulsive forces, while the nonmonotone character of W' at ∞ (given by the term γ) is related to the occurrence of long-range repulsive forces. An extensive presentation of the underlying physical situation is given in [6] to which we refer for more details (see also [5, 15] and Remark 2.2 below).

In the three-dimensional case, the model is also physically relevant since it is closely related with the Cahn-Hilliard equation [8] with nonconstant mobility analyzed in a number of recent papers both in the nondegenerate and in the degenerate case (cf., e.g., [3, 4, 13, 19, 21] and the references therein). Indeed, if ± 1 represent the pure states of the order parameter in the Cahn-Hilliard model, then we can modify f into the form $f(u) \sim (1-u^2)^{-\kappa}$ and accordingly suppose that b degenerates near ± 1 (instead that near 0) as a power of $(1-u^2)$. Then, the so-modified system (1.1)-(1.2) turns out to represent a variant of the Cahn-Hilliard equation with degenerate mobility and singular potentials analyzed in the celebrated paper [13]. Correspondingly, all the results proved here for (1.1)-(1.2) also apply to the Cahn-Hilliard setting with straighforward modifications in the proofs (to be more precise, in the Cahn-Hilliard setting we would even have slightly stronger results since it would no longer be necessary to take care of the growth of b at infinity). This also motivates the choice of considering also the viscous case $\delta > 0$, which is particularly meaningful in the context of Cahn-Hilliard models (cf. [16], see also [20]).

Initial-boundary value problems related to (1.1)-(1.2) have been addressed in a number of recent contributions. In particular, in [6] further qualitative properties of the solutions are proved in the one-dimensional case and the stability properties of the steady states are investigated. The papers [15] (devoted to the one-dimensional case) and [14] (considering space dimensions 2 and 3) analyze problem (1.1)-(1.4) under assumptions on the nonlinear terms very similar to ours. In [14, 15], existence of a solution is proved by means of a nonnegativity-preserving finite-element scheme, which is also effective for a numerical investigation for the model. Finally, we mention the recent work [23], where the long-time behavior of the problem is studied in the one-dimensional setting. In this situation, the authors can prove strict positivity of the solution also in the nonviscous case, which allows them to show existence of a smooth global attractor by relying on the standard theory of infinite-dimensional dynamical systems.

Our first purpose in this paper is to prove existence of at least one weak solution to Problem (1.1)-(1.4) under general assumptions on the data, by using a regularization – a priori estimate – passage to the limit procedure. Compared to the proof given in [14], our method has the advantage to be relatively simple. Moreover, as a byproduct of our procedure we see that the possibly singular solutions to (1.1)-(1.4) can be approximated by the smooth and *positive* solutions of a regular PDE. Actually, if we consider, for instance, the nonviscous equation (i.e., the case $\delta = 0$), then we can construct the ε-approximation taking $\delta_\varepsilon > 0$ (which, as noted above, is physically motivated at least in the Cahn-Hilliard setting) and choosing a sufficiently singular function f_ε, with $\delta_\varepsilon \searrow 0$ and $f_\varepsilon \to f$ in the limit $\varepsilon \searrow 0$. Then, for $\varepsilon > 0$ Theorem 6.1 applies (cf. also Remark 6.3); hence, the approximating solutions u_ε are smooth and positive. Moreover, the very same argument used to pass to the limit in Subsec. 3.3 below shows that any limit point of $\{u_\varepsilon\}$ for $\varepsilon \searrow 0$ solves the original nonviscous problem.

In comparison with [14, 15], we have here the extra assumption that the singularity of $1/b(u)$ at $u = 0$ is not stronger than the singularity of the potential F (the antiderivative of f), cf. (2.3) below. On the one hand, this assumption looks natural and is satisfied for the physically relevant examples of $b(u)$ and $f(u)$. Indeed, the most used non-linearity b in the theory of thin films is (cf., e.g., [6] or [14])

$$b(u) = u^3 + \beta^{3-n} u^n, \qquad n \in (0,3), \tag{1.5}$$

and for the function f we have the following model examples (cf. [6]):

$$f(u) = Au^{-3} - Bu^{-9} \quad \text{or} \quad f(u) = Au^{-3} - Bu^{-4}, \tag{1.6}$$

where $A, B > 0$. Thus, in all the physically relevant cases (2.3) is satisfied. On the other hand, this assumption allows us not only to simplify the proof of the existence of a weak solution and to consider more general functions f and b, but also to use the Moser-type iteration technique for improving the regularity of the constructed weak solution in the viscous case.

Once the existence of a solution is achieved, we show dissipativity of the (multivalued) dynamical process associated to the system. This in particular entails existence of a (weak) trajectory attractor in the sense of Chepyzhov and Vishik [9].

Our subsequent results regard only the viscous case $\delta > 0$ (and in particular can be applied to the Cahn-Hilliard model up to the modifications described above). Then, we can also prove that the trajectory attractor can be intended in the strong sense (i.e., w.r.t. the strong topology of the natural phase space) at least under slightly more restrictive conditions on f. The proof relies on an ad-hoc integration by parts formula and a variant of the so-called energy method (cf. [1], see also [2, 18] and [10, 11, 12] for applications to trajectory attractors). If we furtherly restrict the class of admissible functions f (namely, asking κ to be large enough), then we can also prove that weak solutions become uniformly separated from 0 for any time $t > 0$, so that the degenerate character of the system is actually lost for strictly positive times. This result, which is in our opinion the main achievement of this paper, is shown by means of a suitable version of the Moser iteration scheme which takes time regularization effects into account. As a further consequence of this "separation" property, we can also prove arbitrarily high regularity of weak solutions, as well as uniqueness, at least for $t > 0$. In turn, this permits to interpret the global attractor in the frame of the standard (single-valued) theory [1, 22], rather than in the trajectory sense.

The plan of the paper is as follows. In the next Section 2 we will report our notation and hypotheses and the statement of our existence result. Its proof is divided into several steps and will be presented in Section 3. In Section 4, we will show the existence of weak trajectory attractors. In Section 5, we will prove existence of a strong trajectory attractor in the viscous case $\delta > 0$ by applying the so-called energy method. Finally, in Section 6, we will prove the strict positivity of u in the viscous case under more restrictive growth conditions on f.

2 Existence result

Let Ω be a smooth bounded domain of $\mathbb{R}^d$, $d \in \{2, 3\}$. Let $T > 0$ a given final time and let $Q := \Omega \times (0, T)$. Let $H := L^2(\Omega)$, endowed with the standard scalar product $(\cdot, \cdot)$ and norm $\|\cdot\|$. Let also $V := H^1(\Omega)$. We note by $\|\cdot\|_X$ the norm in the generic Banach space X.

We make the following assumptions on data:

$$b(r) = r^s + \beta r^n, \qquad \beta \geq 0, \quad r \geq 0; \tag{2.1}$$

$$0 \leq n \leq s < 10 \text{ if } d = 3, \qquad 0 \leq n \leq s \text{ if } d = 2; \tag{2.2}$$

$$f(r) = -\frac{1}{r^\kappa}, \qquad \kappa > 1, \quad \kappa \geq s + 1, \quad r > 0; \tag{2.3}$$

$$\gamma \in W^{1,\infty}(\mathbb{R}) \cap L^1(\mathbb{R}); \tag{2.4}$$

$$g \in V \cap L^\infty(\Omega), \qquad \hat{g} := \|g\|_{L^\infty(\Omega)}; \tag{2.5}$$

$$\delta \geq 0. \tag{2.6}$$

Remark 2.1. Our key assumption here is that the singular character of f has to dominate over the degeneracy of b at 0 (cf. (2.3)). Actually, one could see with straighforward modifications in the proofs that in the case $\beta > 0$ (i.e., if $b(r)$ has a lower order degeneration r^n at 0), then it would be enough to ask $\kappa \geq n + 1$ rather than $\kappa \geq s + 1$. We also point out that the requirement $s < 10$ in the three-dimensional case is motivated by the growth of b at ∞ (and not by its degeneration at 0). Consequently, in the application to the Cahn-Hilliard model (where solutions have to stay in between the two barriers $r = \pm 1$), s could in fact be arbitrary also in the case $d = 3$.

We also define, whenever they make sense, the following functions:

$$F(r) := \frac{1}{\kappa - 1} + \int_1^r f(\tau)\, \mathrm{d}\tau, \qquad \Gamma(r) := \int_1^r \gamma(\tau)\, \mathrm{d}\tau, \qquad W(r) := F(r) + \Gamma(r). \tag{2.7}$$

Remark 2.2. According, e.g., to [6], a physically relevant expression for $W' = f + \gamma$ is given by

$$W'(r) \sim -\frac{1}{r^\kappa} + \frac{1}{r^k}, \qquad \text{where } k \in (1, \kappa). \tag{2.8}$$

Actually, this situation is not covered by our assumptions (2.3)-(2.4). However, it is clear that, just with technical modifications in the proofs, one could replace (2.3) with something like

$$c_1 \frac{1}{r^{\kappa+1}} \le f'(r) \le c_2 \frac{1}{r^{\kappa+1}}, \tag{2.9}$$

for all $r > 0$ and some $c_1, c_2 > 0$. Assuming (2.9) and properly choosing γ, it is clear that we can deal with the case (2.8). Nonetheless, we will assume (2.3) in place of (2.9) in order to reduce technical complications in the proofs.

Next, we introduce the energy functional associated to system (1.1)-(1.2):

$$\mathcal{E}(u) := \int_\Omega \Big(\frac{|\nabla u|^2}{2} + W(u) - gu \Big). \tag{2.10}$$

This leads to defining the *energy space*, that will act as a phase space for our system:

$$\mathcal{X} := \big\{ u \in V : \ u \ge 0 \text{ a.e.}, \ u^{1-\kappa} \in L^1(\Omega) \big\}. \tag{2.11}$$

The space $\mathcal{X}$ is endowed with the natural (graph) metric

$$\mathrm{d}_{\mathcal{X}}(u_1, u_2) := \|u_1 - u_2\|_V + \big\| u_1^{1-\kappa} - u_2^{1-\kappa} \big\|_{L^1(\Omega)}, \tag{2.12}$$

which is readily proved to be complete. Given $\mu \in (0, \infty)$, we also define

$$\mathcal{X}_\mu := \big\{ u \in \mathcal{X} : \ u_\Omega = \mu \big\}, \tag{2.13}$$

$(\cdot)_\Omega$ denoting here and below the spatial average over Ω. Actually, integrating (1.1) in space one readily sees that the quantity $u_\Omega(t)$ is conserved in time for any solution u.

Thanks to (2.4), (2.5) and to the above conservation property, a direct computation shows that, for some $\alpha_\mu, c_\mu, C_\mu > 0$ (also depending on g), there holds

$$\alpha_\mu \Big(\|u\|_V^2 + \big\| u^{1-\kappa} \big\|_{L^1(\Omega)} \Big) - C_\mu \le \mathcal{E}(u) \le c_\mu \Big(1 + \|u\|_V^2 + \big\| u^{1-\kappa} \big\|_{L^1(\Omega)} \Big). \tag{2.14}$$

for all $u \in \mathcal{X}_\mu$. In particular, for a function u of assigned spatial mean μ, the finiteness of the energy $\mathcal{E}(u)$ corresponds exactly to the condition $u \in \mathcal{X}_\mu$.

The above notation is sufficient to define the class of weak solutions.

Definition 2.3. *A* weak solution *to problem* (1.1)-(1.4) *is a couple* (u, w)*, with*

$$u \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \qquad \delta^{1/2} u \in H^1(0,T;H), \tag{2.15}$$

$$F(u) \in L^\infty(0,T;L^1(\Omega)), \tag{2.16}$$

$$w \in L^{5/4}(0,T;W^{1,5/4}(\Omega)), \tag{2.17}$$

$$b^{1/2}(u) \nabla w \in L^2(0,T;H), \tag{2.18}$$

$$b(u)\nabla w \in L^{\frac{20}{10+s}}(Q), \qquad u_t \in L^{\frac{20}{10+s}}(0,T;(W^{1,\frac{20}{10-s}})^*(\Omega)), \tag{2.19}$$

$$f(u) \in L^2(0,T;L^1(\Omega)) \cap L^{5/3}(Q). \tag{2.20}$$

such that the following relations hold a.e. *in* $(0,T)$*:*

$$\langle u_t, \phi \rangle + \langle b(u) \nabla w, \nabla \phi \rangle = 0 \qquad \forall\, \phi \in W^{1,\frac{20}{10-s}}(\Omega), \tag{2.21}$$

$$\langle w, \psi \rangle = \delta (u_t, \psi) + \langle \nabla u, \nabla \psi \rangle + \langle f(u) + \gamma(u) - g, \psi \rangle \qquad \forall\, \psi \in V, \tag{2.22}$$

where $\langle \cdot, \cdot \rangle$ *denote suitable duality pairings, and such that, in addition,*

$$u|_{t=0} = u_0, \quad \text{a.e. in } \Omega. \tag{2.23}$$

Remark 2.4. The above regularity framework actually refers to the three-dimensional case. If $d = 2$, (2.17)-(2.20) could be improved a bit (we omit the details).

We can now state our basic existence result, which can be considered as a variant of the theorem proved in [14, Sec. 8] (our assumptions are, indeed, slightly different).

Theorem 2.5. *Let us assume* (2.1)-(2.6) *and let, for some* $\mu \in (0, \infty)$,

$$u_0 \in \mathcal{X}_\mu. \tag{2.24}$$

Then, problem (1.1)-(1.4) *admits* at least *one weak solution.*

3 Proof of Theorem 2.5

We will give the proof in the case $d = 3$ and just point out some minor differences occurring in the two-dimensional case.

3.1 Regularized problem

First of all, we introduce a suitably approximated statement. Namely, given $\varepsilon \in (0, 1)$, we set

$$b_\varepsilon(r) := b((r^2 + \varepsilon^a)^{1/2}), \tag{3.1}$$

$$f_\varepsilon(r) := \begin{cases} f(r) & \text{if } r \geq \varepsilon, \\ f(\varepsilon) + f'(\varepsilon)(r - \varepsilon) = -\dfrac{1}{\varepsilon^\kappa} + \dfrac{\kappa}{\varepsilon^{\kappa+1}}(r - \varepsilon) & \text{if } r < \varepsilon, \end{cases} \tag{3.2}$$

where $a > 0$ will be chosen later on, and it is intended that f_ε is defined for all $r \in \mathbb{R}$. It is then worth noting that

$$f'_\varepsilon(r) := \begin{cases} \dfrac{\kappa}{r^{\kappa+1}} & \text{if } r \geq \varepsilon, \\ \dfrac{\kappa}{\varepsilon^{\kappa+1}} & \text{if } r < \varepsilon. \end{cases} \tag{3.3}$$

Moreover, setting

$$F_\varepsilon(r) := \frac{1}{\kappa - 1} + \int_1^r f_\varepsilon(\tau)\, \mathrm{d}\tau, \tag{3.4}$$

we obtain that

$$F_\varepsilon(r) := \begin{cases} \dfrac{1}{(\kappa - 1)r^{\kappa-1}} & \text{if } r \geq \varepsilon, \\ \dfrac{1}{(\kappa - 1)\varepsilon^{\kappa-1}} - \dfrac{1}{\varepsilon^\kappa}(r - \varepsilon) + \dfrac{\kappa}{2\varepsilon^{\kappa+1}}(r - \varepsilon)^2 & \text{if } r < \varepsilon. \end{cases} \tag{3.5}$$

At this point, we can consider the approximate statement

$$u_{\varepsilon,t} - \operatorname{div}(b_\varepsilon(u_\varepsilon)\nabla w_\varepsilon) = 0, \tag{3.6}$$

$$w_\varepsilon = \delta u_{\varepsilon,t} - \Delta u_\varepsilon + f_\varepsilon(u_\varepsilon) + \gamma(u_\varepsilon) - g, \tag{3.7}$$

coupled with the initial conditions and the no-flux boundary conditions. Then, in analogy with [13, Thms. 2 and 4] (see also [3, Thm. 2.1]), we have the following existence result for approximate solutions.

Theorem 3.1. *Let us assume* (2.1)-(2.6) *and let* b_ε, f_ε *be specified by* (3.1)-(3.2). *Let, in addition,* $u_{0,\varepsilon} \in H^3(\Omega)$ *be defined as the (unique) solution to the elliptic problem*

$$u_{0,\varepsilon} - \varepsilon^2 \Delta u_{0,\varepsilon} = u_0, \qquad \partial_{\boldsymbol{n}} u_{0,\varepsilon}|_{\partial\Omega} = 0. \tag{3.8}$$

Then, there exists at least one couple $(u_\varepsilon, w_\varepsilon)$ *with*

$$u_\varepsilon \in L^\infty(0, T; V) \cap L^2(0, T; H^3(\Omega)), \qquad \delta^{1/2} u_\varepsilon \in H^1(0, T; H), \qquad \delta u_\varepsilon \in H^1(0, T; V), \tag{3.9}$$

$$w_\varepsilon \in L^2(0, T; V), \tag{3.10}$$

satisfying (3.6)-(3.7) *a.e. in* $\Omega \times (0,T)$*, together with*

$$u_\varepsilon|_{t=0} = u_{0,\varepsilon}, \quad \text{a.e. in } \Omega. \tag{3.11}$$

Moreover, in the case $\delta > 0$*, the couple* $(u_\varepsilon, w_\varepsilon)$ *is unique.*

Sketch of proof of Theorem 3.1. We can proceed by following very closely the proofs of the quoted results [13, Thms. 2 and 4], [3, Thm. 2.1]. For this reason, we will just give some very brief highlights. Actually, the main difference here is due to the the growth of b_ε at infinity. Nevertheless, one can of course truncate b_ε near ∞ and replace it by some approximation $b_{\varepsilon,\nu}$ of at most linear growth, such that $b_{\varepsilon,\nu}$ suitably tends to b_ε as $\nu \searrow 0$. Then, existence for $\nu > 0$ is proved similarly as in [13, 3] and it remains to prove suitable a-priori estimates uniform w.r.t. ν. However, for the sake of simplicity we will omit the ν-approximation and rather perform formal estimates on the ε-solution in order to show that it fulfills regularity properties (3.9)-(3.10).

Thus, we can firstly perform the *energy* and *entropy* estimate, as below. Using that $b_\varepsilon \geq \varepsilon^{as/2}$ and the Lipschitz continuity of f_ε, we then obtain

$$\|u_\varepsilon\|_{L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega))} + \delta^{1/2}\|u_\varepsilon\|_{H^1(0,T;H)} \leq c_\varepsilon, \tag{3.12}$$

$$\|b_\varepsilon^{1/2}\nabla w_\varepsilon\|_{L^2(0,T;H)} + \|w_\varepsilon\|_{L^2(0,T;V)} \leq c_\varepsilon. \tag{3.13}$$

Next, we test (3.7) by $\Delta^2 u_\varepsilon$. Using (3.8), (3.12)-(3.13), (2.5) and the global Lipschitz continuity of f_ε and γ, it is then not difficult to obtain

$$\|u_\varepsilon\|_{L^2(0,T;H^3(\Omega))} + \delta^{1/2}\|u_\varepsilon\|_{L^\infty(0,T;H^2(\Omega))} \leq c_\varepsilon. \tag{3.14}$$

In case $\delta > 0$, we can also test (3.7) by $-\Delta u_{\varepsilon,t}$, that yields

$$\delta\|u_\varepsilon\|_{H^1(0,T;V)} \leq c_\varepsilon. \tag{3.15}$$

This gives all desired regularity properties. Finally, to prove uniqueness in the case $\delta > 0$, we can test the difference of (3.7) by the difference of the $u_{\varepsilon,t}$ and the difference of (3.6) by the difference of $w_{\varepsilon,t}$. The details are left to the reader. ∎

3.2 A priori estimates

We now aim to obtain a number of a priori bounds, uniform in ε, with the purpose of removing the approximation.

Energy estimate. We test (3.6) by w_ε, (3.7) by $u_{\varepsilon,t}$, and sum the results. We then obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varepsilon(u_\varepsilon) + \delta\|u_{\varepsilon,t}\|^2 + \int_\Omega b_\varepsilon(u_\varepsilon)|\nabla w_\varepsilon|^2 = 0, \tag{3.16}$$

where

$$\mathcal{E}_\varepsilon(u) := \int_\Omega \Big(\frac{|\nabla u|^2}{2} + F_\varepsilon(u) + \Gamma(u) - gu\Big). \tag{3.17}$$

Remark 3.2. We point out that, even at the approximate level, this formal estimate is not completely justified in the case $\delta = 0$. Actually, w_ε is (only) in $L^2(0,T;V)$, while (3.6) is not an equation in $L^2(0,T;V')$ since b_ε grows fast at infinity. However it is clear that, performing a truncation of b_ε and then passing to the limit, the estimate could be justified.

Then, we integrate (3.16) in time and notice that, by (3.5),

$$\int_\Omega F_\varepsilon(u_{0,\varepsilon}) \leq \int_\Omega F(u_{0,\varepsilon}) \leq \int_\Omega F(u_0) + \big(f(u_{0,\varepsilon}), u_{0,\varepsilon} - u_0\big) \leq \int_\Omega F(u_0), \tag{3.18}$$

the latter inequality following from (3.8) and Green's formula. Thus, owing to (2.24), (3.16) gives

$$\|u_\varepsilon\|_{L^\infty(0,T;V)} + \delta^{1/2}\|u_{\varepsilon,t}\|_{L^2(0,T;H)} \leq c, \tag{3.19}$$

$$\|F_\varepsilon(u_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \leq c, \tag{3.20}$$

$$\|b_\varepsilon^{1/2}(u_\varepsilon)\nabla w_\varepsilon\|_{L^2(0,T;H)} \leq c. \tag{3.21}$$

Here and below, c denotes a positive constant, independent of ε and of time, whose value may vary even inside a single row. We will use the letter α to denote constants used in estimates from below.

Entropy estimate. Let us define, for $r \in (0, \infty)$, the *entropy* M, by setting

$$m(r) := \int_1^r \frac{\mathrm{d}\tau}{b(\tau)}, \qquad M(r) := \int_1^r m(\tau)\ \mathrm{d}\tau. \tag{3.22}$$

Clearly, M is a convex function that grows at most like r for $r \sim \infty$. Moreover, let us introduce its approximate version by taking, for $r \in \mathbb{R}$,

$$m_\varepsilon(r) := \int_1^r \frac{\mathrm{d}\tau}{b_\varepsilon(\tau)}, \qquad M_\varepsilon(r) := \int_1^r m_\varepsilon(\tau)\ \mathrm{d}\tau. \tag{3.23}$$

Clearly, M_ε is a convex function such that $M_\varepsilon \le M$ a.e. in $(0, \infty)$. Moreover, m_ε, M_ε tend to m, M, respectively, uniformly on compact sets of $(0, \infty)$.

Then, we can test (3.6) by $m_\varepsilon(u_\varepsilon)$, (3.7) by $-\Delta u_\varepsilon$, and sum the results. We deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega M_\varepsilon(u_\varepsilon) + \frac{\delta}{2}\|\nabla u_\varepsilon\|^2\Big) + \|\Delta u_\varepsilon\|^2 + \int_\Omega \big(f'_\varepsilon(u_\varepsilon) + \gamma'(u_\varepsilon)\big)|\nabla u_\varepsilon|^2 + \big(g, \Delta u_\varepsilon\big) = 0, \tag{3.24}$$

and we have to control some terms. Firstly, by (2.4) and Hölder's inequality, we have

$$\Big|\int_\Omega \gamma'(u_\varepsilon)|\nabla u_\varepsilon|^2 + \big(g, \Delta u_\varepsilon\big)\Big| \le c\big(1 + \|\nabla u_\varepsilon\|^2\big) + \frac{1}{2}\|\Delta u_\varepsilon\|^2. \tag{3.25}$$

We now observe that, for all $\varepsilon \in (0,1)$, it is $M_\varepsilon(u_{0,\varepsilon}) \le M(u_0)$ (to prove this, proceed as in (3.18)). Moreover, we have that $M(u_0) \in L^1(\Omega)$. Actually, (2.24) entails $u_0^{1-\kappa} \in L^1(\Omega)$ and $M(r)$ grows no faster than r^{2-s} in the neighbourhood of 0, which is good since we assumed $\kappa \ge s+1$ (cf. (2.3)). Thus, integrating (3.25) in time we arrive at

$$\|u_\varepsilon\|_{L^2(0,T;H^2(\Omega))} \le c, \tag{3.26}$$

$$\int_0^T \int_\Omega f'_\varepsilon(u_\varepsilon)|\nabla u_\varepsilon|^2 \le c. \tag{3.27}$$

Control of the nonlinear terms. Here, our aim is to derive ε-uniform bounds in order to pass to the limit in the terms $b_\varepsilon(u_\varepsilon)$ and $f_\varepsilon(u_\varepsilon)$.

First of all, by (3.19), (3.26) and interpolation, we obtain, if $d = 3$,

$$\|u_\varepsilon\|_{L^{10}(\Omega\times(0,T))} \le c, \tag{3.28}$$

whereas for $d = 2$ we have instead

$$\|u_\varepsilon\|_{L^{\mathfrak{p}}(\Omega\times(0,T))} \le c_{\mathfrak{p}} \quad \forall\, \mathfrak{p} \in [1, \infty). \tag{3.29}$$

Let us now set

$$Q^1_\varepsilon := \big\{(x,t) \in Q :\ u_\varepsilon(x,t) < \varepsilon\big\}, \tag{3.30}$$

$$Q^2_\varepsilon := \big\{(x,t) \in Q :\ \varepsilon \le u_\varepsilon(x,t) \le 1\big\}, \tag{3.31}$$

$$Q^3_\varepsilon := \big\{(x,t) \in Q :\ u_\varepsilon(x,t) > 1\big\} \tag{3.32}$$

and notice that, by (3.5) and (3.20),

$$|\Omega^1_\varepsilon(t)| \le c\varepsilon^{\kappa-1} \quad \text{for a.e. } t \in (0,T). \tag{3.33}$$

Here and below, $\Omega^i_\varepsilon(t)$, $i \in \{1,2,3\}$, is the section of Q^i_ε at the generic time $t \in (0,T)$. Also, it is obvious that

$$\|f_\varepsilon(u_\varepsilon)\|_{L^\infty(Q^3_\varepsilon)} \le c. \tag{3.34}$$

Next, we remark (cf. (3.1)) that

$$\frac{1}{b_\varepsilon(r)} \le \begin{cases} \dfrac{c}{r^s+\beta r^n} & \text{if } r \ge \varepsilon, \\ \dfrac{c}{\varepsilon^{as/2}} & \text{if } r < \varepsilon. \end{cases} \tag{3.35}$$

Hence, using (3.5), (3.20), (3.33) and the condition $\kappa \ge s+1$ (cf. (2.3)), taking a small enough in the definition (3.1) of b_ε we obtain

$$\big\| b_\varepsilon^{-1}(u_\varepsilon) \big\|_{L^\infty(0,T;L^1(\Omega))} \le c. \tag{3.36}$$

Setting now

$$z_\varepsilon(x,t) := \max\big\{ u_\varepsilon(x,t), \varepsilon \big\}, \qquad v_\varepsilon(x,t) := z_\varepsilon(x,t)^{\frac{1-\kappa}{2}}, \tag{3.37}$$

for a.e. $t \in (0,t)$ we have

$$\begin{aligned} \|v_\varepsilon(t)\|^2_{L^6(\Omega)} &\le c\big(\|v_\varepsilon(t)\|^2 + \|\nabla v_\varepsilon(t)\|^2 \big) \\ &\le c \int_{\Omega^1_\varepsilon(t)} \frac{1}{\varepsilon^{\kappa-1}} + c \int_{\Omega^2_\varepsilon(t) \cup \Omega^3_\varepsilon(t)} \frac{1}{u_\varepsilon^{\kappa-1}} + c \int_{\Omega^2_\varepsilon(t) \cup \Omega^3_\varepsilon(t)} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^{\kappa+1}} \end{aligned} \tag{3.38}$$

(here and below, we compute the exponents referring to the case $d = 3$; for $d = 2$, they can be improved, of course). Hence, using (3.20), (3.24) and (3.33), and integrating in time, it is not difficult to arrive at

$$\|v_\varepsilon\|_{L^2(0,T;L^6(\Omega))} \le c, \tag{3.39}$$

whence, recalling (3.35) and possibly choosing a smaller a,

$$\big\| b_\varepsilon^{-1}(u_\varepsilon) \big\|_{L^1(0,T;L^3(\Omega))} \le c. \tag{3.40}$$

Then, using (3.21) and either (3.36) or (3.40), we also have

$$\|\nabla w_\varepsilon\|_{L^2(0,T;L^1(\Omega))} + \|\nabla w_\varepsilon\|_{L^1(0,T;L^{3/2}(\Omega))} \le c, \tag{3.41}$$

whence, by interpolation,

$$\|\nabla w_\varepsilon\|_{L^{5/4}(Q)} + \|\nabla w_\varepsilon\|_{L^{5/3}(0,T;L^{15/14}(\Omega))} \le c. \tag{3.42}$$

We are now ready to give an estimate of the term $f_\varepsilon(u_\varepsilon)$. To do this, we first test (3.7) by $u_\varepsilon - \mu$, to obtain

$$\mu \int_\Omega |f_\varepsilon(u_\varepsilon)| = \int_\Omega \big(-\delta u_{\varepsilon,t} + \Delta u_\varepsilon - \gamma(u_\varepsilon) + g \big)(u_\varepsilon - \mu) + \int_\Omega w_\varepsilon (u_\varepsilon - \mu) - \int_\Omega f_\varepsilon(u_\varepsilon) u_\varepsilon \tag{3.43}$$

and the terms on the right hand side are treated as follows:

$$\begin{aligned} \int_\Omega \big(-\delta u_{\varepsilon,t} + \Delta u_\varepsilon - \gamma(u_\varepsilon) + g \big)(\mu - u_\varepsilon) &\le c \big\| -\delta u_{\varepsilon,t} + \Delta u_\varepsilon - \gamma(u_\varepsilon) + g \big\| \|\mu - u_\varepsilon\| \\ &\le c \big\| -\delta u_{\varepsilon,t} + \Delta u_\varepsilon - \gamma(u_\varepsilon) + g \big\| =: \eta_1, \end{aligned} \tag{3.44}$$

thanks to (3.19), where $\|\eta_1\|_{L^2(0,T)} \le c$ by (2.4), (2.5) and (3.26). Next,

$$\int_\Omega w_\varepsilon (u_\varepsilon - \mu) = \int_\Omega \big(w_\varepsilon - (w_\varepsilon)_\Omega \big)(\mu - u_\varepsilon) \le c \|\nabla w_\varepsilon\| =: \eta_2, \tag{3.45}$$

with $\|\eta_2\|_{L^2(0,T)} \le c$ by (3.41). Finally, due to (3.2) it is clear that

$$-\int_\Omega f_\varepsilon(u_\varepsilon) u_\varepsilon \le \frac{\mu}{2} \int_\Omega |f_\varepsilon(u_\varepsilon)| + c_\mu. \tag{3.46}$$

Squaring (3.43), integrating in time, and using (3.44)-(3.46), we easily arrive at

$$\|f_\varepsilon(u_\varepsilon)\|_{L^2(0,T;L^1(\Omega))} \le c. \tag{3.47}$$

Setting now $\phi_\varepsilon := -|f_\varepsilon(r)|^{2/3}$, we test (3.7) by $\phi_\varepsilon(u_\varepsilon) - (\phi_\varepsilon(u_\varepsilon))_\Omega$, and integrate in space and time. This gives

$$\begin{aligned}\int_0^T \int_\Omega f_\varepsilon(u_\varepsilon)\phi_\varepsilon(u_\varepsilon) + \int_0^T \big(\delta u_{\varepsilon,t} + \gamma(u_\varepsilon) - g, \phi_\varepsilon(u_\varepsilon) - (\phi_\varepsilon(u_\varepsilon))_\Omega\big) \\ = \int_0^T \big(w_\varepsilon - (w_\varepsilon)_\Omega, \phi_\varepsilon(u_\varepsilon)\big) + \int_0^T \Big((\phi_\varepsilon(u_\varepsilon))_\Omega \int_\Omega f_\varepsilon(u_\varepsilon)\Big). \qquad (3.48)\end{aligned}$$

Let us first notice that the first term on the left hand side gives

$$\int_0^T \int_\Omega f_\varepsilon(u_\varepsilon)\phi_\varepsilon(u_\varepsilon) = \|f_\varepsilon(u_\varepsilon)\|^{5/3}_{L^{5/3}(Q)}. \qquad (3.49)$$

Next, by Hölder's inequality, we obtain

$$\begin{aligned}\Big| \int_0^T \big(\delta u_{\varepsilon,t} + \gamma(u_\varepsilon) - g, \phi_\varepsilon(u_\varepsilon) - (\phi_\varepsilon(u_\varepsilon))_\Omega\big) \Big| \\ \le \|\delta u_{\varepsilon,t} + \gamma(u_\varepsilon) + g\|^2_{L^2(Q)} + c\|\phi_\varepsilon(u_\varepsilon)\|^2_{L^2(Q)} \le \sigma \|f_\varepsilon(u_\varepsilon)\|^{5/3}_{L^{5/3}(Q)} + c_\sigma. \qquad (3.50)\end{aligned}$$

where σ is a small constant to be chosen below and $c_\sigma > 0$ depends on σ. Notice that (3.19), (2.4) and (2.5) have been used here.

It then remains to control the terms on the right hand side of (3.48). As far as the first one is concerned, recalling (3.34), we notice that, a.e. in $(0,T)$,

$$\begin{aligned}\big(w_\varepsilon - (w_\varepsilon)_\Omega, \phi_\varepsilon(u_\varepsilon)\big) &\le \|\phi_\varepsilon(u_\varepsilon)\|_{L^{5/2}(Q)} \|w_\varepsilon - (w_\varepsilon)_\Omega\|_{L^{5/3}(Q)} \\ &\le \sigma \|f_\varepsilon(u_\varepsilon)\|^{5/3}_{L^{5/3}(\Omega)} + c_\sigma \|\nabla w_\varepsilon\|^{5/3}_{L^{15/14}(\Omega)}. \qquad (3.51)\end{aligned}$$

Finally, let us estimate the latter term in (3.48). Actually, it is clear that

$$\int_0^T (\phi_\varepsilon(u_\varepsilon))_\Omega \int_\Omega f_\varepsilon(u_\varepsilon) \le c\|f_\varepsilon(u_\varepsilon)\|^2_{L^2(0,T;L^1(\Omega))} + c \le c, \qquad (3.52)$$

the latter inequality following from (3.47). Collecting now (3.48)-(3.52) and recalling (3.34), we finally arrive at

$$\|f_\varepsilon(u_\varepsilon)\|_{L^{5/3}(Q)} \le c. \qquad (3.53)$$

3.3 Passage to the limit

We will just consider, for brevity, the case $d = 3$. For simplicity of notation, let us set $\zeta_\varepsilon := b_\varepsilon^{1/2}(u_\varepsilon)\nabla w_\varepsilon$ and let $\|\cdot\|_p$ denote the norm in the space $L^p(Q)$. Then, by (3.21),

$$\|\zeta_\varepsilon\|_2 \le c. \qquad (3.54)$$

Moreover, being $s < 10$, (3.28) guarantees that

$$\big\|b_\varepsilon(u_\varepsilon)\nabla w_\varepsilon\big\|_q = \big\|b_\varepsilon^{1/2}(u_\varepsilon)\zeta_\varepsilon\big\|_q \le c \quad \text{for some } q > 1. \qquad (3.55)$$

Hence, by comparison in (3.6),

$$\|u_{\varepsilon,t}\|_{L^q(0,T;W^{-1,q}(\Omega))} \le c \quad \text{for some } q > 1 \qquad (3.56)$$

(of course, if $\delta > 0$ we have much more, but we want to deal with the most general case here). Consequently, using (3.19), (3.28), the Aubin-Lions lemma, and Lebesgue's theorem,

$$u_\varepsilon \to u \quad \text{strongly in } L^q(\Omega \times (0,T)) \quad \forall\, q \in [1,10) \qquad (3.57)$$

(here and below, all convergence relations are to be intended up to the extraction of non-relabelled subsequences). Let us now notice that

$$b_\varepsilon \to \bar{b} \quad \text{uniformly on compact subsets of } \mathbb{R}, \tag{3.58}$$

where $\bar{b}$ denotes the even extension of b to $\mathbb{R}$. Then, by (3.57), $s < 10$ in (2.1), and Lebesgue's theorem again, we obtain

$$b_\varepsilon(u_\varepsilon) \to \bar{b}(u) \quad \text{strongly in } L^q(\Omega \times (0,T)) \quad \text{for some } q > 1. \tag{3.59}$$

Analogously, by (3.53),

$$f_\varepsilon(u_\varepsilon) \to f(u) \quad \text{strongly in } L^q(\Omega \times (0,T)) \quad \text{for all } q \in [1, 5/3), \tag{3.60}$$

whence, in particular, the limit u is a.e. nonnegative and we can replace $\bar{b}$ with b in (3.59).

Our next aim is to pass to the limit in the product $b_\varepsilon(u_\varepsilon)\nabla w_\varepsilon$. To do this, we first notice that, by (3.57) and $s < 10$,

$$b_\varepsilon(u_\varepsilon)^{1/5} \to b(u)^{1/5} \quad \text{strongly in } L^q(Q) \quad \text{for some } q > 5. \tag{3.61}$$

Thus, using also (3.42) we arrive at

$$b_\varepsilon(u_\varepsilon)^{1/5}\nabla w_\varepsilon \to b(u)^{1/5}\nabla w \quad \text{weakly in } L^q(Q) \quad \text{for some } q > 1. \tag{3.62}$$

Next, interpolating between (3.36) and (3.40), we get

$$\big\|b_\varepsilon(u_\varepsilon)^{-1}\big\|_{5/3} \le c. \tag{3.63}$$

Using (3.63) and (3.54), we then obtain

$$\big\|b_\varepsilon(u_\varepsilon)^{1/5}\nabla w_\varepsilon\big\|_{25/17} \le \|\zeta_\varepsilon\|_2 \big\|b_\varepsilon(u_\varepsilon)^{-3/10}\big\|_{50/9} \le c. \tag{3.64}$$

Hence, by (3.64), (3.61) and (3.62),

$$b_\varepsilon(u_\varepsilon)^{2/5}\nabla w_\varepsilon = b_\varepsilon(u_\varepsilon)^{1/5}\big(b_\varepsilon(u_\varepsilon)^{1/5}\nabla w_\varepsilon\big) \to b(u)^{2/5}\nabla w \quad \text{weakly in } L^{25/22}(Q). \tag{3.65}$$

Then, writing $b_\varepsilon(u_\varepsilon)^{2/5}\nabla w_\varepsilon$ as $\zeta_\varepsilon b_\varepsilon(u_\varepsilon)^{-1/10}$ and using once more (3.54) and (3.63), it is clear that the exponent $25/22$ in (3.65) can be improved. Thus, iterating the above procedure and using once more (3.54), it is not difficult to arrive at

$$\zeta_\varepsilon \to b(u)^{1/2}\nabla w \quad \text{weakly in } L^2(Q), \tag{3.66}$$

whence, using once more (3.59) (and computing explicitly the exponent), we finally obtain

$$b_\varepsilon(u_\varepsilon)\nabla w_\varepsilon \to b(u)\nabla w \quad \text{weakly in } L^{\frac{20}{10+s}}(Q). \tag{3.67}$$

Thus, we can take the limit of all terms in (3.6)-(3.7) and get back (2.21)-(2.22), where the properties required to test functions depend of course on the regularity conditions proved above. Indeed, (2.15)-(2.20) follow as a direct byproduct of the procedure. The proof of Theorem 2.5 is concluded.

4 Weak trajectory attractors

In this section, we construct the so-called weak trajectory attractor for problem (1.1)-(1.2). To avoid technicalities, we will limit ourselves to deal with the (more degenerate) case $\beta = 0$, i.e., $b(u) = u^s$, for $s \in [0, 10)$ (cf. (2.1)).

We start by proving a dissipativity result holding for the weak solutions constructed in the proof of Theorem 2.5.

Theorem 4.1. *Let the assumptions of* Theorem 2.5 *hold, with* $\beta = 0$*, and let*

$$\mathbb{E}_0 := \mathcal{E}(u_0), \tag{4.1}$$

that is finite thanks to (2.24)*. Then, there exist a weak solution* (u, w) *and a monotone function* $Q : [0, \infty) \to [0, \infty)$ *such that*

$$\mathcal{E}(u(t)) + \int_0^t \big(\delta\|u_t\|^2 + \|b^{1/2}(u)\nabla w\|^2\big) \le Q(\mathbb{E}_0) \quad \forall\, t \ge 0. \tag{4.2}$$

More precisely, there exists a set $\mathcal{B}_0$ *bounded with respect to the metric* (2.12)*, such that, for any* $\mathrm{d}_{\mathcal{X}}$*-bounded set* $B \subset \mathcal{X}_\mu$*, there exist a time* $T_B \ge 0$ *such that for any initial datum* $u_0 \in B$ *there exists at least one weak solution* u *starting from* u_0 *and such that* $u(t) \in \mathcal{B}_0$ *for all* $t \ge T_B$*.*

Remark 4.2. Notice that the above result does not claim that dissipativity holds in the whole class of weak solutions, but just that from any admissible initial datum there starts (at least) one weak solution in the dissipative class (cf. Remark 4.7 for further considerations).

PROOF. We integrate (3.16) between 0 and an arbitrary $t > 0$. This gives the ε-equivalent of (4.2). Then, we take the lim inf with respect to $\varepsilon \searrow 0$ and use estimates (3.19), (3.57), (3.66), the fact that F_ε converges to F uniformly on compact sets of $(0, \infty)$, Fatou's Lemma, and the lower semicontinuity of norms with respect to weak or weak star convergences (of course, we will get an energy *inequality*, and not necessarily an equality, in this way). Relation (4.2) is proved.

To show dissipativity, we start considering the case $\kappa > s+1$. Then, it is convenient to rewrite the energy inequality in the differential form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u) + \delta\|u_t\|^2 + \int_\Omega b(u)|\nabla w|^2 \le 0, \tag{4.3}$$

for a.e. $t > 0$. We then notice that, setting $z := u^{-1}$, we have, from (2.14),

$$\sigma\big(\|z^{\kappa-1}\|_{L^1(\Omega)} + \|\nabla u\|^2\big) - c \le \mathcal{E}(u) \le \sigma^{-1}\big(\mu\|z\|^\kappa_{L^\kappa(\Omega)} + \|\nabla u\|^2 + 1\big), \tag{4.4}$$

where $\sigma \in (0, 1)$ depends in particular on $\mu = u_\Omega$.

Now, similarly with (3.43), we test (1.2) by $u - \mu$, obtaining

$$\begin{aligned}\|\nabla u\|^2 + \mu\|z\|^\kappa_{L^\kappa(\Omega)} &\le \int_\Omega (-\delta u_t - \gamma(u) + g)(u - \mu) + \|z^{\kappa-1}\|_{L^1(\Omega)} + \int_\Omega w(u - \mu)\\ &\le \frac{1}{2}\|\nabla u\|^2 + c\big(1 + \delta^2\|u_t\|^2\big) + \frac{\mu}{2}\|z\|^\kappa_{L^\kappa(\Omega)} + c_\mu + c\|w - w_\Omega\|^2_{L^{3/2}(\Omega)}\\ &\le \frac{1}{2}\|\nabla u\|^2 + c\big(1 + \delta^2\|u_t\|^2\big) + \frac{\mu}{2}\|z\|^\kappa_{L^\kappa(\Omega)} + c_\mu + c\|\nabla w\|^2_{L^1(\Omega)}.\end{aligned} \tag{4.5}$$

The last term can be controlled this way:

$$\|\nabla w\|^2_{L^1(\Omega)} \le c\|z^s\|_{L^1(\Omega)} \int_\Omega b(u)|\nabla w|^2. \tag{4.6}$$

Next, being $\kappa > s+1$, using the first inequality in (4.4) and Jensen's inequality, we obtain, for suitable positive constants α,

$$\big(\mathcal{E}(u)\big)^{\frac{s}{\kappa-1}} \ge \alpha\|z^s\|_{L^1(\Omega)} \ge \alpha(z_\Omega)^s \ge \alpha(u_\Omega)^{-s} = \alpha\mu^{-s}. \tag{4.7}$$

From (4.4)-(4.7), we then obtain, for a suitable $c_* > 0$,

$$\mathcal{E}(u) \le c_*\Big(1 + \delta^2\|u_t\|^2 + \big(\mathcal{E}(u)\big)^{\frac{s}{\kappa-1}} \int_\Omega b(u)|\nabla w|^2\Big). \tag{4.8}$$

Thus, assuming $\mathcal{E} \ge 1$, which is of course not restrictive, we can divide by $\mathcal{E}^{s/(\kappa-1)}$ to obtain

$$\big(\mathcal{E}(u)\big)^{\frac{\kappa-1-s}{\kappa-1}} \le c_*\Big(1 + \delta^2\|u_t\|^2 + \int_\Omega b(u)|\nabla w|^2\Big). \tag{4.9}$$

Taking the $(1-\epsilon)$-power for $\epsilon \in (0,1)$, summing to (4.3), and applying Young's inequality, we then have, for some $\alpha > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u) + \alpha\Big(\big(\mathcal{E}(u)\big)^{\frac{(\kappa-1-s)(1-\epsilon)}{\kappa-1}} + \delta\|u_t\|^2 + \int_\Omega b(u)|\nabla w|^2\Big) \le c_\epsilon, \tag{4.10}$$

whence the thesis follows by integrating in time and applying the comparison principle for ODE's. More precisely, we also have a quantitative decay estimate for the energy.

In the case $\kappa = s+1$, we can say a little bit less, but dissipativity still holds. Actually, we can repeat the above procedure up to (4.8). Then, we notice that (4.3) implies in particular that

$$\int_0^\infty \Big(\delta\|u_t\|^2 + \int_\Omega b(u)|\nabla w|^2\Big) \le \mathbb{E}_0 < \infty. \tag{4.11}$$

Consequently, there exists at least one time T_* such that

$$T_* \in [0, 2c_*\mathbb{E}_0], \quad \text{and} \quad \delta\|u_t(T_*)\|^2 + \int_\Omega b(u(T_*))|\nabla w(T_*)|^2 \le \frac{1}{2c_*}. \tag{4.12}$$

Substituting in (4.8), we then have

$$\mathcal{E}(u(T_*)) \le 2c_* + \delta =: C^*. \tag{4.13}$$

Then, we obtain $\mathcal{E}(u(t)) \le C^*$ for any $t \ge T_*$ simply observing that, by (4.3), $\mathcal{E}$ is nonincreasing. ∎

As a next step (see [9] for more details), we need to rewrite the dissipative estimate in such a way that, on the one hand, we will be able to control all the norms which are necessary to pass to the weak limit in the space of solutions of the problem considered (and verify that the limit function is again a solution) and, on the other hand, be sure that the corresponding trajectory phase space will be translation invariant.

The following lemma improves the dissipative estimate (4.2) by adding the terms controlled by the entropy estimate.

Lemma 4.3. *Let the assumptions of* Theorem 4.1 *hold. Then, there exists a solution* (u,w) *of problem* (1.1)-(1.2) *which satisfies the following estimate:*

$$\begin{aligned}\mathcal{E}(u(T)) + \int_T^{T+1} \delta\|u_t(t)\|^2 + \|b^{1/2}(u(t))\nabla w(t)\|^2 + \|\Delta u(t)\|^2 + \big(f'(u(t))\nabla u(t), \nabla u(t)\big)\,\mathrm{d}t \\ \le Q(\mathcal{E}(u(0)))e^{-\alpha T} + C_*, \quad \forall T \ge 0.\end{aligned} \tag{4.14}$$

where the positive constants α *and* C_* *and the monotone function* Q *are independent of* t *and of the concrete choice of the solution* u.

Proof. As a consequence of Theorem 4.1, it is clear that (4.14) holds, for suitable Q and α, without the last two terms in the integral on the left hand side. To control these terms, it is sufficient, in the case $\kappa > s+1$, to sum (3.24) to (4.10) in the preceding proof. In the case $\kappa = s+1$, we know that there exists $T_* = T_*(\mathbb{E}_0)$ such that the energy is smaller than some constant C^* independent of the initial data for any $T \ge T_*$ (cf. (4.13)). Then, it is sufficient to integrate (3.24) over $(T, T+1)$ for $T \ge T_*$ to get

$$\begin{aligned}&\int_T^{T+1} \|\Delta u(t)\|^2 + \big(f'(u(t))\nabla u(t), \nabla u(t)\big)\,\mathrm{d}t \\ &\quad \le c + \int_\Omega M(u(T)) + \frac{\delta}{2}\|\nabla u(T)\|^2 \le Q(\mathcal{E}(T)) \le Q(C^*),\end{aligned} \tag{4.15}$$

as desired. Actually, it is clear that the energy controls from above the terms $M(u)$ and $\|\nabla u\|$. Being pedantic, all these estimates should be done on the level of approximations u_ε with passing to the limit after that; we directly performed the estimate on u just for brevity. ∎

Remark 4.4. Note also that all the norms involved into our definition of a weak solution (see Theorem 2.5) are under control if we assume that the weak solution satisfies (4.14). This fact, which can be verified exactly as in the proof of Theorem 2.5, is crucial in order to be able to pass to the weak limit on the space of weak solutions and establish that the absorbing set for the trajectory semigroup is indeed closed, see below.

We are now able to define the trajectory phase space and trajectory dynamical system associated with problem (1.1)-(1.2).

Definition 4.5. *Let $\mathcal{K}_+ \subset L^\infty(\mathbb{R}_+, \mathcal{E})$ be the set of all solutions u of problem* (1.1)-(1.2) *belonging to the class* (2.15)-(2.20) *which satisfy the following analogue of* (4.14)*:*

$$\mathcal{E}(u(T)) + \int_T^{T+1} \delta\|u_t(t)\|^2 + \|b^{1/2}(u(t))\nabla w(t)\|^2 + \|\Delta u(t)\|^2 + (f'(u(t))\nabla u(t), \nabla u(t))\ \mathrm{d}t \\ \le C_u e^{-\alpha T} + C_*, \quad \forall T \ge 0. \tag{4.16}$$

for some constant C_u depending on the solution u. Then, the shift semigroup $T(h)$, $h \ge 0$, acts on $\mathcal{K}_+$:

$$T(h) : \mathcal{K}_+ \to \mathcal{K}_+, \qquad (T(h)u)(t) := u(t+h). \tag{4.17}$$

We will refer below to $\mathcal{K}_+$ and $T(h) : \mathcal{K}_+ \to \mathcal{K}_+$ as a trajectory phase space and trajectory dynamical system associated with problem (1.1)-(1.2) *respectively.*

Furthermore, in order to be able to introduce the attractor of the trajectory dynamical system, we need to specify the topology on $\mathcal{K}_+$ as well as the class of bounded sets.

Definition 4.6. *We endow the set $\mathcal{K}_+$ with the topology induced by the embedding $\mathcal{K}_+ \subset \Theta_+^{weak} := [L^\infty_{\mathrm{loc}}(\mathbb{R}_+, H^1(\Omega)) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+, H^1(\Omega))]^{w^*}$, where w^* stands for the weak-star topology, and will refer to it as a weak topology on the trajectory phase space $\mathcal{K}_+$.*

A set $B \subset \mathcal{K}_+$ will be called bounded if inequality (4.16) *holds uniformly with respect to all $u \in B$, i.e., if*

$$C_B := \sup_{u \in B} C_u < \infty. \tag{4.18}$$

Remark 4.7. As usual (see [9] for the details), under the general assumptions of Theorem 4.1, we know neither the fact that any weak solution of problem (1.1)-(1.2) satisfies the energy inequality (4.16) nor that the constant C_u in (4.16) can be expressed in terms of $\mathcal{E}(u(0))$. Actually, it may be possible to construct a solution u which satisfies (4.14) for the initial moment $T = 0$ only and be unable to verify its analogue for other initial times. By this reason, attempting to replace (4.16) by (4.14) in the definition of the trajectory phase space $\mathcal{K}_+$, we lose the translation invariance $T(h)\mathcal{K}_+ \subset \mathcal{K}_+$ which is crucial for the attractors theory. However, as we will see below, under the more restrictive assumptions of Theorem 5.1, the answer to both the questions posed above is positive. So, in that case, every reasonably defined weak solution satisfies (4.14) and the boundedness condition is equivalent to the boundedness of $u(0)$ in the energy space.

Finally, we are now able to introduce the trajectory attractor for problem (1.1)-(1.2).

Definition 4.8. *set $\mathcal{A}^{tr} \subset \mathcal{K}_+$ is a (weak) trajectory attractor for problem* (1.1)-(1.2) *if the following conditions are satisfied:*

1) $\mathcal{A}^{tr}$ is compact in Θ_+^{weak};

2) It is strictly invariant with respect to the trajectory semigroup: $T(h)\mathcal{A}^{tr} = \mathcal{A}^{tr}$;

3) It attracts the images of all bounded sets of $\mathcal{K}_+$ as time tends to infinity, i.e., for every bounded subset $B \subset \mathcal{K}_+$ and every neighborhood $\mathcal{O}(\mathcal{A}^{tr})$ (in the topology of Θ_+^{weak}), there exists a time $T = T(B, \mathcal{O})$ such that

$$T(h)B \subset \mathcal{O}(\mathcal{A}^{tr}),$$

for all $h \ge T$.

Next, we can state the existence result for the above introduced object.

Theorem 4.9. *Let the assumptions of* Theorem 4.1 *hold. Then, problem* (1.1)-(1.2) *possesses a trajectory attractor* $\mathcal{A}^{tr}$ *in the sense of the above definition. Moreover, this attractor is generated by all complete (i.e., defined for all* $t \in \mathbb{R}$*) and bounded trajectories for that system. Namely, we have*

$$\mathcal{A}^{tr} := \mathcal{K}\big|_{t\geq 0}, \tag{4.19}$$

where $\mathcal{K} \subset L^\infty(\mathbb{R}, \mathcal{E})$ *is the set of all solutions of* (1.1)-(1.2) *which satisfy*

$$\mathcal{E}(u(T)) + \int_T^{T+1} \delta\|u_t(t)\|^2 + \|b^{1/2}(u(t))\nabla w(t)\|^2 + \|\Delta u(t)\|^2 + (f'(u(t))\nabla u(t), \nabla u(t))\ \mathrm{d}t \leq C_*,$$

for all $T \in \mathbb{R}$ *and some* $C_* > 0$.

Proof. As usual (see [9]), in order to show the attractor existence, we only need to verify the existence of a compact and bounded absorbing set for the trajectory dynamical system $T(h) : \mathcal{K}_+ \to \mathcal{K}_+$ (the continuity of the semigroup in the Θ_+^{weak}-topology is obvious since it is just a translation semigroup). Note that, due to (4.16), the set $\mathcal{B} \subset \mathcal{K}_+$ of solutions u satisfying

$$\mathcal{E}(u(T)) + \int_T^{T+1} \delta\|u_t(t)\|^2 + \|b^{1/2}(u(t))\nabla w(t)\|^2 \\ + \|\Delta u(t)\|^2 + (f'(u(t))\nabla u(t), \nabla u(t))\ \mathrm{d}t \leq 2C_*, \tag{4.20}$$

for all $T \geq 0$ will be an absorbing set for the semigroup $T(h)$ acting on $\mathcal{K}_+$. Obviously, this set is bounded (in the sense of the Definition 4.6). Thus, we only need to verify that it is compact in the Θ_+^{weak} topology.

Indeed, let $\{u_n\} \subset \mathcal{B}$ be a sequence of solutions. Then, due to estimate (4.20), this sequence is precompact in Θ_+^{weak}, so, without loss of generality, we may assume that $u_n \to u \in \Theta_+^{weak}$ in the topology of Θ_+^{weak} and we only need to verify that the limit function u solves (1.1)-(1.2) and satisfies (4.20) as well.

The proof of this fact repeats almost word by word the proof of the existence Theorem 2.5 and is even a bit simpler since we do not need to consider the regular approximations to f and b (note that the uniform estimate (4.20) allows us to control uniformly all of the norms involved into (2.15)-(2.20)). By this reason, we leave the rigorous proof to the reader.

Thus, all of the assumptions of the abstract attractor existence theorem are verified and the theorem is proved. ∎

5 Energy equalities and strong attraction

In the viscous case $\delta > 0$ and under slightly more restrictive assumptions on the growth of f, we can prove that $\mathcal{A}_{\mathrm{tr}}$ is in fact a *strong* trajectory attractor (i.e., it attracts with respect of the strong topology of $\mathcal{X}$). This is the object of our next result:

Theorem 5.1. *Let assumptions* (2.1)-(2.6) *hold and let, additionally,* $\delta > 0$ *and* $\beta = 0$*. In addition, let*

$$\kappa \geq \frac{3s}{2} + 1 \ \text{ if } d = 3, \qquad \text{and } \ \kappa \geq s + 1 \ \text{ if } d = 2. \tag{5.1}$$

Then, any *weak solution of problem* (1.1)-(1.2) *satisfies the additional regularity properties*

$$w \in L^2(0, T; H), \qquad f(u) \in L^2(0, T; H). \tag{5.2}$$

Moreover, a.e. in $(0, \infty)$*, there holds the following energy* equality*:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(u) + \delta\|u_t\|^2 + \int_\Omega b(u)|\nabla w|^2 = 0, \tag{5.3}$$

as well as the following entropy equality *(compare with* (3.24)*):*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega M(u) + \frac{\delta}{2}\|\nabla u\|^2\Big) + \|\Delta u\|^2 + \int_\Omega \big(f'(u) + \gamma'(u)\big)|\nabla u|^2 + \big(g, \Delta u\big) = 0. \tag{5.4}$$

Proof. Let us start proving (5.3) and first deal with the 3D-case. The key step is given by the following integration by parts formula.

Lemma 5.2. *Let $b \in L^p(\Omega)$ for some $p > 1$, with $b \geq 0$ a.e. in Ω. Let also $b^{-1} \in L^q(\Omega)$ for some $q > 3/2$ if $d = 3$ (respectively, for some $q > 1$ if $d = 2$). Let $\phi \in H$ and let w be the (unique) solution to the degenerate elliptic problem*

$$-\operatorname{div}(b\nabla w) + w = \phi, \quad \text{in } \Omega, \qquad (b\nabla w)\cdot \boldsymbol{n} = 0, \quad \text{on } \Gamma. \tag{5.5}$$

Then, $b|\nabla w|^2 \in L^1(\Omega)$ and

$$\big(-\operatorname{div}(b\nabla w), w\big) = \int_\Omega b|\nabla w|^2. \tag{5.6}$$

Remark 5.3. As it will be clear from the proof, the regularity of w is sufficient to state (5.5) in that "strong" form. In particular, since we have

$$b\nabla w \in L^{\frac{2p}{p+1}}(\Omega), \qquad \operatorname{div}(b\nabla w) \in H, \tag{5.7}$$

a suitable trace theorem (cf., e.g., [7, Thm. 2.7.6]) permits to interpret the boundary condition in (5.5) in the sense of trace operators as a relation in the space $W^{-\frac{p+1}{2p},\frac{2p}{p+1}}(\Gamma)$.

Proof of Lemma 5.2. Again, we prove the theorem for $d = 3$ and just point out some minor differences occurring for $d = 2$. Let A be the Laplace operator with 0-Neumann boundary conditions, namely,

$$A : V \to V', \qquad \langle Av, z\rangle := \int_\Omega \nabla v \cdot \nabla z. \tag{5.8}$$

Then, one can see $A + \mathrm{Id}$ as a strictly positive unbounded operator on H and consider fractional powers of it. For $\varepsilon > 0$, we let $b_\varepsilon := \max\{b, \varepsilon\}$. We now consider the approximate problem

$$\varepsilon A^3 w_\varepsilon - \operatorname{div}(b_\varepsilon \nabla w_\varepsilon) + w_\varepsilon = \phi, \qquad (b_\varepsilon\nabla w_\varepsilon)\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma. \tag{5.9}$$

Then, testing (5.9) by w_ε we obtain

$$\varepsilon\|A^{3/2}w_\varepsilon\|^2 + \int_\Omega b_\varepsilon|\nabla w_\varepsilon|^2 + \|w_\varepsilon\|^2 = (\phi, w_\varepsilon). \tag{5.10}$$

Thus, for all $\varepsilon > 0$, we have that $w_\varepsilon \in D((A + \mathrm{Id})^{3/2}) \subset H^3(\Omega)$.

From (5.10), we obtain that w_ε is bounded, independently of ε, in H. In addition, we have

$$\big\|\nabla w_\varepsilon\big\|_{L^{\frac{2q}{q+1}}(\Omega)} \leq \big\|b_\varepsilon^{1/2}\nabla w_\varepsilon\big\|\big\|b_\varepsilon^{-1/2}\big\|_{L^{2q}(\Omega)} \leq c \tag{5.11}$$

and, being $q > 3/2$, it follows $2q/(q+1) > 6/5$ (respectively, if $d = 2$, from $q > 1$ we have $2q/(q+1) > 1$). Thus, by standard compact embedding results, we have, up to a (nonrelabelled) subsequence of $\varepsilon \searrow 0$,

$$w_\varepsilon \to w \quad \text{weakly in } W^{1,\frac{2q}{q+1}}(\Omega) \quad \text{and strongly in } H. \tag{5.12}$$

Moreover, being $p > 1$, we can write

$$\|b_\varepsilon\nabla w_\varepsilon\|_{L^{\frac{2p}{p+1}}(\Omega)} \leq \|b_\varepsilon^{1/2}\nabla w_\varepsilon\|\|b_\varepsilon^{1/2}\|_{L^{2p}(\Omega)} \leq c. \tag{5.13}$$

Thus, $b_\varepsilon\nabla w_\varepsilon$ is bounded in $L^{2p/(p+1)}(\Omega) \subset D((A + \mathrm{Id})^{-1})$ and, consequently,

$$\big\|-\operatorname{div}(b_\varepsilon\nabla w_\varepsilon)\big\|_{D((A+\mathrm{Id})^{-3/2})} \leq c \tag{5.14}$$

and, proceeding similarly with Subsection 3.3, we can also prove that $b_\varepsilon\nabla w_\varepsilon$ tends to $b\nabla w$ weakly in $L^{\frac{2p}{p+1}}(\Omega)$. Moreover, (5.14) tells us that, for any $\varepsilon > 0$, equation (5.9) makes sense at least as a relation in $D((A+\mathrm{Id})^{-3/2})$ (in particular, the obtained regularity $w_\varepsilon \in D((A+\mathrm{Id})^{3/2})$ justifies having used w_ε as a test function in (5.9)).

Now, the obvious fact that $b_\varepsilon \to b$ strongly in $L^p(\Omega)$, the first of (5.12), and Ioffe's theorem (see, e.g., [17]) give

$$\int_\Omega b|\nabla w|^2 \le \liminf_{\varepsilon\searrow 0} \int_\Omega b_\varepsilon |\nabla w_\varepsilon|^2. \tag{5.15}$$

Thus, using (5.10) and the second of (5.12), we can go on as follows:

$$\begin{aligned}\int_\Omega b|\nabla w|^2 &\le \lim_{\varepsilon\searrow 0}(\phi - w_\varepsilon, w_\varepsilon) + \liminf_{\varepsilon\searrow 0}\big(-\varepsilon\|A^{3/2}w_\varepsilon\|^2\big)\\ &\le (\phi - w, w) = \big(-\operatorname{div}(b\nabla w), w\big),\end{aligned} \tag{5.16}$$

where (5.5) has been used to deduce the last equality. Thus, to complete the proof, we have to show the inequality converse to (5.16). Now, $b\nabla w \in L^{2p/(p+1)}(\Omega)$ thanks to (the lim inf of) (5.13). Thus, also $-\operatorname{div}(b\nabla w) \in D((A+\mathrm{Id})^{-3/2})$ so that we can test (5.5) by $w_\varepsilon \in D((A+\mathrm{Id})^{3/2})$ and rigorously integrate by parts to obtain

$$\begin{aligned}\big(-\operatorname{div}(b\nabla w), w_\varepsilon\big) &= \int_\Omega b\nabla w \cdot \nabla w_\varepsilon \le \frac12\int_\Omega b|\nabla w|^2 + \frac12 \int_\Omega b|\nabla w_\varepsilon|^2\\ &\le \frac12\int_\Omega b|\nabla w|^2 + \frac12\int_\Omega b_\varepsilon|\nabla w_\varepsilon|^2\\ &\le \frac12\int_\Omega b|\nabla w|^2 - \frac{\varepsilon}{2}\|A^{3/2}w_\varepsilon\|^2 + \frac12(\phi - w_\varepsilon, w_\varepsilon),\end{aligned} \tag{5.17}$$

where the fact that $b \le b_\varepsilon$ almost everywhere and the equality (5.10) have also been used. Passing to the limit and using the second (5.12) and that $-\operatorname{div}(b\nabla w) \in H$ (as it follows by comparison in (5.5)), we then obtain

$$\big(-\operatorname{div}(b\nabla w), w\big) \le \frac12\int_\Omega b|\nabla w|^2 + \frac12(\phi - w, w) \le \frac12\int_\Omega b|\nabla w|^2 + \frac12\big(-\operatorname{div}(b\nabla w), w\big). \tag{5.18}$$

Namely, we obtained the inequality converse to (5.16), whence the thesis. ∎

We now proceed with the proof of Theorem 5.1 and, precisely, of equality (5.3) under the assumption (5.1). To do this, we first prove (5.2) and, with this purpose, we set $z := u^{-1}$ and observe that equation (1.2) can be rewritten as

$$\delta z_t + z^2\Delta z^{-1} + z^{\kappa+2} = -z^2 w + z^2\phi, \qquad \text{where } \phi := \gamma(u) - g \tag{5.19}$$

and we notice that

$$\|\phi\|_{L^\infty(\Omega\times(0,T))} \le C, \tag{5.20}$$

thanks to (2.4)-(2.5). Here and below, C denotes a constant possibly depending on the "energy" of the initial data (cf. (2.24)) and on the choice of T, while c is an absolute constant (i.e., it does not depend on the initial data or on T). By (2.18), we also know that

$$\|u^{s/2}\nabla w\|_{L^2(\Omega\times(0,T))} \le C. \tag{5.21}$$

Next, by the first of (2.20), a comparison in (1.2) gives also

$$\|w\|_{L^2(0,T;L^1(\Omega))} \le C. \tag{5.22}$$

At this point, we note that, for $d = 3$, thanks to (2.16) and (5.1),

$$\|b^{-1/2}(u)\|_{L^\infty(0,T;L^3(\Omega))} \le C. \tag{5.23}$$

Hence, combining (5.23) with (5.22) and (2.18), we readily arrive at

$$\|w\|_{L^2(Q)} \le c\|w\|_{L^2(0,T;W^{1,6/5}(\Omega))} \le C, \tag{5.24}$$

whence (5.2) follows simply by comparing terms in (1.2) and taking advantage of (2.15)-(2.20). In the case $d = 2$, we only have the $L^\infty(0,T;H)$-norm in (5.23), but (5.2) still follows by using the continuous embedding $W^{1,1}(\Omega) \subset H$ in the analogue of (5.24).

We now proceed with the proof of (5.3). Summing together (1.1) and (1.2), we have

$$-\operatorname{div}(b(u)\nabla w) + w = \phi := (\delta - 1)u_t - \Delta u + f(u) + \gamma(u) - g \tag{5.25}$$

and it is clear from (2.4)-(2.5), (2.15) and (5.2) that, for any $T > 0$, it is $\|\phi\|_{L^2(0,T;H)} \le C_T$.

Moreover, from (2.2) and (the limit of) (3.28) (or (3.29)), we have that, a.e. in $(0, T)$, $b(u) \in L^p(\Omega)$ for a suitable $p > 1$ (e.g., if $d = 3$, we can take $p = 10/s$). Finally, thanks to (5.2), it is clear that (if $d = 3$, the case $d = 2$ being analogous), a.e. in $(0, T)$,

$$\|b^{-1}(u)\|_{L^q(\Omega)} \le \|z^s\|_{L^q(\Omega)} \le c\|z^{\frac{2(\kappa-1)}{3}}\|_{L^q(\Omega)} + c \le C, \tag{5.26}$$

for a suitable $q > 3/2$. Hence, $b = b(u)$ and ϕ defined in (5.25) satisfy, a.e. in $(0, T)$, the assumptions of Lemma 5.2. Consequently, as we test (1.1) by w and (1.2) by u_t, we obtain, thanks to (5.6), the energy equality (5.3), as desired.

Finally, we come to the proof of (5.4). From (2.20) and (5.1), we see that at least $m(u) \in L^2(Q)$, so since due to (2.15) $u_t \in L^2(Q)$ as well, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(M(u), 1) = (u_t, m(u)) = \big(\operatorname{div}(b(u)\nabla w), m(u)\big), \tag{5.27}$$

where the right-hand side is understood as a scalar product in $L^2(Q)$. Thus, we need to verify that

$$\big(\operatorname{div}(b(u)\nabla w), m(u)\big) = -\big(b(u)\nabla w, \nabla m(u)\big) = -(\nabla w, \nabla u) = (w, \Delta u), \tag{5.28}$$

almost everywhere in time. To this end, we note that, according to (2.15) and the embedding $H^2(\Omega) \subset C(\overline{\Omega})$, we have $u(t), b(u(t)) \in C(\overline{\Omega})$ for almost all t. Then, keeping in mind that $b^{1/2}(u)\nabla w \in L^2(Q)$ by (2.18), we conclude that $b(u(t))\nabla w(t) \in L^2(\Omega)$ for almost all t.

Thus, we only need to check that $m(u(t)) \in H^1(\Omega)$ for almost all t. The fact that this function belongs to $L^2(\Omega)$ is already verified, so we need that

$$\nabla m(u(t)) = b^{-1}(u(t))\nabla u(t) \in L^2(\Omega).$$

Since, due to (2.15), we know that $\nabla u(t) \in L^6(\Omega)$ for almost all t, it is sufficient to check that $b^{-1}(u(t)) \in L^3(\Omega)$. Actually, this follows immediately from the proved fact that $f(u) \in L^2(Q)$ (see the proof of the energy equality) and condition (5.1).

Thus, we have verified that, for almost all t, $m(u(t)) \in H^1(\Omega)$ and $b(u)\nabla w(t) \in L^2(\Omega)$. This justifies the first two equalities in (5.28). Note that the last one is obvious since $w(t) \in L^2(\Omega)$ and $\Delta u(t) \in L^2(\Omega)$ for almost all t. Thus, we have verified that

$$\frac{\mathrm{d}}{\mathrm{d}t}(M(u(t)), 1) = (w(t), \Delta u(t))$$

for almost all t. Inserting the expression for $w(t)$ from (1.2) into this identity, we end up with the desired entropy equality (5.4). Theorem 5.1 is proved. ∎

Our next task here is to obtain stronger results on the attraction to the above constructed trajectory attractor under the additional assumptions of Theorem 5.1. We start by stating a couple of corollaries that improve the results of Theorems 2.5 and 4.1 and simplify the construction of the trajectory phase space based on the energy and entropy inequalities obtained above.

Corollary 5.4. *Let the assumptions of* Theorem 5.1 *hold. Then, every weak solution of problem* (1.1)-(1.2) *satisfies the dissipative estimate* (4.14) *and, therefore, every weak solution automatically satisfies* (4.16) *with* $C_u = Q(\mathcal{E}(u(0))$. *Thus, the condition* (4.16) *in the definition of the trajectory phase space* $\mathcal{K}_+$ *can be omitted and we may naturally consider* $\mathcal{K}_+$ *just as the set of all weak solutions of problem* (1.1)-(1.2). *In addition, for every weak solution* u, *we have* $u \in C([0, T], H^1(\Omega))$ *and* $u^{1-\kappa} \in C([0, T], L^1(\Omega))$.

Indeed, we only need the entropy and energy (in)equalities in order to derive the dissipative estimate (4.14). Since these inequalities now hold for every weak solution, we have this estimate for every weak solution as well. The continuity properties stated in the corollary follow immediately from the energy equality.

Corollary 5.5. *Let the assumptions of* Theorem 5.1 *hold. Then, the set B is bounded in $\mathcal{K}_+$, in the sense of* Definition 4.6, *if and only if the set of the initial data $\{u(0),\ u \in B\}$ is bounded in the energy space $\mathcal{X}_\mu$.*

We are now able to state our main result on the strong convergence to the trajectory attractor.

Theorem 5.6. *Let the assumptions of* Theorem 5.1 *hold. Then, the trajectory attractor $\mathcal{A}^{tr}$ of problem* (1.1)-(1.2) *is compact in $C_{loc}(\mathbb{R}_+, \mathcal{X})$ and the attraction property holds in that strong topology as well (remind that the compactness in $C_{loc}(\mathbb{R}_+, \mathcal{X})$ means that the u-component of $\mathcal{A}^{tr}$ is compact in $C_{loc}(\mathbb{R}_+, H^1(\Omega))$ and the $u^{1-\kappa}$-component is compact in $C_{loc}(\mathbb{R}_+, L^1(\Omega))$).*

Proof. Let $\mathcal{B} \subset K_+$ be the absorbing set introduced in the proof of Theorem 4.9. We claim that the set

$$\mathcal{B}_1 := T(1)\mathcal{B}$$

is an absorbing set which is compact in the above mentioned topology (this is clearly enough for the proof of the theorem). Indeed, let $\{u_n\} \subset \mathcal{B}$ be an arbitrary sequence of solutions. Then, since $\mathcal{B}$ is compact in Θ_+^{weak}, we may assume without loss of generality that $u_n \to u$ in Θ_+^{weak}, where u also solves the problem (1.1)-(1.2). To verify the above mentioned compactness, we need to check that

$$u_n \to u \ \text{ in } \ C([1,N], H^1(\Omega)), \qquad u_n^{1-\kappa} \to u^{1-\kappa} \ \text{ in } \ C([1,N], L^1(\Omega)), \tag{5.29}$$

for every $N > 1$. Furthermore, without loss of generality, we may check these convergences for $N = 2$ only.

To this end, we will use the proved energy equality which we will rewrite in the following form:

$$\begin{aligned} T\Big[\frac{1}{2}\|\nabla u_n(T)\|^2 + (F(u_n(T)),1) + (\Gamma(u_n(T)),1) - (g, u_n(T))\Big] \\ + \int_0^T \delta t\|\partial_t u_n(t)\|^2 + t\big(b(u_n(t))\nabla w_n(t), \nabla w_n(t)\big)\ \mathrm{d}t \\ = \int_0^T \frac{1}{2}\|\nabla u_n(t)\|^2 + (F(u_n(t)),1) + (\Gamma(u_n(t)),1) - (g, u_n(t))\ \mathrm{d}t, \end{aligned} \tag{5.30}$$

where $T \in [1,2]$. Our next task is to pass to the limit $n \nearrow \infty$ in this inequality. First of all, thanks to the energy and entropy estimates and to the Aubin-Lions compactness theorem, we have

$$u_n \to u \quad \text{in } \ C_w([1,2]; V), \qquad \text{(strongly) in } \ C([1,2]; H) \cap L^2(1,2;V), \tag{5.31}$$

and pointwise (a.e.). Thus, using the first convergence above and Fatou's Lemma, we see that

$$\frac{1}{2}\|\nabla u(T)\|^2 \le \liminf_{n\nearrow\infty} \frac{1}{2}\|\nabla u_n(T)\|^2, \qquad (F(u(T)),1) \le \liminf_{n\nearrow\infty} (F(u_n(T)),1). \tag{5.32}$$

Next, thanks also to (2.4)-(2.5),

$$(\Gamma(u),1) - (g,u) = \lim_{n\nearrow\infty} (\Gamma(u_n),1) - (g,u_n) \quad \text{strongly in } \ C^0([1,2]). \tag{5.33}$$

Moreover, thanks to lower semicontinuity of norms w.r.t. weak convergence and to Ioffe's theorem, we also have

$$\int_0^T \delta t\|\partial_t u(t)\|^2\ \mathrm{d}t \le \liminf_{n\nearrow\infty} \int_0^T \delta t\|\partial_t u_n(t)\|^2\ \mathrm{d}t, \tag{5.34}$$

$$\int_0^T t\big(b(u(t))\nabla w(t), \nabla w(t)\big)\ \mathrm{d}t \le \liminf_{n\nearrow\infty} \int_0^T t\big(b(u_n(t))\nabla w_n(t), \nabla w_n(t)\big)\ \mathrm{d}t. \tag{5.35}$$

Finally, in order to pass to the limit in (5.30), we only need to prove that $F(u_n) \to F(u)$ *strongly* in $L^1([0,2]\times\Omega)$. Actually, this follows from the uniform L^2-bound of $f(u_n)$, the pointwise convergence

$u_n \to u$ and the generalized Lebesgue theorem. Thus, we can take the supremum limit $n \nearrow \infty$ in (5.30) and obtain the *inequality*:

$$\limsup_{n\nearrow\infty} T\Big[\frac{1}{2}\|\nabla u_n(T)\|^2 + (F(u_n(T)),1)\Big]$$
$$\leq -T\big[(\Gamma(u(T)),1) - (g,u(T))\big] - \int_0^T \delta t\|\partial_t u(t)\|^2 - t\big(b(u(t))\nabla w(t), \nabla w(t)\big)\,\mathrm{d}t$$
$$+ \int_0^T \frac{1}{2}\|\nabla u(t)\|^2 + (F(u(t)),1) + (\Gamma(u(t)),1) - (g,u(t))\,\mathrm{d}t. \tag{5.36}$$

On the other hand, applying the energy equality in the form (5.30) directly to the limit solution u, and comparing with (5.36), we obtain, for all $T \in (1,2)$,

$$\limsup_{n\nearrow\infty} T\Big[\frac{1}{2}\|\nabla u_n(T)\|^2 + (F(u_n(T)),1)\Big] \leq T\Big[\frac{1}{2}\|\nabla u(T)\|^2 + (F(u(T)),1)\Big],$$

whence, recalling (5.32), we infer

$$\|u_n(T)\|_V \to \|u(T)\|_V, \qquad \|F(u_n(T))\|_{L^1(\Omega)} \to \|F(u(T))\|_{L^1(\Omega)}.$$

This, together with the weak convergence $u_n(T) \to u(T)$ in V (cf. (5.31)) and $u_n \to u$ almost everywhere, implies the strong convergence

$$u_n(T) \to u(T) \text{ in } V, \qquad u_n^{1-\kappa}(T) \to u^{1-\kappa}(T) \text{ in } L^1(\Omega), \tag{5.37}$$

for all $T \in [1,2]$. This gives the strong convergence $u_n \to u$ in $\mathcal{X}$ pointwise in time. The desired uniform convergence (5.29) can be easily obtained using the standard contradiction arguments and applying the energy equality for $u_n(T_n)$ instead of $u_n(T)$. Theorem 5.6 is proved. ∎

Corollary 5.7. *Arguing in a similar way (and using also the entropy equality), one can verify the compactness and strong convergence to the trajectory attractor in all spaces involved in* (2.15)-(2.20).

6 Separation from singularities and uniqueness

In this section we prove that, in the viscous case $\delta > 0$, if κ is large enough, then any weak solution becomes uniformly strictly positive for any $t > 0$. This is the object of the following

Theorem 6.1. *Let assumptions* (2.1)-(2.6) *hold and let, additionally, $\delta > 0$ and $\beta = 0$. In addition, let*

$$\kappa > 2s + 3 \text{ if } d = 3, \qquad \text{and } \kappa > s + 1 \geq 2 \text{ if } d = 2. \tag{6.1}$$

Then, there exists a function $Q; [0,\infty)^2 \to [0,\infty)$, monotone in each of its arguments, such that any weak solution u satisfies, for any $\epsilon > 0$, the separation property

$$\|u^{-1}(t)\|_{L^\infty(\Omega)} \leq Q\big(\mathbb{E}_0, \epsilon^{-1}\big) \quad \text{for a.e. } t \geq \epsilon. \tag{6.2}$$

The proof of the theorem will be given later in this section. As a consequence, we also have further time-regularization properties that imply uniqueness for strictly positive times as well:

Theorem 6.2. *Let the assumptions of* Theorem 6.1 *hold (in particular, let $\delta > 0$). Then, for any $\epsilon > 0$ and any weak solution u there holds:*

$$w \in L^2(\epsilon, T; V), \tag{6.3}$$
$$u \in H^1(\epsilon, T; V) \cap L^\infty(\epsilon, \infty; H^2(\Omega)). \tag{6.4}$$

Moreover, in the class of weak solutions uniqueness holds at least for strictly positive times.

Remark 6.3. We point out that (6.3)-(6.4), which *suffice* to prove uniqueness, are however not presumed to be optimal properties. Actually, thanks to (6.2), (1.1) is nondegenerate and (1.2) is nonsingular for strictly positive times. Thus, by means of classical methods, one could easily prove that the solution u becomes, instantaneously in time, arbitrarily regular, provided of course that also the data γ and g are smooth.

As a consequence of uniqueness, we finally have

Corollary 6.4. *Let the assumptions of* Theorem 6.1 *hold (in particular, let $\delta > 0$). Then, the dynamical process generated by weak solutions admits the (strong) global attractor $\mathcal{A}$ in the standard sense (to be more precise, in the sense of semigroups with unique continuation). Namely, $\mathcal{A}$ is a compact and fully invariant subset of $\mathcal{X}_\mu$ such that, for any bounded set $B \subset \mathcal{X}_\mu$ there holds*

$$\lim_{t \nearrow \infty} \mathrm{d}_{\mathcal{X}}(u(t), \mathcal{A}) = 0, \tag{6.5}$$

uniformly with respect to weak solutions u such that $u(0) \in B$.

6.1 Proof of Theorem 6.1 in the 3D-case

We consider equation (1.2) rewritten in the form (5.19), where we assume $\delta = 1$ for simplicity. We also assume $s > 0$, the case $s = 0$ being simpler since one can directly take advantage of the $L^2(0, T; L^6(\Omega))$-regularity of w. Then, the proof is based on a suitable version of the Moser iteration argument, i.e., we will take $\nu > 1$ and test (5.19) by $\nu z^{\nu-1}$ for increasing exponents ν. We have to remark that this procedure, apparently having a formal character since the above test function could grow very fast and, hence, have insufficient regularity, can be easily justified simply by truncating z at some level K and then letting $K \nearrow \infty$. In particular, the argument does not require any approximation of the equation and, hence, works for *all* weak solutions in the class introduced in Definition 2.3

That said, testing (5.19) by $\nu z^{\nu-1}$ and integrating over $Q_\nu := \Omega \times (\tau_\nu, T)$, where the "initial" time τ_ν will be chosen later on, we then have

$$J_\nu^\nu + \iint_{Q_\nu} z^{\kappa+\nu+1} \le \|z(\tau_\nu)\|_{L^\nu(\Omega)}^\nu + \nu \iint_{Q_\nu} \phi z^{\nu+1} - \nu \iint_{Q_\nu} w z^{\nu+1}, \tag{6.6}$$

where we have set

$$J_\nu^\nu := \|z\|_{L^\infty(\tau_\nu,T;L^\nu(\Omega))}^\nu + \|\nabla z^{\nu/2}\|_{L^2(\tau_\nu,T;H)}^2. \tag{6.7}$$

Adding now

$$\|z^{\nu/2}\|_{L^2(\tau_\nu,T;H)}^2 = \|z\|_{L^\nu(Q_\nu)}^\nu \tag{6.8}$$

to both hands sides of (6.6), in order to recover the full V-norm of $z^{\nu/2}$, and setting

$$I_\nu^\nu := \|z\|_{L^\infty(\tau_\nu,T;L^\nu(\Omega))}^\nu + c_\Omega \|z\|_{L^\nu(\tau_\nu,T;L^{3\nu}(\Omega))}^\nu \ge \|z\|_{L^{5\nu/3}(Q_\nu)}^\nu, \tag{6.9}$$

where c_Ω is a suitable embedding constant, we then arrive at

$$I_\nu^\nu + \iint_{Q_\nu} z^{\kappa+\nu+1} \le \|z(\tau_\nu)\|_{L^\nu(\Omega)}^\nu + \nu C \|z\|_{L^{\nu+1}(Q_\nu)}^{\nu+1} + \|z\|_{L^\nu(Q_\nu)}^\nu - \nu \iint_{Q_\nu} w z^{\nu+1}, \tag{6.10}$$

and we have to provide a bound for the last term on the right hand side.

To do this, let (p, p^*) and (q, q^*) be two couples of conjugate exponents with $p \le 2$. Then, using also (5.22), we obtain

$$\begin{aligned} -\nu \iint_{Q_\nu} w z^{\nu+1} &\le \nu \|w\|_{L^p(\tau_\nu,T;L^q(\Omega))} \|z^{\nu+1}\|_{L^{p^*}(\tau_\nu,T;L^{q^*}(\Omega))} \\ &\le \nu \big(\|w\|_{L^p(\tau_\nu,T;L^1(\Omega))} + \|\nabla w\|_{L^p(Q_\nu)} \big) \|z^{\nu+1}\|_{L^{p^*}(\tau_\nu,T;L^{q^*}(\Omega))} \\ &\le \nu \big(C + \|\nabla w\|_{L^p(Q_\nu)} \big) \|z^{\nu+1}\|_{L^{p^*}(\tau_\nu,T;L^{q^*}(\Omega))}, \end{aligned} \tag{6.11}$$

provided that we choose $q = 3p/(3 - p)$, so that $W^{1,p}(\Omega) \subset L^q(\Omega)$ continuously.

Then, the term with ∇w is estimated this way:

$$\begin{aligned}\|\nabla w\|_{L^p(Q_\nu)} &= \|u^{s/2}z^{s/2}\nabla w\|_{L^p(Q_\nu)} \le \|u^{s/2}\nabla w\|_{L^2(Q_\nu)}\|z^{s/2}\|_{L^{\frac{2p}{2-p}}(Q_\nu)}\\ &\le C\|z^{s/2}\|_{L^{\frac{2p}{2-p}}(Q_\nu)},\end{aligned}\tag{6.12}$$

thanks also to (5.21). Thus, collecting (6.11) and (6.12), we have

$$-\nu\iint_{Q_\nu} wz^{\nu+1} \le C\nu\Big(1+\|z^{s/2}\|_{L^{\frac{2p}{2-p}}(Q_\nu)}\Big)\|z\|^{\nu+1}_{L^{p^*(\nu+1)}(\tau_\nu,T;L^{q^*(\nu+1)}(\Omega))}.\tag{6.13}$$

Now, let us assume to know a bound of the term $I_{n-1}=I_{\nu_{n-1}}$ from the preceding step of the iteration. Then, thanks to the last inequality in (6.9), we can use it to estimate the term in brackets so to have

$$-\nu\iint_{Q_\nu} wz^{\nu+1} \le C\nu\big(1+I^{s/2}_{n-1}\big)\|z\|^{\nu+1}_{L^{p^*(\nu+1)}(\tau_\nu,T;L^{q^*(\nu+1)}(\Omega))},\tag{6.14}$$

provided that one chooses p as follows:

$$\frac{p}{2-p}=\frac{5\nu_{n-1}}{3s},\quad \text{i.e.,}\quad \frac{1}{p}=\frac12+\frac{3s}{10\nu_{n-1}}.\tag{6.15}$$

This gives in turn

$$\frac1q=\frac16+\frac{3s}{10\nu_{n-1}},\qquad \frac{1}{p^*}=\frac12-\frac{3s}{10\nu_{n-1}},\qquad \frac{1}{q^*}=\frac56-\frac{3s}{10\nu_{n-1}}.\tag{6.16}$$

Then, we have to take $\nu=\nu_n$ in a way suitable for the next step of the iteration. The choice is dictated by the exponents of the last term in (6.14); namely, $\nu=\nu_n$ should be close enough to ν_{n-1} in order that term be still controlled by I_{n-1}. Using interpolation, we require that, for some $\theta\in[0,1]$,

$$\frac{1}{p^*(\nu_n+1)}=\frac{1-\theta}{\infty}+\frac{\theta}{\nu_{n-1}},\qquad \frac{1}{q^*(\nu_n+1)}=\frac{1-\theta}{\nu_{n-1}}+\frac{\theta}{3\nu_{n-1}}.\tag{6.17}$$

To compute θ, we first take the quotient of the above equalities and then use (6.16). This gives

$$\frac{3-2\theta}{3\theta}=\frac{p^*}{q^*}=\frac{25\nu_{n-1}-9s}{15\nu_{n-1}-9s},\tag{6.18}$$

whence

$$\theta=\frac{15\nu_{n-1}-9s}{35\nu_{n-1}-15s},\qquad \text{and}\qquad \nu_n=\frac{\nu_{n-1}}{p^*\theta}-1=\frac{7\nu_{n-1}-3s-6}{6},\tag{6.19}$$

where the second of (6.16) has also been used.

Thus, it turns out that $\nu_n>\nu_{n-1}$ provided that $\nu_{n-1}>3(s+2)$. Thus, in order the above iteration could be performed, we need to find some $\overline{n}\in\mathbb{N}$ and some $\nu_{\overline{n}}>3(s+2)$ such that, for any $\epsilon\in(0,1)$, there holds

$$I_{\overline{n}}=\Big(\|z\|^{\nu_{\overline{n}}}_{L^\infty(\epsilon,T;L^{\nu_{\overline{n}}}(\Omega))}+c_\Omega\|z\|^{\nu_{\overline{n}}}_{L^{\nu_{\overline{n}}}(\epsilon,T;L^{3\nu_{\overline{n}}}(\Omega))}\Big)^{\frac{1}{\nu_{\overline{n}}}}\le Q(\epsilon^{-1}),\tag{6.20}$$

where Q is a computable monotone function (whose expression can depend on the magnitude of the initial data and of T).

Let us pospone the verification of (6.20) and let us now see that, for $n>\overline{n}$, the induction principle can be applied. Coming back to (6.10), we then have

$$I_n^{\nu_n}\le\|z(\tau_n)\|^{\nu_n}_{L^{\nu_n}(\Omega)}+C\nu_n\big(1+\|z\|^{\nu_n+1}_{L^{\nu_n+1}(Q_n)}\big)+C\nu_n I_{n-1}^{\frac{s}{2}+\nu_n+1},\tag{6.21}$$

where we wrote n in place of ν_n in some subscripts and assumed w.l.o.g. $I_{n-1}\ge1$. Thus, extracting the ν_n-th root and noting that $\nu_n+1\le5\nu_{n-1}/3$, we obtain

$$I_n\le\|z(\tau_n)\|_{L^{\nu_n}(\Omega)}+(C\nu_n)^{\frac{1}{\nu_n}}I_{n-1}^{\eta_n},\quad \text{where }\ \eta_n:=\frac{2\nu_n+s+2}{2\nu_n}\tag{6.22}$$

and C is independent of n. Moreover, for (arbitrarily small) $\epsilon \in (0,1)$, given τ_{n-1} we can choose $\tau_n \in [\tau_{n-1}, \tau_{n-1} + \epsilon n^{-2}]$ such that

$$\|z(\tau_n)\|_{L^{\nu_n}(\Omega)}^{\frac{5\nu_{n-1}}{3}} \le c\|z(\tau_n)\|_{L^{\frac{5\nu_{n-1}}{3}}(\Omega)}^{\frac{5\nu_{n-1}}{3}} \le c\frac{n^2}{\epsilon}\int_{\tau_{n-1}}^{\tau_{n-1}+\epsilon n^{-2}} \|z\|_{L^{\frac{5\nu_{n-1}}{3}}(\Omega)}^{\frac{5\nu_{n-1}}{3}} \le c\frac{n^2}{\epsilon} I_{n-1}^{\frac{5\nu_{n-1}}{3}}. \tag{6.23}$$

Thus, (6.22) can be rewritten as

$$I_n \le \Big[\Big(c\frac{n^2}{\epsilon}\Big)^{\frac{3}{5\nu_{n-1}}} + (C\nu_n)^{\frac{1}{\nu_n}}\Big] I_{n-1}^{\eta_n}, \tag{6.24}$$

whence a standard computation permits to pass to the limit w.r.t. $n \nearrow \infty$. Since $\lim_{n\nearrow\infty} \tau_n$ exists and is less or equal than $c\epsilon$, we then obtain

$$\|u\|_{L^\infty(\Omega\times(\epsilon,T))} \le Q(\epsilon^{-1}), \tag{6.25}$$

for Q as in (6.20), as desired.

Thus, to conclude the proof it only remains to check that (6.20) holds. To do this, we come back to (6.10) and use now the $L^{\kappa+\nu+1}$-norm to estimate the right hand side. Proceeding as above, we still arrive at (6.13), where now we have to take

$$\frac{p}{2-p} = \frac{\kappa+\nu_{n-1}+1}{s} \quad \text{i.e.,} \quad \frac{1}{p} = \frac{1}{2}\Big(1 + \frac{s}{\kappa+\nu_{n-1}+1}\Big). \tag{6.26}$$

Thus, we obtain

$$\frac{1}{q} = \frac{1}{2}\Big(\frac{1}{3} + \frac{s}{\kappa+\nu_{n-1}+1}\Big), \qquad \frac{1}{p^*} = \frac{1}{2}\Big(1 - \frac{s}{\kappa+\nu_{n-1}+1}\Big), \qquad \frac{1}{q^*} = \frac{1}{2}\Big(\frac{5}{3} - \frac{s}{\kappa+\nu_{n-1}+1}\Big), \tag{6.27}$$

whence we get the analogue of (6.14), i.e.,

$$-\nu \iint_{Q_\nu} wz^{\nu+1} \le C\nu\big(1 + \Lambda_{n-1}^{s/2}\big)\|z\|_{L^{p^*(\nu+1)}(\tau_\nu,T;L^{q^*(\nu+1)}(\Omega))}^{\nu+1}, \tag{6.28}$$

where

$$\Lambda_\nu^\nu := \|z\|_{L^\infty(\tau_\nu,T;L^\nu(\Omega))}^\nu + \|\nabla z^{\nu/2}\|_{L^2(\tau_\nu,T;H)}^2 + \|z\|_{L^{\nu+\kappa+1}(Q_\nu)}^{\nu+\kappa+1} \tag{6.29}$$

and $\Lambda_n := \Lambda_{\nu_n}$, as before. Then, we still have to choose $\nu = \nu_n$ in a suitable way. Similarly as before, we require that for some $\theta \in [0,1]$ it is

$$\frac{1}{p^*(\nu_n+1)} = \frac{1-\theta}{\infty} + \frac{\theta}{\kappa+\nu_{n-1}+1}, \qquad \frac{1}{q^*(\nu_n+1)} = \frac{1-\theta}{\nu_{n-1}} + \frac{\theta}{\kappa+\nu_{n-1}+1}. \tag{6.30}$$

To compute θ, we first take the quotient of the above equalities and then use (6.27). This gives

$$\frac{\theta\nu_{n-1} + (1-\theta)(\nu_{n-1}+\kappa+1)}{\theta\nu_{n-1}} = \frac{p^*}{q^*} = \frac{5(\nu_{n-1}+\kappa+1) - 3s}{3(\nu_{n-1}+\kappa+1-s)}, \tag{6.31}$$

whence

$$\frac{1}{\theta} = 1 + \frac{2\nu_{n-1}}{3(\nu_{n-1}+\kappa+1) - 3s} \tag{6.32}$$

and, from the first of (6.30),

$$\nu_n = \frac{\nu_{n-1}+\kappa+1}{\theta p^*} - 1 = \frac{5}{6}\nu_{n-1} + \frac{1}{2}(\kappa - s - 1), \tag{6.33}$$

whence it is clear that $\nu_n > \nu_{n-1}$ if and only if $\nu_{n-1} < 3(\kappa - s - 1)$. Then, proceeding similarly with the previous part of the iteration, if we start knowing a bound of Λ_{ν_0} for some $\nu_0 > 1$, then we can reach, in a finite number $\overline{n}$ of steps, any $\nu_{\overline{n}} < 3(\kappa - s - 1)$. Since we also need $\nu_{\overline{n}} > 3(s+2)$

from before, this leads to the compatibility condition $3(s+2) < 3(\kappa - s - 1)$, that is equivalent to assumption (6.1).

Thus, the proof is concluded provided that we find $\nu_0 > 1$ to start the argument. Actually, we can test (5.19) by z^ι for small $\iota > 0$. We obtain, for $Q = \Omega \times (0, T)$,

$$J_{1+\iota}^{1+\iota} + \iint_Q z^{\kappa+2+\iota} \le \|z_0\|_{L^{1+\iota}(\Omega)}^{1+\iota} + \iint_Q \phi z^{2+\iota} - \iint_Q w z^{2+\iota} \tag{6.34}$$

and, being $\kappa > 3$ by (6.1), it is clear that, at least for $\iota < 1$,

$$\iint_Q \phi z^{2+\iota} - \iint_Q w z^{2+\iota} \le \frac{1}{2} \iint_Q z^{\kappa+2+\iota} + C, \tag{6.35}$$

thanks also to (5.2). Hence, we can take $\nu_0 = 1 + \iota$ for arbitrary $\iota \in (0,1)$, which concludes the proof in the case $d = 3$.

6.2 Proof of Theorem 6.1 in the 2D-case

The proof is carried out by the very same scheme used in the 3D case, the differences being limited to the exponents related to use of interpolation and embeddings. Thus, we limit ourselves to point out these differences. Now, in place of (6.9), we have

$$I_\nu^\nu := \|z\|_{L^\infty(\tau_\nu,T;L^\nu(\Omega))}^\nu + \|z^{\frac{\nu}{2}}\|_{L^2(\tau_\nu,T;V)}^2 \ge \|z\|_{L^{2\nu}(Q_\nu)}^\nu. \tag{6.36}$$

Thus, taking $p \in (1,2)$, we have $q = 2p/(2-p)$, so that, to control the right hand side of (6.14), we need to choose p so that

$$\frac{p}{2-p} = \frac{2\nu_{n-1}}{s}, \quad \text{i.e.,} \quad \frac{1}{p} = \frac{1}{2}\Big(1 + \frac{s}{2\nu_{n-1}}\Big), \tag{6.37}$$

whence we obtain

$$\frac{1}{q} = \frac{s}{4\nu_{n-1}}, \qquad \frac{1}{p^*} = \frac{1}{2}\Big(1 - \frac{s}{2\nu_{n-1}}\Big), \qquad \frac{1}{q^*} = 1 - \frac{s}{4\nu_{n-1}} \tag{6.38}$$

and, correspondingly,

$$\frac{1}{p^*(\nu_n+1)} = \frac{1-\theta}{\infty} + \frac{\theta}{2\nu_{n-1}}, \qquad \frac{1}{q^*(\nu_n+1)} = \frac{1-\theta}{\nu_{n-1}} + \frac{\theta}{2\nu_{n-1}}. \tag{6.39}$$

Thus,

$$\frac{2-\theta}{\theta} = \frac{p^*}{q^*} = \frac{4\nu_{n-1} - s}{2\nu_{n-1} - s}, \tag{6.40}$$

whence

$$\theta = \frac{2\nu_{n-1} - s}{3\nu_{n-1} - s}, \qquad \text{and} \qquad \nu_n = \frac{3}{2}\nu_{n-1} - \frac{s+2}{2}, \tag{6.41}$$

so that we need to find $\nu_{\overline{n}} > s + 2$ in order the procedure works.

To do this, we proceed again as before and, choosing p as in (6.26), the other exponents are then given by

$$\frac{1}{q} = \frac{s}{2(\kappa + \nu_{n-1} + 1)}, \qquad \frac{1}{p^*} = \frac{1}{2}\Big(1 - \frac{s}{\kappa + \nu_{n-1} + 1}\Big), \qquad \frac{1}{q^*} = 1 - \frac{s}{2(\kappa + \nu_{n-1} + 1)}. \tag{6.42}$$

Then, taking $\theta \in [0,1]$ as in (6.30), we now arrive at

$$\frac{\theta\nu_{n-1} + (1-\theta)(\nu_{n-1} + \kappa + 1)}{\theta\nu_{n-1}} = \frac{p^*}{q^*} = \frac{2(\nu_{n-1} + \kappa + 1) - s}{\nu_{n-1} + \kappa + 1 - s}, \tag{6.43}$$

whence

$$\frac{1}{\theta} = 1 + \frac{\nu_{n-1}}{\nu_{n-1} + \kappa + 1 - s}, \qquad \text{and} \qquad \nu_n = \nu_{n-1} + \frac{1}{2}(\kappa - 1 - s), \tag{6.44}$$

so that it is $\nu_n > \nu_{n-1}$ if and only if $\kappa > s + 1$, i.e., (6.1) holds. Thus, we can arrive in some finite number $\overline{n}$ of steps to have $\nu_{\overline{n}} > s + 2$ provided that we can start as before from $\nu_0 = 1 + \iota$ for some (small) $\iota > 0$. Actually, we can now take $\iota = \kappa - 2$, which is strictly positive thanks to (6.1). Thus, (6.34) can be repeated without any variation and, of course, we still have (6.35) thanks to Hölder's and Young's inequalities. The proof is complete.

6.3 Proof of Theorem 6.2 and Corollary 6.4

Again, we just consider the case $d = 3$, the case $d = 2$ being simpler. First of all, we deduce further regularity of weak solutions. Actually, thanks to (6.2), u is uniformly separated from 0 for any time $t \geq \epsilon > 0$, $\epsilon > 0$ being arbitrary. Then, (1.1) becomes in fact nondegenerate and the energy estimate gives the improved regularity (6.3). Moreover, the term $f(u)$ in (1.2) is now smooth and we can apply the linear parabolic theory (or test (1.2) by $-(t-\epsilon)\Delta u_t$ and perform standard computations) to deduce (6.4).

At this point, rewriting (1.1) as a family of time-dependent elliptic problems, namely

$$-\Delta w = \frac{1}{b(u)}\big(- u_t + b'(u)\nabla u \cdot \nabla w\big), \tag{6.45}$$

relations (6.4) and (6.3) permit to see that the right hand side belongs to $L^2(\epsilon, T; L^{3/2}(\Omega))$, whence we obtain

$$w \in L^2(\epsilon, T; W^{2,3/2}(\Omega)) \subset L^2(\epsilon, T; W^{1,3}(\Omega)). \tag{6.46}$$

To prove uniqueness, we can now consider a couple of solutions u_1, u_2, set $u := u_1 - u_2$ (and, correspondingly, $w := w_1 - w_2$) and take the difference of equations (1.1)-(1.2) to obtain

$$u_t - \operatorname{div}(b(u_1)\nabla w) = \operatorname{div}\big((b(u_1) - b(u_2))\nabla w_2\big), \tag{6.47}$$

$$w = \delta u_t - \Delta u + W'(u_1) - W'(u_2), \tag{6.48}$$

where $W' = f + \gamma$ can be thought to be globally Lipschitz in view of the strict positivity of u_1 and u_2. Then, we test (6.47) by w and (6.48) by u_t. We obtain, for some $c, \alpha > 0$,

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 + \alpha\|\nabla w\|^2 + \delta\|u_t\|^2 \\ &\qquad \leq c\|u\|\|u_t\| + \int_\Omega \big|(b(u_1) - b(u_2))\nabla w \cdot \nabla w_2\big| \end{aligned} \tag{6.49}$$

and we can estimate the last term as follows:

$$\begin{aligned} \int_\Omega \big|(b(u_1) - b(u_2))\nabla w \cdot \nabla w_2\big| &\leq \frac{\alpha}{2}\|\nabla w\|^2 + c\|u\|^2_{L^6(\Omega)}\|\nabla w_2\|^2_{L^3(\Omega)} \\ &\leq \frac{\alpha}{2}\|\nabla w\|^2 + c\|u\|^2_V\|\nabla w_2\|^2_{L^3(\Omega)}. \end{aligned} \tag{6.50}$$

Thanks to (6.46), we can then apply Gronwall's Lemma to (6.49), which gives the assert. At this point, Corollary 6.4 is an immediate consequence of the uniqueness property and of the general theory of infinite-dimensional dynamical systems [1, 22].

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First author's address:
Giulio Schimperna

Dipartimento di Matematica, Università degli Studi di Pavia
Via Ferrata, 1, I-27100 Pavia, Italy
E-mail: giusch04@unipv.it

Second author's address:
Sergey Zelik
Department of Mathematics, University of Surrey
Guildford, GU2 7XH, United Kingdom
E-mail: S.Zelik@surrey.ac.uk