NEW OUTLOOK ON MORI THEORY, I

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Abstract. We give a simple and self-contained proof of the finite generation of adjoint rings with big boundaries. As an easy consequence, we show that the canonical ring of a smooth projective variety is finitely generated.

Contents

1. Introduction

The main goal of this paper is to provide a simple proof of the following theorem while avoiding techniques of the Minimal Model Program.

Theorem 1.1. *Let X be a smooth projective variety and let Δ be a $\mathbb{Q}$-divisor with simple normal crossings such that $\lfloor \Delta \rfloor = 0$.*

Then the log canonical ring $R(X, K_X + \Delta)$ is finitely generated.

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Part of this work was written while the second author was a PhD student of A. Corti, who influenced ideas developed here immensely. Part of the paper started as a collaboration with J. M^{c}Kernan. We would like to express our gratitude to both of them for their encouragement, support and continuous inspiration. We thank C. Hacon, J. Hausen, A.-S. Kaloghiros and M. Reid for many useful comments and suggestions. The first author was partially supported by an EPSRC grant. The second author is grateful for support from the University of Cambridge, the Max-Planck-Institut für Mathematik, and the Institut Fourier.

This work supersedes [Laz09], where the results of this paper were first proved without Mori theory by the second author. Several arguments here follow closely those in [Laz09] and, based on these methods, we obtain a streamlined proof which is almost entirely self-contained. We even prove a lifting lemma for adjoint bundles without relying on asymptotic multiplier ideals, assuming only Kawamata-Viehweg vanishing and some elementary arithmetic.

The results presented here were originally proved by extensive use of methods of the Minimal Model Program in [BCHM10, HM10], and an analytic proof of finite generation of the canonical ring for varieties of general type is announced in [Siu06]. By contrast, in this paper we avoid the following tools which are commonly used in the Minimal Model Program: Mori's bend and break, which relies on methods in positive characteristic [Mor82], the Cone and Contraction theorem [KM98], the theory of asymptotic multiplier ideals, which was necessary to prove the existence of flips in [HM10]. Moreover, contrary to the classical Mori theory, we do not need to work with singular varieties.

In [CL10], Corti and the second author recently proved that the Cone and Contraction theorem, and the main result of [BCHM10], follow quickly from one of our main results, Theorem A. Therefore, this paper and [CL10] together give a completely new organisation of Mori theory.

We now briefly describe the strategy of the proof. As part of the induction, we prove the following two theorems.

Theorem A. Let X be a smooth projective variety of dimension n. Let $B_1, \dots, B_k$ be $\mathbb{Q}$-divisors on X such that $\lfloor B_i \rfloor = 0$ for all i, and such that the support of $\sum_{i=1}^k B_i$ has simple normal crossings. Let A be an ample $\mathbb{Q}$-divisor on X, and denote $D_i = K_X + A + B_i$ for every i.

Then the adjoint ring

$$R(X; D_1, \dots, D_k) = \bigoplus_{(m_1,\dots,m_k)\in\mathbb{N}^k} H^0\big(X, \mathcal{O}_X\big(\lfloor \sum m_i D_i \rfloor\big)\big)$$

is finitely generated.

Theorem B. Let $(X, \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where $S_1, \dots, S_p$ are distinct prime divisors. Let $V = \sum_{i=1}^p \mathbb{R} S_i \subseteq \operatorname{Div}_{\mathbb{R}}(X)$, and let A be an ample $\mathbb{Q}$-divisor on X.

Then

$$\mathcal{E}_A(V) = \{B \in \mathcal{L}(V) \mid |K_X + A + B|_{\mathbb{R}} \neq \emptyset\}$$

is a rational polytope.

For definitions of various terms involved in the statements of the theorems, see Section 2. In the sequel, "Theorem A_n" stands for "Theorem A in dimension n," and so forth.

In Section 2 we lay the foundation for the remainder of the paper: we discuss basic properties of asymptotic invariants of divisors, convex geometry and Diophantine approximation, and we introduce divisorial rings graded by monoids of higher rank and present basic consequences of finite generation of these rings. Basic references for asymptotic invariants of divisors are [Nak04, ELM$^+$06]. The first systematic use of Diophantine approximation in Mori theory was initiated by Shokurov in [Sho03], and our arguments at several places in this paper are inspired by some of the techniques introduced there.

In Section 3 we give a simplified proof of a version of the lifting lemma from [HM10]. The proof in [HM10] is based on methods initiated in [Siu98], which also inspired a systematic use of multiplier ideals in the theory. We want to emphasise that our proof, even though ultimately following the same path, is much simpler and uses only Kawamata-Viehweg vanishing and some elementary arithmetic.

In Section 4 we prove that one of the sets which naturally appears in the theory is a rational polytope. Some steps in the proof are close in spirit to Hacon's ideas in the proof of [HK10, Theorem 9.16]. The proof is an application of the lifting result from Section 3.

In Section 5 we prove Theorem B_n, assuming Theorems A_{n-1} and B_{n-1}. Certain steps of the proof here are almost the same as in [BCHM10, Section 6]. Lemma 5.3 was obtained in [Pău08] by analytic methods, without assuming Theorems A_{n-1} and B_{n-1}.

Finally, in Section 6, we prove Theorem A_n, assuming Theorems A_{n-1} and B_n, therefore completing the induction step. This part of the proof is close in spirit to that of the finite generation of the restricted algebra when grading is by non-negative integers, see [Cor05, Lemma 2.3.6].

The paper [Cor10] is an introduction to some of the ideas presented in this work.

2. Preliminary results

2.1. **Notation and conventions.** In this paper all algebraic varieties are defined over $\mathbb{C}$. We denote by $\mathbb{R}_+$ and $\mathbb{Q}_+$ the sets of non-negative real and rational numbers. For any $x, y \in \mathbb{R}^N$, we denote by $[x, y]$ the segment joining x and y. Given subsets $A, B \subseteq \mathbb{R}^N$, we denote $A + B = \{a + b \mid a \in A, b \in B\}$. We denote by $\overline{\mathcal{C}}$ the topological closure of a set $\mathcal{C} \subset \mathbb{R}^N$.

Let X be a smooth projective variety and $\mathbf{R} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. We denote by $\mathrm{Div}_{\mathbf{R}}(X)$ the group of $\mathbf{R}$-divisors on X, and $\sim_{\mathbf{R}}$ and $\equiv$ denote the $\mathbf{R}$-linear and numerical equivalence of $\mathbb{R}$-divisors. If $A = \sum a_i C_i$ and $B = \sum b_i C_i$ are two $\mathbb{R}$-divisors on X, $\lfloor A \rfloor$ is the round-down of A, $\lceil A \rceil$ is the round-up of A, $\{A\} = A - \lfloor A \rfloor$ is the fractional part of A, $\|A\| = \max\limits_i \{|a_i|\}$ is the sup-norm of A, and

$$A \wedge B = \sum \min\{a_i, b_i\} C_i.$$

Given $D \in \mathrm{Div}_{\mathbb{R}}(X)$ and $x \in X$, $\mathrm{mult}_x D$ is the order of vanishing of D at x. If S is a prime divisor, $\mathrm{mult}_S D$ is the order of vanishing of D at the generic point of S.

In this paper, a *log pair* (X, Δ) consist of a smooth variety X and an $\mathbb{R}$-divisor $\Delta \geq 0$. We say that (X, Δ) is *log smooth* if $\mathrm{Supp}\,\Delta$ has simple normal crossings. A projective birational morphism $f\colon Y \longrightarrow X$ is a *log resolution* of the pair (X, Δ) if Y is smooth, $\mathrm{Exc}(f)$ is a divisor and the support of $f_*^{-1}(\Delta) + \mathrm{Exc}(f)$ has simple normal crossings.

Definition 2.1. Let (X, Δ) be a log pair with $\lfloor \Delta \rfloor = 0$. Then (X, Δ) has *klt* (respectively *canonical*, *terminal*) singularities if for every log resolution $f\colon Y \longrightarrow X$, if we write $E = K_Y + f_*^{-1}\Delta - f^*(K_X + \Delta)$, we have $\lceil E \rceil \geq 0$ (respectively $E \geq 0$; $E \geq 0$ and $\mathrm{Supp}\,E = \mathrm{Exc}\,f$).

The following result is standard.

Lemma 2.2. *Let $(X, S + B)$ be a log smooth projective pair, where S is a prime divisor and B is a $\mathbb{Q}$-divisor such that $\lfloor B \rfloor = 0$ and $S \not\subseteq \mathrm{Supp}\,B$. Then there exist a log resolution $f\colon Y \longrightarrow X$ of (X, Δ) and $\mathbb{Q}$-divisors $C, E \geq 0$ on Y with no common components such that the components of C are disjoint, E is f-exceptional, and if $T = f_*^{-1}S$, then*

$$K_Y + T + C = f^*(K_X + S + B) + E.$$

Proof. By [KM98, Proposition 2.36], there exists a birational morphism $f\colon Y \longrightarrow X$, and $\mathbb{Q}$-divisors $C, E \geq 0$ on Y with no common components, such that the components of C are disjoint, E is f-exceptional, and

$$K_Y + C = f^*(K_X + B) + E.$$

Moreover, f is a sequence of blow-ups along intersections of components of B. Since $(X, S + B)$ is log smooth, it follows that if a collection of components of B intersect, then no irreducible component of their intersection is contained in S. Thus $T = f^*S$, and the lemma follows. □

If D is an $\mathbb{R}$-divisor on X, we denote

$$|D|_{\mathbf{R}} = \{D' \geq 0 \mid D \sim_{\mathbf{R}} D'\} \qquad \text{and} \qquad \mathbf{B}(D) = \bigcap_{D' \in |D|_{\mathbb{R}}} \mathrm{Supp}\,D',$$

and we call $\mathbf{B}(D)$ the *stable base locus* of D. We set $\mathbf{B}(D) = X$ if $|D|_{\mathbb{R}} = \emptyset$. The following result shows that this is compatible with the usual definition.

Lemma 2.3. *Let X be a smooth projective variety and let D be a $\mathbb{Q}$-divisor. Then $\mathbf{B}(D) = \bigcap_q \mathrm{Bs}\,|qD|$ for all q sufficiently divisible.*

Proof. Fix a point $x \in X \setminus \mathbf{B}(D)$. Then there exist an $\mathbb{R}$-divisor $F \geq 0$, real numbers $r_1, \dots, r_k$ and rational functions $f_1, \dots, f_k \in k(X)$ such that $F = D + \sum_{i=1}^k r_i(f_i)$ and $x \notin \mathrm{Supp}\,F$. Let $W \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ be the subspace spanned by the components of D, D' and all (f_i). Let $W_0 \subseteq W$ be the subspace of divisors $\mathbb{R}$-linearly equivalent to zero, and note that W_0 is a rational subspace of W. Consider the quotient map $\pi\colon W \longrightarrow W/W_0$. Then the set $\{G \in \pi^{-1}(\pi(D)) \mid G \geq 0\}$ is not empty as it contains F, and it is cut out from W by rational hyperplanes. Thus, it contains a $\mathbb{Q}$-divisor $D' \geq 0$ such that $D \sim_{\mathbb{Q}} D'$ and $x \notin \operatorname{Supp} D'$. $\square$

Definition 2.4. Let $(X, S + \sum_{i=1}^p S_i)$ be a log smooth projective pair, where S and all S_i are distinct prime divisors, let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \operatorname{Div}_{\mathbb{R}}(X)$, and let A be a $\mathbb{Q}$-divisor on X. We define

$$\begin{aligned} \mathcal{L}(V) &= \{B = \textstyle\sum b_i S_i \in V \mid 0 \leq b_i \leq 1 \text{ for all } i\}, \\ \mathcal{E}_A(V) &= \{B \in \mathcal{L}(V) \mid |K_X + A + B|_{\mathbb{R}} \neq \emptyset\}, \\ \mathcal{B}_A^S(V) &= \{B \in \mathcal{L}(V) \mid S \not\subseteq \mathbf{B}(K_X + S + A + B)\}. \end{aligned}$$

If D is an integral divisor, $\operatorname{Fix}(D)$ and $\operatorname{Mob}(D)$ denote the *fixed* and *mobile* parts of D. Hence $|D| = |\operatorname{Mob}(D)| + \operatorname{Fix}(D)$, and the base locus of $|\operatorname{Mob}(D)|$ contains no divisors. More generally, if V is any linear system on X, $\operatorname{Fix}(V)$ denotes the fixed divisor of V. If S is a prime divisor on X such that $S \not\subseteq \operatorname{Fix}(D)$, then $|D|_S$ denotes the image of the linear system $|D|$ under restriction to S.

Definition 2.5. Let X be a smooth projective variety and let S be a prime divisor on X. Let C and D be $\mathbb{Q}$-divisors on X such that $|C|_{\mathbb{Q}} \neq \emptyset$, $|D|_{\mathbb{Q}} \neq \emptyset$ and $S \not\subseteq \mathbf{B}(D)$. Then we define

$$\mathbf{Fix}(C) = \liminf \frac{1}{k} \operatorname{Fix}|kC| \qquad \text{and} \qquad \mathbf{Fix}_S(D) = \liminf \frac{1}{k} \operatorname{Fix}|kD|_S$$

for all k sufficiently divisible.

2.2. **Diophantine approximation and convex geometry.**

Definition 2.6. Let $\mathcal{C} \subseteq \mathbb{R}^N$ be a convex set. A subset $F \subseteq \mathcal{C}$ is a *face* of $\mathcal{C}$ if F is convex, and whenever $u + v \in F$ for $u, v \in \mathcal{C}$, then $u, v \in F$. Note that $\mathcal{C}$ is itself a face of $\mathcal{C}$. We say that $x \in \mathcal{C}$ is an *extreme point* of $\mathcal{C}$ if $\{x\}$ is a face of $\mathcal{C}$. For $y \in \mathcal{C}$, the minimal face of $\mathcal{C}$ which contains y is denoted by $\operatorname{face}(\mathcal{C}, y)$.

A *polytope* in $\mathbb{R}^N$ is a compact set which is the intersection of finitely many half spaces; equivalently, it is the convex hull of finitely many points in $\mathbb{R}^N$. A polytope is *rational* if it is an intersection of rational half spaces; equivalently, it is the convex hull of finitely many rational points in $\mathbb{R}^N$. A *rational polyhedral cone* in $\mathbb{R}^N$ is a convex cone spanned by finitely many rational vectors.

Remark 2.7. Given a smooth projective variety X, we often consider subspaces $V \subseteq \operatorname{Div}_{\mathbb{R}}(X)$ which are spanned by a finite set of prime divisors. Thus, these divisors implicitly define an isomorphism between V and $\mathbb{R}^N$ for some N. In particular, with notation from Definition 2.4, $\mathcal{L}(V)$ is a rational polytope.

Definition 2.8. Let $\mathcal{C} \subseteq \mathbb{R}^N$ be a convex set and let $\Phi\colon \mathcal{C} \longrightarrow \mathbb{R}^M$ be a function. Then Φ is *convex* if $\Phi\big(tx + (1-t)y\big) \leq t\Phi(x) + (1-t)\Phi(y)$ for any $x, y \in \mathcal{C}$ and any

$t\in[0,1]$. If $\mathcal{C}$ is a rational polytope, then Φ is *rationally piecewise affine* if there exists a finite decomposition $\mathcal{C}=\bigcup_{i=1}^{\ell}\mathcal{C}_i$ into rational polytopes such that $\Phi_{|\mathcal{C}_i}$ is a rational affine map for all i. If $\mathcal{C}$ is a cone, then Φ is *homogeneous of degree one* if $\Phi(tx)=t\Phi(x)$ for any $x\in\mathcal{C}$ and $t\in\mathbb{R}_+$.

Lemma 2.9. *Let $\mathcal{H}\subseteq\mathbb{R}^N$ be a rational affine hyperplane which does not contain the origin, and let $\mathcal{P}\subseteq\mathcal{H}$ be a rational polytope. Let $\mathcal{P}_\mathbb{Q}=\mathcal{P}\cap\mathbb{Q}^N$, and let $f\colon\mathcal{P}_\mathbb{Q}\longrightarrow\mathbb{R}$ be a bounded convex function. Assume that there exist $x_1,\dots,x_q\in\mathcal{P}_\mathbb{Q}$ with $f(x_i)\in\mathbb{Q}_+$ for all i, and that for any $x\in\mathcal{P}_\mathbb{Q}$ there exists $(r_1,\dots,r_q)\in\mathbb{R}^q_+$ such that $x=\sum r_ix_i$ and $f(x)=\sum r_if(x_i)$.*

Then f can be extended to a rational piecewise affine function on $\mathcal{P}$.

Proof. Since $\mathcal{P}\subseteq\mathcal{H}$, for any $x\in\mathcal{P}_\mathbb{Q}$ and $(r_1,\dots,r_q)\in\mathbb{R}^q_+$ such that $x=\sum r_ix_i$, we have $\sum r_i=1$. Pick $C\in\mathbb{Q}_+$ such that $-C\leq f(x)\leq C$ for all $x\in\mathcal{P}_\mathbb{Q}$.

Let $\mathcal{Q}\subseteq\mathbb{R}^{N+1}$ be the convex hull of all points $\big(x_i,f(x_i)\big)$ and (x_i,C), and set $\mathcal{Q}'=\{(x,y)\in\mathcal{P}_\mathbb{Q}\times\mathbb{R}_+\mid f(x)\leq y\leq C\}$. Since f is convex, and all $\big(x_i,f(x_i)\big)$ and (x_i,C) are contained in $\mathcal{Q}'$, it follows that $\mathcal{Q}\cap\mathbb{Q}^{N+1}\subseteq\mathcal{Q}'$. Now, fix $(u,v)\in\mathcal{Q}'$. Then there exists $t\in[0,1]$ such that $v=tf(u)+(1-t)C$, and as $u\in\mathcal{P}_\mathbb{Q}$, there exist $r_i\in\mathbb{R}_+$ such that $\sum r_i=1$, $u=\sum r_ix_i$ and $f(u)=\sum r_if(x_i)$. Therefore

$$(u,v)=\textstyle\sum tr_i\big(x_i,f(x_i)\big)+\sum(1-t)r_i(x_i,C),$$

and hence $(u,v)\in\mathcal{Q}$. This yields $\mathcal{Q}=\overline{\mathcal{Q}'}$. Define $F\colon\mathcal{P}\longrightarrow[-C,C]$ as

$$F(x)=\min\{y\in[-C,C]\mid(x,y)\in\mathcal{Q}\}.$$

Then F extends f, and it is rational piecewise affine as $\mathcal{Q}$ is a rational polytope. $\square$

We use the following result from Diophantine approximation.

Lemma 2.10. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^N$, let $\mathcal{P}\subseteq\mathbb{R}^N$ be a rational polytope and let $x\in\mathcal{P}$. Fix a positive integer k and a positive real number ε.*

Then there are finitely many $x_i\in\mathcal{P}$ and positive integers k_i divisible by k, such that k_ix_i/k are integral, $\|x-x_i\|<\varepsilon/k_i$, and x is a convex combination of x_i.

Proof. See [BCHM10, Lemma 3.7.7]. $\square$

2.3. **Nakayama-Zariski decomposition.** We need several definitions and results from [Nak04].

Definition 2.11. Let X be a smooth projective variety, let A be an ample $\mathbb{R}$-divisor, and let Γ be a prime divisor. If $D\in\operatorname{Div}_\mathbb{R}(X)$ is a big divisor, define

$$\sigma'_\Gamma(D)=\inf\{\operatorname{mult}_\Gamma D'\mid D'\in|D|_\mathbb{R}\}.$$

If $D\in\operatorname{Div}_\mathbb{R}(X)$ is pseudo-effective, set

$$\sigma_\Gamma(D)=\lim_{\varepsilon\to0}\sigma'_\Gamma(D+\varepsilon A)\qquad\text{and}\qquad N_\sigma(D)=\textstyle\sum_\Gamma\sigma_\Gamma(D)\cdot\Gamma,$$

where the sum runs over all prime divisors Γ on X.

Lemma 2.12. *Let X be a smooth projective variety, let A be an ample $\mathbb{R}$-divisor, let D be a pseudo-effective $\mathbb{R}$-divisor, and let Γ be a prime divisor. Then $\sigma_\Gamma(D)$ exists as a limit, it is independent of the choice of A, it depends only on the numerical equivalence class of D, and $\sigma_\Gamma(D) = \sigma'_\Gamma(D)$ if D is big. The function σ_Γ is homogeneous of degree one, convex and lower semi-continuous on the cone of pseudo-effective divisors on X, and it is continuous on the cone of big divisors.*

Furthermore, $N_\sigma(D)$ is an $\mathbb{R}$-divisor on X, $D - N_\sigma(D)$ is pseudo-effective, and for any $\mathbb{R}$-divisor $0 \leq F \leq N_\sigma(D)$ we have $N_\sigma(D-F) = N_\sigma(D) - F$.

Proof. See [Nak04, §III.1]. □

Lemma 2.13. *Let X be a smooth projective variety, let D be a pseudo-effective $\mathbb{R}$-divisor, and let A be an ample $\mathbb{Q}$-divisor.*

If $D \not\equiv N_\sigma(D)$, then there exist a positive integer k and a positive rational number β such that kA is integral and

$$h^0(X, \mathcal{O}_X(\lfloor mD \rfloor + kA)) > \beta m \quad \textit{for all} \quad m \gg 0.$$

Proof. Replacing D by $D - N_\sigma(D)$, we may assume that $N_\sigma(D) = 0$. Now apply [Nak04, Theorem V.1.11]. □

Lemma 2.14. *Let X be a smooth projective variety, let D be a pseudo-effective $\mathbb{R}$-divisor on X, and let $\Gamma_1, \dots, \Gamma_\ell$ be distinct prime divisors such that $\sigma_{\Gamma_i}(D) > 0$ for all i. Then for any $\gamma_j \in \mathbb{R}_+$ we have $\sigma_{\Gamma_i}(\sum_{j=1}^{\ell} \gamma_j \Gamma_j) = \gamma_i$ for every i. In particular, if $D \geq 0$ and if $\sigma_\Gamma(D) > 0$ for every component Γ of D, then $D = N_\sigma(D)$.*

Proof. This is [Nak04, Proposition III.1.10]. □

Lemma 2.15. *Let X be a smooth projective variety and let Γ be a prime divisor. Let D be a pseudo-effective $\mathbb{R}$-divisor such that $\sigma_\Gamma(D) = 0$ and let A be an ample $\mathbb{Q}$-divisor. Then $\Gamma \not\subseteq \mathbf{B}(D + A)$.*

Proof. Note that $\sigma_\Gamma(D + \frac{1}{2}A) \leq \sigma_\Gamma(D) = 0$. Thus there exists $0 \leq D' \sim_\mathbb{R} D + \frac{1}{2}A$ such that $\gamma = \operatorname{mult}_\Gamma D' \ll 1$ and $\frac{1}{2}A + \gamma\Gamma$ is ample. Pick $A' \sim_\mathbb{R} \frac{1}{2}A + \gamma\Gamma$ such that $A' \geq 0$ and $\operatorname{mult}_\Gamma A' = 0$. Then

$$D + A \sim_\mathbb{R} D' - \gamma\Gamma + A' \geq 0 \qquad \text{and} \qquad \operatorname{mult}_\Gamma(D' - \gamma\Gamma + A') = 0.$$

This proves the lemma. □

2.4. **Divisorial rings.** Now we establish properties of finite generation of (divisorial) graded rings that we use in the paper.

Definition 2.16. Let X be a smooth projective variety and let $\mathcal{S} \subseteq \operatorname{Div}_\mathbb{Q}(X)$ be a finitely generated monoid. Then

$$R(X, \mathcal{S}) = \bigoplus_{D \in \mathcal{S}} H^0\big(X, \mathcal{O}_X\big(\lfloor D \rfloor\big)\big)$$

is the *divisorial $\mathcal{S}$-graded ring*. If $D_1,\dots,D_\ell$ are generators of $\mathcal{S}$ and if $D_i\sim_{\mathbb{Q}} k_i(K_X+\Delta_i)$, where $\Delta_i\geq 0$ and $k_i\in\mathbb{Q}_+$ for every i, the algebra $R(X,\mathcal{S})$ is the *adjoint ring associated to* $\mathcal{S}$; furthermore, the *adjoint ring associated to the sequence* $D_1,\dots,D_\ell$ is

$$R(X;D_1,\dots,D_\ell)=\bigoplus_{(m_1,\dots,m_\ell)\in\mathbb{N}^\ell} H^0\big(X,\mathcal{O}_X(\lfloor\sum m_iD_i\rfloor)\big).$$

Note that then there is a natural projection map $R(X;D_1,\dots,D_\ell)\longrightarrow R(X,\mathcal{S})$.

If $\mathcal{C}\subseteq\operatorname{Div}_{\mathbb{R}}(X)$ is a rational polyhedral cone, then $\mathcal{S}=\mathcal{C}\cap\operatorname{Div}(X)$ is a finitely generated monoid, and we define the algebra $R(X,\mathcal{C})$, the *adjoint ring associated to* $\mathcal{C}$, to be $R(X,\mathcal{S})$.

Definition 2.17. Let $(X,S+D)$ be a projective pair, where X is smooth, S is a prime divisor and $D\geq 0$ is integral, and fix $\eta\in H^0(X,\mathcal{O}_X(S))$ such that $\operatorname{div}\eta=S$. From the exact sequence

$$0\longrightarrow H^0(X,\mathcal{O}_X(D-S))\xrightarrow{\cdot\eta} H^0(X,\mathcal{O}_X(D))\xrightarrow{\rho_S} H^0(S,\mathcal{O}_S(D))$$

we define $\operatorname{res}_S H^0(X,\mathcal{O}_X(D))=\operatorname{Im}(\rho_S)$, and for $\sigma\in H^0(X,\mathcal{O}_X(D))$, denote $\sigma_{|S}=\rho_S(\sigma)$. Note that

$$\operatorname{Ker}(\rho_S)=H^0(X,\mathcal{O}_X(D-S))\cdot\eta,$$

and that $\operatorname{res}_S H^0(X,\mathcal{O}_X(D))=0$ if $S\subseteq\operatorname{Bs}|D|$.

If $\mathcal{S}\subseteq\operatorname{Div}_{\mathbb{Q}}(X)$ is a monoid generated by divisors $D_1,\dots,D_\ell$, the *restriction of* $R(X,\mathcal{S})$ *to* S is the $\mathcal{S}$-graded ring

$$\operatorname{res}_S R(X,\mathcal{S})=\bigoplus_{D\in\mathcal{S}}\operatorname{res}_S H^0\big(X,\mathcal{O}_X(\lfloor D\rfloor)\big),$$

and similarly for $\operatorname{res}_S R(X;D_1,\dots,D_\ell)$. These rings are not necessarily divisorial.

Definition 2.18. Let $\mathcal{S}\subseteq\mathbb{N}^r$ be a finitely generated monoid and let $R=\bigoplus_{s\in\mathcal{S}}R_s$ be an $\mathcal{S}$-graded algebra. If $\mathcal{S}'\subseteq\mathcal{S}$ is a finitely generated submonoid, then $R'=\bigoplus_{s\in\mathcal{S}'}R_s$ is a *Veronese subring* of R. If there exists a subgroup $\mathbb{L}\subset\mathbb{Z}^r$ of finite index such that $\mathcal{S}'=\mathcal{S}\cap\mathbb{L}$, then R' is a *Veronese subring of finite index* of R.

Lemma 2.19. *Let $\mathcal{S}\subseteq\mathbb{N}^r$ be a finitely generated monoid and let $R=\bigoplus_{s\in\mathcal{S}}R_s$ be an $\mathcal{S}$-graded algebra. Let $\mathcal{S}'\subseteq\mathcal{S}$ be a finitely generated submonoid and let $R'=\bigoplus_{s\in\mathcal{S}'}R_s$.*

(1) If R is finitely generated over R_0, then R' is finitely generated over R_0.
(2) If additionally R_0 is Noetherian, R' is a Veronese subring of finite index of R, and R' is finitely generated over R_0, then R is finitely generated over R_0.

Proof. Part (1) is [ADHL10, Proposition 1.1.6].

To prove (2), let $\mathbb{L}\subseteq\mathbb{Z}^r$ be a subgroup of index d such that $\mathcal{S}'=\mathcal{S}\cap\mathbb{L}$. Then for any $f\in R$ we have $f^d\in R'$, so R is an integral extension of R'. Now the claim follows from the theorem of E. Noether on finiteness of integral closure. $\square$

The proof of the following result was kindly communicated to us by J. Hausen.

Lemma 2.20. *Let $\mathcal{S} = \sum_{i=1}^{\ell} \mathbb{N}s_i \subseteq \mathbb{N}^r$ be a finitely generated monoid and let $R = \bigoplus_{s\in\mathcal{S}} R_s$ be an $\mathcal{S}$-graded algebra such that R is an integral domain and R_0 is Noetherian. Set $\mathcal{T} = \bigoplus_{i=1}^{\ell} \mathbb{N}s_i$, let $\varphi\colon \mathcal{T} \longrightarrow \mathcal{S}$ be the projection map, and consider the $\mathcal{T}$-graded algebra $R' = \bigoplus_{t\in\mathcal{T}} R_{\varphi(t)}$.*

If R is finitely generated over R_0, then R' is finitely generated over R_0.

Proof. If $\widehat{\mathcal{S}} = \sum_{i=1}^{\ell} \mathbb{R}s_i$ and $\widehat{\mathcal{T}} = \bigoplus_{i=1}^{\ell} \mathbb{R}s_i$ are the associated groups, then φ extends to the projection $\widehat{\varphi}\colon \widehat{\mathcal{T}} \longrightarrow \widehat{\mathcal{S}}$. Define $R_s = 0$ for $s \in \widehat{\mathcal{S}} \setminus \mathcal{S}$, and set $\widehat{R} = \bigoplus_{s\in\widehat{\mathcal{S}}} R_s$ and $\widehat{R}' = \bigoplus_{t\in\widehat{\mathcal{T}}} R_{\widehat{\varphi}(t)}$. Then $\widehat{R}$ is finitely generated, and R' is a Veronese subring of $\widehat{R}'$. Therefore, by Lemma 2.19(1) and by replacing $\mathcal{S}$ by $\widehat{\mathcal{S}}$ and $\mathcal{T}$ by $\widehat{\mathcal{T}}$, we may assume that $\mathcal{S}$ is a group and that $\mathcal{T} = \mathbb{Z}^{\ell}$.

Denote $R'_t = R_{\varphi(t)}$ for every $t \in \mathcal{T}$, and set $\mathcal{T}_0 = \varphi^{-1}(0)$. Since $\mathcal{T}_0$ is a subgroup of $\mathbb{Z}^{\ell}$, there is a basis $e_1, \dots, e_\ell$ of $\mathbb{Z}^{\ell}$ and positive integers $a_1, \dots, a_k$ such that $a_1e_1, \dots, a_ke_k$ is a basis of $\mathcal{T}_0$. Let $\mathcal{T}_1 = \bigoplus_{i=k+1}^{\ell} \mathbb{Z}e_i \subseteq \mathbb{Z}^{\ell}$. This gives Veronese subrings of R':

$$R'_0 = \bigoplus_{t\in\mathcal{T}_0} R'_t, \qquad R'_1 = \bigoplus_{t\in\mathcal{T}_1} R'_t, \qquad R'_\infty = \bigoplus_{t\in\mathcal{T}_0\oplus\mathcal{T}_1} R'_t.$$

Note that R'_∞ is a Veronese subring of finite index of R', and that it is generated by R'_1 and by the elements $f_i \in R'_{a_ie_i}$ and $g_i \in R'_{-a_ie_i}$ corresponding to $1 \in R_0$. Observe that R'_1 is isomorphic to the Veronese subring $\bigoplus_{t\in\mathcal{T}_1} R_{\varphi(t)}$ of R, and thus it is finitely generated by Lemma 2.19(1). Therefore R'_∞ is finitely generated, and hence so is R' by Lemma 2.19(2). $\square$

Corollary 2.21. *Let X be a smooth projective variety and let $D_1, \dots, D_\ell \in \mathrm{Div}(X)$. Let $\mathcal{C} = \sum_{i=1}^{\ell} \mathbb{R}_+ D_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$, and assume that $R(X, \mathcal{C})$ is finitely generated.*

Then the ring $R = R(X; D_1, \dots, D_\ell)$ is finitely generated.

Proof. The monoid $\mathcal{S} = \sum_{i=1}^{\ell} \mathbb{N}D_i \subseteq \mathrm{Div}(X)$ is a submonoid of $\mathcal{C} \cap \mathrm{Div}(X)$, and thus $R(X, \mathcal{S})$ is finitely generated by Lemma 2.19(1). But then $R(X; D_1, \dots, D_\ell)$ is finitely generated by Lemma 2.20. $\square$

A stronger version of the following result can be found in [ELM+06], see [CL10, Theorem 3.5].

Lemma 2.22. *Let X be a smooth projective variety and let $D_1, \dots, D_\ell \in \mathrm{Div}_{\mathbb{Q}}(X)$ be such that $|D_i|_{\mathbb{Q}} \neq \emptyset$ for each i. Let $V \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ be the subspace spanned by the components of $D_1, \dots, D_\ell$, and let $\mathcal{P} \subseteq V$ be the convex hull of $D_1, \dots, D_\ell$. Assume that the ring $R(X; D_1, \dots, D_\ell)$ is finitely generated. Then:*

1. ***Fix** extends to a rational piecewise affine function on $\mathcal{P}$;*
2. *there exists a positive integer k such that for every $D \in \mathcal{P}$ and every $m \in \mathbb{N}$, if $\frac{m}{k}D \in \mathrm{Div}(X)$, then $\mathbf{Fix}(D) = \frac{1}{m}\,\mathrm{Fix}\,|mD|$.*

Proof. Pick a prime divisor $S\in\operatorname{Div}(X)\setminus V$ and a rational function $\eta\in k(X)$ such that $\operatorname{mult}_S\operatorname{div}\eta=1$. Then, setting $D_i'=D_i+\operatorname{div}\eta\sim_{\mathbb{Q}}D_i$, we have $\operatorname{mult}_S D_i'=1$ and $R(X;D_1,\dots,D_\ell)\simeq R(X;D_1',\dots,D_\ell')$. If $\mathcal{P}'\subseteq\operatorname{Div}_{\mathbb{R}}(X)$ is the convex hull of $D_1',\dots,D_\ell'$, it suffices to prove claims (1) and (2) on $\mathcal{P}'$. Therefore, after replacing D_i by D_i', we may assume that $\mathcal{P}$ belongs to a rational affine hyperplane which does not contain the origin. Denote $\mathcal{P}_{\mathbb{Q}}=\mathcal{P}\cap\operatorname{Div}_{\mathbb{Q}}(X)$.

Fix a prime divisor $G\in V$. For all $D\in\mathcal{P}_{\mathbb{Q}}$ and all $m\in\mathbb{N}$ sufficiently divisible, let $\varphi_m(D)=\frac{1}{m}\operatorname{mult}_G\operatorname{Fix}|mD|$, and set $\varphi(D)=\operatorname{mult}_G\mathbf{Fix}(D)$. Then, in order to show (1), it suffices to prove that φ is rational piecewise affine.

For every $D\in\mathcal{P}_{\mathbb{Q}}$, the ring $R(X,D)$ is finitely generated by Lemma 2.19(1), and so by [Bou89, III.1.2], there exists a positive integer d such that $R(X,dD)$ is generated by $H^0(X,\mathcal{O}_X(dD))$. Thus $\varphi(D)=\varphi_d(D)$, and in particular $\varphi(D)\in\mathbb{Q}$.

If $\sigma_1,\dots,\sigma_q$ are generators of $R(X;D_1,\dots,D_\ell)$, then there are $G_i\in\mathcal{P}$ and $m_i\in\mathbb{Q}_+$ such that $\sigma_i\in H^0\big(X,\mathcal{O}_X(\lfloor m_iG_i\rfloor)\big)$. Fix $D\in\mathcal{P}_{\mathbb{Q}}$. Let m be a sufficiently divisible positive integer such that $mD\in\sum\mathbb{N}D_i\cap\operatorname{Div}(X)$, and let $\sigma\in H^0(X,\mathcal{O}_X(mD))$ be such that $\varphi_m(D)=\frac{1}{m}\operatorname{mult}_G\operatorname{div}\sigma$. Then σ is a polynomial in σ_i, and thus there are $\alpha_i\in\mathbb{N}$ such that $mD=\sum\alpha_im_iG_i$ and $\operatorname{mult}_G\operatorname{div}\sigma=\sum\alpha_i\operatorname{mult}_G\operatorname{div}\sigma_i$. Denote $t_{m,i}=\frac{\alpha_im_i}{m}$, and note that $\operatorname{mult}_G\operatorname{div}\sigma_i\geq\varphi(m_iG_i)=m_i\varphi(G_i)$. Then we have

$$D=\sum t_{m,i}G_i\qquad\text{and}\qquad\varphi(D)=\inf_m\varphi_m(D)\geq\inf_m\sum t_{m,i}\varphi(G_i).$$

However, for all $t_i\in\mathbb{Q}_+$ with $D=\sum t_iG_i$, by convexity we have $\sum t_i\varphi(G_i)\geq\varphi(D)$. Therefore

$$\varphi(D)=\inf\sum t_i\varphi(G_i),$$

where the infimum is taken over all $(t_1,\dots,t_q)\in\mathbb{R}_+^q$ such that $D=\sum t_iG_i$. By compactness, there exists $(r_1,\dots,r_q)\in\mathbb{R}_+^q$ such that $D=\sum r_iG_i$ and $\varphi(D)=\sum r_i\varphi(G_i)$. Thus, φ is rational piecewise affine by Lemma 2.9, and (1) follows.

Now we show (2). After decomposing $\mathcal{P}$, we may assume that $\mathbf{Fix}$ is rational linear on $\mathbb{R}_+\mathcal{P}$. By Gordan's lemma the monoid $\mathcal{S}=\mathbb{R}_+\mathcal{P}\cap\operatorname{Div}(X)$ is finitely generated, and let $F_1,\dots,F_p$ be its generators. By the first part of the proof, there exists a positive integer k such that $\mathbf{Fix}(F_i)=\frac{1}{k}\operatorname{Fix}|kF_i|$ for all i. Let $D\in\mathcal{P}\cap\operatorname{Div}_{\mathbb{Q}}(X)$, and let $m,\alpha_i\in\mathbb{N}$ be such that $\frac{m}{k}D=\sum\alpha_iF_i\in\mathcal{S}$. Then

$$\sum\alpha_i\,\mathbf{Fix}(F_i)=\tfrac{m}{k}\,\mathbf{Fix}(D)\leq\tfrac{1}{k}\operatorname{Fix}|mD|\leq\tfrac{1}{k}\sum\alpha_i\operatorname{Fix}|kF_i|=\sum\alpha_i\,\mathbf{Fix}(F_i),$$

and hence all inequalities are equalities. This completes the proof. □

3. Lifting sections

We will need the following easy consequence of Kawamata-Viehweg vanishing:

Lemma 3.1. *Let $(X,S+B)$ be a log smooth projective pair of dimension n, where S is a prime divisor and B is a $\mathbb{Q}$-divisor such that $\lfloor B\rfloor=0$ and $S\not\subseteq\operatorname{Supp}B$. Let A be a nef and big $\mathbb{Q}$-divisor on X.*

(1) If $G \in \operatorname{Div}(X)$ is such that $G \sim_{\mathbb{Q}} K_X + S + A + B$, then $|G_{|S}| = |G|_S$.
(2) Let $f\colon X \longrightarrow Y$ be a birational morphism to a projective variety Y, and let $U \subseteq X$ be an open set such that $f_{|U}$ is an isomorphism. Let H' be a very ample divisor on Y and let $H = f^*H'$. If $F \in \operatorname{Div}(X)$ is such that $F \sim_{\mathbb{Q}} K_X + (n+1)H + A + B$, then $|F|$ is basepoint free at every point of U.

Proof. Considering the exact sequence

$$0 \longrightarrow \mathcal{O}_X(G-S) \longrightarrow \mathcal{O}_X(G) \longrightarrow \mathcal{O}_S(G) \longrightarrow 0,$$

Kawamata-Viehweg vanishing implies $H^1(X, \mathcal{O}_X(G-S)) = 0$. In particular, the map $H^0(X, \mathcal{O}_X(G)) \longrightarrow H^0(S, \mathcal{O}_S(G))$ is surjective. This proves (1).

We prove (2) by induction on n. Let $x \in U$, and pick a general element $T \in |H|$ which contains x. Then $(X, T+B)$ is log smooth, and since $F_{|T} \sim_{\mathbb{Q}} K_T + nH_{|T} + A_{|T} + B_{|T}$, by induction $F_{|T}$ is free at x. Considering the exact sequence

$$0 \longrightarrow \mathcal{O}_X(F-T) \longrightarrow \mathcal{O}_X(F) \longrightarrow \mathcal{O}_T(F) \longrightarrow 0,$$

Kawamata-Viehweg vanishing implies that $H^1(X, \mathcal{O}_X(F-T)) = 0$. In particular, the map $H^0(X, \mathcal{O}_X(F)) \longrightarrow H^0(T, \mathcal{O}_T(F))$ is surjective, and (2) follows. □

Lemma 3.2. *Let $(X, S+B)$ be a projective pair, where X is smooth, S is a smooth prime divisor and B is a $\mathbb{Q}$-divisor such that $S \not\subseteq \operatorname{Supp} B$. Let A be a nef and big $\mathbb{Q}$-divisor on X. Assume that $D \in \operatorname{Div}(X)$ is such that $D \sim_{\mathbb{Q}} K_X + S + A + B$, and let $\Sigma \in |D|$. Let $\Phi \in \operatorname{Div}_{\mathbb{Q}}(S)$ be such that the pair (S, Φ) is klt and $B_{|S} \leq \Sigma + \Phi$.*

Then $\Sigma \in |D|_S$.

Proof. Let $f\colon Y \longrightarrow X$ be a log resolution of the pair $(X, S+B)$, and write $T = f_*^{-1}S$. Then there are divisors $\Gamma \geq 0$ and $E \geq 0$ on Y with no common components such that $T \not\subseteq \operatorname{Supp}\Gamma$, E is f-exceptional, and

$$K_Y + T + \Gamma = f^*(K_X + S + B) + E.$$

Let $C = \Gamma - E$ and $G = f^*D - \lfloor C \rfloor = f^*D - \lfloor \Gamma \rfloor + \lceil E \rceil$. Note that

$$G - (K_Y + T + \{C\}) \sim_{\mathbb{Q}} f^*(K_X + S + A + B) - (K_Y + T + C) = f^*A$$

is nef and big, and Lemma 3.1(1) implies that $|G_{|T}| = |G|_T$.

Denote $g = f_{|T}\colon T \longrightarrow S$. Then

$$K_T + C_{|T} = g^*(K_S + B_{|S}) \quad \text{and} \quad K_T + \Psi = g^*(K_S + \Phi),$$

for some divisor Ψ on S, and note that $\lfloor \Psi \rfloor \leq 0$ since (S, Φ) is klt. Therefore

$$g^*(B_{|S} - \Phi) = C_{|T} - \Psi.$$

By assumption we have that $B_{|S} \leq \Sigma + \Phi$, that $g^*\Sigma$ is integral, and that the support of $C + T$ has normal crossings, hence

$$\begin{aligned} g^*\Sigma &\geq g^*\Sigma + \lfloor \Psi \rfloor = \lfloor g^*\Sigma + \Psi \rfloor \geq \lfloor g^*(B_{|S} - \Phi) + \Psi \rfloor \\ &= \lfloor C_{|T} \rfloor = \lfloor C \rfloor_{|T} = (f^*D)_{|T} - G_{|T}. \end{aligned}$$

Thus, since $g^*\Sigma\in|(f^*D)_{|T}|$, if we denote

$$R=G_{|T}-(f^*D)_{|T}+g^*\Sigma\geq 0,$$

then $R\in|G_{|T}|=|G|_T$. Moreover, since $E\geq 0$ is f-exceptional, we have

$$|G|_T+\lfloor\Gamma\rfloor_{|T}=|f^*D-\lfloor\Gamma\rfloor+\lceil E\rceil|_T+\lfloor\Gamma\rfloor_{|T}\subseteq|f^*D+\lceil E\rceil|_T=|f^*D|_T+\lceil E\rceil_{|T}.$$

Therefore $R+\lfloor\Gamma\rfloor_{|T}\in|f^*D|_T+\lceil E\rceil_{|T}$, and

$$g^*\Sigma=R+(f^*D)_{|T}-G_{|T}=R+\lfloor\Gamma\rfloor_{|T}-\lceil E\rceil_{|T}\in|f^*D|_T,$$

hence the claim follows. □

Lemma 3.3. *Let $(X,S+\sum_{i=1}^p S_i)$ be a log smooth projective pair, where S and all S_i are distinct prime divisors. Let $V=\sum_{i=1}^p\mathbb{R}S_i\subseteq\mathrm{Div}_{\mathbb{R}}(X)$, and let $B\in\mathcal{L}(V)$ and $0\leq D\in V$ be $\mathbb{Q}$-divisors with no common components. Let P be a nef $\mathbb{Q}$-divisor and denote $\Delta=S+B+P$. Assume that*

$$K_X+\Delta\sim_{\mathbb{Q}}D.$$

Let k be a positive integer such that kP and kB are integral, and write $\Omega=(B+P)_{|S}$.

Then there is a very ample divisor H such that for all divisors $\Sigma\in|k(K_S+\Omega)|$ and $U\in|H_{|S}|$, and for every positive integer l we have

$$l\Sigma+U\in|lk(K_X+\Delta)+H|_S.$$

Proof. For any $m\geq 0$, let $l_m=\lfloor\frac{m}{k}\rfloor$ and $r_m=m-l_mk\in\{0,1,\dots,k-1\}$, define $B_m=\lceil mB\rceil-\lceil(m-1)B\rceil$, and set $P_m=kP$ if $r_m=0$, and otherwise $P_m=0$. Let

$$D_m=\sum_{i=1}^m(K_X+S+P_i+B_i)=m(K_X+S)+l_mkP+\lceil mB\rceil,$$

and note that D_m is integral and

$$D_m=l_mk(K_X+\Delta)+D_{r_m}.$$

By Serre vanishing, we can pick a very ample divisor H on X such that:

(1) D_j+H is basepoint free for every $0\leq j\leq k-1$,
(2) $|D_k+H|_S=|(D_k+H)_{|S}|$.

We claim that for all divisors $\Sigma\in|k(K_S+\Omega)|$ and $U_m\in|(D_{r_m}+H)_{|S}|$ we have

$$l_m\Sigma+U_m\in|D_m+H|_S.$$

The case $r_m=0$ immediately implies the lemma.

We prove the claim by induction on m. The case $m=k$ is covered by (2). Now let $m>k$, and pick a rational number $0<\delta\ll 1$ such that $D_{r_{m-1}}+H+\delta B_m$ is ample. Note that $0\leq B_m\leq\lceil B\rceil$, that $(X,S+B+D)$ is log smooth, and that D and $S+B$ have no common components. Thus, there exists a rational number $0<\varepsilon\ll 1$ such that, if we define

$$F=(1-\varepsilon\delta)B_m+l_{m-1}k\varepsilon D,$$

then $(X, S+F)$ is log smooth and $\lfloor F \rfloor = 0$. In particular, if W is a general element of the free linear system $|(D_{r_{m-1}} + H)_{|S}|$ and

$$\Phi = F_{|S} + (1-\varepsilon)W,$$

then (S, Φ) is klt.

By induction, there is a divisor $\Upsilon \in |D_{m-1} + H|$ which does not contain S in its support and such that

$$\Upsilon_{|S} = l_{m-1}\Sigma + W.$$

Denoting $C = (1-\varepsilon)\Upsilon + F$, we have

$$C \sim_{\mathbb{Q}} (1-\varepsilon)(D_{m-1} + H) + (1-\varepsilon\delta)B_m + l_{m-1}k\varepsilon D$$

and

$$C_{|S} = (1-\varepsilon)\Upsilon_{|S} + F_{|S} \leq l_{m-1}\Sigma + \Phi \leq (l_m\Sigma + U_m) + \Phi.$$

By the choice of δ and since P_m is nef, the $\mathbb{Q}$-divisor

$$A = \varepsilon(D_{r_{m-1}} + H + \delta B_m) + P_m$$

is ample. Since

$$\begin{aligned} D_m + H &= K_X + S + D_{m-1} + B_m + P_m + H \\ &= K_X + S + (1-\varepsilon)D_{m-1} + l_{m-1}k\varepsilon(K_X + \Delta) + \varepsilon D_{r_{m-1}} + B_m + P_m + H \\ &\sim_{\mathbb{Q}} K_X + S + A + (1-\varepsilon)D_{m-1} + l_{m-1}k\varepsilon D + (1-\varepsilon\delta)B_m + (1-\varepsilon)H \\ &\sim_{\mathbb{Q}} K_X + S + A + C, \end{aligned}$$

we deduce $l_m\Sigma + U_m \in |D_m + H|_S$ by Lemma 3.2. □

Theorem 3.4. *Let $(X, S + \sum_{i=1}^p S_i)$ be a log smooth projective pair, where S and all S_i are distinct prime divisors. Let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$, and let $B \in \mathcal{L}(V)$ be a $\mathbb{Q}$-divisor such that $(S, B_{|S})$ is canonical. Let A be an ample $\mathbb{Q}$-divisor on X and denote $\Delta = S + A + B$. Let $C \geq 0$ be a $\mathbb{Q}$-divisor on S, and let m be a positive integer such that mA, mB and mC are integral.*

Assume that there exists a positive integer $q \gg 0$ such that qA is very ample, $S \not\subseteq \mathrm{Bs}\,|qm(K_X + \Delta + \frac{1}{m}A)|$ and

$$C \leq B_{|S} - B_{|S} \wedge \frac{1}{qm}\,\mathrm{Fix}\,|qm(K_X + \Delta + \tfrac{1}{m}A)|_S.$$

Then

$$|m(K_S + A_{|S} + C)| + m(B_{|S} - C) \subseteq |m(K_X + \Delta)|_S.$$

In particular, if $|m(K_S + A_{|S} + C)| \neq \emptyset$, then $|m(K_X + \Delta)|_S \neq \emptyset$, and

$$\mathrm{Fix}\,|m(K_S + A_{|S} + C)| + m(B_{|S} - C) \geq \mathrm{Fix}\,|m(K_X + \Delta)|_S \geq m\,\mathbf{Fix}_S(K_X + \Delta).$$

Proof. Let $f\colon Y\longrightarrow X$ be a log resolution of the pair $(X,S+B)$ and of the linear system $|qm(K_X+\Delta+\frac{1}{m}A)|$, and write $T=f_*^{-1}S$. Then there are divisors $B',E\geq 0$ on Y with no common components, such that E is f-exceptional and

$$K_Y+T+B'=f^*(K_X+S+B)+E.$$

Let $\Gamma=T+B'+f^*A$, and define

$$F_q=\tfrac{1}{qm}\operatorname{Fix}|qm(K_Y+\Gamma+\tfrac{1}{m}f^*A)|\qquad\text{and}\qquad B'_q=B'-B'\wedge F_q.$$

Set $\Gamma_q=T+B'_q+f^*A$. Since $(Y,T+B'+F_q)$ is log smooth and $\operatorname{Mob}\big(qm(K_Y+\Gamma+\frac{1}{m}f^*A)\big)$ is basepoint free, by Bertini's theorem there exists a $\mathbb{Q}$-divisor $D\geq 0$ such that

$$K_Y+\Gamma_q+\tfrac{1}{m}f^*A\sim_{\mathbb{Q}}D,$$

the pair $(Y,T+B'_q+D)$ is log smooth, and D does not contain any component of $T+B'_q$. Let $g=f_{|T}\colon T\longrightarrow S$. Since $(S,B_{|S})$ is canonical and $C\leq B_{|S}$, there is a g-exceptional divisor $F\geq 0$ on T such that

$$K_T+C'=g^*(K_S+C)+F,$$

where $C'=g_*^{-1}C$. We claim that $C'\leq B'_{q|T}$. Assuming the claim, let us show how it implies the theorem.

By Lemma 3.3, there exists a very ample divisor H' on Y such that for all divisors $\Sigma'\in|qm(K_T+(B'_q+(1+\frac{1}{m})f^*A)_{|T})|$ and $U'\in|H'_{|T}|$, and for every positive integer k we have

$$k\Sigma'+U'\in|kqm(K_Y+\Gamma_q+\tfrac{1}{m}f^*A)+H'|_T.$$

Pick a $\mathbb{Q}$-divisor $G\in V$ such that $B+\frac{1}{m}G\geq 0$, $\lfloor B+\frac{1}{m}G\rfloor=0$ and $A-G$ is ample. In particular, $(S,(B+\frac{1}{m}G)_{|S})$ is klt. Let $H=f_*H'$, and let $W_1\in|qA_{|S}|$ and $W'_2\in|H'_{|T}|$ be general sections. Pick a positive integer $k\gg 0$ such that, if we denote $l=kq$, $W=kW_1+g_*W'_2$ and $\Phi=B_{|S}+\frac{1}{m}G_{|S}+\frac{1}{l}W$, then the divisor $A_0=\frac{1}{m}(A-G)-\frac{m-1}{ml}H$ is ample and the pair (S,Φ) is klt.

Pick $\Sigma\in|m(K_S+A_{|S}+C)|$. Since $C'\leq B'_{q|T}$, we have that

$$qg^*\Sigma+qm(F+B'_{q|T}-C')+g^*W_1\in|qm(K_T+(B'_q+(1+\tfrac{1}{m})f^*A)_{|T})|.$$

Then there exists $\Upsilon'\in|lm(K_Y+\Gamma_q+\frac{1}{m}f^*A)+H'|$, which does not contain T in its support, such that

$$\Upsilon'_{|T}=lg^*\Sigma+lm(F+B'_{q|T}-C')+kg^*W_1+W'_2.$$

Pushing forward via g and denoting $\Upsilon=f_*\Upsilon'\in|lm(K_X+f_*\Gamma_q+\frac{1}{m}A)+H|$, we have

$$\Upsilon_{|S}=l\Sigma+lm(g_*B'_{q|T}-C)+W.$$

Denoting $B_0 = \frac{m-1}{ml}\Upsilon + (m-1)(\Delta - f_*\Gamma_q) + B + \frac{1}{m}G$, and noting that $\Delta - f_*\Gamma_q = B - f_*B'_q$, we have

$$B_{0|S} = \frac{m-1}{m}\Sigma + (m-1)\big(g_*B'_{q|T} - C + (\Delta - f_*\Gamma_q)_{|S}\big) \\ + \frac{m-1}{ml}W + B_{|S} + \frac{1}{m}G_{|S} \leq \Sigma + m(B_{|S} - C) + \Phi,$$

and since

$$\begin{aligned} m(K_X + \Delta) &= K_X + S + (m-1)(K_X + \Delta + \tfrac{1}{m}A) + \tfrac{1}{m}A + B \\ &\sim_{\mathbb{Q}} K_X + S + \tfrac{m-1}{ml}\Upsilon + (m-1)(\Delta - f_*\Gamma_q) + \tfrac{1}{m}A - \tfrac{m-1}{ml}H + B \\ &= K_X + S + A_0 + B_0, \end{aligned}$$

we deduce $\Sigma + m(B_{|S} - C) \in |m(K_X + \Delta)|_S$ by Lemma 3.2. The lemma follows.

Now we prove the claim stated above. As

$$K_T + B'_{|T} = g^*(K_S + B_{|S}) + E_{|T},$$

and since $(Y, T + B' + F_q)$ is log smooth and B' and E do not have common components, it follows that $B'_{|T}$ and $E_{|T}$ do not have common components, and in particular, $E_{|T}$ is g-exceptional and $g_*B'_{|T} = B_{|S}$.

Since $\operatorname{Mob}\big(qm(K_Y + \Gamma + \frac{1}{m}f^*A)\big)$ is basepoint free and T is not a component of F_q, it follows that $\frac{1}{qm}\operatorname{Fix}|qm(K_Y + \Gamma + \frac{1}{m}f^*A)|_T = F_{q|T}$ and

$$B'_{q|T} = B'_{|T} - (B' \wedge F_q)_{|T} = B'_{|T} - B'_{|T} \wedge \frac{1}{qm}\operatorname{Fix}|qm(K_Y + \Gamma + \tfrac{1}{m}f^*A)|_T.$$

Furthermore, we have

$$g_*\operatorname{Fix}|qm(K_Y + \Gamma + \tfrac{1}{m}f^*A)|_T = \operatorname{Fix}|qm(K_X + \Delta + \tfrac{1}{m}A)|_S,$$

so

$$g_*C' = C \leq B_{|S} - B_{|S} \wedge \frac{1}{qm}\operatorname{Fix}|qm(K_X + \Delta + \tfrac{1}{m}A)|_S = g_*B'_{q|T}.$$

Therefore $C' \leq B'_{q|T}$, since $B'_{q|T} \geq 0$ and $C' = g_*^{-1}C$. $\square$

Corollary 3.5. *Let $(X, S + \sum_{i=1}^p S_i)$ be a log smooth projective pair, where S and all S_i are distinct prime divisors. Let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \operatorname{Div}_{\mathbb{R}}(X)$, and let $B \in \mathcal{L}(V)$ be a $\mathbb{Q}$-divisor such that $(S, B_{|S})$ is canonical. Let A be an ample $\mathbb{Q}$-divisor on X and denote $\Delta = S + A + B$. Let m be a positive integer such that mA and mB are integral, and such that $S \not\subseteq \operatorname{Bs}|m(K_X + \Delta)|$. Denote $\Phi_m = B_{|S} - B_{|S} \wedge \frac{1}{m}\operatorname{Fix}|m(K_X + \Delta)|_S$.*

Then

$$|m(K_S + A_{|S} + \Phi_m)| + m(B_{|S} - \Phi_m) = |m(K_X + \Delta)|_S.$$

Proof. The inclusion $|m(K_S + A_{|S} + \Phi_m)| + m(B_{|S} - \Phi_m) \subseteq |m(K_X + \Delta)|_S$ follows from Theorem 3.4, whereas the inverse inclusion is obvious since $m(B_{|S} - \Phi_m) \leq \operatorname{Fix}|m(K_X + \Delta)|_S$. $\square$

Lemma 3.6. *Let X be a smooth projective variety and let S be a smooth prime divisor on X. Let $D \subseteq \operatorname{Div}(X)$ be such that $S \nsubseteq \mathbf{B}(D)$, and let A be an ample $\mathbb{Q}$-divisor. Then there exists a sufficiently divisible positive integer q such that*

$$\frac{1}{q}\operatorname{Fix}|q(D+A)|_S \leq \mathbf{Fix}_S(D).$$

Proof. Let P be a prime divisor on S and let $\gamma = \operatorname{mult}_P \mathbf{Fix}_S(D)$. It is enough to show that there exists a sufficiently divisible positive integer q such that

$$\operatorname{mult}_P \frac{1}{q}\operatorname{Fix}|q(D+A)|_S \leq \gamma.$$

Assume first that $\gamma > 0$. Let $\varepsilon > 0$ be a rational number such that $\varepsilon D + A$ is ample, and pick a positive integer m such that

$$\frac{1-\varepsilon}{m}\operatorname{mult}_P \operatorname{Fix}|mD|_S \leq \gamma.$$

Let q be a sufficiently divisible positive integer such that the divisor $q(\varepsilon D + A)$ is very ample, and $\frac{1}{q(1-\varepsilon)}\operatorname{Fix}|q(1-\varepsilon)D|_S \leq \frac{1}{m}\operatorname{Fix}|mD|_S$. Then

$$\begin{aligned}\frac{1}{q}\operatorname{mult}_P \operatorname{Fix}|q(D+A)|_S &= \frac{1}{q}\operatorname{mult}_P \operatorname{Fix}|q(1-\varepsilon)D + q(\varepsilon D + A)|_S\\ &\leq \frac{1}{q}\operatorname{mult}_P \operatorname{Fix}|q(1-\varepsilon)D|_S \leq \frac{1-\varepsilon}{m}\operatorname{mult}_P \operatorname{Fix}|mD|_S \leq \gamma.\end{aligned}$$

Now assume that $\gamma = 0$. Let $n = \dim X$ and let H be a very ample divisor on X. Pick a positive integer q such that qA and qD are integral, and such that

$$C = qA - K_X - S - nH$$

is ample. By Lemma 2.3, there exists a $\mathbb{Q}$-divisor $D' \geq 0$ such that $D' \sim_{\mathbb{Q}} D$, $S \nsubseteq \operatorname{Supp} D'$ and $\operatorname{mult}_P(D'_{|S}) < \frac{1}{q}$. Let $f\colon Y \longrightarrow X$ be a log resolution of $(X, S+D')$ which is obtained as a sequence of blowups along smooth centres. Let $T = f_*^{-1}S$, and let $E \geq 0$ be the f-exceptional integral divisor such that

$$K_Y + T = f^*(K_X + S) + E.$$

Then, denoting $F = qf^*(D+A) - \lfloor qf^*D' \rfloor + E$, we have

$$F \sim_{\mathbb{Q}} qf^*A + \{qf^*D'\} + E = K_Y + T + f^*(nH + C) + \{qf^*D'\},$$

and in particular $|F_{|T}| = |F|_T$ by Lemma 3.1(1). Denote $g = f_{|T}\colon T \longrightarrow S$ and let $P' = g_*^{-1}P$. Since $F_{|T} \sim_{\mathbb{Q}} K_T + g^*(nH_{|S}) + g^*(C_{|S}) + \{qf^*D'\}_{|T}$ and g is an isomorphism at the generic point of P', Lemma 3.1(2) implies that $F_{|T}$ is free at the generic point of P'. In particular, if $V \in |F|$ is a general element, then $P \nsubseteq \operatorname{Supp} f_*V$.

Let $U = V + \lfloor qf^*D' \rfloor \in |qf^*(D+A)+E|$. Since E is f-exceptional, this implies that $f_*U \in |q(D+A)|$, and since $f_*\lfloor qf^*D' \rfloor \leq qD'$, we have

$$\operatorname{mult}_P(f_*U)_{|S} = \operatorname{mult}_P(f_*V)_{|S} + \operatorname{mult}_P(f_*\lfloor qf^*D' \rfloor)_{|S} \leq \operatorname{mult}_P qD'_{|S} < 1.$$

Thus, $\operatorname{mult}_P(f_*U)_{|S} = 0$ and the lemma follows. □

4. $\mathcal{B}_A^S(V)$ is a rational polytope

In all results of this section we work in the following setup, and we write "Setup 4.1_n" to denote "Setup 4.1 in dimension n."

Setup 4.1. Let $(X, S + \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S and all S_i are distinct prime divisors. Let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \operatorname{Div}_{\mathbb{R}}(X)$, let A be an ample $\mathbb{Q}$-divisor on X, and let $W \subseteq \operatorname{Div}_{\mathbb{R}}(S)$ be the subspace spanned by the components of $\sum S_{i|S}$. For $\mathbb{Q}$-divisors $E \in \mathcal{E}_{A_{|S}}(W)$ and $B \in \mathcal{B}_A^S(V)$, let

$$\mathbf{F}(E) = \mathbf{Fix}(K_S + A_{|S} + E) \quad \text{and} \quad \mathbf{F}_S(B) = \mathbf{Fix}_S(K_X + S + A + B).$$

Denote

$$\Phi_m(B) = B_{|S} - B_{|S} \wedge \tfrac{1}{m} \operatorname{Fix}|m(K_X + S + A + B)|_S$$

for every sufficiently divisible positive integer m, and let $\mathbf{\Phi}(B) = B_{|S} - B_{|S} \wedge \mathbf{F}_S(B)$. Note that $\mathbf{\Phi}(B) = \limsup \Phi_m(B)$.

Lemma 4.2. *Let the assumptions of Setup 4.1_n hold. Then $\Phi_m(B) \in \mathcal{E}_{A_{|S}}(W)$ and $\Phi_m(B) \wedge \mathbf{F}(\Phi_m(B)) = 0$.*

Proof. Trivially $\Phi_m(B) \in \mathcal{E}_{A_{|S}}(W)$. For the second claim, note that

$$|m(K_S + A_{|S} + \Phi_m(B))| + m(B_{|S} - \Phi_m(B)) \supseteq |m(K_X + S + A + B)|_S,$$

so

$$\operatorname{Fix}|m(K_S + A_{|S} + \Phi_m(B))| + m(B_{|S} - \Phi_m(B)) \leq \operatorname{Fix}|m(K_X + S + A + B)|_S.$$

Thus, if T is a component of $\Phi_m(B)$, then by definition

$$\operatorname{mult}_T \Phi_m(B) = \operatorname{mult}_T B_{|S} - \tfrac{1}{m} \operatorname{mult}_T \operatorname{Fix}|m(K_X + S + A + B)|_S,$$

and therefore $\operatorname{mult}_T \operatorname{Fix}|m(K_S + A_{|S} + \Phi_m(B))| = 0$. Hence $\operatorname{mult}_T \operatorname{Fix}|km(K_S + A_{|S} + \Phi_m(B))| = 0$ for every $k \in \mathbb{N}$, which implies

$$\Phi_m(B) \wedge \tfrac{1}{km} \operatorname{Fix}|km(K_S + A_{|S} + \Phi_m(B))| = 0.$$

Letting $k \longrightarrow \infty$ yields the lemma. □

Lemma 4.3. *Let the assumptions of Setup 4.1_n hold. Let $0 < \varepsilon \ll 1$ be a rational number such that $D + \frac{1}{4}A$ is ample for any $D \in V$ with $\|D\| < \varepsilon$, and $\varepsilon(K_X + S + A + B) + \frac{1}{4}A$ is ample for any $B \in \mathcal{L}(V)$.*

Let $\mathcal{F} \subseteq \mathcal{E}_{A_{|S}}(W)$ be a rational polytope such that $E \wedge \mathbf{F}(E) = 0$ for any $E \in \mathcal{F}$, and assume that there exists a positive integer k such that $\mathbf{F}(E) = \frac{1}{m} \operatorname{Fix}|m(K_S +$

$A_{|S}+E)|$ for all $E\in\mathcal{F}$ and all positive integers m such that mE/k is integral. Let $\mathcal{C}=\{(L,E)\in\mathcal{L}(V)\times\mathcal{F}\mid E\leq L_{|S}\}$.

Let $(B,C)\in\mathcal{C}$ and $(\Gamma,\Psi)\in\operatorname{face}\big(\mathcal{C},(B,C)\big)$, and assume there exist a positive integer m and a rational number $0<\phi\leq 1$ such that mA/k, $m\Gamma/k$ and $m\Psi/k$ are integral, and $\|\Gamma-B\|<\frac{\phi\varepsilon}{2m}$ and $\|\Psi-C\|<\frac{\phi\varepsilon}{2m}$. For every $0<\delta\ll\frac{\varepsilon}{m}$, let $(B_\delta,C_\delta)\in\mathcal{C}$ be a rational point such that $\|B-B_\delta\|<\frac{\delta}{2}$, $\|C-C_\delta\|<\frac{\delta}{2}$, $B_\delta\in\mathcal{B}_A^S(V)$ and $0\leq C_\delta\leq\mathbf{\Phi}(B_\delta)$. Assume that $(S,\Gamma_{|S})$ and all $(S,B_{\delta|S})$ are terminal, and that for any prime divisor T on S we have

$$\operatorname{mult}_T(B_{|S}-C)>\phi\qquad\text{or}\qquad\operatorname{mult}_T(B_{|S}-C)\leq\operatorname{mult}_T(\Gamma_{|S}-\Psi).$$

Then $\Gamma\in\mathcal{B}_A^S(V)$ and $\Psi\leq\mathbf{\Phi}(\Gamma)$.

Proof. Let T be a prime divisor on S. We claim that $S\not\subseteq\mathbf{B}(K_X+S+A+\Gamma+\frac{1}{2m}A)$ and that for all δ we have

$$\operatorname{mult}_T\mathbf{Fix}_S(K_X+S+A+\Gamma+\tfrac{1}{2m}A)\leq\operatorname{mult}_T\big(\Gamma_{|S}-\Psi\big)+\operatorname{mult}_T\mathbf{F}(C_\delta)+\delta.$$

Assuming the claim, let us show how it implies the lemma.

Note that $\mathcal{C}$ is a rational polytope. If T is a component of Ψ, then T is a component of C as $(\Gamma,\Psi)\in\operatorname{face}\big(\mathcal{C},(B,C)\big)$. Thus $T\subseteq\operatorname{Supp}C_\delta$ for $\delta\ll 1$, and so $\operatorname{mult}_T\mathbf{F}(C_\delta)=0$ since $C_\delta\in\mathcal{F}$. Hence, letting $\delta\longrightarrow 0$ in the claim, we get

$$\Gamma_{|S}\wedge\mathbf{Fix}_S(K_X+S+A+\Gamma+\tfrac{1}{2m}A)\leq\Gamma_{|S}-\Psi.$$

By Lemma 3.6, there exists a sufficiently divisible positive integer ℓ such that

$$\Gamma_{|S}\wedge\tfrac{1}{\ell}\operatorname{Fix}|\ell(K_X+S+A+\Gamma+\tfrac{1}{m}A)|_S\leq\Gamma_{|S}\wedge\mathbf{Fix}_S(K_X+S+A+\Gamma+\tfrac{1}{2m}A),$$

and moreover, we may assume that $S\not\subseteq\operatorname{Bs}|\ell(K_X+S+A+\Gamma+\frac{1}{m}A)|$. Then Theorem 3.4 implies that $\Gamma\in\mathcal{B}_A^S(V)$ as $\Psi\in\mathcal{E}_{A_{|S}}(W)$, and furthermore,

$$m\,\mathbf{F}(\Psi)+m(\Gamma_{|S}-\Psi)=\operatorname{Fix}|m(K_S+A_{|S}+\Psi)|+m(\Gamma_{|S}-\Psi)\geq m\,\mathbf{F}_S(\Gamma).$$

Since $\Psi\in\mathcal{F}$, it follows that $\Psi\wedge\mathbf{F}(\Psi)=0$. Therefore $\Gamma_{|S}-\Psi\geq\Gamma_{|S}\wedge\mathbf{F}_S(\Gamma)$, and so $\Psi\leq\mathbf{\Phi}(\Gamma)$.

Now we prove the claim. Fix δ. Since $\|\Gamma-B_\delta\|\leq\frac{\varepsilon}{m}$, the divisors $H=\Gamma-B_\delta+\frac{1}{4m}A$ and $G=\frac{\varepsilon}{m}(K_X+S+A+B_\delta)+\frac{1}{4m}A$ are ample. By assumption, and by Lemmas 2.3 and 3.6, there exists a sufficiently divisible positive integer q such that $S\not\subseteq\operatorname{Bs}|q(K_X+S+A+B_\delta)|$, $\frac{1}{q}\operatorname{Fix}|q(K_S+A_{|S}+C_\delta)|=\mathbf{F}(C_\delta)$, and

$$\tfrac{1}{q}\operatorname{Fix}|q(K_X+S+A+B_\delta+H+\tfrac{1}{2m}A)|_S\leq\mathbf{Fix}_S(K_X+S+A+B_\delta+\tfrac{1}{2m}A).$$

By Lemma 3.6, there is an integer $w\gg 0$ such that

$$\tfrac{1}{wq}\operatorname{Fix}|wq(K_X+S+A+B_\delta+\tfrac{1}{q}A)|_S\leq\mathbf{F}_S(B_\delta),$$

so, by assumption, we have

$$C_\delta\leq\mathbf{\Phi}(B_\delta)\leq B_{\delta|S}-B_{\delta|S}\wedge\tfrac{1}{wq}\operatorname{Fix}|wq(K_X+S+A+B_\delta+\tfrac{1}{q}A)|_S.$$

Since the pair $(S, B_{\delta|S})$ is terminal, Theorem 3.4 implies

$$|q(K_S + A_{|S} + C_\delta)| + q(B_{\delta|S} - C_\delta) \subseteq |q(K_X + S + A + B_\delta)|_S,$$

thus

$$\begin{aligned}\mathbf{F}_S(B_\delta) &\leq \tfrac{1}{q}\operatorname{Fix}|q(K_X + S + A + B_\delta)|_S \\ &\leq B_{\delta|S} - C_\delta + \tfrac{1}{q}\operatorname{Fix}|q(K_S + A_{|S} + C_\delta)| = B_{\delta|S} - C_\delta + \mathbf{F}(C_\delta).\end{aligned}$$

As $\Gamma + \frac{1}{2m}A = B_\delta + H + \frac{1}{4m}A$, we have $\mathbf{B}(K_X + S + A + \Gamma + \frac{1}{2m}A) \subseteq \mathbf{B}(K_X + S + A + B_\delta)$, and so $S \nsubseteq \mathbf{B}(K_X + S + A + \Gamma + \frac{1}{2m}A)$. Furthermore,

$$\begin{aligned}\mathbf{Fix}_S(K_X + S + A + \Gamma + \tfrac{1}{2m}A) &\leq \tfrac{1}{q}\operatorname{Fix}|q(K_X + S + A + B_\delta + H + \tfrac{1}{4m}A)|_S \\ &\leq \mathbf{Fix}_S\big((1 - \tfrac{\varepsilon}{m})(K_X + S + A + B_\delta) + G\big) \\ &\leq \big(1 - \tfrac{\varepsilon}{m}\big)\,\mathbf{F}_S(B_\delta) \leq (1 - \tfrac{\varepsilon}{m})(B_{\delta|S} - C_\delta) + \mathbf{F}(C_\delta),\end{aligned}$$

and since $(1 - \frac{\varepsilon}{m})\operatorname{mult}_T(B_{\delta|S} - C_\delta) \leq (1 - \frac{\varepsilon}{m})\operatorname{mult}_T(B_{|S} - C) + \delta$, it is enough to show that

$$\big(1 - \tfrac{\epsilon}{m}\big)\operatorname{mult}_T(B_{|S} - C) \leq \operatorname{mult}_T(\Gamma_{|S} - \Psi).$$

This is obvious if $\operatorname{mult}_T(B_{|S} - C) \leq \operatorname{mult}_T(\Gamma_{|S} - \Psi)$. Otherwise, $\operatorname{mult}_T(B_{|S} - C) > \phi$ and $\operatorname{mult}_T(B_{|S} - C) \leq \operatorname{mult}_T(\Gamma_{|S} - \Psi) + \frac{\phi\varepsilon}{m}$ by assumption, so we have

$$\begin{aligned}(1 - \tfrac{\varepsilon}{m})\operatorname{mult}_T(B_{|S} - C) &\leq \operatorname{mult}_T(\Gamma_{|S} - \Psi) + \tfrac{\phi\varepsilon}{m} - \tfrac{\varepsilon}{m}\operatorname{mult}_T(B_{|S} - C) \\ &\leq \operatorname{mult}_T(\Gamma_{|S} - \Psi) - \tfrac{\varepsilon}{m}\big(\operatorname{mult}_T(B_{|S} - C) - \phi\big) \leq \operatorname{mult}_T(\Gamma_{|S} - \Psi).\end{aligned}$$

This completes the proof. □

The main result of this section is:

Theorem 4.4. *Assume Theorem A_{n-1} and Theorem B_{n-1}, and let the assumptions of Setup 4.1$_n$ hold. Let $\mathcal{G}$ be a rational polytope contained in the interior of $\mathcal{L}(V)$, and assume that $(S, G_{|S})$ is terminal for every $G \in \mathcal{G}$. Denote $\mathcal{P} = \mathcal{G} \cap \mathcal{B}^S_A(V)$. Then:*

(1) $\mathcal{P}$ is a rational polytope,

(2) $\mathbf{\Phi}$ extends to a rational piecewise affine function on $\mathcal{P}$, and there exists a positive integer ℓ such that $\mathbf{\Phi}(P) = \Phi_m(P)$ for every $P \in \mathcal{P}$ and every positive integer m such that mP/ℓ is integral.

Proof. Theorem B$_{n-1}$ implies that $\mathcal{E}_{A_{|S}}(W)$ is a rational polytope, and hence, if $E_1, \dots, E_d$ are its extreme points, the ring $R(S; K_S + A_{|S} + E_1, \dots, K_S + A_{|S} + E_d)$ is finitely generated by Theorem A$_{n-1}$. Therefore, by Lemma 2.22(1), $\mathbf{F}$ extends to a rational piecewise affine function on $\mathcal{E}_{A_{|S}}(W)$, and there exists a positive integer k such that $\mathbf{F}(E) = \frac{1}{m}\operatorname{Fix}|m(K_S + A_{|S} + E)|$ for every $E \in \mathcal{E}_{A_{|S}}(W)$ and every $m \in \mathbb{N}$ such that mE/k is integral. Let

$$\mathcal{F} = \{E \in \mathcal{E}_{A_{|S}}(W) \mid E \wedge \mathbf{F}(E) = 0\}.$$

Then $\mathcal{F}$ is cut out from $\mathcal{E}_{A_{|S}}(W)$ by a finite number of rational hyperplanes and rational half-spaces, thus there are finitely many rational polytopes $\mathcal{F}_i$ such that $\mathcal{F}=\bigcup_i \mathcal{F}_i$. For every i, set

$$\mathcal{Q}'_i=\{(P,F)\in\mathcal{P}\times\mathcal{F}_i\mid P\in\operatorname{Div}_{\mathbb{Q}}(X),F\in\operatorname{Div}_{\mathbb{Q}}(S),F\leq\mathbf{\Phi}(P)\},$$

and let $\mathcal{Q}_i$ be the convex hull of $\mathcal{Q}'_i$. We first show that $\mathcal{Q}'_i$ is dense in $\mathcal{Q}_i$. Let $(P_0,F_0),(P_1,F_1)\in\mathcal{Q}'_i$, and for a rational number $0\leq t\leq 1$, let $P_t=(1-t)P_0+tP_1$ and $F_t=(1-t)F_0+tF_1$. Let T be a prime divisor in W. If $\operatorname{mult}_T F_t=0$ for some $0<t<1$, then $\operatorname{mult}_T F_t=0$ for all rational $t\in[0,1]$, and in particular $\operatorname{mult}_T F_t\leq \operatorname{mult}_T\mathbf{\Phi}(P_t)$. Otherwise, we have $\operatorname{mult}_T F_t>0$ for all $0<t<1$, and it follows from the definition of $\mathcal{F}$ that $\operatorname{mult}_T\mathbf{F}(F_t)=0$ for all $t\in[0,1]$. Let m be a positive integer such that mP_j/k and mF_j/k are integral. By Lemma 3.6, there exists a sufficiently divisible positive integer q such that $\frac{1}{q}\operatorname{Fix}|q(K_X+A+P_j+\frac{1}{m}A)|_S\leq\mathbf{F}_S(P_j)$. Then Theorem 3.4 implies

$$m\,\mathbf{F}(F_j)+m(P_{j|S}-F_j)=\operatorname{Fix}|m(K_S+A_{|S}+F_j)|+m(P_{j|S}-F_j)\geq m\,\mathbf{F}_S(P_j),$$

and therefore $\operatorname{mult}_T\big(P_{j|S}-\mathbf{F}_S(P_j)\big)\geq\operatorname{mult}_T F_j$. Convexity of $\mathbf{F}_S$ yields $\operatorname{mult}_T F_t\leq \operatorname{mult}_T\big(P_{t|S}-\mathbf{F}_S(P_t)\big)\leq\operatorname{mult}_T\mathbf{\Phi}(P_t)$ for all t, and thus $(P_t,F_t)\in\mathcal{Q}'_i$.

We claim that $\mathcal{Q}_i$ is a rational polytope. Granting the claim, let us show how it implies the theorem.

Let $\mathcal{P}_i\subseteq V$ be the image of $\mathcal{Q}_i$ through the first projection. For any $\mathbb{Q}$-divisor $P\in\mathcal{P}$ and for any sufficiently divisible positive integer m, we have $\big(P,\Phi_m(P)\big)\in\bigcup_i\mathcal{Q}_i$ by Lemma 4.2. Compactness implies $\big(P,\mathbf{\Phi}(P)\big)\in\bigcup_i\mathcal{Q}_i$, and therefore $\mathcal{P}=\bigcup_i\mathcal{P}_i$, so (1) follows.

For (2), denote $\mathcal{P}_{\mathbb{Q}}=\mathcal{P}\cap\operatorname{Div}_{\mathbb{Q}}(X)$ and $\mathcal{P}_S=S+\mathcal{P}_{\mathbb{Q}}$, and note that $\mathcal{P}_S$ lies in the hyperplane $(S=1)\subseteq\mathbb{R}S+V$. Fix a prime divisor $T\in W$, and consider the map $\mathbf{\Phi}_T\colon\mathcal{P}_S\longrightarrow[-1,0]$ defined by $\mathbf{\Phi}_T(S+P)=-\operatorname{mult}_T\mathbf{\Phi}(P)$ for every $P\in\mathcal{P}_{\mathbb{Q}}$. Let $\mathcal{R}_T$ be the closure of the set

$$\mathcal{R}'_T=\{S+P\in\mathcal{P}_S\mid\mathbf{\Phi}_T(S+P)\neq 0\}\subseteq\mathcal{P}_S.$$

Note that the condition $\mathbf{\Phi}_T(S+P)\neq 0$ implies $\mathbf{\Phi}_T(S+P)=-\operatorname{mult}_T\big(P_{|S}-\mathbf{F}_S(P)\big)$, and since $\mathbf{F}_S$ is a convex map on $\mathcal{P}$, the set $\mathcal{R}_T$ is convex and $\mathbf{\Phi}_T$ is convex on $\mathcal{R}_T$.

We first show that $\mathcal{R}_T$ is a union of some $S+\mathcal{P}_i$, and therefore that it is a rational polytope since it is convex. Fix $P\in\mathcal{P}_{\mathbb{Q}}$ such that $S+P\in\mathcal{R}'_T$. Then $\big(P,\mathbf{\Phi}(P)\big)\in\mathcal{Q}'_i$ for some i, and since $\operatorname{mult}_T\mathbf{\Phi}(P)\neq 0$, we have $\operatorname{mult}_T C\neq 0$ for every point (B,C) in the relative interior of $\mathcal{Q}_i$. Therefore $\operatorname{mult}_T\mathbf{F}(C)=0$ for all $(B,C)\in\mathcal{Q}_i$ by the definition of $\mathcal{F}$. Fix $(B,C)\in\mathcal{Q}'_i$, and let m be a positive integer such that all mB/k and mC/k are integral. By Lemma 3.6, there exists a sufficiently divisible positive integer q such that $\frac{1}{q}\operatorname{Fix}|q(K_X+A+B+\frac{1}{m}A)|_S\leq\mathbf{F}_S(B)$, so Theorem 3.4 implies

$$m\,\mathbf{F}(C)+m(B_{|S}-C)=\operatorname{Fix}|m(K_S+A_{|S}+C)|+m(B_{|S}-C)\geq m\,\mathbf{F}_S(B),$$

and hence $\operatorname{mult}_T\big(B_{|S}-\mathbf{F}_S(C)\big)\geq\operatorname{mult}_T C\geq 0$. Therefore, for every $B\in\mathcal{P}_i$ we have $\mathbf{\Phi}_T(S+B)=-\operatorname{mult}_T\big(B_{|S}-\mathbf{F}_S(B)\big)$, and $S+\mathcal{P}_i\subseteq\mathcal{R}_T$.

Now, let (P_j,F_j) be the extreme points of all $\mathcal{Q}_i$ for which $S+\mathcal{P}_i\subseteq\mathcal{R}_T$. Since $\mathcal{Q}_i$ is the convex hull of $\mathcal{Q}'_i$, it follows that $(P_j,F_j)\in\bigcup\mathcal{Q}'_i$, and in particular $\operatorname{mult}_T F_j\leq\operatorname{mult}_T\mathbf{\Phi}(P_j)=-\mathbf{\Phi}_T(S+P_j)$. Fix $P\in\mathcal{P}_{\mathbb{Q}}$ such that $S+P\in\mathcal{R}'_T$. Then $\big(P,\mathbf{\Phi}(P)\big)\in\mathcal{Q}'_i$ for some i by the argument above, hence there exist $r_j\in\mathbb{R}_+$ such that $\sum r_j=1$ and $\big(P,\mathbf{\Phi}(P)\big)=\sum r_j(P_j,F_j)$. Thus $\mathbf{\Phi}_T(S+P)=-\sum r_j\operatorname{mult}_T F_j$, so by convexity we have

$$\sum r_j\mathbf{\Phi}_T(S+P_j)\geq\mathbf{\Phi}_T(S+P)=-\sum r_j\operatorname{mult}_T F_j\geq\sum r_j\mathbf{\Phi}_T(S+P_j).$$

Therefore $\mathbf{\Phi}_T(S+P_j)=-\operatorname{mult}_T F_j\in\mathbb{Q}$ and $\mathbf{\Phi}_T(S+P)=\sum r_j\mathbf{\Phi}_T(S+P_j)$. By Lemma 2.9, $\mathbf{\Phi}_T$ extends to a rational piecewise affine map on $\mathcal{R}_T$, and thus on $\mathcal{P}$. Therefore, $\mathbf{\Phi}$ also extends to a rational piecewise affine map on $\mathcal{P}$.

In particular, $\mathbf{\Phi}(P)\in\mathbb{Q}$ for every $P\in\mathcal{P}_{\mathbb{Q}}$, and by subdividing $\mathcal{P}$, we may assume that $\mathbf{\Phi}$ extends to a rational affine map on $\mathcal{P}$. By Gordan's lemma, the monoid $\mathbb{R}_+\mathcal{P}_S\cap\operatorname{Div}(X)$ is finitely generated, and let $q_i(S+Q_i)$ be its generators for some $q_i\in\mathbb{Q}_+$ and $Q_i\in\mathcal{P}_{\mathbb{Q}}$. Pick a positive integer w such that $wq_i\mathbf{\Phi}(Q_i)\in\operatorname{Div}(S)$ for every i, and set $\ell=wk$.

Fix $B\in\mathcal{P}_{\mathbb{Q}}$ and a positive integer m such that $\frac{m}{\ell}B\in\operatorname{Div}(X)$. If $\alpha_i\in\mathbb{N}$ are such that $\frac{m}{\ell}(S+B)=\sum\alpha_iq_i(S+Q_i)$, then $\frac{\ell}{m}\sum\alpha_iq_i=1$ as $\mathcal{P}_S\subseteq(S=1)$, and therefore $\frac{m}{\ell}\mathbf{\Phi}(B)=\sum\alpha_iq_i\mathbf{\Phi}(Q_i)$. Hence $\frac{m}{k}\mathbf{\Phi}(B)=\sum\alpha_iwq_i\mathbf{\Phi}(Q_i)\in\operatorname{Div}(S)$, so $\mathbf{F}(\mathbf{\Phi}(B))=\frac{1}{m}\operatorname{Fix}|m(K_S+A_{|S}+\mathbf{\Phi}(B))|$ by assumption. In particular,

$$\mathbf{\Phi}(B)\wedge\operatorname{Fix}|m(K_S+A_{|S}+\mathbf{\Phi}(B))|=0$$

as $\mathbf{\Phi}(B)\in\bigcup_i\mathcal{Q}_i$. By Lemma 3.6, there exists a positive integer q such that $\mathbf{\Phi}(B)\leq B_{|S}-B_{|S}\wedge\frac{1}{qm}\operatorname{Fix}|qm(K_X+S+A+B+\frac{1}{m}A)|_S$, and thus

$$\begin{aligned}\operatorname{Fix}|m(K_S+A_{|S}+\mathbf{\Phi}(B))|+m(B_{|S}-\mathbf{\Phi}(B))&\geq\operatorname{Fix}|m(K_X+S+A+B)|_S\\&\geq m(B_{|S}\wedge\tfrac{1}{m}\operatorname{Fix}|m(K_X+S+A+B)|_S)=m(B_{|S}-\Phi_m(B))\end{aligned}$$

by Theorem 3.4. This implies $\Phi_m(B)\geq\mathbf{\Phi}(B)$. But, by definition, $\mathbf{\Phi}(B)\geq\Phi_m(B)$, and (2) follows.

Now we prove the claim stated above. Let ε be as in Lemma 4.3. The set

$$\mathcal{C}_i=\{(L,F)\in\mathcal{G}\times\mathcal{F}_i\mid F\leq L_{|S}\}$$

is a rational polytope which contains $\mathcal{Q}_i$. Fix a point $(B,C)\in\overline{\mathcal{Q}_i}$, and let Π be the set of prime divisors T on S such that $\operatorname{mult}_T(B_{|S}-C)>0$. If $\Pi\neq\emptyset$, pick a positive rational number

$$\phi<\min\{\operatorname{mult}_T(B_{|S}-C)\mid P\in\Pi\}\leq 1,$$

and set $\phi=1$ if $\Pi=\emptyset$. By Lemma 2.10, there exist points $(\Gamma_j,\Psi_j)\in\operatorname{face}\big(\mathcal{C}_i,(B,C)\big)$ and positive integers m_j divisible by k, such that m_jA/k, $m_j\Gamma_j/k$ and $m_j\Psi_j/k$ are integral, (B,C) is a convex combination of all (Γ_j,Ψ_j), and

$$\|B-\Gamma_j\|<\frac{\phi\varepsilon}{2m_j}\qquad\text{and}\qquad\|C-\Psi_j\|<\frac{\phi\varepsilon}{2m_j}.$$

Note that $\Psi_j\in\mathcal{F}_i$ since $(\Gamma_j,\Psi_j)\in\mathcal{C}_i$. By Lemma 4.3, $\Gamma_j\in\mathcal{B}^S_A(V)$ and $\Psi_j\leq\mathbf{\Phi}(\Gamma_j)$. Thus $(\Gamma_j,\Psi_j)\in\mathcal{Q}_i$, and so $(B,C)\in\mathcal{Q}_i$. Therefore, $\mathcal{Q}_i$ is a closed set and moreover, every extreme point of $\mathcal{Q}_i$ is rational.

Assume that there exist infinitely many extreme points $v_\ell=(B_\ell,C_\ell)$ of $\mathcal{Q}_i$, with $\ell\in\mathbb{N}$. Since $\mathcal{Q}_i$ is compact and $\mathcal{C}_i$ is a rational polytope, by passing to a subsequence there exist $v_\infty=(B_\infty,C_\infty)\in\mathcal{Q}_i$ and a positive dimensional face F of $\mathcal{C}_i$ such that

$$v_\infty=\lim_{\ell\to\infty}v_\ell\qquad\text{and}\qquad\operatorname{face}(\mathcal{C}_i,v_\ell)=F\quad\text{for all }\ell\in\mathbb{N}.$$

In particular, $v_\infty\in F$. Let Π_∞ be the set of all prime divisors T on S such that $\operatorname{mult}_T(B_{\infty|S}-C_\infty)>0$. If $\Pi_\infty\neq\emptyset$, pick a positive rational number

$$\phi<\min\{\operatorname{mult}_T(B_{\infty|S}-C_\infty)\mid P\in\Pi_\infty\}\leq 1,$$

and set $\phi=1$ if $\Pi_\infty=\emptyset$. Then, by Lemma 2.10 there exist $m\in\mathbb{N}$ divisible by k, and $v'_\infty=(B'_\infty,C'_\infty)\in\operatorname{face}(\mathcal{C}_i,(B_\infty,C_\infty))$ such that $\frac{m}{k}v'_\infty$ is integral and $\|B_\infty-B'_\infty\|<\frac{\phi\varepsilon}{2m}$ and $\|C_\infty-C'_\infty\|<\frac{\phi\varepsilon}{2m}$. Lemma 4.3 yields $v'_\infty\in\mathcal{Q}_i$. Pick $j\gg 0$ so that

$$\|v_j-v'_\infty\|\leq\|v_j-v_\infty\|+\|v_\infty-v'_\infty\|<\frac{\phi\varepsilon}{2m},$$

and that $\operatorname{mult}_T(B_j,C_j)>\phi$ if $T\in\Pi_\infty$. Note that v_j is contained in the interior of F. Therefore, there exists a positive integer $m'\gg 0$ divisible by k, such that $\frac{m+m'}{k}v_j$ is integral, and such that if we define

$$v'_j=\frac{m+m'}{m'}v_j-\frac{m}{m'}v'_\infty\in v'_\infty+\mathbb{R}_+(v_j-v'_\infty),$$

then $v'_j=(B'_j,C'_j)\in F$. Note that $\frac{m'}{k}v'_j$ is integral, $v_j=\frac{m'}{m+m'}v'_j+\frac{m}{m+m'}v'_\infty$, and

$$\|v'_j-v_j\|=\frac{m}{m'}\|v'_j-v'_\infty\|<\frac{\phi\varepsilon}{2m'}.$$

Furthermore, if T is a prime divisor on S such that $T\notin\Pi$, then

$$\operatorname{mult}_T(B_{j|S}-C_j)=\frac{m'}{m+m'}\operatorname{mult}_T(B'_{j|S}-C'_j)\leq\operatorname{mult}_T(B'_{j|S}-C'_j).$$

Thus, $v'_j\in\mathcal{Q}_i$ by Lemma 4.3, and since v_j is in the interior of $[v'_j,v'_\infty]$, we have that v_j is not an extreme point of $\mathcal{Q}_i$, a contradiction which proves the claim. □

Corollary 4.5. *Assume Theorem A_{n-1} and Theorem B_{n-1}.*

Let $(X,S+\sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S and all S_i are distinct prime divisors. Let $V=\sum_{i=1}^p\mathbb{R}S_i\subseteq\operatorname{Div}_\mathbb{R}(X)$ and let A be an ample $\mathbb{Q}$-divisor on X. Then $\mathcal{B}^S_A(V)$ is a rational polytope, and

$$\mathcal{B}^S_A(V)=\{B\in\mathcal{L}(V)\mid\sigma_S(K_X+S+A+B)=0\}.$$

Proof. Fix $B \in \overline{\mathcal{B}_A^S(V)}$. Let $G \in V$ be a $\mathbb{Q}$-divisor such that $B - G$ is contained in the interior of $\mathcal{L}(V)$ and $A + G$ is ample. Then, in order to prove the first claim, it is enough to show that $B - G \in \mathcal{B}_{A+G}^S(V)$, and by compactness, that $\mathcal{B}_{A+G}^S(V)$ is locally a rational polytope around $B - G$. Thus, after replacing B by $B - G$ and A by $A + G$, we can assume that B is in the interior of $\mathcal{L}(V)$.

By Lemma 2.2, there exist a log resolution $f\colon Y \longrightarrow X$ of $(X, S + B)$ and $\mathbb{Q}$-divisors $C, E \geq 0$ on Y with no common components such that the components of C are disjoint, $\lfloor C \rfloor = 0$, $T = f_*^{-1}S \nsubseteq \operatorname{Supp} C$, and

$$K_Y + T + C = f^*(K_X + S + B) + E.$$

Let $W' \subseteq \operatorname{Div}_{\mathbb{R}}(Y)$ be the subspace spanned by the components of C and by all f-exceptional prime divisors. Then there exists an f-exceptional $\mathbb{Q}$-divisor $F \geq 0$ such that $f^*A - F$ is ample, $C + F$ lies in the interior of $\mathcal{L}(W')$ and $(T, (C + F)_{|T})$ is terminal. Then it is enough to show that $\mathcal{B}_{f^*A-F}^T(W')$ is locally a rational polytope around $C + F$, so after replacing X by Y, S by T, B by $C + F$, A by $f^*A - F$ and V by W', we may assume that there exists $0 < \eta \ll 1$ such that $(S, (B + \Theta)_{|S})$ is terminal for every $\Theta \in V$ with $\|\Theta\| \leq \eta$. Let $\mathcal{P} = \{B' \in \mathcal{L}(V) \mid \|B' - B\| \leq \eta\}$, and note that $\mathcal{P}$ is a rational polytope since we are working with the sup-norm. Then $\mathcal{P} \cap \mathcal{B}_A^S(V)$ is a rational polytope by Theorem 4.4. In particular, it is closed, and thus $B \in \mathcal{B}_A^S(V)$. This proves the first claim.

Now we prove the second claim. Denoting $\mathcal{Q} = \{B \in \mathcal{L}(V) \mid \sigma_S(K_X + S + A + B) = 0\}$, clearly $\mathcal{Q} \supseteq \mathcal{B}_A^S(V)$. For the inverse inclusion, fix $B \in \mathcal{Q}$, and let H be a very ample divisor which is general in the linear system $|H|$. Then $(X, S + \sum_{i=1}^p S_i + H)$ is log smooth and $H \nsubseteq \operatorname{Supp}(S + \sum_{i=1}^p S_i)$. Let $V' = \mathbb{R}H + V \subseteq \operatorname{Div}_{\mathbb{R}}(X)$. Then $B + tH \in \mathcal{B}_A^S(V')$ for any rational $0 < t < 1$ by Lemma 2.15, hence $B \in \mathcal{B}_A^S(V')$ since $\mathcal{B}_A^S(V')$ is closed by the first part of the proof. Therefore $B \in \mathcal{B}_A^S(V)$. $\square$

5. Effective non-vanishing

Lemma 5.1. *Let (X, B) be a log smooth pair, where B is a $\mathbb{Q}$-divisor such that $\lfloor B \rfloor = 0$. Let A be a nef and big $\mathbb{Q}$-divisor, and assume that there exists an $\mathbb{R}$-divisor $D \geq 0$ such that $K_X + A + B \equiv D$.*

Then there exists a $\mathbb{Q}$-divisor $D' \geq 0$ such that $K_X + A + B \sim_{\mathbb{Q}} D'$.

Proof. Let $V \subseteq \operatorname{Div}(X)_{\mathbb{R}}$ be the vector space spanned by the components of K_X, A, B and D, and let $\phi\colon V \longrightarrow N^1(X)_{\mathbb{R}}$ be the linear map sending a divisor to its numerical class. Since $\phi^{-1}(\phi(K_X + A + B))$ is a rational affine subspace of V, we can assume that D is an effective $\mathbb{Q}$-divisor.

Let $f\colon Y \longrightarrow X$ be a log resolution of $(X, B + D)$. Then there exist $\mathbb{Q}$-divisors $B', E \geq 0$ with no common components such that

$$K_Y + B' = f^*(K_X + B) + E.$$

After replacing (X,B) by (Y,B'), A by f^*A, and D by f^*D+E, we may assume that $(X,B+D)$ is log smooth.

Let m be a positive integer such that $m(A+B)$ and mD are integral. Denoting $F=(m-1)D+B$, $L=m(K_X+A+B)-\lfloor F\rfloor$ and $L'=mD-\lfloor F\rfloor$, we have

$$L\equiv L'=D-B+\{F\}\equiv K_X+A+\{F\}.$$

Thus, Kawamata-Viehweg vanishing implies that $H^i(X,L)=H^i(X,L')=0$ for all $i>0$, and since the Euler characteristic is a numerical invariant, this yields $h^0(X,L)=h^0(X,L')$. As mD is integral and $\lfloor B\rfloor=0$, it follows that

$$L'=mD-\lfloor (m-1)D+B\rfloor=\lceil D-B\rceil\geq 0,$$

and thus $h^0\big(X,m(K_X+A+B)\big)=h^0(X,L+\lfloor F\rfloor)\geq h^0(X,L)=h^0(X,L')>0.$ □

Lemma 5.2. *Let X be a smooth projective variety of dimension n and let $x\in X$. Let $D\in\operatorname{Div}(X)$ and assume that s is a positive integer such that $h^0(X,D)>\binom{s+n}{n}$.*

Then there exists $D'\in|D|$ such that $\operatorname{mult}_x D'>s$.

Proof. Let $\mathfrak{m}\subseteq\mathcal{O}_X$ be the ideal sheaf of x. Then we have

$$h^0(X,\mathcal{O}_X/\mathfrak{m}^{s+1})=\dim_{\mathbb{C}}\mathbb{C}[x_1,\dots,x_n]/(x_1,\dots,x_n)^{s+1}=\binom{s+n}{n},$$

hence $h^0(X,D)>h^0(X,\mathcal{O}_X/\mathfrak{m}^{s+1})$. Therefore the exact sequence

$$0\longrightarrow\mathfrak{m}^{s+1}\otimes\mathcal{O}_X(D)\longrightarrow\mathcal{O}_X(D)\longrightarrow(\mathcal{O}_X/\mathfrak{m}^{s+1})\otimes\mathcal{O}_X(D)\simeq\mathcal{O}_X/\mathfrak{m}^{s+1}\longrightarrow 0$$

yields $h^0\big(X,\mathfrak{m}^{s+1}\otimes\mathcal{O}_X(D)\big)>0$, so there exists a divisor $D'\in|D|$ with multiplicity at least $s+1$ at x. □

Lemma 5.3. *Assume Theorem A_{n-1} and Theorem B_{n-1}.*

Let (X,B) be a log smooth pair, where B is an $\mathbb{R}$-divisor such that $\llcorner B\lrcorner=0$. Let A be an ample $\mathbb{Q}$-divisor on X, and assume that K_X+A+B is a pseudo-effective divisor such that $K_X+A+B\not\equiv N_\sigma(K_X+A+B)$.

Then there exists an $\mathbb{R}$-divisor $F\geq 0$ such that $K_X+A+B\sim_{\mathbb{R}}F$.

Proof. Denote $\Delta=A+B$. By Lemma 2.13, there exist sufficiently divisible positive integers m and k such that $h^0(X,\lfloor mk(K_X+\Delta)\rfloor+kA)>\binom{nk+n}{n}$. Fix a point $x\in X\setminus\operatorname{Supp}N_\sigma(K_X+\Delta)$. Then, by Lemma 5.2 there exists an $\mathbb{R}$-divisor $G\geq 0$ such that $G\sim_{\mathbb{R}}mk(K_X+\Delta)+kA$ and $\operatorname{mult}_x G>nk$, so setting $D=\frac{1}{mk}G$, we have

$$D\sim_{\mathbb{R}}K_X+\Delta+\frac{1}{m}A\qquad\text{and}\qquad\operatorname{mult}_x D>\frac{n}{m}.$$

For any $t\in[0,m]$, define $A_t=\frac{m-t}{m}A$ and $\Psi_t=B+tD$, so that

$$(1+t)(K_X+\Delta)\sim_{\mathbb{R}}K_X+A+B+t\big(D-\tfrac{1}{m}A\big)=K_X+A_t+\Psi_t.$$

Let $f\colon Y \longrightarrow X$ be a log resolution of $(X, B+D)$ constructed by first blowing up X at x. Then for every $t \in [0,m]$, there exist $\mathbb{R}$-divisors $C_t, E_t \geq 0$ with no common components such that E_t is f-exceptional and

$$K_Y + C_t = f^*(K_X + \Psi_t) + E_t.$$

The exceptional divisor of the initial blowup gives a prime divisor $P \subseteq Y$ which is not contained in $\operatorname{Supp} N_\sigma(f^*(K_X+\Delta))$, such that $\operatorname{mult}_P(K_Y - f^*K_X) = n-1$ and $\operatorname{mult}_P f^*\Psi_t = \operatorname{mult}_x \Psi_t$. Since $\operatorname{mult}_x \Psi_m > n$, it follows that $\operatorname{mult}_P E_m = 0$ and $\operatorname{mult}_P C_m > 1$. Note that $\lfloor C_0 \rfloor = 0$, and denote

$$B_t = C_t - C_t \wedge N_\sigma(K_Y + f^*A_t + C_t).$$

Observe that

$$\begin{aligned} N_\sigma(K_Y + f^*A_t + C_t) &= N_\sigma(f^*(K_X + A_t + \Psi_t)) + E_t \\ &= (1+t)N_\sigma\big(f^*(K_X+\Delta)\big) + E_t, \end{aligned}$$

hence B_t is a continuous function in t. Moreover $P \not\subseteq \operatorname{Supp} N_\sigma(K_Y + f^*A_m + B_m)$, and in particular $\operatorname{mult}_P B_m > 1$. Pick $0 < \varepsilon \ll 1$ such that $\operatorname{mult}_P B_{m-\varepsilon} > 1$, and let $H \geq 0$ be an f-exceptional $\mathbb{Q}$-divisor on Y such that $\lfloor B_0 + H \rfloor = 0$ and $f^*A_{m-\varepsilon} - H$ is ample. Then there exists a minimal $\lambda < m - \varepsilon$ such that $\lfloor B_\lambda + H \rfloor$ contains a prime divisor S. Note that $B_\lambda \wedge N_\sigma(K_X + f^*A_\lambda + B_\lambda) = 0$, and since $\lfloor H \rfloor = 0$, it follows that $S \not\subseteq \operatorname{Supp} N_\sigma(K_Y + f^*A_\lambda + B_\lambda)$.

Let $A' = f^*A_\lambda - H = f^*(\frac{m-\varepsilon-\lambda}{m}A) + (f^*A_{m-\varepsilon} - H)$. Then A' is ample, and since $\sigma_S(K_Y + A' + B_\lambda + H) = \sigma_S(K_Y + f^*A_\lambda + B_\lambda) = 0$, Corollary 4.5 implies that $S \not\subseteq \mathbf{B}(K_Y + A' + B_\lambda + H) = \mathbf{B}(K_Y + f^*A_\lambda + B_\lambda)$. In particular, there exists an $\mathbb{R}$-divisor $F' \geq 0$ such that $K_Y + f^*A_\lambda + B_\lambda \sim_\mathbb{R} F'$, and thus

$$K_X + \Delta \sim_\mathbb{R} \frac{1}{1+\lambda} f_*(K_Y + f^*A_\lambda + C_\lambda) \sim_\mathbb{R} \frac{1}{1+\lambda} f_*(F' + C_\lambda - B_\lambda) \geq 0.$$

This finishes the proof. □

Lemma 5.4. *Assume Theorem A_{n-1} and Theorem B_{n-1}.*

Let $(X, S + \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S and all S_i are distinct prime divisors. Let A be an ample $\mathbb{Q}$-divisor on X, let $W = \mathbb{R}S + \sum_{i=1}^p \mathbb{R}S_i \subseteq \operatorname{Div}_\mathbb{R}(X)$, and assume $\Upsilon \in \mathcal{L}(W)$ and $0 \leq \Sigma \in W$ are such that $\operatorname{mult}_S \Upsilon = 1$, $\operatorname{mult}_S \Sigma > 0$, $\sigma_S(K_X + A + \Upsilon) = 0$ *and* $K_X + A + \Upsilon \sim_\mathbb{R} \Sigma$. *Let $\Upsilon_m \in W$ be a sequence such that $K_X + A + \Upsilon_m$ is pseudo-effective and* $\lim \Upsilon_m = \Upsilon$.

Then for infinitely many m there exists $\Upsilon'_m \in W$ such that Υ_m is contained in the interior of $[\Upsilon, \Upsilon'_m]$ and $K_X + A + \Upsilon'_m$ is pseudo-effective.

Proof. Denote $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \operatorname{Div}_\mathbb{R}(X)$. Let $\Sigma_m = \Sigma + \Upsilon_m - \Upsilon \sim_\mathbb{R} K_X + A + \Upsilon_m$, and let $\Gamma_m = \Sigma_m - \sigma_S(\Sigma_m)S$. Then Γ_m is pseudo-effective. Pick $0 < \varepsilon \ll 1$ such that $A + \Phi$ is ample for every $\Phi \in W$ with $\|\Phi\| \leq \varepsilon$, and let

$$Z = \sum_{\operatorname{mult}_{S_i} \Upsilon = 1} S_i - \sum_{\operatorname{mult}_{S_j} \Upsilon = 0} S_j.$$

Then the divisor $A' = A + \varepsilon Z$ is ample, and Corollary 4.5 implies that $\Upsilon - \varepsilon Z - S \in \mathcal{B}^S_{A'}(V)$. Note that $K_X + S + A' \sim_{\mathbb{R}} \Sigma - (\Upsilon - \varepsilon Z - S)$, and let

$$\mathcal{P} = \Sigma - (\Upsilon - \varepsilon Z - S) + \mathcal{B}^S_{A'}(V) \subseteq W \qquad \text{and} \qquad \mathcal{D} = \mathbb{R}_+ \mathcal{P} \subseteq W.$$

Then $\Sigma \in \mathcal{P}$, and Corollary 4.5 implies that $\mathcal{P}$ is a rational polytope. Moreover, $\operatorname{mult}_S D = \operatorname{mult}_S \Sigma > 0$ for any $D \in \mathcal{P}$ and, in particular, $\mathcal{P}$ does not contain the origin. Thus $\mathcal{D}$ is a rational polyhedral cone.

We claim that, after passing to a subsequence, we have that $\Gamma_m \in \mathcal{D}$ for all $m > 0$ and $\lim \Gamma_m = \Sigma$. Granting the claim, let us show how it implies the lemma.

Since $\mathcal{D}$ is a rational polyhedral cone, for any $m \gg 0$ there exist $\Psi_m \in \mathcal{D}$ and $0 < \mu_m < 1$ such that $\Gamma_m = \mu_m \Sigma + (1 - \mu_m)\Psi_m$. In particular, Ψ_m is pseudo-effective, and hence so is the divisor $\Sigma'_m = \Psi_m + \frac{1}{1-\mu_m}(\Sigma_m - \Gamma_m)$. Note that $\Sigma_m = \mu_m \Sigma + (1 - \mu_m)\Sigma'_m$, and let $\Upsilon'_m \in W$ be such that $\Upsilon_m = \mu_m \Upsilon + (1 - \mu_m)\Upsilon'_m$. Then $K_X + A + \Upsilon'_m \sim_{\mathbb{R}} \Sigma'_m$ is pseudo-effective as desired.

Now we prove the claim. Note that $\{\Sigma + \Theta \mid \Theta \in \mathcal{L}(V), \|\Theta\| \leq \varepsilon\} \subseteq \mathcal{D}$, and therefore the dimension of $\mathcal{D}$ is equal to $\dim W$. If Σ belongs to the interior of $\mathcal{D}$, then $\Sigma_m \in \mathcal{D}$ for $m \gg 0$ and, in particular, $\sigma_S(\Sigma_m) = 0$. Therefore, $\Gamma_m = \Sigma_m$ and the claim follows. Otherwise, Σ belongs to the boundary of $\mathcal{D}$. Let $\mathcal{H}_i$ be the supporting hyperplanes of maximal faces of $\mathcal{D}$ containing Σ, for $i = 1, \dots, \ell \leq \dim W - 1$. Let $\mathcal{W}_i \supseteq \mathcal{D}$ be half-spaces bounded by $\mathcal{H}_i$, and denote $\mathcal{Q} = \bigcap_{i=1}^{\ell} \mathcal{W}_i$. If $\Sigma_m \in \mathcal{Q}$ for infinitely many m, then $\Sigma_m \in \mathcal{D}$, and again $\Gamma_m = \Sigma_m$. Thus, after taking a subsequence, we may assume that $\Sigma_m \notin \mathcal{Q}$ for all m.

Since $\operatorname{mult}_S \Sigma > 0$, for every m there exist $\beta_m \in \mathbb{R}_{>0}$ and $\delta_m < 1$ such that

$$\beta_m \operatorname{mult}_S \Gamma_m - \delta_m \operatorname{mult}_S \Sigma = 0 \qquad \text{and} \qquad \|\beta_m \Gamma_m - \delta_m \Sigma\| < \varepsilon,$$

and let $R_m = \Upsilon + \beta_m \Gamma_m - \delta_m \Sigma$. Then $\operatorname{mult}_S R_m = \operatorname{mult}_S \Upsilon = 1$, $R_m - \varepsilon Z - S$ belongs to the interior of $\mathcal{L}(V)$, and note that

$$(1 - \delta_m)\Sigma + \beta_m \Gamma_m \sim_{\mathbb{R}} K_X + A + R_m = K_X + S + A' + (R_m - \varepsilon Z - S).$$

Since $\sigma_S\big((1 - \delta_m)\Sigma + \beta_m \Gamma_m\big) \leq (1 - \delta_m)\sigma_S(\Sigma) + \beta_m \sigma_S(\Gamma_m) = 0$, Corollary 4.5 implies that $R_m - \epsilon Z - S \in \mathcal{B}^S_{A'}(V)$ and in particular $(1 - \delta_m)\Sigma + \beta_m \Gamma_m \in \mathcal{D}$. As $\Sigma \in \mathcal{H}_i$ for every i, the convex cone $\mathbb{R}_{>0}\Sigma + \mathbb{R}_{>0}\Gamma_m$ intersects $\mathcal{W}_i$ for every i. This implies that $\Gamma_m \in \mathcal{W}_i$, and thus $\Gamma_m \in \mathcal{Q}$. Therefore, after passing to a subsequence we may assume that there is $i_0 \in \{1, \dots, \ell\}$, such that for all m there exists $P_m \in [\Sigma_m, \Gamma_m] \cap \mathcal{H}_{i_0}$. In particular $\lim P_m = \Sigma$, and thus $P_m \in \mathcal{D}$ for $m \gg 0$. This implies $\sigma_S(P_m) = 0$, and finally $\Gamma_m = P_m \in \mathcal{D}$ and $\lim \Gamma_m = \Sigma$. □

Theorem 5.5. *Theorem A_{n-1} and Theorem B_{n-1} imply Theorem B_n.*

Proof. Let

$$\mathcal{P}_A(V) = \{B \in \mathcal{L}(V) \mid K_X + A + B \equiv D \text{ for some } \mathbb{R}\text{-divisor } D \geq 0\}.$$

We claim that $\mathcal{P}_A(V)$ is a rational polytope. Assuming the claim, let $B_1, \dots, B_q$ be the extreme points of $\mathcal{P}_A(V)$, and choose $\varepsilon > 0$ such that $A + \varepsilon B_i$ is ample for every i. Since $K_X + A + B_i = K_X + (A + \varepsilon B_i) + (1 - \varepsilon)B_i$ and $\lfloor (1-\varepsilon)B_i \rfloor = 0$, Lemma 5.1 implies that there exist $\mathbb{Q}$-divisors $D_i \geq 0$ such that $K_X + A + B_i \sim_{\mathbb{Q}} D_i$. Thus $B_i \in \mathcal{E}_A(V)$ for every i, and therefore $\mathcal{P}_A(V) \subseteq \mathcal{E}_A(V)$ as $\mathcal{E}_A(V)$ is convex. Since obviously $\mathcal{E}_A(V) \subseteq \mathcal{P}_A(V)$, the theorem follows.

In order to prove the claim, we first show that $\mathcal{P}_A(V)$ is closed. Next, we prove that it is a polytope around every point in it, and compactness then implies that it is a polytope. Finally, at the end of the proof we show that it is a *rational* polytope.

Fix $B \in \overline{\mathcal{P}_A(V)}$ and denote $\Delta = A + B$. In particular, $K_X + \Delta$ is pseudo-effective. Pick a divisor $G \in V$ such that $A + G$ is ample and $B - G$ is in the interior of $\mathcal{L}(V)$. Then it is enough to show that $\mathcal{P}_{A+G}(V)$ is a polytope around $B - G$, so after replacing A by $A + G$ and B by $B - G$, we may assume that $\lfloor B \rfloor = 0$.

If $K_X + \Delta \not\equiv N_\sigma(K_X + \Delta)$, then by Lemma 5.3 there exists an $\mathbb{R}$-divisor $F \geq 0$ such that $K_X + \Delta \sim_{\mathbb{R}} F$, and in particular $B \in \mathcal{P}_A(V)$. If $K_X + \Delta \equiv N_\sigma(K_X + \Delta)$, then it follows immediately that $B \in \mathcal{P}_A(V)$. This implies that $\mathcal{P}_A(V)$ is compact.

We now show that $\mathcal{P}_A(V)$ is a polytope around B. We distinguish two cases. Let us first assume that $K_X + \Delta \not\equiv N_\sigma(K_X + \Delta)$. Let $F \geq 0$ be an $\mathbb{R}$-divisor such that $K_X + \Delta \sim_{\mathbb{R}} F$, and let $f \colon Y \longrightarrow X$ be a log resolution of $(X, B + F)$. Then there are divisors $C, E \geq 0$ on Y with no common components such that

$$K_Y + C = f^*(K_X + B) + E.$$

Let $G \geq 0$ be an f-exceptional divisor such that $f^*A - G$ is ample and $\lfloor C + G \rfloor = 0$. Let $V' \subseteq \operatorname{Div}_{\mathbb{R}}(Y)$ be the vector space spanned by the components of C, E, f^*F and G. It suffices to show that $\mathcal{P}_{f^*A-G}(V')$ is a polytope around $C + G$. Thus after replacing X by Y, A by $f^*A - G$, B by $C + G$, F by $f^*F + E$ and V by V', we may assume that $(X, B + F)$ is log smooth and that $F \in V$.

Let us assume there exists an infinite sequence of distinct extreme points $B_m \in \mathcal{P}_A(V)$ such that $\lim B_m = B$. For any $t \geq 0$, define $\Phi_t = B + tF$, so that

$$(1 + t)(K_X + \Delta) \sim_{\mathbb{R}} K_X + A + B + tF = K_X + A + \Phi_t.$$

Note that $\lfloor \Phi_0 \rfloor = 0$ and

$$N_\sigma(K_X + A + \Phi_t) = (1 + t)N_\sigma(K_X + \Delta).$$

Thus, if we denote

$$\Upsilon_t = \Phi_t - \Phi_t \wedge N_\sigma(K_X + A + \Phi_t),$$

then Υ_t is a continuous function in t. Write $F = \sum_{j=1}^{\ell} f_j F_j$, where F_j are prime divisors and $f_j > 0$ for all j. Since $F \not\equiv N_\sigma(F)$, by Lemma 2.14 there exists $j \in \{1, \dots, \ell\}$ such that $\sigma_{F_j}(F) = 0$. Thus $\operatorname{mult}_{F_j} \Upsilon_t = \operatorname{mult}_{F_j} B + tf_j$, so there exists a minimal $\mu > 0$ such that $\lfloor \Upsilon_\mu \rfloor$ contains a prime divisor S such that $S \subseteq \operatorname{Supp} F$, and note that $\sigma_S(K_X + A + \Upsilon_\mu) = 0$. Let

$$\Sigma = (1+\mu)F - \Phi_\mu \wedge N_\sigma\big((1+\mu)F\big).$$

Then we have $\Sigma \geq 0$, $\operatorname{mult}_S \Sigma > 0$ and $K_X + A + \Upsilon_\mu \sim_{\mathbb{R}} \Sigma$.

For every $m \in \mathbb{N}$, define $\Phi_{\mu,m} = B_m + \mu(F + B_m - B)$. Then

$$\lim_{m\to\infty} \Phi_{\mu,m} = \Phi_\mu \qquad \text{and} \qquad (1+\mu)(K_X + A + B_m) \sim_{\mathbb{R}} K_X + A + \Phi_{\mu,m}.$$

Let

$$\Lambda = \Phi_\mu \wedge N_\sigma(K_X + A + \Phi_\mu) \qquad \text{and} \qquad \Lambda_m = \Phi_{\mu,m} \wedge \sum_{Z \subseteq \operatorname{Supp} \Lambda} \sigma_Z(K_X + A + \Phi_{\mu,m})Z.$$

Note that $0 \leq \Lambda_m \leq N_\sigma(K_X + A + \Phi_{\mu,m})$, and therefore $K_X + A + \Phi_{\mu,m} - \Lambda_m$ is pseudo-effective. By Lemma 2.12 we have $\Lambda \leq \liminf \Lambda_m$, and in particular, $\operatorname{Supp} \Lambda_m = \operatorname{Supp} \Lambda$ for $m \gg 0$. Therefore, there exists an increasing sequence of rational numbers $\varepsilon_m > 0$ such that $\lim \varepsilon_m = 1$ and $\Lambda_m \geq \varepsilon_m \Lambda$.

Define $\Upsilon_{\mu,m} = \Phi_{\mu,m} - \varepsilon_m \Lambda$. Note that $K_X + A + \Upsilon_{\mu,m}$ is pseudo-effective and

$$\lim_{m\to\infty} \Upsilon_{\mu,m} = \Phi_\mu - \Lambda = \Upsilon_\mu.$$

Therefore, by Lemma 5.4 there exist $m \gg 0$, $\Upsilon'_m \in V$ and $0 < \alpha_m \ll \varepsilon_m$ such that

$$K_X + A + \Upsilon'_m \quad \text{is pseudo-effective} \qquad \text{and} \qquad \Upsilon_{\mu,m} = \alpha_m \Upsilon_\mu + (1-\alpha_m)\Upsilon'_m.$$

Setting $B'_m = \frac{1}{1-\alpha_m}(B_m - \alpha_m B)$, we have $B_m = \alpha_m B + (1-\alpha_m)B'_m$, and an easy calculation shows that

$$K_X + A + B'_m \sim_{\mathbb{R}} \frac{1}{1+\mu}\Big(K_X + A + \Upsilon'_m + \frac{\varepsilon_m - \alpha_m}{1-\alpha_m}\Lambda\Big).$$

In particular, $K_X + A + B'_m$ is pseudo-effective. Since $\mathcal{L}(V)$ is a rational polytope, we may assume that $B'_m \in \mathcal{L}(V)$. In particular, $B'_m \in \mathcal{P}_A(V)$ and B_m is not an extreme point of $\mathcal{P}_A(V)$, a contradiction. Therefore, $\mathcal{P}_A(V)$ is a polytope in a neighbourhood of B.

Now assume that $K_X + \Delta \equiv N_\sigma(K_X + \Delta)$. If $\mathcal{P}_A(V)$ is not a polytope around B, then there exists an infinite sequence of distinct extreme points $B_m \in \mathcal{P}_A(V)$ such that $\lim B_m = B$. Since $\mathcal{L}(V)$ is a rational polytope, there exists $0 < \delta \ll 1$ such that $C_{m,t} = B_m + t(B_m - B) \in \mathcal{L}_A(V)$ for any $m \gg 0$ and $0 \leq t \leq \delta$. Let $D_m \geq 0$ be $\mathbb{R}$-divisors such that $K_X + A + B_m \equiv D_m$. There exists an ample $\mathbb{R}$-divisor H such that

$$\operatorname{Supp} N_\sigma(K_X + \Delta) \subseteq \mathbf{B}(K_X + \Delta + H),$$

and since $H + (K_X + \Delta - D_m) \equiv H + (B - B_m)$ is ample for all $m \gg 0$, we have

$$\operatorname{Supp} N_\sigma(K_X + \Delta) \subseteq \mathbf{B}\left(D_m + H + (K_X + \Delta - D_m)\right) \subseteq \mathbf{B}(D_m) \subseteq \operatorname{Supp} D_m.$$

Thus, if m is sufficiently large, there exists $0 < t \ll 1$ such that

$$K_X + A + C_{m,t} \equiv (1+t)D_m - t(K_X + \Delta) \equiv (1+t)D_m - tN_\sigma(K_X + \Delta) \geq 0,$$

and in particular $C_{m,t} \in \mathcal{P}_A(V)$. But then $B_m = \frac{1}{1+t}C_{m,t} + \frac{t}{1+t}B$ implies that B_m is not an extreme point of $\mathcal{P}_A(V)$, a contradiction.

Therefore $\mathcal{P}_A(V)$ is a polytope. Let $B_1, \dots, B_q$ be its extreme points. Then there exist $\mathbb{R}$-divisors $D_i \geq 0$ such that $K_X + A + B_i \equiv D_i$. Let $W \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ be the vector space spanned by V and by the components of $K_X + A$ and $\sum D_i$. Note that for every $\tau = (t_1, \dots, t_q) \in \mathbb{R}^q_+$ such that $\sum t_i = 1$, we have $B_\tau = \sum t_i B_i \in \mathcal{P}_A(V)$ and $K_X + A + B_\tau \equiv \sum t_i D_i \in W$. Let $\phi\colon W \longrightarrow N^1(X)_{\mathbb{R}}$ be the linear map sending a divisor to its numerical class. Then $W_0 = \phi^{-1}(0)$ is a rational affine subspace of W and

$$\mathcal{P}_A(V) = \{B \in \mathcal{L}(V) \mid B = -K_X - A + D + R, \text{ where } 0 \leq D \in W, R \in W_0\}.$$

Therefore, $\mathcal{P}_A(V)$ is cut out from $\mathcal{L}(V) \subseteq W$ by finitely many rational half-spaces, and thus is a rational polytope. □

6. Finite generation

Lemma 6.1. *Let $(X, \sum_{i=1}^p S_i)$ be a log smooth projective pair, and denote $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$. Let $\mathcal{C} \subseteq V$ be a rational polyhedral cone, let $\mathcal{C} = \bigcup_{j=1}^p \mathcal{C}_j$ be a rational polyhedral decomposition, and denote $\mathcal{S} = \mathcal{C} \cap \mathrm{Div}(X)$ and $\mathcal{S}_j = \mathcal{C}_j \cap \mathrm{Div}(X)$ for all j. Assume that:*

(1) there exists $M > 0$ such that, if $\sum \alpha_i S_i \in \mathcal{C}_j$ for some j and some $\alpha_i \in \mathbb{N}$ with $\sum \alpha_i \geq M$, then $\sum \alpha_i S_i - S_j \in \mathcal{C}$;

(2) the ring $\mathrm{res}_{S_j} R(X, \mathcal{S}_j)$ is finitely generated for every $j = 1, \dots, p$.

Then the divisorial ring $R(X, \mathcal{S})$ is finitely generated.

Proof. For every $i = 1, \dots, p$, let $\sigma_i \in H^0(X, \mathcal{O}_X(S_i))$ be a section such that $\mathrm{div}\,\sigma_i = S_i$. Let $\mathfrak{R} \subseteq R(X; S_1, \dots, S_p)$ be the ring spanned by $R(X, \mathcal{S})$ and $\sigma_1, \dots, \sigma_p$, and note that $\mathfrak{R}$ is graded by $\sum_{i=1}^p \mathbb{N}S_i$. By Lemma 2.19(1), it is enough to show that $\mathfrak{R}$ is finitely generated.

For any $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{N}^p$, denote $D_\alpha = \sum \alpha_i S_i$ and $\deg(\alpha) = \sum \alpha_i$, and for a section $\sigma \in H^0(X, \mathcal{O}_X(D_\alpha))$, set $\deg(\sigma) = \deg(\alpha)$. For each $j = 1, \dots, p$, there exists a finite set $\mathcal{H}_j \subseteq R(X, \mathcal{S}_j)$ such that the ring $\mathrm{res}_{S_j} R(X, \mathcal{S}_j)$ is generated by the set $\{\sigma_{|S_j} \mid \sigma \in \mathcal{H}_j\}$. Since the vector space $H^0(X, \mathcal{O}_X(D_\alpha))$ is finitely generated for every $\alpha \in \mathbb{N}^p$, there is a finite set $\mathcal{H} \subseteq \mathfrak{R}$ such that $\{\sigma_1, \dots, \sigma_p\} \cup \mathcal{H}_1 \cup \dots \cup \mathcal{H}_p \subseteq \mathcal{H}$, and that $H^0(X, D_\alpha) \subseteq \mathbb{C}[\mathcal{H}]$ for every $\alpha \in \mathbb{N}^p$ with $\deg(\alpha) \leq M$, where $\mathbb{C}[\mathcal{H}]$ is the ring of polynomials in the elements of $\mathcal{H}$. Observe that $\mathbb{C}[\mathcal{H}] \subseteq \mathfrak{R}$, and it is enough to show that $\mathfrak{R} = \mathbb{C}[\mathcal{H}]$.

Let $\chi \in \mathfrak{R}$. By definition of $\mathfrak{R}$, we may write $\chi = \sum_i \sigma_1^{\lambda_{1,i}} \dots \sigma_p^{\lambda_{p,i}} \chi_i$, where $\chi_i \in H^0(X, \mathcal{O}_X(D_{\alpha_i}))$ for some $D_{\alpha_i} \in \mathcal{S}$ and $\lambda_{j,i} \in \mathbb{N}$. Thus, it is enough to show that $\chi_i \in \mathbb{C}[\mathcal{H}]$ and after replacing χ by χ_i, we may assume that $\chi \in H^0(X, D_\alpha)$, for some $D_\alpha \in \mathcal{S}$. The proof is by induction on $\deg \chi$. If $\deg \chi \leq M$, then $\chi \in \mathbb{C}[\mathcal{H}]$ by definition of $\mathcal{H}$. Now assume $\deg \chi > M$. Then there exists $1 \leq j \leq p$ such that

$D_\alpha \in \mathcal{S}_j$, and thus, by definition of $\mathcal{H}$, there are $\theta_1, \dots, \theta_z \in \mathcal{H}$ and a polynomial $\varphi \in \mathbb{C}[X_1, \dots, X_z]$ such that $\chi_{|S_j} = \varphi(\theta_{1|S_j}, \dots, \theta_{z|S_j})$. Therefore, from the exact sequence

$$0 \longrightarrow H^0(X, \mathcal{O}_X(D_\alpha - S_j)) \xrightarrow{\cdot \sigma_j} H^0(X, \mathcal{O}_X(D_\alpha)) \longrightarrow H^0(S_j, \mathcal{O}_{S_j}(D_\alpha))$$

we obtain

$$\chi - \varphi(\theta_1, \dots, \theta_z) = \sigma_j \cdot \chi'$$

for some $\chi' \in H^0(X, \mathcal{O}_X(D_\alpha - S_j))$. Note that $D_\alpha - S_j \in \mathcal{S}$ by assumption, and since $\deg \chi' < \deg \chi$, by induction we have $\chi' \in \mathbb{C}[\mathcal{H}]$. Therefore $\chi = \sigma_j \cdot \chi' + \varphi(\theta_1, \dots, \theta_z) \in \mathbb{C}[\mathcal{H}]$, and we are done. $\square$

Lemma 6.2. *Assume Theorem A_{n-1} and Theorem B_{n-1}.*

Let $(X, S + \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S and all S_i are distinct prime divisors. Let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_\mathbb{R}(X)$, let A be an ample $\mathbb{Q}$-divisor on X, let $B_1, \dots, B_m \in \mathcal{E}_{S+A}(V)$ be $\mathbb{Q}$-divisors, and denote $D_i = K_X + S + A + B_i$.

Then the ring $\mathrm{res}_S R(X; D_1, \dots, D_m)$ is finitely generated.

Proof. For every i, there is a $\mathbb{Q}$-divisor $G_i \in V$ such that $A - G_i$ is ample and $B_i + G_i$ is in the interior of $\mathcal{L}(V)$. Let A' be an ample $\mathbb{Q}$-divisor such that every $A - G_i - A'$ is also ample, and pick $\mathbb{Q}$-divisors $A_i \geq 0$ such that $A_i \sim_\mathbb{Q} A - G_i - A'$, $\lfloor A_i \rfloor = 0$, $(X, S + \sum_{i=1}^p S_i + \sum_{i=1}^m A_i)$ is log smooth, and the support of $\sum_{i=1}^m A_i$ does not contain any of the divisors $S, S_1, \dots, S_p$. Let $V' \subseteq \mathrm{Div}_\mathbb{R}(X)$ be the vector space spanned by V and by the components of $\sum_{i=1}^m A_i$. Let $\varepsilon > 0$ be a rational number such that $A'' = A' - \varepsilon \sum_{i=1}^m A_i$ is ample, and such that every $B_i' = B_i + G_i + A_i + \varepsilon \sum_{i=1}^m A_i$ is in the interior of $\mathcal{L}(V')$. Let $D_i' = K_X + S + A'' + B_i'$. Then $D_i \sim_\mathbb{Q} D_i'$, and after replacing A by A'', B_i by B_i', and V by V', by Lemma 2.19 we may assume that B_i is in the interior of $\mathcal{L}(V)$ for every i.

Let $B \geq 0$ be a $\mathbb{Q}$-divisor such that $\lfloor B \rfloor = 0$ and $B \geq B_i$ for all i. By Lemma 2.2, there exists a log resolution $f\colon Y \longrightarrow X$ such that

$$K_Y + T + C = f^*(K_X + S + B) + E,$$

where $C, E \geq 0$ have no common components, E is f-exceptional, the components of C are disjoint, and $T = f_*^{-1}S \nsubseteq \mathrm{Supp}\, C$. In particular, there are $\mathbb{Q}$-divisors $0 \leq C_i \leq C$ and f-exceptional divisors $E_i \geq 0$ such that

$$K_Y + T + C_i = f^*(K_X + S + B_i) + E_i.$$

Let $V' \subseteq \mathrm{Div}_\mathbb{R}(Y)$ be the subspace spanned by the components of C and by all f-exceptional prime divisors. There exists an f-exceptional $\mathbb{Q}$-divisor $F \geq 0$ such that $f^*A - F$ is ample, $C + F$ is in the interior of $\mathcal{L}(V')$ and $(T, (C + F)_{|T})$ is terminal. After replacing X by Y, S by T, A by $f^*A - F$, B_i by $C_i + F$ and V by V', we may assume that all $(S, B_{i|S})$ are terminal.

Let $\mathcal{G} \subseteq \mathcal{E}_{S+A}(V)$ be the convex hull of all B_i, and note that $(S, G_{|S})$ is terminal for every $G \in \mathcal{G}$. Denote $\mathcal{F} = \mathbb{R}_+(K_X + S + A + \mathcal{G})$. Then, by Corollary 2.21 it suffices to prove that $\operatorname{res}_S R(X, \mathcal{F})$ is finitely generated.

Let Φ_m and $\mathbf{\Phi}$ be the functions defined in Setup 4.1. By Theorem 4.4, $\mathcal{P} = \mathcal{G} \cap \mathcal{B}_A^S(V)$ is a rational polytope, and $\mathbf{\Phi}$ extends to a rational piecewise affine function on $\mathcal{P}$. Thus, there exists a finite decomposition $\mathcal{P} = \bigcup \mathcal{P}_i$ into rational polytopes such that $\mathbf{\Phi}$ is rational affine on each $\mathcal{P}_i$. Denote $\mathcal{C} = \mathbb{R}_+(K_X + S + A + \mathcal{P})$ and $\mathcal{C}_i = \mathbb{R}_+(K_X + S + A + \mathcal{P}_i)$. Since $\operatorname{res}_S H^0(X, D) = 0$ for every $D \in \mathcal{F} \setminus \mathcal{C}$, and as $\mathcal{C}$ is a rational polyhedral cone, it suffices to show that $\operatorname{res}_S R(X, \mathcal{C})$ is finitely generated, and therefore, to prove that $\operatorname{res}_S R(X, \mathcal{C}_i)$ is finitely generated for each i. Hence, after replacing $\mathcal{G}$ by $\mathcal{P}_i$, we can assume that $\mathbf{\Phi}$ is rational affine on $\mathcal{G}$.

For $i = 1, \dots, q$, let $F_i = g_i(K_X + S + A + G_i)$ be generators of $\mathcal{F}$, where $G_i \in \mathcal{G}$ and $g_i \in \mathbb{Q}_+$. By Theorem 4.4, there exists a positive integer ℓ such that $\Phi_m(G) = \mathbf{\Phi}(G)$ for every $G \in \mathcal{G}$ and every $m \in \mathbb{N}$ such that $\frac{m}{\ell} G \in \operatorname{Div}(X)$. Pick a positive integer k such that all $\frac{kg_i}{\ell} \in \mathbb{N}$ and $\frac{kg_i}{\ell} G_i \in \operatorname{Div}(X)$. For each $\alpha = (\alpha_1, \dots, \alpha_q) \in \mathbb{N}^q$, denote

$$g_\alpha = \sum \alpha_i g_i, \quad G_\alpha = \frac{1}{g_\alpha} \sum \alpha_i g_i G_i, \quad F_\alpha = \sum \alpha_i F_i = g_\alpha(K_X + S + A + G_\alpha),$$

and note that $\frac{kg_\alpha}{\ell} G_\alpha \in \operatorname{Div}(X)$ and $\mathbf{\Phi}(G_\alpha) = \frac{1}{g_\alpha} \sum_i \alpha_i g_i \mathbf{\Phi}(G_i)$. Then, by Corollary 3.5 we have

$$\begin{aligned} \operatorname{res}_S H^0(X, mkF_\alpha) &= H^0\big(S, mkg_\alpha(K_S + A_{|S} + \Phi_{mkg_\alpha}(G_\alpha))\big) \\ &= H^0\big(S, mkg_\alpha(K_S + A_{|S} + \mathbf{\Phi}(G_\alpha))\big) \end{aligned}$$

for all $\alpha \in \mathbb{N}^q$ and $m \in \mathbb{N}$, and thus

$$\operatorname{res}_S R(X; kF_1, \dots, kF_q) = R(S; kg_1 F_1', \dots, kg_q F_q'),$$

where $F_i' = K_S + A_{|S} + \mathbf{\Phi}(G_i)$. Since the last ring is a Veronese subring of the adjoint ring $R(S; F_1', \dots, F_q')$, it is finitely generated by Theorem A_{n-1} and by Lemma 2.19(1). Therefore $\operatorname{res}_S R(X; F_1, \dots, F_q)$ is finitely generated by Lemma 2.19(2), and since there is the natural projection of this ring onto $\operatorname{res}_S R(X, \mathcal{F})$, this proves the lemma. □

Theorem 6.3. *Theorem A_{n-1} and Theorem B_n imply Theorem A_n.*

Proof. Let $V \subseteq \operatorname{Div}_{\mathbb{R}}(X)$ be the subspace spanned by the components of all B_i, and let $\mathcal{P} \subseteq V$ be the convex hull of all B_i. By Theorem B_n, $\mathcal{P}' = \mathcal{P} \cap \mathcal{E}_A(V)$ is a rational polytope, and denote $\mathcal{G} = \mathbb{R}_+(K_X + A + \mathcal{P})$ and $\mathcal{H} = \mathbb{R}_+(K_X + A + \mathcal{P}')$. By Gordan's lemma, there are generators $H_i = h_i(K_X + A + B_i')$ of the monoid $\mathcal{H} \cap \operatorname{Div}(X)$ for $i = 1, \dots, \ell$, where $h_i \in \mathbb{Q}_+$ and $B_i \in \mathcal{P}' \cap \operatorname{Div}_{\mathbb{Q}}(X)$. Then, since $H^0(X, D) = 0$ for every $D \in \mathcal{G} \setminus \mathcal{H}$, the ring $R(X, \mathcal{G})$ is finitely generated if and only if $R(X, \mathcal{H})$ is, and there is the natural projection map $R(X; H_1, \dots, H_\ell) \longrightarrow R(X, \mathcal{H})$. Therefore, by Lemma 2.19 and Corollary 2.21, it is enough to show that $R(X; H_1', \dots, H_\ell')$ is finitely generated, where $H_i' = K_X + A + B_i'$. After replacing $B_1, \dots, B_k$ by $B_1', \dots, B_\ell'$, we may assume that there exist divisors $F_i \geq 0$ such that $F_i \sim_{\mathbb{Q}} K_X + A + B_i$ for all i. By Lemma 2.19, it suffices to prove that $R(X; F_1, \dots, F_k)$ is finitely generated.

Let $f\colon Y \longrightarrow X$ be a log resolution of $\big(X, \sum_i (B_i + F_i)\big)$. For every i, there are $\mathbb{Q}$-divisors $C_i, E_i \geq 0$ with no common components such that E_i is f-exceptional and

$$K_Y + C_i = f^*(K_X + B_i) + E_i.$$

Let $H \geq 0$ be an f-exceptional $\mathbb{Q}$-divisor such that $f^*A - H$ is ample and $\lfloor C_i + H \rfloor = 0$ for all i. Pick $A' \sim_{\mathbb{Q}} f^*A - H$ such that $A' \geq 0$, $\operatorname{Supp} A'$ is a prime divisor, and $\operatorname{Supp} A' \not\subseteq \operatorname{Supp} \sum_i (f^*F_i + E_i)$. Then

$$K_Y + A' + (C_i + H) \sim_{\mathbb{Q}} f^*(K_X + A + B_i) + E_i \sim_{\mathbb{Q}} f^*F_i + E_i,$$

so, by Lemma 2.19, after replacing X by Y, A by A', B_i by $C_i + H$, and F_i by $f^*F_i + E_i$, we may assume that $\big(X, \sum_i (B_i + F_i)\big)$ is log smooth, $A \geq 0$, $\operatorname{Supp} A$ is a prime divisor, and $\operatorname{Supp} A \not\subseteq \operatorname{Supp} \sum_i F_i$.

Let W be the subspace of $\operatorname{Div}_{\mathbb{R}}(X)$ spanned by V and the components of all F_i, and let $S_1, \dots, S_p$ be the prime divisors in W. Denote by $\mathcal{T} = \{(t_1, \dots, t_k) \mid t_i \geq 0, \sum t_i = 1\} \subseteq \mathbb{R}^k$ the standard simplex, and for each $\tau = (t_1, \dots, t_k) \in \mathcal{T}$, set

$$B_\tau = \sum_{i=1}^k t_i B_i \qquad \text{and} \qquad F_\tau = \sum_{i=1}^k t_i F_i \sim_{\mathbb{R}} K_X + A + B_\tau.$$

Denote

$$\mathcal{B} = \{F_\tau + B \mid \tau \in \mathcal{T}, 0 \leq B \in W, B_\tau + B \in \mathcal{L}(W)\},$$

and for every $j = 1, \dots, p$, let

$$\mathcal{B}_j = \{F_\tau + B \mid 0 \leq B \in W, B_\tau + B \in \mathcal{L}(W), S_j \subseteq \lfloor B_\tau + B \rfloor\}.$$

Then $\mathcal{C} = \mathbb{R}_+\mathcal{B}$ and $\mathcal{C}_j = \mathbb{R}_+\mathcal{B}_j$ are rational polyhedral cones, and denote $\mathcal{S} = \mathcal{C} \cap \operatorname{Div}(X)$ and $\mathcal{S}_j = \mathcal{C}_j \cap \operatorname{Div}(X)$. We claim that:

(1) $\mathcal{C} = \bigcup_{j=1}^p \mathcal{C}_j$,
(2) there exists $M > 0$ such that, if $\sum \alpha_i S_i \in \mathcal{C}_j$ for some j and some $\alpha_i \in \mathbb{N}$ with $\sum \alpha_i \geq M$, then $\sum \alpha_i S_i - S_j \in \mathcal{C}$;
(3) the ring $\operatorname{res}_{S_j} R(X, \mathcal{S}_j)$ is finitely generated for every $j = 1, \dots, p$.

This claim readily implies the theorem: indeed, Lemma 6.1 shows that $R(X, \mathcal{S})$ is finitely generated. Pick divisors $F_{k+1}, \dots, F_m$ such that $F_1, \dots, F_m$ are generators of $\mathcal{S}$. Then $R(X; F_1, \dots, F_m)$ is finitely generated by Corollary 2.21, and finally Lemma 2.19(1) implies that $R(X; F_1, \dots, F_k)$ is finitely generated.

We now prove the claim. In order to see (1), fix $G \in \mathcal{C} \backslash \{0\}$. Then, by definition, there exist $\tau \in \mathcal{T}, B \in W$ and $r > 0$ such that $B \geq 0$, $B_\tau + B \in \mathcal{L}(W)$ and $G = r(F_\tau + B)$. Setting

$$\lambda = \max\{t \geq 1 \mid B_\tau + tB + (t-1)F_\tau \in \mathcal{L}(W)\}$$

and $B' = \lambda B + (\lambda - 1)F_\tau$, we have

$$\lambda G = r(F_\tau + B'),$$

and there exists j_0 such that $S_{j_0} \subseteq \lfloor B_\tau + B' \rfloor$. Therefore $G \in \mathcal{C}_{j_0}$, which proves (1).

For (2), note first that there exists $\varepsilon > 0$ such that $\|B_i\| \leq 1 - \varepsilon$ for all i, and thus $\|B_\tau\| \leq 1 - \varepsilon$ for any $\tau \in \mathcal{T}$. Since the polytopes $\mathcal{B}_j \subseteq W$ are compact, there is a positive constant C such that $\|H\| \leq C$ for any $H \in \mathcal{B}_j$, and for all $j = 1, \dots, p$. Let M be a positive integer such that $M \geq qC/\varepsilon$. Let $G = \sum \alpha_i S_i \in \mathcal{S}_j$ be such that $\sum \alpha_i \geq M$. Since $q\|G\| \geq \sum \alpha_i$, we have

$$\|G\| \geq \frac{M}{q} \geq \frac{C}{\varepsilon}.$$

By definition, $G = rG'$ with $G' \in \mathcal{B}_j$ and $r > 0$. In particular, $r = \|G\|/\|G'\| \geq \frac{1}{\varepsilon}$ since $\|G'\| \leq C$ by definition of C. Furthermore, $G' = F_\tau + B$ for some $\tau \in \mathcal{T}$ and $0 \leq B \in W$ such that $B_\tau + B \in \mathcal{L}(W)$ and $S_j \subseteq \lfloor B + B_\tau \rfloor$. In particular,

$$\operatorname{mult}_{S_j} B = 1 - \operatorname{mult}_{S_j} B_\tau \geq \varepsilon \geq \frac{1}{r},$$

and thus

$$G - S_j = r\big(F_\tau + B - \tfrac{1}{r} S_j\big) \in \mathcal{C}.$$

Finally, to show (3), fix $1 \leq j \leq p$, and let $\{E_1, \dots, E_\ell\}$ be a set of generators of $\mathcal{S}_j$. Then, for every $i = 1, \dots, \ell$ there exist $k_i \in \mathbb{Q}_+$, $\tau_i \in \mathcal{T} \cap \mathbb{Q}^k$ and $0 \leq B_i \in W$ such that $B_{\tau_i} + B_i \in \mathcal{L}(W)$, $S_j \subseteq \lfloor B_{\tau_i} + B_i \rfloor$ and

$$E_i = k_i(F_{\tau_i} + B_i) \sim_{\mathbb{Q}} k_i(K_X + A + B_{\tau_i} + B_i).$$

Denote $E_i' = K_X + A + B_{\tau_i} + B_i$. Then the ring $\operatorname{res}_{S_j} R(X; E_1', \dots, E_\ell')$ is finitely generated by Lemma 6.2, and it has a Veronese subring of finite index which is isomorphic to a Veronese subring of finite index of $\operatorname{res}_{S_j} R(X; E_1, \dots, E_\ell)$. Since there is the natural projection $\operatorname{res}_{S_j} R(X; E_1, \dots, E_\ell) \longrightarrow \operatorname{res}_{S_j} R(X, \mathcal{S}_j)$, we conclude by Lemma 2.19. $\square$

Finally, we have:

Proof of Theorem 1.1. By [FM00, Theorem 5.2] and by induction on $\dim X$, we may assume that $K_X + \Delta$ is big. Write $K_X + \Delta \sim_{\mathbb{Q}} A + B$, where A is an ample $\mathbb{Q}$-divisor and $B \geq 0$. Pick a rational number $0 < \varepsilon \ll 1$. Setting $\Delta' = (\Delta + \varepsilon B) + \varepsilon A$, we have $K_X + \Delta' \sim_{\mathbb{Q}} (\varepsilon + 1)(K_X + \Delta)$. Therefore, the rings $R(X, K_X + \Delta)$ and $R(X, K_X + \Delta')$ have isomorphic truncations, so the result follows from Theorem A and Lemma 2.19. $\square$

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