ENVELOPING ALGEBRAS OF SLODOWY SLICES AND GOLDIE RANK

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Abstract. Let $U(\mathfrak{g}, e)$ be the finite W-algebra associated with a nilpotent element e in a complex simple Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and let I be a primitive ideal of the enveloping algebra $U(\mathfrak{g})$ whose associated variety equals the Zariski closure of the nilpotent orbit $(\mathrm{Ad}\, G)\, e$. Then it is known that $I = \mathrm{Ann}_{U(\mathfrak{g})}\big(Q_e \otimes_{U(\mathfrak{g},\, e)} V\big)$ for some *finite dimensional* irreducible $U(\mathfrak{g}, e)$-module V, where Q_e stands for the generalised Gelfand–Graev $\mathfrak{g}$-module associated with e. The main goal of this paper is to prove that the Goldie rank of the primitive quotient $U(\mathfrak{g})/I$ always divides $\dim V$. For $\mathfrak{g} = \mathfrak{sl}_n$, we use a result of Joseph on the Gelfand–Kirillov conjecture for primitive quotients of $U(\mathfrak{g})$ to show that the Goldie rank of $U(\mathfrak{g})/I$ equals $\dim V$.

1. Introduction

1.1. Denote by G a simple, simply connected algebraic group over $\mathbb{C}$, let (e, h, f) be a nontrivial $\mathfrak{sl}_2$-triple in the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, and denote by $(\,\cdot\,,\,\cdot\,)$ the G-invariant bilinear form on $\mathfrak{g}$ for which $(e, f) = 1$. Let $\chi \in \mathfrak{g}^*$ be such that $\chi(x) = (e, x)$ for all $x \in \mathfrak{g}$ and write $U(\mathfrak{g}, e)$ for the quantisation of the Slodowy slice $e + \mathrm{Ker}\,\mathrm{ad}\, f$ to the adjoint orbit $\mathcal{O} := (\mathrm{Ad}\, G)e$; see [21, 9]. Recall that $U(\mathfrak{g}, e) = (\mathrm{End}_{\mathfrak{g}}\, Q_e)^{\mathrm{op}}$, where Q_e is the generalised Gelfand–Graev $\mathfrak{g}$-module associated with the triple (e, h, f). The module Q_e is induced from a 1-dimensional module $\mathbb{C}_\chi$ over of a nilpotent subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ whose dimension equals $d(e) := \frac{1}{2}\dim \mathcal{O}$. The Lie subalgebra $\mathfrak{m}$ is $(\mathrm{ad}\, h)$-stable, all eigenvalues of $\mathrm{ad}\, h$ on $\mathfrak{m}$ are negative, and χ vanishes on $[\mathfrak{m}, \mathfrak{m}]$. The action of $\mathfrak{m}$ on $\mathbb{C}_\chi = \mathbb{C}1_\chi$ is given by $x(1_\chi) = \chi(x)1_\chi$ for all $x \in \mathfrak{m}$. The algebra $U(\mathfrak{g}, e)$ is also known as the finite W-algebra associated with the pair $(\mathfrak{g}, e)$ and it shares many remarkable features with the universal enveloping algebra $U(\mathfrak{g})$.

From now on we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by using the G-equivariant Killing isomorphism $\mathfrak{g} \ni x \mapsto (x, \cdot\,) \in \mathfrak{g}^*$. Given a primitive ideal I of $U(\mathfrak{g})$ we write $\mathcal{VA}(I)$ for the associated variety of I. By a classical result of Lie Theory, proved by Borho–Brylinski in special cases and by Joseph in general, the variety $\mathcal{VA}(I)$ coincides with the closure of a nilpotent orbit in $\mathfrak{g}$. If V is a finite dimensional $U(\mathfrak{g}, e)$-module, then it follows from Skryabin's theorem [28] that the $\mathfrak{g}$-module $Q_e \otimes_{U(\mathfrak{g},\, e)} V$ is simple and hence the annihilator $I_V := \mathrm{Ann}_{U(\mathfrak{g})}\big(Q_e \otimes_{U(\mathfrak{g},\, e)} V\big)$ is a primitive ideal of $U(\mathfrak{g})$. According to [22, Thm. 3.1(ii)], the variety $\mathcal{VA}(I_V)$ coincides with Zariski closure of $\mathcal{O}$.

In [22], the author conjectured that the converse is also true, i.e. for any primitive ideal I of $U(\mathfrak{g})$ with $\mathcal{VA}(I) = \overline{\mathcal{O}}$ there exists a finite dimensional irreducible $U(\mathfrak{g}, e)$-module M such that $I = I_M$. This conjecture was proved by the author in [23] under a mild technical assumption on the central character of I (removed in [25]) and by Losev [16] in general. Yet another proof of the conjecture was later found by Ginzburg

Mathematics Subject Classification (2000 *revision*). Primary 17B35. Secondary 17B63, 17B81.

[11]. Losev's proof employed his new construction of $U(\mathfrak{g},e)$ via equivariant Fedosov quantization, whilst Ginzburg's proof was based of the notion of Harish-Chandra bimodules for quantized Slodowy slices introduced and studied in [11]. The author's proof relied almost entirely on characteristic p methods.

1.2. Write $\mathcal{X}_{\mathcal{O}}$ for the set of all primitive ideals I of $U(\mathfrak{g})$ with $\mathcal{VA}(I) = \overline{\mathcal{O}}$ an denote by $\operatorname{Irr} U(\mathfrak{g},e)$ the set of all isoclasses of finite dimensional irreducible $U(\mathfrak{g},e)$-modules. It is well known that the group $C(e) := G_e \cap G_f$ is reductive and its finite quotient $\Gamma(e) := C(e)/C(e)^\circ$ identifies naturally with the component group of the nilpotent centraliser G_e (here $G_x := \{g \in G \mid (\operatorname{Ad} g)\, x = x\}$). From the realisation of $U(\mathfrak{g},e)$ obtained by Gan–Ginzburg [9] it is immediate that the algebraic group $C(e)$ acts on $U(\mathfrak{g},e)$ as algebra automorphisms. Thus, we can twist the module structure $U(\mathfrak{g},e)\times M \to M$ of any $U(\mathfrak{g},e)$-module M by an element $g \in C(e)$ to obtain a new $U(\mathfrak{g},e)$-module, ${}^gM$, with underlying vector space M and the $U(\mathfrak{g},e)$-action given by $u \cdot m = g(u)\cdot m$ for all $u \in U(\mathfrak{g},e)$ and $m \in M$. It turns out that if the $U(\mathfrak{g},e)$-module M is irreducible and $g \in C(e)$, then $I_M = I_{{}^gM}$, so that the primitive ideal I_M depends only on the isomorphism class of M; see [25, 4.8], for example. We thus obtain a natural surjective map $\varphi_e\colon \operatorname{Irr} U(\mathfrak{g},e) \twoheadrightarrow \mathcal{X}_{\mathcal{O}}$ which assigns to an isoclass $[M] \in \operatorname{Irr} U(\mathfrak{g},e)$ the primitive ideal $I_M \in \mathcal{X}_{\mathcal{O}}$, where M is any representative in $[M]$. The above discussion shows that the map φ_e is well defined and its fibres are stable under the action of $C(e)$.

By [22, Lemma 2.4], there is an algebra embedding $\Theta\colon U(\operatorname{Lie} C(e)) \hookrightarrow U(\mathfrak{g},e)$ such that the differential of the rational action of $C(e)$ on $U(\mathfrak{g},e)$ coincides with $(\operatorname{ad} \circ \Theta)_{|\operatorname{Lie}(C(e))}$. As a consequence, every two-sided ideal of $U(\mathfrak{g},e)$ is stable under the action of the connected group $C(e)^\circ$. Applying this to the primitive ideals of finite codimension in $U(\mathfrak{g},e)$ it is easy to observe that the identity component $C(e)^\circ$ of $C(e)$ acts trivially on $\operatorname{Irr} U(\mathfrak{g},e)$. We thus obtain a natural action of the finite group $\Gamma(e)$ on the set $\operatorname{Irr} U(\mathfrak{g},e)$.

1.3. Confirming another conjecture of the author (first circulated around 2007) Losev proved that each fibre of φ_e is a single $\Gamma(e)$-orbit; see [17, Thm. 1.2.2]. This result shows that a generalised Gelfand–Graev model of $I \in \mathcal{X}_{\mathcal{O}}$ is *almost* unique; in particular, if $I_M = I = I_{M'}$ for two finite dimensional irreducible $U(\mathfrak{g},e)$-modules M and M', then necessarily $\dim M = \dim M'$. The main goal of this paper is to relate the latter number with the Goldie rank of the primitive quotient $U(\mathfrak{g})/I$.

Let $\mathcal{A}$ be a prime Noetherian ring. An element of $\mathcal{A}$ is called *regular* if it is not a zero divisor in $\mathcal{A}$. By Goldie's theory, the set S of all regular elements of $\mathcal{A}$ is multiplicative and satisfies the left and right Ore conditions. Therefore, it can be used to form a classical ring of fractions $\mathcal{Q}(\mathcal{A}) = S^{-1}\mathcal{A}$; see [7, 3.6] for more detail. The ring $\mathcal{Q}(\mathcal{A})$ is prime Artinian, hence isomorphic to $\operatorname{Mat}_n(\mathcal{D})$ for some $n \in \mathbb{N}$ and some skew-field $\mathcal{D}$. We write $n = \operatorname{rk}(\mathcal{A})$ and call n the *Goldie rank* of $\mathcal{A}$. The division ring $\mathcal{D}$ is called the *Goldie field* of $\mathcal{A}$. It is well known that $\operatorname{rk}(\mathcal{A}) = 1$ if and only if $\mathcal{A}$ is a domain. More generally, it follows from the Feith–Utumi theorem that the Goldie rank of $\mathcal{A}$ coincides with the maximum value of $k \in \mathbb{N}$ for which there is an $x \in \mathcal{A}$ with $x^k = 0$ and $x^{k-1} \neq 0$ (we adopt the standard convention that $x^0 = 1$ for any $x \in \mathcal{A}$). This is an elegant *internal* characterisation of Goldie rank, but it is not very useful in practice.

Since $U(\mathfrak{g})$ is a Noetherian domain, its classical ring of fractions $\mathcal{Q}(U(\mathfrak{g}))$ is a division ring (or a skew-field). It is sometimes referred to as the *Lie field* of $\mathfrak{g}$ and denoted by $K(\mathfrak{g})$. In [16], Losev proved that for every finite dimensional irreducible $U(\mathfrak{g}, e)$-module M the inequality $\mathrm{rk}\big(U(\mathfrak{g})/I_M\big) \le \dim M$ holds. Our first theorem strengthens this result:

Theorem A. *Let M be a finite dimensional irreducible $U(\mathfrak{g}, e)$-module and let $I_M = \mathrm{Ann}_{U(\mathfrak{g})}\big(Q_e \otimes_{U(\mathfrak{g},e)} M\big)$ be the corresponding primitive ideal in $\mathcal{X}_{\mathcal{O}}$. Then the Goldie rank of the primitive quotient $U(\mathfrak{g})/I_M$ divides* $\dim M$.

Since Theorem A can be restated by saying that $q_M := (\dim M)/\mathrm{rk}\big(U(\mathfrak{g})/I_M\big)$ is an integer, the following question arises:

Question. *Is it always true that the positive integer q_M divides the order of the component group $\Gamma(e)$?*

Our proof of Theorem A relies on reduction modulo $\mathfrak{P}$ in the spirit of [23] and [25, Sect. 4] and makes use of the techniques introduced in [24, Sect. 2].

Notably, there are three nilpotent orbits $\mathcal{O}$ in $\mathfrak{g}$ with the property that for $e \in \mathcal{O}$ the equality $\mathrm{rk}\big(U(\mathfrak{g})/I_M\big) = \dim M$ holds for any finite dimensional irreducible $U(\mathfrak{g}, e)$-module M. Firstly, the zero orbit has this property because $U(\mathfrak{g}, 0) = U(\mathfrak{g})$ and all primitive ideals in $\mathcal{X}_{\{0\}}$ have finite codimension in $U(\mathfrak{g})$. Secondly, if e lies in the regular nilpotent orbit in $\mathfrak{g}$, then classical results of Kostant on Whittaker modules show that that the algebra $U(\mathfrak{g}, e)$ is isomorphic to the centre of $U(\mathfrak{g})$ and $\mathrm{rk}\big(U(\mathfrak{g})/I_M\big) = \dim M = 1$ for any $M \in \mathrm{Irr}\, U(\mathfrak{g}, e)$; see [15]. Thirdly, the minimal nonzero nilpotent orbit of $\mathfrak{g}$ enjoys the above property by [22, Thm. 1.2(v)]. Our second theorem indicates that the same could be true for many (if not all) nilpotent orbits in finite dimensional simple Lie algebras.

Let $\mathcal{D}_M$ stand for the Goldie field of the primitive quotient $U(\mathfrak{g})/I_M$. When $\mathfrak{g} = \mathfrak{sl}_n$, A. Joseph proved that $\mathcal{D}_M$ is isomorphic to a Weyl skew-field, more precisely, to the Goldie field of the Weyl algebra $\mathbf{A}_{d(e)}(\mathbb{C})$; see [14, Thm. 10.3].

Theorem B. *If $\mathcal{D}_M$ is isomorphic to the Goldie field of $\mathbf{A}_{d(e)}(\mathbb{C})$, then* $\mathrm{rk}\big(U(\mathfrak{g})/I_M\big) = \dim M$ *for any finite dimensional irreducible $U(\mathfrak{g}, e)$-module M.*

Combining Theorem B with the result of Joseph mentioned above we see that for $\mathfrak{g} = \mathfrak{sl}_n$ the equality $\mathrm{rk}\big(U(\mathfrak{g})/I_M\big) = \dim M$ holds for all nilpotent elements $e \in \mathfrak{g}$ and all finite dimensional irreducible $U(\mathfrak{g}, e)$-modules M. In view of our earlier remarks this enables us to classify the completely prime primitive ideals I of $U(\mathfrak{sl}_n)$ with $\mathcal{VA}(I) = \overline{\mathcal{O}}$ as exactly those $I = I_M$ for which M is a one-dimensional $U(\mathfrak{g}, e)$-module (one should also keep in mind here that in type A the component group $\Gamma(e)$ acts trivially on $\mathrm{Irr}\, U(\mathfrak{g}, e)$). This description differs from the classical one which is due to Mœglin [19]. Mœglin's classification of the completely prime primitive ideals of $U(\mathfrak{sl}_n)$ stems from her confirmation of a long-standing conjecture of Dixmier according to which any completely prime primitive ideal of $U(\mathfrak{g})$, for $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$, coincides with the annihilator of a $\mathfrak{g}$-module induced from a one-dimensional representation of a parabolic subalgebra of $\mathfrak{g}$. We remark that for any nilpotent element $e \in \mathfrak{g} = \mathfrak{sl}_n$ a complete description of one-dimensional $U(\mathfrak{g}, e)$-modules can be deduced from [25, 3.8] which, in turn, relies on the Brundan–Kleshchev description of the finite W-algebras for $\mathfrak{gl}_n$ as truncated shifted Yangians; see [5].

More generally, using Theorem B and arguing as in [25, 4.9] it is straightforward to see that for $\mathfrak{g} = \mathfrak{sl}_n$ and any $d \in \mathbb{N}$ the set $\mathcal{X}_d := \{I \in \mathcal{X} \,|\, \mathrm{rk}\big(U(\mathfrak{g})/I\big) = d\}$ has a natural structure of a quasi-affine algebraic variety. There is some hope that in the future one would be able to combine Theorem B with the main results of [6] to determine the scale factors of all Goldie rank polynomials for $\mathfrak{g} = \mathfrak{sl}_n$.

At this point it should be mentioned that a conjecture of Joseph (put forward in 1976) asserts that the Goldie field of a primitive quotient of $U(\mathfrak{g})$ is *always* isomorphic to a Weyl skew-field; see [13, 1.2] and references therein. Unfortunately, this conjecture is wide open for all simple Lie algebras except $\mathfrak{sl}_n$ and $\mathfrak{sp}_4$ (to the best of my knowledge, some details of the proof for $\mathfrak{g} = \mathfrak{sp}_4$ remain unpublished). It is needless to say that Joseph's conjecture was inspired by the famous Gelfand–Kirillov conjecture (from 1966) on the structure of the Lie field $K(\mathfrak{g})$. Curiously, the latter conjecture fails for $\mathfrak{g}$ simple outside types A_n, C_n and G_2 (see [24, Thm. 1]) and is still open in types C_n and G_2 (in type A the conjecture was proved by Gelfand and Kirillov themselves who made use of very special properties of the so-called *mirabolic* subalgebras of $\mathfrak{sl}_n$; see [10]).

Having said that, at the present time there is no evidence that the structure of $K(\mathfrak{g})$ has a serious impact on the structure of the Goldie field of $U(\mathfrak{g})/I$. Furthermore, Joseph's version of the Gelfand–Kirillov conjecture is known to hold for many primitive quotients outside type A; see [13, 14]. If it does hold in general, then the conclusion of Theorem B would be true for *all* primitive ideals $I = I_M$ of $U(\mathfrak{g})$. Of course, in that case one would be able, among other things, to classify all completely prime primitive ideals of $U(\mathfrak{g})$.

Regardless of the outcome of this story our proof of Theorem B underlines the importance of finding explicit presentations for the Goldie fields of the primitive quotients of $U(\mathfrak{g})$ (and for the Lie field $K(\mathfrak{g})$ itself!) in the spirit of the Gelfand–Kirillov conjecture.

Acknowledgements. The results presented here were announced in my talks at the conferences on symmetric spaces (Levico Terme, 2010), noncommutative algebras (Canterbury, 2010) and algebraic groups (Nagoya, 2010). I would like to thank Toby Stafford for his interest and many useful discussions.

2. Reducing modulo $\mathfrak{P}$ certain A-forms of primitive quotients

2.1. Let G be a simple, simply connected algebraic group over $\mathbb{C}$, and $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and Φ the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. Choose a basis of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ in Φ, let Φ^+ be the corresponding positive system in Φ, and put $\Phi^- := -\Phi^+$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the corresponding triangular decomposition of $\mathfrak{g}$ and choose a Chevalley basis $\mathcal{B} = \{e_\gamma \,|\, \gamma \in \Phi\} \cup \{h_\alpha \,|\, \alpha \in \Pi\}$ in $\mathfrak{g}$. Set $\mathcal{B}^\pm := \{e_\alpha \,|\, \alpha \in \Phi^\pm\}$. Let $\mathfrak{g}_\mathbb{Z}$ and $U_\mathbb{Z}$ denote the Chevalley $\mathbb{Z}$-form of $\mathfrak{g}$ and the Kostant $\mathbb{Z}$-form of $U(\mathfrak{g})$ associated with $\mathcal{B}$. Given a $\mathbb{Z}$-module V and a $\mathbb{Z}$-algebra A, we write $V_A := V \otimes_\mathbb{Z} A$.

Take a nonzero nilpotent element $e \in \mathfrak{g}_\mathbb{Z}$ and choose $f, h \in \mathfrak{g}_\mathbb{Q}$ such that (e, h, f) is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_\mathbb{Q}$. Denote by $(\,\cdot\,,\,\cdot\,)$ a scalar multiple of the Killing form κ of $\mathfrak{g}$ for which $(e, f) = 1$ and define $\chi \in \mathfrak{g}^*$ by setting $\chi(x) = (e, x)$ for all $x \in \mathfrak{g}$. Given $x \in \mathfrak{g}$ we set $\mathcal{O}(x) := (\mathrm{Ad}\, G) \cdot x$ and $d(x) := \frac{1}{2} \dim \mathcal{O}(x)$.

Following [23, 25] we call a a finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{C}$ *admissible* if $\kappa(e,f) \in A^\times$ and all bad primes of the root system of G and the determinant of the Gram matrix of $(\,\cdot\,,\,\cdot\,)$ relative to a Chevalley basis of $\mathfrak{g}$ are invertible in A. Every admissible ring is a Noetherian domain. Moreover, it is well known (and easy to see) that for every $\mathfrak{P} \in \operatorname{Specm} A$ the residue field $A/\mathfrak{P}$ is isomorphic to $\mathbb{F}_q$, where q is a p-power depending on $\mathfrak{P}$. We denote by $\Pi(A)$ the set of all primes $p \in \mathbb{N}$ that occur this way. It follows from Hilbert's Nullstellensatz, for example, that the set $\Pi(A)$ contains almost all primes in $\mathbb{N}$ (see the proof of Lemma 4.4 in [25] for more detail).

Let $\mathfrak{g}(i) = \{x \in \mathfrak{g} \,|\, [h,x] = ix\}$. Then $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$, by the $\mathfrak{sl}_2$-theory, and all subspaces $\mathfrak{g}(i)$ are defined over $\mathbb{Q}$. Also, $e \in \mathfrak{g}(2)$ and $f \in \mathfrak{g}(-2)$. We define a (nondegenerate) skew-symmetric bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathfrak{g}(-1)$ by setting $\langle x,y\rangle := (e,[x,y])$ for all $x,y \in \mathfrak{g}(-1)$. There exists a basis $B = \{z'_1,\dots,z'_s,z_1,\dots,z_s\}$ of $\mathfrak{g}(-1)$ contained in $\mathfrak{g}_\mathbb{Q}$ and such that

$$\langle z'_i, z_j\rangle = \delta_{ij}, \qquad \langle z_i, z_j\rangle = \langle z'_i, z'_j\rangle = 0 \qquad (1 \le i,j \le s).$$

As explained in [23, 4.1], after enlarging A, possibly, one can assume that $\mathfrak{g}_A = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_A(i)$, that each $\mathfrak{g}_A(i) := \mathfrak{g}_A \cap \mathfrak{g}(i)$ is a freely generated over A by a basis of the vector space $\mathfrak{g}(i)$, and that B is a free basis of the A-module $\mathfrak{g}_A(-1)$.

Put $\mathfrak{m} := \mathfrak{g}(-1)^0 \oplus \sum_{i\le -2} \mathfrak{g}(i)$ where $\mathfrak{g}(-1)^0$ denotes the $\mathbb{C}$-span of $z'_1,\dots,z'_s$. Then $\mathfrak{m}$ is a nilpotent Lie subalgebra of dimension $d(e)$ in $\mathfrak{g}$ and χ vanishes on the derived subalgebra of $\mathfrak{m}$; see [21] for more detail. It follows from our assumptions on A that $\mathfrak{m}_A = \mathfrak{g}_A \cap \mathfrak{m}$ is a free A-module and a direct summand of $\mathfrak{g}_A$. More precisely, $\mathfrak{m}_A = \mathfrak{g}_A(-1)^0 \oplus \sum_{i\le -2} \mathfrak{g}_A(i)$, where $\mathfrak{g}_A(-1)^0 = \mathfrak{g}_A \cap \mathfrak{g}(-1) = Az'_1 \oplus \cdots \oplus Az'_s$. Enlarging A further we may assume that $e,f \in \mathfrak{g}_A$ and that $[e,\mathfrak{g}_A(i)]$ and $[f,\mathfrak{g}_A(i)]$ are direct summands of $\mathfrak{g}_A(i+2)$ and $\mathfrak{g}_A(i-2)$, respectively. Then $\mathfrak{g}_A(i+2) = [e,\mathfrak{g}_A(i)]$ for all $i \ge 0$.

Write $\mathfrak{g}_e = \operatorname{Lie}(G_e)$ for the centraliser of e in $\mathfrak{g}$. As in [21 4.2, 4.3] we choose a basis $x_1,\dots,x_r,x_{r+1},\dots,x_m$ of the free A-module $\mathfrak{p}_A := \bigoplus_{i\ge 0} \mathfrak{g}_A(i)$ such that

(a) $x_i \in \mathfrak{g}_A(n_i)$ for some $n_i \in \mathbb{Z}_+$;

(b) $x_1,\dots,x_r$ is a free basis of the A-module $\mathfrak{g}_A \cap \mathfrak{g}_e$;

(c) $x_{r+1},\dots,x_m \in [f,\mathfrak{g}_A]$.

2.2. Let Q_e be the generalised Gelfand-Graev $\mathfrak{g}$-module associated to e. Recall that $Q_e = U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi$, where $\mathbb{C}_\chi = \mathbb{C}1_\chi$ is a 1-dimensional $\mathfrak{m}$-module such that $x \cdot 1_\chi = \chi(x)1_\chi$ for all $x \in \mathfrak{m}$. Given $(\mathbf{a},\mathbf{b}) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^s$ we let $x^\mathbf{a} z^\mathbf{b}$ denote the monomial $x_1^{a_1}\cdots x_m^{a_m} z_1^{b_1}\cdots z_s^{b_s}$ in $U(\mathfrak{g})$. Set $Q_{e,A} := U(\mathfrak{g}_A) \otimes_{U(\mathfrak{m}_A)} A_\chi$, where $A_\chi = A1_\chi$. Note that $Q_{e,A}$ is a $\mathfrak{g}_A$-stable A-lattice in Q_e with $\{x^\mathbf{i} z^\mathbf{j} \otimes 1_\chi,\mid (\mathbf{i},\mathbf{j}) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^s\}$ as a free basis. Given $(\mathbf{a},\mathbf{b}) \in \mathbb{Z}_+^m \times Z_+^s$ we set

$$|(\mathbf{a},\mathbf{b})|_e := \textstyle\sum_{i=1}^m a_i(n_i+2) + \sum_{i=1}^s b_i.$$

For $\mathbf{i} = (i_1,\dots,i_k) \in \mathbb{Z}_+^k$ set $|\mathbf{i}| := \sum_{j=1}^k i_j$. By [21, Thm. 4.6], the algebra $U(\mathfrak{g},e) := (\operatorname{End}_\mathfrak{g} Q_e)^{\mathrm{op}}$ is generated over $\mathbb{C}$ by endomorphisms $\Theta_1,\dots,\Theta_r$ such that

$$\Theta_k(1_\chi) = \Big(x_k + \sum_{0<|(\mathbf{i},\mathbf{j})|_e \le n_k+2} \lambda^k_{\mathbf{i},\mathbf{j}}\, x^\mathbf{i} z^\mathbf{j}\Big) \otimes 1_\chi, \qquad 1 \le k \le r, \tag{1}$$

where $\lambda^k_{\mathbf{i},\mathbf{j}} \in \mathbb{Q}$ and $\lambda^k_{\mathbf{i},\mathbf{j}} = 0$ if either $|(\mathbf{i},\mathbf{j})|_e = n_k + 2$ and $|\mathbf{i}| + |\mathbf{j}| = 1$ or $\mathbf{i} \neq \mathbf{0}$, $\mathbf{j} = \mathbf{0}$, and $i_l = 0$ for $l > r$. The monomials $\Theta_1^{i_1} \cdots \Theta_r^{i_r}$ with $(i_1, \ldots, i_r) \in \mathbb{Z}_+^r$ form a basis of the vector space $U(\mathfrak{g}, e)$.

The monomial $\Theta_1^{i_1} \cdots \Theta_r^{i_r}$ is said to have *Kazhdan degree* $\sum_{i=1}^r a_i(n_i + 2)$. For $k \in \mathbb{Z}_+$ we let $U(\mathfrak{g}, e)_k$ denote the $\mathbb{C}$-span of all monomials $\Theta_1^{i_1} \cdots \Theta_r^{i_r}$ of Kazhdan degree $\leq k$. The subspaces $U(\mathfrak{g}, e)_k$, $k \geq 0$, form an increasing exhaustive filtration of the algebra $U(\mathfrak{g}, e)$ called the *Kazhdan filtration*; see [21]. The corresponding graded algebra $\operatorname{gr} U(\mathfrak{g}, e)$ is a polynomial algebra in $\operatorname{gr}\Theta_1, \ldots, \operatorname{gr}\Theta_r$. It follows from [21, Thm. 4.6] that there exist polynomials $F_{ij} \in \mathbb{Q}[X_1, \ldots, X_r]$, where $1 \leq i < j \leq r$, such that

$$[\Theta_i, \Theta_j] = F_{ij}(\Theta_1, \ldots, \Theta_r) \qquad (1 \leq i < j \leq r). \tag{2}$$

Moreover, if $[x_i, x_j] = \sum_{k=1}^r \alpha_{ij}^k x_k$ in $\mathfrak{g}_e$, then

$$F_{ij}(\Theta_1, \ldots, \Theta_r) \equiv \sum_{k=1}^r \alpha_{ij}^k \Theta_k + q_{ij}(\Theta_1, \ldots, \Theta_r) \quad \big(\operatorname{mod} U(\mathfrak{g}, e)_{n_i+n_j}\big),$$

where the initial form of $q_{ij} \in \mathbb{Q}[X_1, \ldots, X_r]$ has total degree ≥ 2 whenever $q_{ij} \neq 0$. By [23, Lemma 4.1], the algebra $U(\mathfrak{g}, e)$ is generated by $\Theta_1, \ldots, \Theta_r$ subject to the relations (2). In what follows we assume that our admissible ring A contains all $\lambda^k_{\mathbf{i},\mathbf{j}}$ in (1) and all coefficients of the F_{ij}'s in (2) (due to the above PBW theorem for $U(\mathfrak{g}, e)$ we can view the F_{ij}'s as polynomials in $r = \dim \mathfrak{g}_e$ variables with coefficients in $\mathbb{Q}$).

2.3. Let N_χ denote the left ideal of $U(\mathfrak{g})$ generated by all $x - \chi(x)$ with $x \in \mathfrak{m}$. Then $Q_e \cong U(\mathfrak{g})/N_\chi$ as $\mathfrak{g}$-modules. As N_χ is a $(U(\mathfrak{g}), U(\mathfrak{m}))$-bimodule, the fixed point space $(U(\mathfrak{g})/N_\chi)^{\operatorname{ad}\mathfrak{m}}$ carries a natural algebra structure given by $(x + N_\chi) \cdot (y + N_\chi) = xy + N_\chi$ for all $x, y \in U(\mathfrak{g})$. Moreover, $U(\mathfrak{g})/N_\chi \cong Q_e$ as $\mathfrak{g}$-modules via the $\mathfrak{g}$-module map sending $1 + N_\chi$ to 1_χ, and $(U(\mathfrak{g})/N_\chi)^{\operatorname{ad}\mathfrak{m}} \cong U(\mathfrak{g}, e)$ as algebras. Any element of $U(\mathfrak{g}, e)$ is uniquely determined by its effect on the generator $1_\chi \in Q_e$ and the canonical isomorphism between $(U(\mathfrak{g})/N_\chi)^{\operatorname{ad}\mathfrak{m}}$ and $U(\mathfrak{g}, e)$ is given by $u \mapsto u(1_\chi)$ for all $u \in (U(\mathfrak{g})/N_\chi)^{\operatorname{ad}\mathfrak{m}}$. This isomorphism is defined over A. In what follows we shall often identify Q_e with $U(\mathfrak{g})/N_\chi$ and $U(\mathfrak{g}, e)$ with $(U(\mathfrak{g})/N_\chi)^{\operatorname{ad}\mathfrak{m}}$.

Let $U(\mathfrak{g}) = \bigcup_{j \in \mathbb{Z}} \mathsf{K}_j U(\mathfrak{g})$ be the Kazhdan filtration of $U(\mathfrak{g})$; see [9, 4.2]. Recall that $\mathsf{K}_j U(\mathfrak{g})$ is the $\mathbb{C}$-span of all products $x_1 \cdots x_t$ with $x_i \in \mathfrak{g}(n_i)$ and $\sum_{i=1}^t (n_i + 2) \leq j$. The Kazhdan filtration on Q_e is defined by $\mathsf{K}_j Q_e := \pi(\mathsf{K}_j U(\mathfrak{g}))$ where $\pi\colon U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})/\mathcal{I}_\chi$ is the canonical homomorphism. It turns Q_e into a filtered $U(\mathfrak{g})$-module. The Kazhdan grading of $\operatorname{gr} Q_e$ has no negative components, and the Kazhdan filtration of $U(\mathfrak{g}, e)$ defined in 2.2 is nothing but the filtration of $U(\mathfrak{g}, e) = (U(\mathfrak{g})/N_\chi)^{\operatorname{ad}\mathfrak{m}}$ induced from the Kazhdan filtration of Q_e through the embedding $(U(\mathfrak{g})/N_\chi)^{\operatorname{ad}\mathfrak{m}} \hookrightarrow Q_e$; see [9] for more detail.

Let $U(\mathfrak{g}_A, e)$ denote the A-span of all monomials $\Theta_1^{i_1} \cdots \Theta_r^{i_r}$ with $(i_1, \ldots, i_r) \in \mathbb{Z}_+^r$. Our assumptions on A guarantee that $U(\mathfrak{g}_A, e)$ is an A-subalgebra of $U(\mathfrak{g}, e)$ contained in $(\operatorname{End}_{\mathfrak{g}_A} Q_{e,A})^{\operatorname{op}}$. It is immediate from the above discussion that $Q_{e,A}$ identifies with the $\mathfrak{g}_A$-module $U(\mathfrak{g}_A)/N_{\chi,A}$, where $N_{\chi,A}$ stands for the left ideal of

$U(\mathfrak{g}_A)$ generated by all $x-\chi(x)$ with $x\in\mathfrak{m}_A$. Hence $U(\mathfrak{g}_A,e)$ embeds into the A-algebra $\big(U(\mathfrak{g}_A)/N_{\chi,A}\big)^{\mathrm{ad}\,\mathfrak{m}_A}$. As $Q_{e,A}$ is a free A-module with basis consisting of all $x^{\mathbf{i}}z^{\mathbf{j}}\otimes 1_\chi$ with $(\mathbf{i},\mathbf{j})\in\mathbb{Z}_+^m\times\mathbb{Z}_+^s$ we have that

$$U(\mathfrak{g}_A,e)\,=\,(\mathrm{End}_{\mathfrak{g}_A}\,Q_{e,A})^{\mathrm{op}}\,\cong\,\big(U(\mathfrak{g}_A)/N_{\chi,A}\big)^{\mathrm{ad}\,\mathfrak{m}_A}. \tag{3}$$

Also, $Q_{\chi,A}$ is free as a right $U(\mathfrak{g}_A,e)$-module; see [25, 2.3] for detail.

2.4. We now pick $p\in\Pi(A)$ and denote by $\Bbbk$ an algebraic closure of $\mathbb{F}_p$. Since the form $(\,\cdot\,,\,\cdot\,)$ is A-valued on $\mathfrak{g}_A$, it induces a symmetric bilinear form on the Lie algebra $\mathfrak{g}_\Bbbk\cong\mathfrak{g}_A\otimes_A\Bbbk$. We use the same symbol to denote this bilinear form on $\mathfrak{g}_\Bbbk$. Let $G_\Bbbk$ be the simple, simply connected algebraic $\Bbbk$-group with hyperalgebra $U_\Bbbk=U_\mathbb{Z}\otimes_Z\Bbbk$. Note that $\mathfrak{g}_\Bbbk=\mathrm{Lie}(G_\Bbbk)$ and the form $(\,\cdot\,,\,\cdot\,)$ is $(\mathrm{Ad}\,G_\Bbbk)$-invariant and nondegenerate. For $x\in\mathfrak{g}_A$ we set $\bar{x}:=x\otimes 1$, an element of $\mathfrak{g}_\Bbbk$. To ease notation we identify e,f with the nilpotent elements $\bar{e},\bar{f}\in\mathfrak{g}_\Bbbk$ and χ with the linear function $(e,\,\cdot\,)$ on $\mathfrak{g}_\Bbbk$.

The Lie algebra $\mathfrak{g}_\Bbbk=\mathrm{Lie}(G_\Bbbk)$ carries a natural $[p]$-mapping $x\mapsto x^{[p]}$ equivariant under the adjoint action of $G_\Bbbk$. The subalgebra of $U(\mathfrak{g}_\Bbbk)$ generated by all $x^p-x^{[p]}\in U(\mathfrak{g}_\Bbbk)$ is called the *p-centre* of $U(\mathfrak{g}_\Bbbk)$ and denoted $Z_p(\mathfrak{g}_\Bbbk)$ or Z_p for short. It is immediate from the PBW theorem that Z_p is isomorphic to a polynomial algebra in $\dim\mathfrak{g}$ variables and $U(\mathfrak{g}_\Bbbk)$ is a free Z_p-module of rank $p^{\dim\mathfrak{g}}$. For every maximal ideal J of Z_p there is a unique linear function $\eta=\eta_J\in\mathfrak{g}_\Bbbk^*$ such that

$$J=\,\langle x^p-x^{[p]}-\eta(x)^p1\mid\, x\in\mathfrak{g}_\Bbbk\rangle.$$

Since the Frobenius map of $\Bbbk$ is bijective, this enables us to identify the maximal spectrum $\mathrm{Specm}\,Z_p$ with $\mathfrak{g}_\Bbbk^*$.

Given $\xi\in\mathfrak{g}_\Bbbk^*$ we denote by I_ξ the two-sided ideal of $U(\mathfrak{g}_\Bbbk)$ generated by all $x^p-x^{[p]}-\xi(x)^p1$ with $x\in\mathfrak{g}_\Bbbk$, and set $U_\xi(\mathfrak{g}_\Bbbk):=U(\mathfrak{g}_\Bbbk)/I_\xi$. The algebra $U_\xi(\mathfrak{g}_\Bbbk)$ is called the *reduced enveloping algebra* of $\mathfrak{g}_\Bbbk$ associated to ξ. The preceding remarks imply that $\dim_\Bbbk U_\xi(\mathfrak{g}_\Bbbk)=p^{\dim\mathfrak{g}}$ and $I_\xi\cap Z_p=J_\xi$, the maximal ideal of Z_p associated with ξ. Every irreducible $\mathfrak{g}_\Bbbk$-module is a module over $U_\xi(\mathfrak{g}_\Bbbk)$ for a unique $\xi=\xi_V\in\mathfrak{g}_\Bbbk^*$. The linear function ξ_V is called the *p-character* of V; see [20] for more detail. By [20], any irreducible $U_\xi(\mathfrak{g}_\Bbbk)$-module has dimension divisible by $p^{(\dim\mathfrak{g}-\dim\mathfrak{z}_\xi)/2}$, where $\mathfrak{z}_\xi=\{x\in\mathfrak{g}_\Bbbk\mid\xi([x,\mathfrak{g}_\Bbbk])=0\}$ is the stabiliser of ξ in $\mathfrak{g}_\Bbbk$. We denote by $Z_{G_\Bbbk}(\xi)$ the coadjoint stabiliser of ξ in $G_\Bbbk$.

2.5. For $i\in\mathbb{Z}$, set $\mathfrak{g}_\Bbbk(i):=\mathfrak{g}_A(i)\otimes_A\Bbbk$ and put $\mathfrak{m}_\Bbbk:=\mathfrak{m}_A\otimes_A\Bbbk$. Our assumptions on A yield that the elements $\bar{x}_1,\dots,\bar{x}_r$ form a basis of the centraliser $(\mathfrak{g}_\Bbbk)_e$ of e in $\mathfrak{g}_\Bbbk$ and that $\mathfrak{m}_\Bbbk$ is a nilpotent subalgebra of dimension $d(e)$ in $\mathfrak{g}_\Bbbk$. Set $Q_{e,\Bbbk}:=U(\mathfrak{g}_\Bbbk)\otimes_{U(\mathfrak{m}_\Bbbk)}\Bbbk_\chi$, where $\Bbbk_\chi=A_\chi\otimes_A\Bbbk\,=\,\Bbbk1_\chi$. Clearly, $\Bbbk1_\chi$ is a 1-dimensional $\mathfrak{m}_\Bbbk$-module with the property that $x(1_\chi)=\chi(x)1_\chi$ for all $x\in\mathfrak{m}_\Bbbk$. It follows from our discussion in 2.2 and 2.3 that $Q_{e,\Bbbk}\cong Q_{e\,A}\otimes_A\Bbbk$ as modules over $\mathfrak{g}_\Bbbk$ and $Q_{e,\Bbbk}$ is a free right module over the $\Bbbk$-algebra

$$U(\mathfrak{g}_\Bbbk,e):=U(\mathfrak{g}_A,e)\otimes_A\Bbbk.$$

Thus we may identify $U(\mathfrak{g}_\Bbbk,e)$ with a subalgebra of $\widehat{U}(\mathfrak{g}_\Bbbk,e):=\big(\mathrm{End}_{\mathfrak{g}_\Bbbk}\,Q_{e,\Bbbk}\big)^{\mathrm{op}}$. The algebra $U(\mathfrak{g}_\Bbbk,e)$ has $\Bbbk$-basis consisting of all monomials $\bar{\Theta}_1^{i_1}\cdots\bar{\Theta}_r^{i_r}$ with $(i_1,\dots,i_r)\in\mathbb{Z}_+^r$, where $\bar{\Theta}_i:=\Theta_i\otimes 1\in U(\mathfrak{g}_A,e)\otimes_A\Bbbk$. Given $g\in A[X_1,\dots,X_n]$ we write ${}^pg$ for the image of g in the polynomial algebra $\Bbbk[X_1,\ldots,X_n] = A[X_1,\ldots,X_n]\otimes_A \Bbbk$. Since all polynomials F_{ij} are in $A[X_1,\ldots,X_r]$, it follows from the relations (2) that

$$[\bar\Theta_i,\bar\Theta_j] = {}^pF_{ij}(\bar\Theta_1,\ldots,\bar\Theta_r) \qquad (1\le i<j\le r). \tag{4}$$

By [25, Lemma 2.1], the algebra $U(\mathfrak{g}_\Bbbk, e)$ is generated by the elements $\bar\Theta_1,\ldots,\bar\Theta_r$ subject to the relations (4).

Let $\mathfrak{g}_A^*$ be the A-module dual to $\mathfrak{g}_A$ and let $\mathfrak{m}_A^\perp$ denote the set of all linear functions on $\mathfrak{g}_A$ vanishing on $\mathfrak{m}_A$. By our assumptions on A, this is a free A-submodule and a direct summand of $\mathfrak{g}_A^*$. Note that $\mathfrak{m}_A^\perp\otimes_A\mathbb{C}$ and $\mathfrak{m}_A^\perp\otimes_A\Bbbk$ identify naturally with with the annihilators $\mathfrak{m}^\perp := \{f\in\mathfrak{g}^*\mid f(\mathfrak{m})=0\}$ and $\mathfrak{m}_\Bbbk^\perp := \{f\in\mathfrak{g}_\Bbbk^*\mid f(\mathfrak{m}_\Bbbk)=0\}$, respectively.

Following [25], for $\eta\in\chi+\mathfrak{m}_\Bbbk^\perp$ we set $Q_e^\eta := Q_{e,\Bbbk}/I_\eta Q_{e,\Bbbk}$. By construction, Q_e^η is a $\mathfrak{g}_\Bbbk$-module with p-character η. Each $\mathfrak{g}_\Bbbk$-endomorphism $\bar\Theta_i$ of $Q_{e,\Bbbk}$ preserves $I_\eta Q_{e,\Bbbk}$, hence induces a $\mathfrak{g}_\Bbbk$-endomorphism of Q_e^η which we denote by θ_i. We write $U_\eta(\mathfrak{g}_\Bbbk,e)$ for the algebra $\big(\mathrm{End}_{\mathfrak{g}_\Bbbk} Q_e^\eta\big)^{\mathrm{op}}$. Since the restriction of η to $\mathfrak{m}_\Bbbk$ coincides with that of χ, the left ideal of $U(\mathfrak{g}_\Bbbk)$ generated by all $x-\eta(x)$ with $x\in\mathfrak{m}_\Bbbk$ equals $N_{\chi,\Bbbk} := N_{\chi,A}\otimes_A\Bbbk$ and $\Bbbk_\chi = \Bbbk_\eta$ as $\mathfrak{m}_\Bbbk$-modules. We denote by $N_{\eta,\chi}$ the left ideal of $U_\eta(\mathfrak{g}_\Bbbk)$ generated by all $x-\chi(x)$ with $x\in\mathfrak{m}_\Bbbk$. The following are proved in [25, 2.6]:

(a) $Q_e^\eta \cong U_\eta(\mathfrak{g}_\Bbbk)\otimes_{U_\eta(\mathfrak{m}_\Bbbk)}\Bbbk_\chi$ as $\mathfrak{g}_\Bbbk$-modules;

(b) $U_\eta(\mathfrak{g}_\Bbbk,e) \cong \big(U_\eta(\mathfrak{g}_\Bbbk)/U_\eta(\mathfrak{g}_\Bbbk)N_{\eta,\chi}\big)^{\mathrm{ad}\,\mathfrak{m}_\Bbbk}$;

(c) Q_e^η is a projective generator for $U_\eta(\mathfrak{g}_\Bbbk)$ and $U_\eta(\mathfrak{g}_\Bbbk)\cong \mathrm{Mat}_{p^{d(e)}}\big(U_\eta(\mathfrak{g}_\Bbbk,e)\big)$;

(d) the monomials $\theta_1^{i_1}\cdots\theta_r^{i_r}$ with $0\le i_k\le p-1$ form a $\Bbbk$-basis of $U_\eta(\mathfrak{g}_\Bbbk,e)$.

Moreover, a Morita equivalence between $U_\eta(\mathfrak{g}_\Bbbk,e)$-$\mathsf{mod}$ and $U_\eta(\mathfrak{g}_\Bbbk)$-$\mathsf{mod}$ in part (b) is given explicitly by the functor that sends a finite dimensional $U_\eta(\mathfrak{g}_\Bbbk,e)$-module W to the $U_\eta(\mathfrak{g}_\Bbbk)$-module $\widetilde{W} = Q_e^\eta\otimes_{U_\eta(\mathfrak{g}_\Bbbk,e)} W$, whilst the quasi-inverse functor from $U_\eta(\mathfrak{g}_\Bbbk)$-$\mathsf{mod}$ to $U_\eta(\mathfrak{g}_\Bbbk,e)$-$\mathsf{mod}$ sends a $U_\eta(\mathfrak{g}_\Bbbk)$-module $\widetilde{W}$ to its subspace

$$W = \mathrm{Wh}_\eta\,\widetilde{W} := \{v\in\widetilde{W}\mid x.v=\eta(x)v\ \text{ for all }\ x\in\mathfrak{m}_\Bbbk\}.$$

Recall from 2.1 the A-basis $\{x_1,\ldots,x_r,x_{r+1},\ldots,x_m\}$ of $\mathfrak{p}_A$ and set

$$X_i = \begin{cases} z_i & \text{if } \ 1\le i\le s,\\ x_{r-s+i} & \text{if } \ s+1\le i\le m-r+s.\end{cases}$$

For $\mathbf{a}\in\mathbb{Z}_+^{d(e)}$, put $X^{\mathbf{a}} := X_1^{a_1}\cdots X_{d(e)}^{a_{d(e)}}$ and $\bar X^{\mathbf{a}} := \bar X_1^{a_1}\cdots \bar X_{d(e)}^{a_{d(e)}}$, elements of $U(\mathfrak{g}_A)$ and $U(\mathfrak{g}_\Bbbk)$, respectively. By [23, Lemma 4.2(i)], the vectors $X^{\mathbf{a}}\otimes 1_\chi$ with $\mathbf{a}\in\mathbb{Z}^{d(e)}$ form a free basis of the right $U(\mathfrak{g}_A,e)$-module $Q_{e,A}$. Let $\mathfrak{a}_\Bbbk$ be the $\Bbbk$-span of $\bar X_1,\ldots,\bar X_{d(e)}$ in $\mathfrak{g}_\Bbbk$ and put $\widetilde{\mathfrak{a}}_\Bbbk := \mathfrak{a}_\Bbbk\oplus\mathfrak{z}_\chi$. Note that $\mathfrak{a}_\Bbbk = \{x\in\widetilde{\mathfrak{a}}_\Bbbk\mid (x,\mathrm{Ker}\,\mathrm{ad}\,f)=0\}$. Since χ vanishes on $\widetilde{\mathfrak{a}}_\Bbbk$, we may identify the symmetric algebra $S(\widetilde{\mathfrak{a}}_\Bbbk)$ with the coordinate ring $\Bbbk[\chi+\mathfrak{m}_\Bbbk^\perp]$ by setting $x(\eta) := \eta(x)$ for all $x\in\widetilde{\mathfrak{a}}_\Bbbk$ and $\eta\in\chi+\mathfrak{m}_\Bbbk^\perp$ and extending to $S(\widetilde{\mathfrak{a}}_\Bbbk)$ algebraically.

Given a subspace $V\subseteq\mathfrak{g}_\Bbbk$ we denote by $Z_p(V)$ the subalgebra of the p-centre $Z(\mathfrak{g}_\Bbbk)$ generated by all $x^p-x^{[p]}$ with $x\in V$. Clearly, $Z_p(V)$ is isomorphic to a polynomial algebra in $\dim_\Bbbk V$ variables. Let $\rho_\Bbbk$ denote the representation of $U(\mathfrak{g}_\Bbbk)$ in $\mathrm{End}_\Bbbk Q_{e,\Bbbk}$.

In [25, 2.7] we proved the following:

Theorem 2.1. *The algebra $\widehat{U}(\mathfrak{g}_{\Bbbk}, e)$ is generated by $U(\mathfrak{g}_{\Bbbk}, e)$ and $\rho_{\Bbbk}(Z_p) \cong Z_p(\widetilde{\mathfrak{a}}_{\Bbbk})$. Moreover, $\widehat{U}(\mathfrak{g}_{\Bbbk}, e)$ is a free $\rho_{\Bbbk}(Z_p)$-module with basis $\{\bar{\Theta}_1^{a_1} \cdots \bar{\Theta}_r^{a_r} \mid 0 \le a_i \le p-1\}$ and $\widehat{U}(\mathfrak{g}_{\Bbbk}, e) \cong U(\mathfrak{g}_{\Bbbk}, e) \otimes Z_p(\mathfrak{a}_{\Bbbk})$ as $\Bbbk$-algebras.*

Combining [25, Thm. 2.1(ii)] with [25, Lemma 2.2(iv)] it is straightforward to see that $Q_{e,\Bbbk}$ is a free right $\widehat{U}(\mathfrak{g}_{\Bbbk}, e)$-module with basis $\{\bar{X}_1^{a_1} \cdots \bar{X}_{d(e)}^{a_{d(e)}} \otimes 1_\chi \mid 0 \le a_i \le p-1\}$ and $U_\eta(\mathfrak{g}_{\Bbbk}, e) \cong \widehat{U}(\mathfrak{g}_{\Bbbk}, e) \otimes_{Z_p(\widetilde{\mathfrak{a}}_{\Bbbk})} \Bbbk_\eta$ for every $\eta \in \chi + \mathfrak{m}_{\Bbbk}^{\perp}$. (The algebra $Z_p(\widetilde{\mathfrak{a}}_{\Bbbk})$ acts on $\Bbbk_\eta = \Bbbk 1_\eta$ by the rule $(x^p - x^{[p]})(1_\eta) = \eta(x)^p$ for all $x \in \widetilde{\mathfrak{a}}_{\Bbbk}$.)

2.6. From now on we fix a primitive ideal $\mathcal{I}$ of $U(\mathfrak{g})$ with $\mathcal{VA}(\mathcal{I}) = \overline{\mathcal{O}}$. The affine variety $\mathcal{VA}(\mathcal{I})$ is the zero locus in $\mathfrak{g}^* \cong \mathfrak{g}$ of the $(\operatorname{Ad} G)$-invariant ideal $\operatorname{gr} \mathcal{I}$ of $S(\mathfrak{g}) = \operatorname{gr} U(\mathfrak{g})$. As we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by using the Killing isomorphism κ, our assumption on $\mathcal{I}$ simply means that the open $(\operatorname{Ad}^* G)$-orbit of $\mathcal{VA}(\mathcal{I})$ contains χ. We know from [16, Thm. 1.2.2], [25, Thm. 4.2] and [11, Thm. 4.5.2] that $\mathcal{I} = \operatorname{Ann}_{U(\mathfrak{g})} \big(Q_e \otimes_{U(\mathfrak{g},\, e)} M \big)$ for some finite dimensional $U(\mathfrak{g}, e)$-module M. We choose a $\mathbb{C}$-basis basis $E = \{m_1, \ldots, m_l\}$ of M and denote by $\tilde{A}$ the A-subalgebra of $\mathbb{C}$ generated by the coefficients of the coordinate vectors of all $\Theta_i(m_j) \in M$ with respect to E. By construction, the ring $\tilde{A}$ is admissible and the $\tilde{A}$-span of E is a $U(\mathfrak{g}_A, e)$-stable $\tilde{A}$-lattice in M. Thus, after replacing A by $\tilde{A}$ if need be, we may assume that the lattice $V_A := Am_1 \oplus \cdots \oplus Am_l$ in M is $U(\mathfrak{g}_A, e)$-stable. We write τ_A for the corresponding representation of $U(\mathfrak{g}_A, e)$ in $\operatorname{End} M_A$. Our discussion in 2.3 and 2.5 then shows that the $\mathfrak{g}$-module $\widetilde{M} := Q_e \otimes_{U(\mathfrak{g},\, e)} M$ contains a $\mathfrak{g}_A$-stable A-lattice with basis $\{X^{\mathbf{a}} \otimes m_i \mid \mathbf{a} \in \mathbb{Z}_+^{d(e)},\ 1 \le i \le l\}$; we call it $\widetilde{M}_A$. Note that $\widetilde{M}_A \cong Q_{e,\, A} \otimes_{U(\mathfrak{g}_A, e)} M_A$ as $\mathfrak{g}_A$-modules. For $p \in \Pi(A)$, the $\mathfrak{g}_{\Bbbk}$-module $\widetilde{M}_{\Bbbk}$ has $\Bbbk$-basis $\{\bar{X}^{\mathbf{a}} \otimes \bar{m}_i \mid \mathbf{a} \in \mathbb{Z}_+^{d(e)},\ 1 \le i \le l\}$, where $\bar{m}_i = m_i \otimes 1$. Also, $\widetilde{M}_{\Bbbk} \cong Q_{e,\, \Bbbk} \otimes_{U(\mathfrak{g}_{\Bbbk}, e)} M_{\Bbbk}$ as $\mathfrak{g}_{\Bbbk}$-modules.

For $1 \le i, j \le l$ denote by $E_{i,j}$ the endomorphism of M such that $E_{i,j}(m_k) = \delta_{j,k} m_i$ for all $1 \le k \le l$. As M is an irreducible $U(\mathfrak{g}, e)$-module, we may assume, after enlarging A further if necessary, that all $E_{i,j}$'s are in the image of $U(\mathfrak{g}_A, e)$ in $\operatorname{End} M$. Thus we may assume that for every $p \in \Pi(A)$ the $U(\mathfrak{g}_{\Bbbk}, e)$-module $M_{\Bbbk}$ is irreducible. We mention that $U(\mathfrak{g}_{\Bbbk}, e)$ acts on $M_{\Bbbk}$ via the representation $\tau_{\Bbbk} = \tau_A \otimes 1$. By Theorem 2.1, $\widehat{U}(\mathfrak{g}_{\Bbbk}, e) \cong U(\mathfrak{g}_{\Bbbk}, e) \otimes_{\Bbbk} Z_p(\mathfrak{a}_{\Bbbk})$ as $\Bbbk$-algebras. Therefore, for any linear function ψ on $\mathfrak{a}_{\Bbbk}$ there is a unique representation $\widehat{\tau}_{\Bbbk,\, \psi} \colon \widehat{U}(\mathfrak{g}_{\Bbbk}, e) \to \operatorname{End} M_{\Bbbk}$ with $\widehat{\tau}_{\Bbbk,\, \psi}(x^p - x^{[p]}) = \psi(x)^p \mathrm{Id}$ for all $x \in \mathfrak{a}_{\Bbbk}$ whose restriction to $U(\mathfrak{g}_{\Bbbk}, e) \hookrightarrow \widehat{U}(\mathfrak{g}_{\Bbbk}\, e)$ coincides with $\tau_{\Bbbk}$. Since the representation $\widetilde{\tau}_{\Bbbk,\, \psi}$ is irreducible and $Z_p(\widetilde{\mathfrak{a}}_{\Bbbk})$ is a central subalgebra of $\widehat{U}(\mathfrak{g}_{\Bbbk}, e)$, the linear function ψ extends uniquely to a linear function Ψ on $\widetilde{\mathfrak{a}}_{\Bbbk}$ such that $\widehat{\tau}_{\Bbbk,\, \psi}(x^p - x^{[p]}) = \Psi(x)^p \mathrm{Id}$ for all $x \in \widetilde{\mathfrak{a}}_{\Bbbk}$. As $\mathfrak{g}_{\Bbbk} = \mathfrak{m}_{\Bbbk} \oplus \widetilde{\mathfrak{a}}_{\Bbbk}$, we can extend Ψ to a linear function on $\mathfrak{g}_{\Bbbk}$ by setting $\Psi(x) = \chi(x)$ for all $x \in \mathfrak{m}_{\Bbbk}$. By construction, $\Psi \in \chi + \mathfrak{m}_{\Bbbk}^{\perp}$ and $\Psi|_{\mathfrak{a}_{\Bbbk}} = \psi$.

We now set $\widetilde{M}_{\Bbbk,\, \Psi} := \widetilde{M}_{\Bbbk} / I_\Psi \widetilde{M}_{\Bbbk}$, a $\mathfrak{g}_{\Bbbk}$-module with p-character Ψ. The definition of Ψ and our discussion in 2.5 show that

$$\begin{aligned} \widetilde{M}_{\Bbbk,\, \Psi} &\cong \widetilde{M}_{\Bbbk} \otimes_{Z_p(\mathfrak{g}_{\Bbbk})} \Bbbk_\Psi \,=\, \big(Q_{e,\, \Bbbk} \otimes_{U(\mathfrak{g}_{\Bbbk}, e)} M_{\Bbbk} \big) \otimes_{Z_p(\mathfrak{m}_{\Bbbk}) \otimes Z_p(\widetilde{\mathfrak{a}}_{\Bbbk})} \Bbbk_\Psi \\ &\cong \big(Q_{e,\, \Bbbk} \otimes_{U(\mathfrak{g}_{\Bbbk}, e)} M_{\Bbbk} \big) \otimes_{Z_p(\widetilde{\mathfrak{a}}_{\Bbbk})} \Bbbk_\Psi \,\cong\, Q_{e,\, \Bbbk} \otimes_{\widehat{U}(\mathfrak{g}_{\Bbbk}, e)} M_{\Bbbk} \,\cong\, Q_e^{\Psi} \otimes_{U_\Psi(\mathfrak{g}_{\Bbbk}, e)} M_{\Bbbk}, \end{aligned}$$

where we view $M_{\Bbbk}$ as a $\widehat{U}(\mathfrak{g}_{\Bbbk}, e)$-module via the representation $\widehat{\tau}_{\Bbbk,\,\psi}$. This implies that under our assumptions on A and Ψ the $U_{\Psi}(\mathfrak{g}_{\Bbbk})$-module $\widetilde{M}_{\Bbbk,\,\Psi}$ is irreducible and has dimension $lp^{d(e)}$; see 2.5 for more detail.

Remark 2.1. One can prove that the linear functions Ψ constructed in this subsection form a single orbit under the action of the connected unipotent subgroup $\mathcal{M}_{\Bbbk}$ of $G_{\Bbbk}$ such that $\operatorname{Ad}\mathcal{M}_{\Bbbk}$ is generated by all linear operators $\exp\operatorname{ad}x$ with $x \in \mathfrak{m}_{\Bbbk}$. Indeed, the group $\mathcal{M}_{\Bbbk}$ preserves the left ideal $U(\mathfrak{g}_{\Bbbk})N_{\chi,\,\Bbbk}$ and hence acts on both $Z_p(\widetilde{\mathfrak{a}}_{\Bbbk}) = \rho_{\Bbbk}(Z_p(\mathfrak{g}_{\Bbbk}))$ and $\widehat{U}(\mathfrak{g}_{\Bbbk}, e) = (U(\mathfrak{g}_{\Bbbk})/U(\mathfrak{g}_{\Bbbk})N_{\chi,\,\Bbbk})^{\operatorname{ad}\mathfrak{m}_{\Bbbk}}$. The rational action of $\mathcal{M}_{\Bbbk}$ on $Q_{e,\,\Bbbk}$ is obtained by reducing modulo $\mathfrak{P}$ the natural action on $Q_{e,\,A}$ of the unipotent subgroup $\mathcal{M}_A$ of G such that $\operatorname{Ad}\mathcal{M}_A$ is generated by all inner automorphisms $\exp\operatorname{ad}x$ with $x \in \mathfrak{m}_A$. From this it follows that $U(\mathfrak{g}_{\Bbbk}, e) \subseteq \widehat{U}(\mathfrak{g}_{\Bbbk}, e)^{\mathcal{M}_{\Bbbk}}$ (one should keep in mind here that $U(\mathfrak{g}_{\Bbbk}, e)$ is generated by $\bar{\Theta}_1, \ldots, \bar{\Theta}_r$ and $p \gg 0$). As we identify $S(\widetilde{\mathfrak{a}}_{\Bbbk})$ with $\Bbbk[\chi + \mathfrak{m}_{\Bbbk}^{\perp}]$, we may regard the $\mathcal{M}_{\Bbbk}$-algebra $Z_p(\widetilde{\mathfrak{a}}_{\Bbbk})$ as the coordinate algebra of the Frobenius twist $(\chi + \mathfrak{m}_{\Bbbk}^{\perp})^{(1)} \subset (\mathfrak{g}_{\Bbbk}^*)^{(1)}$ of $\chi + \mathfrak{m}_{\Bbbk}^{\perp}$; see [24, 3.4] for more detail. The natural action of $\mathcal{M}_{\Bbbk}$ on $(\chi + \mathfrak{m}_{\Bbbk}^{\perp})^{(1)}$ is a Frobenius twist of the coadjoint action of $\mathcal{M}_{\Bbbk}$ on $\chi + \mathfrak{m}_{\Bbbk}^{\perp}$. By Theorem 2.1, $\widehat{U}(\mathfrak{g}_{\Bbbk}, e)$ is a free $Z_p(\widetilde{\mathfrak{a}}_{\Bbbk})$-module with basis consisting of elements from $U(\mathfrak{g}_{\Bbbk}, e)$. From this it is immediate that $\widehat{U}(\mathfrak{g}_{\Bbbk}, e)^{\mathcal{M}_{\Bbbk}} = U(\mathfrak{g}_{\Bbbk}, e)$ and $Z_p(\widetilde{\mathfrak{a}}_{\Bbbk}) \cap \widehat{U}(\mathfrak{g}_{\Bbbk}, e) = Z_p(\widetilde{\mathfrak{a}}_{\Bbbk})^{\mathcal{M}_{\Bbbk}}$. On the other hand, [25, Lemma 3.2] entails that each fibre of the categorical quotient $\chi + \mathfrak{m}_{\Bbbk}^{\perp} \to (\chi + \mathfrak{m}_{\Bbbk}^{\perp})/\!\!/\mathcal{M}_{\Bbbk}$ induced by inclusion $\Bbbk[\chi + \mathfrak{m}_{\Bbbk}^{\perp}]^{\mathcal{M}_{\Bbbk}} \hookrightarrow \Bbbk[\chi + \mathfrak{m}_{\Bbbk}^{\perp}]$ is a single $\mathcal{M}_{\Bbbk}$-orbit. As the maximal spectrum of $Z_p(\widetilde{\mathfrak{a}}_{\Bbbk})$ is isomorphic to $(\chi + \mathfrak{m}_{\Bbbk}^{\perp})^{(1)}$ as $\mathcal{M}_{\Bbbk}$-varieties by our earlier remarks, each fibre of the categorical quotient

$$\alpha\colon \operatorname{Specm} Z_p(\widetilde{\mathfrak{a}}_{\Bbbk}) \longrightarrow \big(\operatorname{Specm} Z_p(\widetilde{\mathfrak{a}}_{\Bbbk})\big)/\!\!/\mathcal{M}_{\Bbbk}$$

is a single $\mathcal{M}_{\Bbbk}$-orbit as well. Now let Ψ_i, $i = 1, 2$, be two linear functions as above, denote by ψ_i the restriction of Ψ_i to $\mathfrak{a}_{\Bbbk}$, and consider the corresponding representations $\widehat{\tau}_{\Bbbk,\,\psi_i}\colon \widehat{U}(\mathfrak{g}_{\Bbbk}, e) \to \operatorname{End} M_{\Bbbk}$. Since $\widehat{\tau}_{\Bbbk,\,\psi_1}$ and $\widehat{\tau}_{\Bbbk,\,\psi_2}$ agree on $Z_p(\widetilde{\mathfrak{a}}_{\Bbbk})^{\mathcal{M}_{\Bbbk}} \subset U(\mathfrak{g}_{\Bbbk}, e)$, it must be that $\alpha(\Psi_1) = \alpha(\Psi_2)$. But then Ψ_1 and Ψ_2 are in the same $\mathcal{M}_{\Bbbk}$-orbit, as claimed.

2.7. Put $\mathcal{I}_A := \operatorname{Ann}_{U(\mathfrak{g}_A)} \widetilde{M}_A$ and denote by $\operatorname{gr}(\mathcal{I}_A)$ the corresponding graded ideal of $S(\mathfrak{g}_A)$. Define $R := U(\mathfrak{g})/\mathcal{I}$, $\operatorname{gr}(R) := S(\mathfrak{g})/\operatorname{gr}(\mathcal{I})$, $R_A := U(\mathfrak{g}_A)/\mathcal{I}_A$, and $\operatorname{gr}(R_A) = S(\mathfrak{g}_A)/\operatorname{gr}(\mathcal{I}_A)$. Clearly, $\operatorname{gr}(R_A) = \bigoplus_{n\ge 0}(\operatorname{gr}(R_A))(n)$ is a finitely generated graded A-algebra and each $(\operatorname{gr}(R_A))(n)$ is a finitely generated A-module. Also, A is a commutative Noetherian domain. If $b \in A \setminus \{0\}$, then $\operatorname{gr}(\mathcal{I}_{A[b^{-1}]}) = \operatorname{gr}(\mathcal{I}_A) \otimes_A A[b^{-1}]$ and

$$\begin{aligned}\operatorname{gr}(R_{A[b^{-1}]}) &= S(\mathfrak{g}_{A[b^{-1}]})/\operatorname{gr}(\mathcal{I}_{A[b^{-1}]}) \cong \big(S(\mathfrak{g}_A) \otimes_A A[b^{-1}]\big)/\big(\operatorname{gr}(\mathcal{I}_A) \otimes_A A[b^{-1}]\big)\\ &\cong \operatorname{gr}(R_A) \otimes_A A[b^{-1}];\end{aligned}$$

see [3, Ch. II, 2.4], for example. Since $\operatorname{gr}(R) = \bigoplus_{n\ge 0}(\operatorname{gr}(R))(n)$ is a graded Noetherian algebra of Krull dimension $2d(e) = \dim \mathcal{O}$ with $(\operatorname{gr}(R))(0) = \mathbb{C}$, we have that $2d(e) = \dim \operatorname{gr}(R) = 1 + \deg P_R(t)$, where $P_{\operatorname{gr}(R)}(t)$ is the Hilbert polynomial of $\operatorname{gr}(R)$; see [8, Corollary 13.7].

Denote by F the quotient field of A. Since $\operatorname{gr}(R_F) := \operatorname{gr}(R_A) \otimes_A F$ is a finitely generated algebra over a field, the Noether Normalisation Theorem says that there exist homogeneous, algebraically independent $y_1, \ldots, y_{2d(e)} \in \operatorname{gr}(R)_F$, such that $\operatorname{gr}(R_F)$ is a finitely generated module over its graded polynomial subalgebra $F[y_1, \dots, y_{2d(e)}]$; see [8, Thm. 13.3]. Let $v_1, \dots, v_D$ be a generating set of the $F[y_1, \dots, y_{2d(e)}]$-module $\mathrm{gr}(R_F)$ and let $r_1, \dots, r_N$ be a generating set of the A-algebra $\mathrm{gr}(R_A)$. Then

$$\begin{aligned} v_i \cdot v_j &= \textstyle\sum_{k=1}^{D} p_{i,j}^k(y_1, \dots, y_d) v_k \qquad (1 \le i, j \le D) \\ r_i &= \textstyle\sum_{j=1}^{D} q_{i,j}(y_1, \dots, y_d) v_j \qquad (1 \le i \le N) \end{aligned}$$

for some polynomials $p_{i,j}^k,\ q_{i,j} \in F[X_1, \dots, X_{2d(e)}]$. The algebra $\mathrm{gr}(R_A)$ contains an F-basis of $\mathrm{gr}(R_F)$. The coordinate vectors of the r_i's, y_i's and v_i's relative to this basis and the coefficients of the polynomials $q_{i,j}$ and $p_{i,j}^k$ involve only finitely many scalars in F. Replacing A by $A[b^{-1}]$ for a suitable $0 \ne b \in A$ if necessary, we may assume that all y_i and v_i are in $\mathrm{gr}(R_A)$ and all $p_{i,j}^k$ and $q_{i,j}$ are in $A[X_1, \dots, X_{2d(e)}]$. In conjunction with our earlier remarks this shows that no generality will be lost by assuming that

$$\mathrm{gr}(R_A) = A[y_1, \dots, y_{2d(e)}]v_1 + \dots + A[y_1, \dots, y_{2d(e)}]v_D \tag{5}$$

is a finitely generated module over the polynomial algebra $A[y_1, \dots, y_{2d(e)}]$.

Since $\mathrm{gr}(R_A)$ is a finitely generated $A[y_1, \dots, y_{d(e)}]$-module and A is a Noetherian domain, a graded version of the Generic Freeness Lemma shows that there exists a nonzero element $a_1 \in A$ such that each $\big(\mathrm{gr}(R_A)(n)\big)[a_1^{-1}]$ is a free $A[a_1^{-1}]$-module of finite rank; see (the proof of) Theorem 14.4 in [8]. Since $\big(\mathrm{gr}(R_A)(n)\big)[a_1^{-1}] \cong \big(\mathrm{gr}(R_{A[a_1^{-1}]})\big)(n)$ for all n by our earlier remarks, we see that there exists an admissible ring $A \subset \mathbb{C}$ such that all graded components of $\mathrm{gr}(R_A)$ are free A-modules of finite rank.

Since $S(\mathfrak{g}_A)$ is a finitely generated A-algebra, we can also apply the proof of Theorem 14.4 in [8] to the graded ideal $\mathrm{gr}(\mathcal{I}_A)$ of $S(\mathfrak{g}_A)$ to deduce that there exists a nonzero $a_2 \in A$ such that all graded components of $\big(\mathrm{gr}(\mathcal{I}_A)\big)[a_2^{-1}]$ are free $A[a_2^{-1}]$-modules of finite rank. As $\big(\mathrm{gr}(\mathcal{I}_A)\big)[a_2^{-1}] \cong \mathrm{gr}(\mathcal{I}_{A[a_2^{-1}]})$ by [3, Ch. II, 2.4], we may (and we will) assume that all graded components of $\mathrm{gr}(\mathcal{I}_A)$ are free A-modules of finite rank. A standard filtered-graded argument then shows that the A-modules $\mathcal{I}_A$ and R_A are free as well.

2.8. Note that $\widetilde{M}_F = \widetilde{M}_A \otimes_A F$ is a module over the split Lie algebra $\mathfrak{g}_F$. Since $\widetilde{M} \cong \widetilde{M}_F \otimes_F \mathbb{C}$, each subspace $\mathcal{I} \cap U_k(\mathfrak{g})$ is defined over F (here $U_k(\mathfrak{g})$ stands for the kth component of the canonical filtration of $U(\mathfrak{g})$). Since the algebra $U(\mathfrak{g})$ is Noetherian, the ideal $\mathcal{I}$ is generated by its F-subspace $\mathcal{I}_{F,N'} := U_{N'}(\mathfrak{g}_F) \cap \mathcal{I}$. Since $\mathcal{I}$ is a two-sided ideal of $U(\mathfrak{g})$, all subspaces $\mathcal{I} \cap U_k(\mathfrak{g})$ are invariant under the adjoint action of G on $U(\mathfrak{g})$. Hence the F-subspaces $\mathcal{I}_{F,N'}$ are invariant under the adjoint action of the distribution algebra $U_F := U_{\mathbb{Z}} \otimes_{\mathbb{Z}} F$. Since $\mathfrak{h}_F := \mathfrak{h} \cap \mathfrak{g}_F$ is a split Cartan subalgebra of $\mathfrak{g}_F$, the adjoint $\mathfrak{g}_F$-module $\mathcal{I}_{F,N'}$ decomposes into a finite direct sum of absolutely irreducible $\mathfrak{g}_F$-modules with integral dominant highest weights. Therefore, the $\mathfrak{g}_F$ module $\mathcal{I}_{F,N'}$ possesses a $\mathbb{Z}$-form invariant under the adjoint action of the Kostant $\mathbb{Z}$-form $U_{\mathbb{Z}}$; we call it $\mathcal{I}_{\mathbb{Z},N'}$.

Let $\{u_i \mid i \in I\}$ be any basis of the free $\mathbb{Z}$-module $\mathcal{I}_{\mathbb{Z},N'}$. Expressing the u_i via the PBW basis of $U(\mathfrak{g}_F)$ associated with the Chevalley basis $\mathcal{B}$ involves only finitely many scalars in F. Enlarging A further if need be we may assume that all u_i are in

$U(\mathfrak{g}_A)$ and hence that the ideal $\mathcal{I}_A$ of $U(\mathfrak{g}_A)$ is invariant under the adjoint action of the Hopf $\mathbb{Z}$-algebra $U_{\mathbb{Z}}$. Thus, from now on we may assume that for any maximal ideal $\mathfrak{P}$ of A the two-sided ideal $\mathcal{I}_{\Bbbk} := \mathcal{I}_A \otimes_A \Bbbk_{\mathfrak{P}}$ of $U(\mathfrak{g}_{\Bbbk})$ is stable under the adjoint action of the simple algebraic $\Bbbk$-group $G_{\Bbbk}$ with hyperalgebra $U_{\Bbbk} = U_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$.

3. Introducing certain finite subsets of regular elements in R

3.1. Let $\mathcal{B} = \{g_1, \ldots, g_n\}$ be our Chevalley basis of $\mathfrak{g}_{\mathbb{Z}}$ and identify $\mathcal{B}$ with its image in R. Denote by R_k the kth component of the filtration of R induced by the canonical filtration of $U(\mathfrak{g})$ and let S be the Ore set of all regular elements in R. Since $\mathcal{Q}(P) = S^{-1}R \cong \mathrm{Mat}_{l'}(\mathcal{D}_M)$, where $l' = \mathrm{rk}(R)$, there exists a unital subalgebra $\mathfrak{C}$ in $\mathcal{Q}(R)$ isomorphic to $\mathrm{Mat}_{l'}(\mathbb{C})$ and such that $\mathcal{Q}(R) \cong \mathfrak{C} \otimes \mathfrak{D}$, where $\mathfrak{D} \cong \mathcal{D}_M$ is the centraliser of $\mathfrak{C}$ in R. Fix a set $\{e_{ij} \mid 1 \le i, j \le l'\}$ of matrix units in $\mathfrak{C}$, so that

$$e_{ij}e_{tk} = \delta_{jt}e_{ik} \qquad (1 \le i,j,t,k \le l'); \tag{6}$$

$$\textstyle\sum_{i,j} e_{ij} = 1. \tag{7}$$

There exist $s_{ij}, s'_{ij} \in S$ and $E_{ij}, E'_{ij} \in R$ such that

$$s_{ij}^{-1}E_{ij} = e_{ij} = E'_{ij}(s'_{ij})^{-1}. \tag{8}$$

Then in R we have the following relations

$$E_{ij}s'_{ij} = s_{ij}E'_{ij} \qquad (1 \le i,j \le l'). \tag{9}$$

As $\mathcal{Q}(R) = \mathfrak{C} \otimes \mathfrak{D}$, there exist $c_{ij}^k \in R$, where $1 \le k \le n$, such that

$$g_k = \textstyle\sum_{i,j} e_{ij}c_{ij}^k \qquad (1 \le k \le n); \tag{10}$$

$$c_{ij}^k e_{th} = e_{th}c_{ij}^k \qquad (1 \le i,j,t,h \le l';\ 1 \le k \le n). \tag{11}$$

For each $k \le l'$ we can find $a_{ij}^k \in S$ and $C_{ij}^k \in R$ such that $c_{ij}^k = (a_{ij}^k)^{-1}C_{ij}^k$. Since S is an Ore set, there are $r_{ij,\,tk}, r_{ij,\,th}^k, a_{ij,\,th}^k \in S$ and $E_{ij,\,tk}, E_{ij,\,th}^k, C_{ij,\,th}^k \in R$ such that

$$r_{ij,\,tk}E_{ij} = E_{ij,\,tk}s_{tk} \qquad (1 \le i,j,t,k \le l'); \tag{12}$$

$$r_{ij,\,th}^k C_{ij}^k = E_{ij,\,th}^k s_{th} \qquad (1 \le i,j,t,h \le l',\ 1 \le k \le l'); \tag{13}$$

$$C_{ij}^k a_{ij,\,th}^k = a_{ij}^k s'_{th} C_{ij,\,th}^k \qquad (1 \le i,j,t,h \le l',\ 1 \le k \le l'). \tag{14}$$

Since $s_{ij}^{-1}E_{ij}s_{tk}^{-1}E_{tk} = \delta_{jt}E'_{ik}(s'_{ik})^{-1}$ by (6) and (9), applying (12) we obtain that $s_{ij}^{-1}r_{ij,\,tk}^{-1}E_{ij,tk}E_{tk} = \delta_{jt}E'_{ik}(s'_{ik})^{-1}$. This yields

$$E_{ij,\,tk}E_{ik}s'_{ik} = \delta_{jt}r_{ij,\,tk}s_{ij}E'_{ik} \qquad (1 \le i,j,t,k \le l'). \tag{15}$$

Similarly, since $(a_{ij}^k)^{-1}C_{ij}^k s_{th}^{-1}E_{th} = E'_{th}(s'_{th})^{-1}(a_{ij}^k)^{-1}C_{ij}^k$ by (11) and (9), applying (13) and (14) yields $(a_{ij}^k)^{-1}(r_{ij,\,th}^k)^{-1}E_{ij,\,th}^k E_{th} = E'_{th}C_{ij,\,th}^k(a_{ij,\,th}^k)^{-1}$. We thus get

$$E_{ij,\,th}^k E_{th} a_{ij,th}^k = r_{ij,\,th}^k a_{ij}^k E'_{th} C_{ij,\,th}^k \qquad (1 \le i,j,t,h \le l',\ 1 \le k \le l'). \tag{16}$$

Let $p(1), \ldots, p(l'^2)$ be all elements in the lexicographically ordered set $\{(i,j) \mid 1 \le i,j \le l'^2\}$ and denote by $e_{p(k)}$, $E_{p(k)}$ and $s_{p(k)}$ the corresponding elements in R. Then (7) can be rewritten as $1 = \sum_{i=1}^{l'^2} e_{p(i)} = \sum_{i=1}^{l'^2} s_{p(k)}^{-1}E_{p(i)}$. Multiplying both sides by $s_{p(1)}$ on the left we get

$$s_{p(1)} = E_{p(1)} + \textstyle\sum_{i=2}^{l'^2} s_{p(1)}s_{p(i)}^{-1}E_{p(i)}. \tag{17}$$

There exist $s_{p(1),p(2)} \in S$ and $q_{p(2)} \in R$ such that $s_{p(1),p(2)} s_{p(1)} = q_{p(2)} s_{p(2)}$. Multiplying both sides of (17) by $s_{p(1),p(2)}$ on the left we then obtain

$$(18) \qquad s_{p(1),p(2)} s_{p(1)} \;=\; s_{p(1),p(2)} E_{p(1)} + q_{p(2)} E_{p(2)} + \textstyle\sum_{i=3}^{l'^2} s_{p(1),p(2)} s_{p(1)} s_{p(i)}^{-1} E_{p(i)}.$$

For $3 \leq k \leq l'^2$, we select (recursively) some $s_{p(1),\ldots,p(k)} \in S$ and $q_{p(k)} \in R$ such that

$$(19) \qquad \textstyle\prod_{i=1}^{k} s_{p(1),\ldots p(k-i+1)} \;=\; q_{p(k)} s_{p(k)}.$$

For convenience, we set $q_{p(1)} = 1$. At the end of the process started with (17) and (18) we get rid of all denominators and arrive at the relation

$$(20) \qquad \prod_{k=1}^{l'^2} s_{p(1),\ldots,p(l'^2-k+1)} \;=\; \sum_{k=1}^{l'^2} \Big(\prod_{i=1}^{l'^2-k} s_{p(1),\ldots,p(l'^2-i+1)} \Big) q_{p(k)} E_{p(k)}$$

which holds in R.

Since e_{ij} commutes with c_{ij}^k we can rewrite (10) as

$$(21) \qquad g_k \;=\; \textstyle\sum_{i,j} (a_{ij}^k)^{-1} C_{ij}^k s_{ij}^{-1} E_{ij} \qquad\qquad (1 \leq k \leq n).$$

For $1 \leq k \leq l'$, there exist $D_{ij}^k, T_{ij}^k \in R$ and $s_{ij}^k, s_{ij;\,k} \in S$ such that

$$(22) \qquad D_{ij}^k s_{ij} \;=\; s_{ij}^k C_{ij}^k, \quad T_{ij}^k \;=\; D_{ij}^k E_{ij}, \quad s_{ij;\,k} \;=\; s_{ij}^k a_{ij}^k \qquad\qquad (1 \leq i, j \leq l').$$

Then we can rewrite (21) as follows:

$$(23) \qquad g_k \;=\; \textstyle\sum_{i,j} s_{ij;\,k}^{-1} T_{ij}^k \;=\; \sum_{i=1}^{l'^2} s_{p(i);\,k}^{-1} T_{p(i)}^k \qquad\qquad (1 \leq k \leq n).$$

Multiplying both sides of (23) by $s_{p(1);\,k}$ on the left we get

$$(24) \qquad s_{p(1);\,k} \cdot g_k \;=\; T_{p(1)}^k + \textstyle\sum_{i=2}^{l'^2} s_{p(1);\,k} s_{p(i);\,k}^{-1} T_{p(i)}.$$

There are $s_{p(1),p(2);\,k} \in S$ and $q_{p(2)}^k \in R$ such that $s_{p(1),p(2);\,k} s_{p(1);\,k} = q_{p(2)}^k s_{p(2);\,k}$. Multiplying both sides of (24) by $s_{p(1),p(2);\,k}$ on the left we get

$$(s_{p(1),p(2);\,k} s_{p(1);\,k}) g_k \;=\; s_{p(1),p(2);\,k} T_{p(1)}^k + q_{p(2)}^k T_{p(2)}^k + \textstyle\sum_{i=3}^{l'^2} s_{p(1),p(2);\,k} s_{p(1);\,k} s_{p(i);\,k}^{-1} T_{p(i)}^k.$$

For $3 \leq j \leq l'^2$, we choose (recursively) some $s_{p(1),\ldots,p(j);\,k} \in S$ and $q_{p(j)}^k \in R$ such that

$$(25) \qquad \textstyle\prod_{i=1}^{j} s_{p(1),\ldots p(j-i+1);\,k} \;=\; q_{p(j)}^k s_{p(j);\,k},$$

and set $q_{p(1)}^k = 1$. As before, at the end of the process just started we arrive at the relations

$$(26) \qquad \Big(\prod_{j=1}^{l'^2} s_{p(1),\ldots,p(l'^2-j+1);\,k} \Big) g_k \;=\; \sum_{j=1}^{l'^2} \Big(\prod_{i=1}^{l'^2-j} s_{p(1),\ldots,p(l'^2-i+1);\,k} \Big) q_{p(j)}^k T_{p(j)}^k$$

which hold in R, where $1 \leq k \leq n$.

3.2. In this subsection we assume that $\mathfrak{D}$ is a Weyl skew-field, more precisely, $\mathfrak{D} \cong \mathfrak{Q}(\mathbf{A}_{d(e)}(\mathbb{C}))$. We follow closely the exposition in [24, Sect. 2] and adopt (with some minor modifications) the notation introduced there.

Set $d := d(e)$. If a pair $(a, b) \in \{(i, j) \mid 1 \leq i, j \leq l'\}$ occupies the kth place in our lexicographical ordering, the we write $c^s_{p(k)}$, $a^s_{p(k)}$ and $C^s_{p(k)}$ for c^s_{ab}, a^s_{ab} and C^s_{ab}, respectively. There exist $w_1, \ldots, w_{2d} \in \mathfrak{D}$ such that

$$[w_i, w_j] = [w_{d+i}, w_{d+j}] = 0 \qquad (1 \leq i, j \leq d); \tag{27}$$

$$[w_i, w_{d+j}] = \delta_{i,j} \qquad (1 \leq i, j \leq d); \tag{28}$$

$$Q^s_{p(k)} \cdot c^s_{p(k)} = P^s_{p(k)}, \qquad (1 \leq k \leq l'^2;\ 1 \leq s \leq n) \tag{29}$$

for some nonzero polynomials $P^s_{p(k)}, Q^s_{p(k)}$ in $w_1, \ldots, w_{2d}$ with coefficients in $\mathbb{C}$. (One should keep in mind here that the monomials $w_1^{a_1} w_2^{a_2} \cdots w_{2d}^{a_{2d}}$ with $a_i \in \mathbb{Z}_+$ form a basis of the $\mathbb{C}$-subalgebra of D generated by $w_1, \ldots, w_{2d}$.)

Since every nonzero element of $\mathfrak{D}$ is regular in $\mathfrak{Q}(R)$, there exist $Q^s_{1;\,p(k)}, Q^s_{2;\,p(k)} \in S$ such that

$$Q^s_{p(k)} Q^s_{1;\,p(k)} = Q^s_{2;\,p(k)} \qquad (1 \leq k \leq l'^2;\ 1 \leq s \leq n). \tag{30}$$

Since $w_i = v_i^{-1} u_i$ for some elements $v_i \in S$ and $u_i \in R$, we can rewrite (27) and (28) as follows:

$$v_i^{-1} u_i \cdot v_j^{-1} u_j = v_j^{-1} u_j \cdot v_i^{-1} u_i; \tag{31}$$

$$v_{d+i}^{-1} u_{d+i} \cdot v_{d+j}^{-1} u_{d+j} = v_{d+j}^{-1} u_{d+j} \cdot v_{d+i}^{-1} u_{d+i}; \tag{32}$$

$$v_i^{-1} u_i \cdot v_{d+j}^{-1} u_{d+j} - v_{d+j}^{-1} u_{d+j} \cdot v_i^{-1} u_i = \delta_{i,j} \qquad (1 \leq i, j \leq d). \tag{33}$$

As S is an Ore set, there are $v_{i,j} \in S$ and $u_{i,j} \in R$ such that

$$v_{i,j} u_i = u_{i,j} v_j \qquad (1 \leq i, j \leq 2d). \tag{34}$$

Thus we can rewrite (31), (32) and (33) in the form

$$v_i^{-1} v_{i,j}^{-1} \cdot u_{i,j} u_j = v_j^{-1} v_{j,i}^{-1} \cdot u_{j,i} u_i \qquad (1 \leq i, j \leq d \text{ or } d \leq i, j \leq 2d) \tag{35}$$

$$v_i^{-1} v_{i,d+j}^{-1} \cdot u_{i,d+j} u_{d+j} = \delta_{ij} + v_{d+j}^{-1} v_{d+j,i}^{-1} \cdot u_{d+j,i} u_{d+i} \qquad (1 \leq i, j \leq d). \tag{36}$$

There exist $b_{i,j} \in S$ and $b'_{i,j} \in R$ such that

$$b_{i,j} v_{i,j} v_i = b'_{i,j} v_{j,i} v_j \qquad (1 \leq i, j \leq 2d). \tag{37}$$

Since $v_{i,j} v_i (v_{j,i} v_j)^{-1} = b_{i,j}^{-1} b'_{i,j}$, we see that (35) and (36) give rise to the relations

$$b_{i,j} u_{i,j} u_j = b'_{i,j} u_{j,i} u_i \qquad (1 \leq i, j \leq d \text{ or } d \leq i, j \leq 2d) \tag{38}$$

$$b_{i,d+j} u_{i,d+j} u_{d+j} = \delta_{ij} b_{i,d+j} v_{i,d+j} v_i + b'_{i,d+j} u_{d+j,i} u_i \qquad (1 \leq i, j \leq d) \tag{39}$$

which hold in R.

For an m-tuple $\mathbf{i} = (i(1), i(2), \ldots, i(m))$ with $1 \leq i(1) \leq i(2) \leq \cdots \leq i(m) \leq 2d$ and $m \geq 3$ we select (recursively) some $u_{i(1),\ldots,i(k)} \in R$ and $v_{i(1),\ldots,i(k)} \in S$, where $3 \leq k \leq m$, such that

$$v_{i(1),\ldots,i(k)} u_{i(1),\ldots,i(k-1)} u_{i(k-1)} = u_{i(1),\ldots,i(k)} v_{i(k)}. \tag{40}$$

Write $w^{\mathbf{i}} := w_{i(1)} \cdot w_{i(2)} \cdot \ldots \cdot w_{i(m)} = \prod_{k=1}^{m} v_{i(k)}^{-1} u_{i(k)}$. Then

$$\begin{aligned} w^{\mathbf{i}} &= v_{i(1)}^{-1} u_{i(1)} \cdot v_{i(2)}^{-1} u_{i(2)} \cdot \prod_{k=3}^{m} v_{i(k)}^{-1} u_{i(k)} \\ &= v_{i(1)}^{-1} v_{i(1),i(2)}^{-1} u_{i(1),i(2)} u_{i(2)} \cdot v_{i(3)}^{-1} u_{i(3)} \cdot \prod_{k=4}^{m} v_{i(k)}^{-1} u_{i(k)} \\ &= v_{i(1)}^{-1} v_{i(1),i(2)}^{-1} v_{i(1),i(2),i(3)}^{-1} u_{i(1),i(2),i(3)} u_{i(3)} \cdot \prod_{k=4}^{m} v_{i(k)}^{-1} u_{i(k)} \\ &= \cdots = \Big(\prod_{k=1}^{m} v_{i(1),\ldots,i(m-k+1)} \Big)^{-1} \cdot u_{i(1),\ldots,i(m)} u_{i(m)}. \end{aligned}$$

Then we set $v_{\mathbf{i}} := \prod_{k=1}^{m} v_{i(1),\ldots,i(m-k+1)}$, an element of S, and $u_{\mathbf{i}} := u_{i(1),\ldots,i(m)} u_{i(m)}$, an element of R.

Let $\{\mathbf{i}(1), \ldots, \mathbf{i}(N)\}$ be the set of all tuples as above with $\sum_{\ell=1}^{N} i(\ell) \leq \Delta$, where $\Delta = \max\big\{ \deg P_{p(k)}^{s}, \deg Q_{p(k)}^{s} \mid 1 \leq k \leq l'^2,\ 1 \leq s \leq n \big\}$. Clearly, $P_{p(k)}^{s} = \sum_{j=1}^{N} \lambda_{j,k}^{s} w^{\mathbf{i}(j)}$ and $Q_{p(k)}^{s} = \sum_{j=1}^{N} \mu_{j,k}^{s} w^{\mathbf{i}(j)}$ for some $\lambda_{j,k}^{s}, \mu_{j,k}^{s} \in \mathbb{C}$, where $1 \leq k \leq l'^2$ and $1 \leq s \leq n$. By the above, we have that $P_{p(k)}^{s} = \sum_{j=1}^{N} \lambda_{j,k}^{s} v_{\mathbf{i}(j)}^{-1} u_{\mathbf{i}(j)}$ and $Q_{p(k)}^{s} = \sum_{i=1}^{N} \mu_{j,k}^{s} v_{\mathbf{i}(j)}^{-1} u_{\mathbf{i}(j)}$.

Set $v_{\mathbf{i}(j)}(0) := v_{\mathbf{i}(j)}$ and $u_{\mathbf{i}(j)}(0) = u_{\mathbf{i}(j)}$. For each pair (j,t) of positive integers satisfying $N \geq j > t > 0$ we select (recursively) some $v_{\mathbf{i}(j)}(t) \in S$ and $u_{\mathbf{i}(j)}(t) \in R$ such that

$$v_{\mathbf{i}(j)}(t) v_{\mathbf{i}(t)}(t-1) \;=\; u_{\mathbf{i}(j)}(t) v_{\mathbf{i}(j)}(t-1). \tag{41}$$

Multiplying both sides of (46) by $v_{\mathbf{i}(1)}$ on the left and applying (47) with $t = 1$ we obtain that

$$\begin{aligned} v_{\mathbf{i}(1)} P_{p(k)}^{s} &= \lambda_{1,k}^{s} u_{\mathbf{i}(1)} + \sum_{j=2}^{N} \lambda_{j,k}^{s} v_{\mathbf{i}(1)} v_{\mathbf{i}(j)}^{-1} u_{\mathbf{i}(j)} \\ &= \lambda_{1,k}^{s} u_{\mathbf{i}(1)} + \sum_{j=2}^{N} \lambda_{j,k}^{s} v_{\mathbf{i}(j)}(1)^{-1} u_{\mathbf{i}(j)}(1) u_{\mathbf{i}(j)}. \end{aligned}$$

Multiplying both sides of this equality by $v_{\mathbf{i}(2)}(1)$ on the left and applying (47) with $s = 2$ we get

$$\begin{aligned} v_{\mathbf{i}(2)}(1) v_{\mathbf{i}(1)} P_{p(k)}^{s} &= \lambda_{1,k}^{s} v_{\mathbf{i}(2)}(1) u_{\mathbf{i}(1)} + \lambda_{2,k}^{s} u_{\mathbf{i}(2)}(1) u_{\mathbf{i}(1)} \\ &+ \sum_{j=3}^{N} \lambda_{j,k}^{s} v_{\mathbf{i}(j)}(2)^{-1} u_{\mathbf{i}(j)}(2) u_{\mathbf{i}(j)}(1) u_{\mathbf{i}(j)}. \end{aligned}$$

Repeating this process N times we arrive at the relation

$$\Big(\prod_{\ell=1}^{N} v_{\mathbf{i}(N-\ell+1)} \Big) P_{p(k)}^{s} \;=\; \sum_{j=1}^{N} \lambda_{j,k}^{s} \cdot \Big(\prod_{\ell=1}^{N-j} v_{\mathbf{i}(N-\ell+1)}(N-\ell) \cdot \prod_{\ell=1}^{j} u_{\mathbf{i}(j-\ell+1)}(j-\ell) \Big) \tag{42}$$

which holds in R (at the ℓ-th step of the process we multiply the preceding equality by $v_{\mathbf{i}(\ell)}(\ell-1)$ on the left and then apply (47) with $s=\ell$). Similarly, we have that

$$(43)\qquad \Big(\prod_{\ell=1}^{N} v_{\mathbf{i}(N-\ell+1)}\Big) Q^s_{p(k)} \;=\; \sum_{j=1}^{N} \mu^s_{j,k}\cdot\Big(\prod_{\ell=1}^{N-j} v_{\mathbf{i}(N-\ell+1)}(N-\ell)\cdot\prod_{\ell=1}^{j} u_{\mathbf{i}(j-\ell+1)}(j-\ell)\Big).$$

We denote the left-hand sides of (42) and (43) by $\widetilde{P}^s_{p(k)}$ and $\widetilde{Q}^s_{p(k)}$, respectively, and set $\tilde{v} := \prod_{\ell=1}^{N} v_{\mathbf{i}(N-\ell+1)}$. Note that $\tilde{v}\in S$. Then

$$(44)\qquad \tilde{v}^{-1}\widetilde{P}^s_{p(k)} \;=\; P^s_{p(k)},\qquad \tilde{v}^{-1}\widetilde{Q}^s_{p(k)} \;=\; Q^s_{p(k)}\qquad (1\le k\le N;\ 1\le s\le l'^2).$$

Now (29) can be rewritten as

$$(45)\qquad \widetilde{Q}^s_{p(k)}(a^s_{p(k)})^{-1}C^s_{p(k)} \;=\; \widetilde{P}^s_{p(k)}\qquad (1\le k\le N;\ 1\le s\le l'')."$$

Choosing $\tilde{a}^s_{p(k)}\in S$ and $\tilde{q}^s_{p(k)}\in R$ such that

$$(46)\qquad \tilde{a}^s_{p(k)}\widetilde{Q}^s_{p(k)} \;=\; a^s_{p(k)}\tilde{q}^s_{p(k)}\qquad (1\le k\le N;\ 1\le s\le l'^2)$$

we can rewrite (45) as follows:

$$(47)\qquad \tilde{q}^s_{p(k)}C^s_{p(k)} \;=\; \tilde{a}^s_{p(k)}\widetilde{P}^s_{p(k)}\qquad (1\le k\le N;\ 1\le s\le l'^2).$$

This relation holds in R. In view of (30) we have that

$$Q^s_{p(k)} \;=\; Q^s_{2;\,p(k)}(Q^s_{1;\,p(k)})^{-1}\qquad (1\le k\le N;\ 1\le s\le l'^2).$$

Combining this with (44) we obtain

$$(48)\qquad \widetilde{Q}^s_{p(k)}Q^s_{1;\,p(k)} \;=\; \tilde{v}Q^s_{2;\,p(k)}\qquad (1\le k\le N;\ 1\le s\le l'^2).$$

This relation holds in R as well.

Finally, in view of (29) and (30) we can replace (11) by the following relation:

$$(49)\qquad e_{ij}w_t \;=\; w_te_{ij}\qquad (1\le i,j\le l';\ 1\le t\le 2d).$$

The latter can be rewritten as

$$s_{ij}^{-1}E_{ij}v_t^{-1}u_t \;=\; u_tv_t^{-1}s_{ij}^{-1}E_{ij}\qquad (1\le i,j\le l';\ 1\le t\le 2d).$$

There exists $v_{ij;\,t}, b_{ij,\,t}\in S$ and $E_{ij;\,t}, D_{ij;\,t}\in R$ such that

$$(50)\qquad v_{ij;\,t}E_{ij} \;=\; E_{ij;\,t}v_t;$$
$$(51)\qquad s_{ij}v_tD_{ij;\,t} \;=\; E_{ij}b_{ij;\,t}$$

for all $1\le i,j\le l'$ and $1\le t\le 2d$. Then (49) gives rise to the relations

$$(52)\qquad E_{ij;\,t}u_tb_{ij;\,t} \;=\; v_{ij;\,t}s_{ij}u_tD_{ij;\,t}\qquad (1\le i,j\le l';\ 1\le t\le 2d)$$

which hold in R.

3.3. Let $X \subset R$ and $Y \subset S$ be the finite subsets introduced in 3.1 and 3.2. Obviously, they lie in R_m for some $m \gg 0$, hence involve only finitely many scalars in $\mathbb{C}$. From now on we shall always assume that those scalars are in A and hence $X \cup Y \subset R_A$. It will be crucial for us in what follows to work with those admissible rings A for which the images of the elements of Y in $R_{\Bbbk} = (R_A/\mathfrak{P}R_A) \otimes_{A/\mathfrak{P}} \Bbbk$ remain regular for all maximal ideals $\mathfrak{P}$ of A. Our next result ensures that such admissible rings do exist.

Lemma 3.1. *Let s be a regular element of R contained in R_A and assume that A satisfies the conditions imposed in 2.7. Then there exists an admissible extension B of A such that for every $\mathfrak{P} \in \operatorname{Specm} B$ the element $s \otimes 1$ is regular in $R_B \otimes_B \Bbbk_{\mathfrak{P}} \cong (R_B/\mathfrak{P}R_B) \otimes_{B/\mathfrak{P}} \Bbbk$.*

Proof. Since $s \cdot R_A$ is a right ideal of R_A, the graded A-module $\operatorname{gr}(s \cdot R_A)$ is an ideal of the commutative Noetherian ring $\operatorname{gr}(R_A)$. Hence $\operatorname{gr}(s \cdot R_A)$ is a finitely generated $\operatorname{gr}(R_A)$-module. As A is a Noetherian domain, applying [8, Thm. 14.4] shows that there is a nonzero $a_1 \in A$ such that each $\big(\operatorname{gr}(s \cdot R_A)(n)\big)[a_1^{-1}]$ is a free $A[a_1^{-1}]$-module of finite rank. Since $\big(\operatorname{gr}(s \cdot R_A)(n)\big)[a_1^{-1}] \cong \big(\operatorname{gr}(s \cdot R_{A[a_1^{-1}]})\big)(n)$ for all n, we see that there exists an admissible ring $\tilde{A} \subset \mathbb{C}$ containing A such that all graded components of $\operatorname{gr}(s \cdot R_{\tilde{A}})$ are free $\tilde{A}$-modules of finite rank. Since we can repeat this argument with the left ideal $R_A \cdot s$ in place of $s \cdot R_A$, it can be assumed, after enlarging $\tilde{A}$ possibly, that all graded components of $\operatorname{gr}(R_{\tilde{A}} \cdot s)$ are free $\tilde{A}$-modules of finite rank as well.

Since $\operatorname{gr}(R_A)$ is a finitely generated A-algebra, we can also apply Theorem 14.4 in [8] to the graded $\operatorname{gr}(R_A)$ module $\operatorname{gr}(R_A/s \cdot R_A) \cong \operatorname{gr}(R_A)/\operatorname{gr}(s \cdot R_A)$ to deduce that there is a nonzero $a_2 \in A$ such that all graded components of

$$\operatorname{gr}(R_A/s \cdot R_A)[a_2^{-1}] \;\cong\; \big(\operatorname{gr}(R_A)/\operatorname{gr}(s \cdot R_A)\big)[a_2^{-1}] \;\cong\; \operatorname{gr}(R_{A[a_2^{-1}]})/\operatorname{gr}(s \cdot R_{A[a_2^{-1}]})$$

are free $A[a_2^{-1}]$-modules of finite rank. Replacing $s \cdot R_A$ by $R_A \cdot s$ in this argument we observe that the same applies to all graded components of $\operatorname{gr}(R_{A[a_3^{-1}]})/\operatorname{gr}(R_{A[a_3^{-1}]} \cdot s)$ for a suitable nonzero $a_3 \in A$.

We conclude that there exists an admissible extension B of A such that all graded components of $\operatorname{gr}(s \cdot R_B)$, $\operatorname{gr}(R_B \cdot s)$, $\operatorname{gr}(R_B)/\operatorname{gr}(s \cdot R_B)$ and $\operatorname{gr}(R_B)/\operatorname{gr}(R_B \cdot s)$ are free B-modules of finite rank. Straightforward induction on filtration degree now shows that the free B-modules $s \cdot R_B \cong R_B$ and $R_B \cdot s \cong R_B$ are direct summands of R_B. Let R'_B and R''_B be B-submodules of R_B such that $R_B = (s \cdot R_B) \oplus R'_B$ and $R_B = (R_B \cdot s) \oplus R''_B$.

We now take any maximal ideal $\mathfrak{P}$ of B, denote by $\mathfrak{f}$ the finite field $B/\mathfrak{P}$, and write $\bar{x}$ for the image of $x \in R_B$ in $R_{\Bbbk} = (R_B/\mathfrak{P}R_B) \otimes_{\mathfrak{f}} \Bbbk$. Note that $R_{\mathfrak{f}} := R_B/\mathfrak{P}R_B$ is an $\mathfrak{f}$-form of the $\Bbbk$-vector space $R_{\Bbbk}$. Suppose $\bar{s} \cdot \bar{u} = 0$ for some $u \in R_B$. Then

$$\begin{aligned} s \cdot u \;&\in\; (s \cdot R_B) \cap \mathfrak{P}R_B \;=\; (s \cdot R_B) \cap \big(\mathfrak{P}(s \cdot R_B) \oplus \mathfrak{P}R'_B\big) \\ &=\; (s \cdot R_B) \cap \big(s \cdot \mathfrak{P}R_B) \oplus \mathfrak{P}R'_B\big) \;=\; s \cdot \mathfrak{P}R_B. \end{aligned}$$

Therefore, $s \cdot u = s \cdot u'$ for some $u' \in \mathfrak{P}R_B$. Since s is a regular element of R and $s \cdot (u - u') = 0$, we deduce that $u = u' \in \mathfrak{P}R_B$. This yields $\bar{u} = 0$. If $\bar{v} \cdot \bar{s} = 0$ for some $v \in R_B$, then we use the decomposition $R_B = (R_B \cdot s) \oplus R''_B$ and argue as before to deduce that $\bar{v} = 0$. Hence $\bar{s}$ is a regular element of $R_{\mathfrak{f}}$.

Let $l_{\bar{s}}\colon R_{\Bbbk} \to R_{\Bbbk}$ and $r_{\bar{s}}\colon R_{\Bbbk} \to R_{\Bbbk}$ denote the left and right multiplication by $\bar{s}$, respectively. Denote by $(R_{\Bbbk})_j$ the jth component of the filtration of $R_{\Bbbk}$ induced by the canonical filtration of $U(\mathfrak{g}_{\Bbbk})$ and set $(R_{\mathfrak{f}})_j := (R_{\Bbbk})_j \cap R_{\mathfrak{f}}$. We know that $\bar{s} \in (R_{\mathfrak{f}})_\ell$ for some ℓ, whereas the regularity of $\bar{s}$ in $R_{\mathfrak{f}}$ yields that the $\mathfrak{f}$-linear maps $l_{\bar{s}}\colon (R_{\mathfrak{f}})_j \to (R_{\mathfrak{f}})_{j+\ell}$ and $r_{\bar{s}}\colon (R_{\mathfrak{f}})_j \to (R_{\mathfrak{f}})_{j+\ell}$ are injective for all $j \in \mathbb{Z}_+$. Standard linear algebra then shows that so are all $\Bbbk$-linear maps $l_{\bar{s}}\colon (R_{\Bbbk})_j \to (R_{\Bbbk})_{j+\ell}$ and $r_{\bar{s}}\colon (R_{\Bbbk})_j \to (R_{\Bbbk})_{j+\ell}$. In other words, $\bar{s}$ is regular in $R_{\Bbbk}$ as claimed. □

4. Proving the main results

4.1. From now on we assume that for every $s \in Y$ the element $s \otimes 1$ is regular in $R_{\Bbbk} = (R_A/\mathfrak{P}R_A) \otimes_{\mathfrak{f}} \Bbbk$ for every $\mathfrak{P} \in \operatorname{Specm} A$ (here $\mathfrak{f} = A/\mathfrak{P}$). Since Y is a finite set, this is a valid assumption thanks to Lemma 3.1. We also assume that our admissible ring A satisfies all requirements mentioned in Sect. 2. The discussion in 2.8 then shows that the simple algebraic group $G_{\Bbbk}$ acts on $R_{\Bbbk}$ as algebra automorphisms and preserves the filtration of $R_{\Bbbk}$ induced by the canonical filtration of $U(\mathfrak{g}_{\Bbbk})$.

Since $U(\mathfrak{g}_{\Bbbk})$ is a finite module over its centre, so is its homomorphic image $R_{\Bbbk} = \big(U(\mathfrak{g}_A)/\mathcal{I}_A\big) \otimes_{\mathfrak{f}} \Bbbk \cong U(\mathfrak{g}_{\Bbbk})/\mathcal{I}_{\Bbbk}$. Being a homomorphic image of $U(\mathfrak{g}_{\Bbbk})$, the ring $R_{\Bbbk}$ is Noetherian and, moreover, an affine PI-algebra over $\Bbbk$. Let $I_1 \ldots, I_\nu$ be the minimal primes of $R_{\Bbbk}$ and $N_{\Bbbk} := \cap_{j=1}^{\nu} I_j$. Then $\nu = \nu(\mathfrak{P}) \in \mathbb{N}$ and $N_{\Bbbk}$ is the maximal nilpotent ideal of $R_{\Bbbk}$; see [27, Theorem 2]. In particular, $\bar{R}_{\Bbbk} := R_{\Bbbk}/N_{\Bbbk}$ is a semiprime Noetherian ring. By Goldie's theory, the set $\bar{S}_{\Bbbk}$ of all regular elements of $\bar{R}_{\Bbbk}$ is an Ore set in $\bar{R}_{\Bbbk}$ and the quotient ring $\mathcal{Q}(\bar{R}_{\Bbbk}) = \bar{S}_{\Bbbk}^{-1}\bar{R}_{\Bbbk}$ is semisimple and Artinian.

Write $Z(\bar{R}_{\Bbbk})$ for the centre of $\bar{R}_{\Bbbk}$ and $\mathcal{C}(Z(\bar{R}_{\Bbbk}))$ for the set of all elements of $Z(\bar{R}_{\Bbbk})$ which are regular in $\bar{R}_{\Bbbk}$. Since $\bar{R}_{\Bbbk}$ is a finite module over the image of the p-centre of $U(\mathfrak{g}_{\Bbbk})$ in $\bar{R}_{\Bbbk}$, it is algebraic over $Z(\bar{R}_{\Bbbk})$. Applying [1, Theorem 2] now yields that $\mathcal{Q}(\bar{R}_{\Bbbk})$ is obtained from $\bar{R}_{\Bbbk}$ by inverting the elements from $\mathcal{C}(Z(\bar{R}_{\Bbbk}))$ (the latter is obviously an Ore set in $\bar{R}_{\Bbbk}$).

Proposition 4.1. *There exists a unital subalgebra $\mathfrak{C}_{\Bbbk}$ of $\mathcal{Q}(\bar{R}_{\Bbbk})$ isomorphic to* $\mathrm{Mat}_{l'}(\Bbbk)$ *and such that $\mathcal{Q}(\bar{R}_{\Bbbk}) \cong \mathfrak{C}_{\Bbbk} \otimes \mathfrak{D}_{\Bbbk}$ where $\mathfrak{D}_{\Bbbk}$ is the centraliser of $\mathfrak{C}_{\Bbbk}$ in $\mathcal{Q}(\bar{R}_{\Bbbk})$.*

Proof. The ring theoretic notation used below will follow that of [18]. Given a two-sided ideal I of the ring $R_{\Bbbk}$ we write $\mathcal{C}'(I)$ for the set of all elements $r \in R_{\Bbbk}$ for which the coset $r + I$ is left regular in the ring $R_{\Bbbk}/I$ (the latter means that $r \cdot x \in I$ for $x \in R_{\Bbbk}$ implies $x \in I$). As we know, for each $y \in Y$ the element $y \otimes 1$ is regular in $R_{\Bbbk}$. In particular, $y \otimes 1 \in \mathcal{C}'(0)$. To ease notation we now let $\bar{x}$ denote the image of $x \in R_A$ in $\bar{R}_{\Bbbk} = R_{\Bbbk}/N_{\Bbbk}$. As the ring $R_{\Bbbk}$ is right Noetherian, it follows from [12, 2.3, 2.5] or from [18, Prop. 4.1.3(iii)] that $\mathcal{C}'(0) \subseteq \mathcal{C}(N_{\Bbbk})$. This shows that for every $y \in Y$ the element $\bar{y}$ is regular in $\bar{R}_{\Bbbk}$.

The subset $\bar{X} \cup \bar{Y}$ of $\bar{R}_{\Bbbk}$ contains elements satisfying the relations (9), (12), (13), (14), (15), (16), (19), (20), (22), (25), (26). Since all elements of $\bar{Y}$ involved in these relations remain regular in $\bar{R}_{\Bbbk}$ and each step of the procedure described in 3.1 is reversible, we can find elements $\bar{e}_{ij}$ and $\bar{c}_{ij}^k$ in $\mathcal{Q}(\bar{R}_{\Bbbk})$, where $1 \le i, j \le l'$ and $1 \le k \le n$, satisfying the relations (6), (7), (10), (11). We denote by $\mathfrak{C}_{\Bbbk}$ the $\Bbbk$-span of the $\bar{e}_{ij}$'s. Thanks to (6) and (7), it is a homomorphic image of $\mathrm{Mat}_{l'}(\Bbbk)$ and a unital subalgebra of $\mathcal{Q}(\bar{R}_{\Bbbk})$. Therefore, $\mathfrak{C}_{\Bbbk} \cong \mathrm{Mat}_{l'}(\Bbbk)$ as $\Bbbk$-algebras.

In view of (11) all elements $\bar{c}_{ij}^k$ commute with $\mathfrak{C}_{\Bbbk}$, whilst (10) implies that the $\bar{g}_k$'s lie in $\mathfrak{C}_{\Bbbk} \cdot \mathfrak{D}_{\Bbbk}$ where $\mathfrak{D}_{\Bbbk}$ is the centraliser of $\mathfrak{C}_{\Bbbk}$ in $\mathcal{Q}(\bar{R}_{\Bbbk})$. As the inverses of the elements from $\mathcal{C}(Z(\bar{R}_{\Bbbk}))$ lie in $\mathfrak{D}_{\Bbbk}$ as well and $\mathcal{Q}(\bar{R}_{\Bbbk}) = \bar{S}_{\Bbbk}^{-1}\bar{R}_{\Bbbk} = \big(\mathcal{C}(Z(\bar{R}_{\Bbbk}))\big)^{-1}\bar{R}_{\Bbbk}$ by our earlier remarks, we deduce that $\mathcal{Q}(\bar{R}_{\Bbbk}) = \mathfrak{C}_{\Bbbk} \cdot \mathfrak{D}_{\Bbbk}$. As a consequence, there exists a surjective algebra homomorphism $\psi\colon \mathfrak{C}_{\Bbbk} \otimes \mathfrak{D}_{\Bbbk} \twoheadrightarrow \mathcal{Q}(\bar{R}_{\Bbbk})$. Since $\mathfrak{C}_{\Bbbk}$ is a matrix algebra, it is straightforward to see that ψ is injective. This completes the proof. □

4.2. Let $Z(\bar{R}_{\Bbbk})$ be the centre of $\bar{R}_{\Bbbk}$ and denote by $Z_p(\bar{R}_{\Bbbk})$ the image of the p-centre $Z_p(\mathfrak{g}_{\Bbbk})$ in $\bar{R}_{\Bbbk}$. Recall from (5) that the commutative A-algebra $\mathrm{gr}(R_A)$ is generated by D graded elements over a graded polynomial subalgebra $A[y_1, \ldots, y_{2d}] \subset \mathrm{gr}(R_A)$, where $d = d(e)$.

Lemma 4.1. *There exists a $\Bbbk$-subalgebra $\bar{Z}_0$ of $Z_p(\bar{R}_{\Bbbk})$ generated by $2d$ elements and such that $\bar{R}_{\Bbbk}$ is generated as a $\bar{Z}_0$-module by Dp^{2d} elements.*

Proof. We follow the proof of [23, Lemma 3.2] very closely. Write $(R_A)_j$ (resp. $(R_{\Bbbk})_j$) for the image in R_A (resp. $R_{\Bbbk}$) of the jth component of the canonical filtration of $U(\mathfrak{g}_A)$ (resp. $U(\mathfrak{g}_{\Bbbk})$).

Suppose that y_i has degree a_i, where $1 \le i \le 2d$, and v_k has degree l_k, where $1 \le k \le D$, and let $\Phi_A\colon S(\mathfrak{g}_A) \twoheadrightarrow \mathrm{gr}(R_A)$ denote the canonical homomorphism. For $1 \le i \le 2d$ (resp. $1 \le k \le D$) choose $u_i \in U(\mathfrak{g}_A)$ (resp. $w_k \in U_{l_k}(\mathfrak{g}_A)$) such that $\Phi_A(\mathrm{gr}_{a_i} u_i) = y_i$ (resp. $\Phi_A(\mathrm{gr}_{l_k} w_k) = v_k$). Let $\bar{u}_i$ (resp. $\bar{w}_k$) denote the image of u_i (resp. w_k) in $R_{\Bbbk} = (U(\mathfrak{g}_A)/\mathcal{I}_A) \otimes_A \Bbbk_{\mathfrak{P}}$. For every $n \in \mathbb{Z}_+$ the set

$$\{w_k u_1^{i_1} \cdots u_{2d}^{i_{2d}} \mid l_k + \textstyle\sum_{j=1}^{2d} i_j a_j \le n;\ 1 \le k \le D\}$$

spans the A-module $(R_A)_n$. In view of our earlier remarks this implies that the set

$$\{\bar{w}_k \bar{u}_1^{i_1} \cdots \bar{u}_{2d}^{i_{2d}} \mid l_k + \textstyle\sum_{j=1}^{2d} i_j a_j \le n;\ 1 \le k \le D\}$$

spans the $\Bbbk$-space $(R_{\Bbbk})_n$. Since $\mathrm{gr}_{pa_i}(\bar{u}_i^p) = (\mathrm{gr}_{a_i} \bar{u}_i)^p$ is a pth power in $S(\mathfrak{g}_{\Bbbk})$, for every $i \le 2d$ there exists a $z_i \in Z_p(\mathfrak{g}_{\Bbbk}) \cap U_{a_i}(\mathfrak{g}_{\Bbbk})$ such that $\bar{u}_i^p - z_i \in U_{pa_i - 1}(\mathfrak{g}_{\Bbbk})$. We let Z_0 be the $\Bbbk$-subalgebra of $Z_p(\mathfrak{g}_{\Bbbk})$ generated by $z_1, \ldots, z_{2d}$ and denote by $\bar{Z}_0$ the image of Z_0 in $\bar{R}_{\Bbbk} = R_{\Bbbk}/N_{\Bbbk}$.

Let $R'_{\Bbbk}$ the Z_0-submodule of $R_{\Bbbk}$ generated by all $\bar{w}_k \bar{u}_1^{i_1} \cdots \bar{u}_{2d}^{i_{2d}}$ with $0 \le i_j \le p-1$ and $1 \le k \le D$. Using the preceding remarks and induction on n we now obtain that $(R_{\Bbbk})_n \subset R'_{\Bbbk}$ for all $n \in \mathbb{Z}_+$. But then $R_{\Bbbk} = R'_{\Bbbk}$, implying that the set

$$\Lambda := \{\bar{w}_k \bar{u}_1^{i_1} \cdots \bar{u}_{2d}^{i_{2d}} \mid 0 \le i_j \le p-1;\, 1 \le k \le D\}$$

generates $R_{\Bbbk}$ as an Z_0-module. Obviously, $|\Lambda| \le Dp^{2d}$. As $\bar{R}_{\Bbbk}$ is a homomorphic image of $R_{\Bbbk}$ and the action of $\bar{Z}_0$ on $\bar{R}_{\Bbbk}$ is induced by that of $Z_0 \subset Z_p(\mathfrak{g}_{\Bbbk})$, the result follows. □

Corollary 4.1. *Every irreducible $\bar{R}_{\Bbbk}$-module has dimension $\le \sqrt{D} \cdot p^d$.*

Proof. This is an immediate consequence of Lemma 4.1, because the central elements of $\bar{R}_{\Bbbk}$ act on any irreducible $\bar{R}_{\Bbbk}$-module V as scalar operators and the image of Λ in $\operatorname{End} V$ spans $\operatorname{End} V$. □

Proposition 4.2. *The centre $Z(\bar{R}_{\Bbbk})$ is an affine algebra over $\Bbbk$ and*

$$\dim\, Z(\bar{R}_{\Bbbk}) = \dim\, Z_p(\bar{R}_{\Bbbk}) = 2d.$$

Proof. By Lemma 4.1, $\bar{R}_{\Bbbk}$ is a finitely generated $\bar{Z}_0$-module. Since $\bar{Z}_0$ is an affine $\Bbbk$-algebra, $\bar{R}_{\Bbbk}$ is a Noetherian $\bar{Z}_0$-module. But then $Z(\bar{R}_{\Bbbk})$ and $Z_p(\bar{R}_{\Bbbk})$ are finitely generated $\bar{Z}_0$-modules. From this it is follows that the $\Bbbk$-algebra $Z(\bar{R}_{\Bbbk})$ is affine (of course, the same is true for $Z_p(\bar{R}_{\Bbbk})$, as it is a homomorphic image of $Z_p(\mathfrak{g}_{\Bbbk})$). Both $Z(\bar{R}_{\Bbbk})$ and $Z_p(\bar{R}_{\Bbbk})$ being integral over $\bar{Z}_0$, the inclusions $\bar{Z}_0 \hookrightarrow Z_p(\bar{R}_{\Bbbk})$ and $\bar{Z}_0 \hookrightarrow Z(\bar{R}_{\Bbbk})$ give rise to finite morphisms $\operatorname{Specm} \bar{Z}_0 \twoheadrightarrow \operatorname{Specm} Z_p(\bar{R}_{\Bbbk})$ and $\operatorname{Specm} \bar{Z}_0 \twoheadrightarrow \operatorname{Specm} Z(\bar{R}_{\Bbbk})$. Since $\bar{Z}_0$ is a homomorphic image of the polynomial algebra $\Bbbk[X_1, \ldots, X_{2d}]$, we now obtain

$$\dim\, Z(\bar{R}_{\Bbbk}) = \dim\, Z_p(\bar{R}_{\Bbbk}) = \dim\, \bar{Z}_0 \leq 2d. \tag{53}$$

Recall from 4.1 that the simple algebraic group $G_{\Bbbk}$ acts rationally on $\bar{R}_{\Bbbk}$. More precisely, the canonical homomorphism $c\colon\ U(\mathfrak{g}_{\Bbbk}) \twoheadrightarrow R_{\Bbbk} = U(\mathfrak{g}_{\Bbbk})/\mathcal{I}_{\Bbbk}$ is $G_{\Bbbk}$-equivariant. Since the inverse image under c of the unique maximal nilpotent ideal $N_{\Bbbk}$ of $R_{\Bbbk}$ is $G_{\Bbbk}$-stable, both $Z_p(\bar{R}_{\Bbbk}) \cong Z_p(\mathfrak{g}_{\Bbbk})/\big(Z_p(\mathfrak{g}_{\Bbbk}) \cap c^{-1}(N_{\Bbbk})\big)$ and $Z(\bar{R}_{\Bbbk})$ are stable under the action of $G_{\Bbbk}$ on $\bar{R}_{\Bbbk}$. Since $Z_p(\bar{R}_{\Bbbk})$ is a homomorphic image of $Z_p(\mathfrak{g}_{\Bbbk})$, the maximal spectrum $\mathcal{V}_{\mathfrak{P}}(M) := \operatorname{Specm} Z_p(\bar{R}_{\Bbbk})$ identifies with a Zariski closed subset of $\mathfrak{g}_{\Bbbk}^*$ (see 2.4 for more detail). By our discussion in 2.6, the affine $G_{\Bbbk}$-variety $\mathcal{V}_{\mathfrak{P}}(M)$ contains a linear function $\Psi \in \chi + \mathfrak{m}_{\Bbbk}^{\perp}$.

Given $j \in \mathbb{Z}^+$ we define $\Xi_j := \{\eta \in \mathfrak{g}_{\Bbbk}^* \,|\, \dim\, \mathfrak{z}_{\xi} \leq 2j\}$, a Zariski closed, conical subset of $\mathfrak{g}_{\Bbbk}^*$. There is a cocharacter $\lambda\colon\ \Bbbk^{\times} \to G_{\Bbbk}$ such that $(\operatorname{Ad} \lambda(t))(x) = t^i x$ for all $x \in \mathfrak{g}_k(i)$ and $t \in \Bbbk^{\times}$. Let $\rho_e\colon\ \Bbbk^{\times} \to \mathrm{GL}(\mathfrak{g}_{\Bbbk}^*)$ denote the composite of $\operatorname{Ad}^* \lambda$ with the scalar cocharacter $\xi \mapsto t^{-2}\xi$, where $\xi \in \mathfrak{g}_{\Bbbk}^*$ and $t \in \Bbbk^{\times}$. Obviously, ρ_e induces a contracting $\Bbbk^{\times}$-action on $\chi + \mathfrak{m}_{\Bbbk}^{\perp}$ with centre at χ. Since for any j the Zariski closed set $(\chi + \mathfrak{m}_{\Bbbk}^{\perp}) \cap \Xi_j$ is $\rho_e(\Bbbk^{\times})$-stable and $\dim \mathfrak{z}_{\chi} = 2d$, we see that $(\chi + \mathfrak{m}_{\Bbbk}^{\perp}) \cap \Xi_j = \emptyset$ for all $j < 2d$. This implies that $\dim\, \mathfrak{z}_{\Psi} \geq 2d$.

Since $(\operatorname{Ad}^* G_{\Bbbk})\, \Psi \subset \mathcal{V}_{\mathfrak{P}}(M)$, we now deduce that $\dim\, \mathcal{V}_{\mathfrak{P}}(M) \geq 2d$. In conjunction with (53) this gives $\dim\, Z(\bar{R}_{\Bbbk}) = \dim\, Z_p(\bar{R}_{\Bbbk}) = 2d$, as stated. □

Remark 4.1. It follows from the proof of Proposition 4.2 that $\dim\, \mathfrak{z}_{\Psi} = 2d$ and the orbit $(\operatorname{Ad}^* G_{\Bbbk})\, \Psi$ is open in the variety $\mathcal{V}_{\mathfrak{P}}(M)$. Moreover, arguing as in [25, 3.6] it is easy to observe that χ and Ψ belong to the same sheet of $\mathfrak{g}_{\Bbbk}^*$.

4.3. In this subsection we assume that $\mathcal{D}_M$ is a Weyl skew-field and we adopt the notation and conventions of 4.1. By Proposition 4.1, there is a unital subalgebra $\mathfrak{C}_{\Bbbk} \cong \mathrm{Mat}_{l'}(\Bbbk)$ of $\mathcal{Q}(\bar{R}_{\Bbbk})$ such that $\mathcal{Q}(\bar{R}_{\Bbbk}) \cong \mathfrak{C}_{\Bbbk} \otimes \mathfrak{D}_{\Bbbk}$ where $\mathfrak{D}_{\Bbbk}$ is the centraliser of $\mathfrak{C}_{\Bbbk}$ in $\mathcal{Q}(\bar{R}_{\Bbbk})$.

Proposition 4.3. *Suppose $\mathcal{D}_M \cong \mathcal{Q}(\mathbf{A}_d(\mathbb{C}))$ and the admissible ring A satisfies all the requirements of 4.1. Then the $\Bbbk$-algebra $\mathfrak{D}_{\Bbbk}$ is isomorphic to the ring of fractions $\mathcal{Q}(\mathbf{A}_d(\Bbbk))$ and $\mathcal{Q}(\bar{R}_{\Bbbk}) \cong \mathrm{Mat}_{l'}\big(\mathcal{Q}(\mathbf{A}_d(\Bbbk))\big)$.*

Proof. First recall from 4.1 that given $x \in R_A$ we write $\bar{x}$ for the image of $x \otimes 1$ in $\bar{R}_{\Bbbk} = \big(R_A \otimes_A \Bbbk_{\mathfrak{P}})\big)/N_{\Bbbk}$. Repeating the argument used at the beginning of the proof of Proposition 4.1 we observe that for every $y \in Y$ the element $\bar{y}$ is regular in $\bar{R}_{\Bbbk}$.

The subset $\bar{X} \cup \bar{Y}$ of $\bar{R}_{\Bbbk}$ contains elements satisfying the relations (34), (37), (38), (39), (40), (41), (42), (43), (44), (46), (47), (48), (50), (51), (52). Since all elements of $\bar{Y}$ involved in these relations are regular and each step of the procedure described in 3.2 is reversible, we can find elements $w_1, \ldots, w_{2d}$ in $\mathcal{Q}(\bar{R}_{\Bbbk})$ satisfying the relations

(27) and (28). We denote by $\mathcal{D}'_{\Bbbk}$ the $\Bbbk$-subalgebra of $\mathcal{Q}(\bar{R}_{\Bbbk})$ generated by the w_i's. Clearly, $\mathcal{D}'_{\Bbbk}$ is a homomorphic image of the Weyl algebra $\mathbf{A}_d(\Bbbk)$.

By (49), we have the inclusion $\mathcal{D}'_{\Bbbk} \subset \mathfrak{D}_{\Bbbk}$. Since the images of the $\widetilde{Q}^s_{i;\,p(k)}$'s with $i = 1, 2$ are regular in $\bar{R}_{\Bbbk}$ and $\mathcal{Q}(\bar{R}_{\Bbbk}) = (\mathcal{C}(Z(\bar{R}_{\Bbbk}))^{-1}\bar{R}_{\Bbbk}$ by our earlier remarks, we can combine (30), (29), (10) and (11) with the equality $\mathcal{Q}(\bar{R}_{\Bbbk}) = \mathfrak{C}_{\Bbbk}\cdot\mathfrak{D}_{\Bbbk}$ to obtain

$$\mathfrak{D}_{\Bbbk} = (\mathcal{C}(Z(\bar{R}_{\Bbbk}))^{-1}\mathcal{D}'_{\Bbbk}. \tag{54}$$

Since it follows from Proposition 4.1 that $\mathfrak{D}_{\Bbbk}$ is a semiprime ring, (54) yields that $\mathcal{D}'_{\Bbbk}$ has no nonzero nilpotent ideals, i.e. the ring $\mathcal{D}'_{\Bbbk}$ is semiprime, too.

Let $\mathcal{C}(Z(\mathcal{D}'_{\Bbbk}))$ denote the set of all regular elements of $\mathcal{D}'_{\Bbbk}$ contained in the centre of $\mathcal{D}'_{\Bbbk}$. It is immediate from (54) that $\mathcal{C}(Z(\mathcal{D}'_{\Bbbk})) \subseteq \mathcal{C}(Z(\mathfrak{D}_{\Bbbk}))$. So $\mathcal{C}(Z(\mathcal{D}'_{\Bbbk}))$ is a multiplicative subset of regular elements of $\mathcal{Q}(\bar{R}_{\Bbbk})$ satisfying the left and right Ore condition.

Being a homomorphic image of $\mathbf{A}_d(\Bbbk)$ the $\Bbbk$-algebra $\mathcal{D}'_{\Bbbk}$ is finitely generated as a module over its centre. As $\mathcal{D}'_{\Bbbk}$ is a semiprime ring, applying [1, Theorem 2] yields that $\mathcal{Q}(\mathcal{D}'_{\Bbbk})$ is obtained from $\mathcal{D}'_{\Bbbk}$ by inverting the elements from $\mathcal{C}(Z(\mathcal{D}'_{\Bbbk}))$. Combining this with (30) and (29) we now deduce that $\bar{c}^k_{ij} \in (\mathcal{C}(Z(\mathcal{D}'_{\Bbbk})))^{-1}\mathcal{D}'_{\Bbbk}$ for all $1 \le i, j \le l'$ and $1 \le k \le n$. But then (10) forces $\bar{g}_k \in (\mathcal{C}(Z(\mathcal{D}'_{\Bbbk})))^{-1}\mathfrak{C}_{\Bbbk}\cdot\mathcal{D}'_{\Bbbk}$ for all $1 \le k \le n$. This, in turn, yields that $(\mathcal{C}(Z(\mathcal{D}'_{\Bbbk})))^{-1}\mathcal{D}'_{\Bbbk}$ contains the centraliser of $\mathfrak{C}_{\Bbbk}$ in $\bar{R}_{\Bbbk}$. Now our remarks earlier in the proof show that

$$\mathfrak{D}_{\Bbbk} = (\mathcal{C}(Z(\mathcal{D}'_{\Bbbk})))^{-1}\mathcal{D}'_{\Bbbk} = \mathcal{Q}(\mathcal{D}'_{\Bbbk}). \tag{55}$$

Let $Z_d(\Bbbk)$ denote the centre of the Weyl algebra $\mathbf{A}_d(\Bbbk)$. It is well known and easy to check that $Z_d(\Bbbk)$ is a polynomial algebra in $2d$ variables over $\Bbbk$ and $\mathbf{A}_d(\Bbbk)$ is a free $Z_d(\Bbbk)$-module of rank p^{2d}. Furthermore, every two-sided ideal of $\mathbf{A}_d(\Bbbk)$ is centrally generated. Since $\mathcal{D}'_{\Bbbk}$ is a homomorphic image of $\mathbf{A}_d(\Bbbk)$, its centre, $\bar{Z}_d$, is a homomorphic image of $Z_d(\Bbbk)$. We let $\beta\colon Z_d(\Bbbk) \twoheadrightarrow \bar{Z}_d$ denote the corresponding homomorphism of $\Bbbk$-algebras.

Recall from 4.1 that $N_{\Bbbk} = \cap_{i=1}^{\nu} I_i$ where $I_1, \ldots, I_\nu$ are the minimal primes of $R_{\Bbbk}$. By the theory of semiprime Noetherian PI-algebras finite over their centres, all quotients $R_{\Bbbk}/I_j$ are prime and $\mathcal{Q}(\bar{R}_{\Bbbk}) \cong \mathcal{Q}(R_{\Bbbk}/I_1) \oplus \cdots \oplus \mathcal{Q}(R_{\Bbbk}/I_\nu)$ as $\Bbbk$-algebras. Moreover, each direct summand $\mathcal{Q}(R_{\Bbbk}/I_j)$ is a simple algebra finite dimensional over its centre $\mathcal{Q}(Z(R_{\Bbbk}/I_j))$; see [26], [1]. In particular, this shows that $\mathcal{Q}(Z(\bar{R}_{\Bbbk})) = (\mathcal{C}(Z(\bar{R}_{\Bbbk}))^{-1}Z(\bar{R}_{\Bbbk})$ injects into $\prod_{j=1}^{\nu}\mathcal{Q}(\bar{R}_{\Bbbk}/I_j)$, a direct product of fields.

On the other hand, the algebra $Z(\bar{R}_{\Bbbk})$ being reduced, $\mathcal{Q}(Z(\bar{R}_{\Bbbk}))$ itself is a direct product of fields. Furthermore, Proposition 4.2 implies that at least one of the fields involved as direct factors of $\mathcal{Q}(Z(\bar{R}_{\Bbbk}))$ has transcendence degree over $\Bbbk$ equal to $2d$. It follows that

$$\operatorname{tr.deg}_{\Bbbk}\mathcal{Q}(Z(R_{\Bbbk}/I_\ell)) \ge 2d \quad \text{for some } \ \ell \le \nu. \tag{56}$$

Since $\mathcal{Q}(\bar{R}_{\Bbbk}) \cong \mathfrak{C}_{\Bbbk}\otimes\operatorname{Mat}_{p^d}(\mathcal{Q}(\bar{Z}_d)) \cong \operatorname{Mat}_{l'p^d}(\mathcal{Q}(\bar{Z}_d))$ by our discussion earlier in the proof, we have that $\mathcal{Q}(Z(\bar{R}_{\Bbbk})) \cong \mathcal{Q}(\bar{Z}_d)$ as $\Bbbk$-algebras. As the algebra $\mathcal{Q}(\bar{R}_{\Bbbk})$ is semiprime, its centre $\bar{Z}_d$ is reduced and hence the ring of fractions $\mathcal{Q}(\bar{Z}_d)$ is a direct product of fields. If $\beta\colon Z_d(\Bbbk) \twoheadrightarrow \bar{Z}_d$ is not injective, then $\dim \bar{Z}_d < 2d$ and hence all fields involved as direct factors of $\mathcal{Q}(\bar{Z}_d)$ have transcendence degree over $\Bbbk$ less than $2d$. Since this contradicts (56), the map β must be injective. Then $\bar{Z}_d \cong Z_d(\Bbbk)$, implying that $\mathcal{Q}(\mathfrak{D}_\Bbbk) \cong \mathcal{Q}(\mathbf{A}_d(\Bbbk))$ and $\mathcal{Q}(\bar{R}_\Bbbk) \cong \mathfrak{C}_\Bbbk \otimes \mathcal{Q}(\mathbf{A}_d(\Bbbk)) \cong \mathrm{Mat}_{l'}(\mathbf{A}_d(\Bbbk))$, as claimed. □

Corollary 4.2. *If $\mathcal{D}_M \cong \mathcal{Q}(\mathbf{A}_d(\mathbb{C}))$ as $\mathbb{C}$-algebras and the admissible ring A satisfies all the requirements of 4.1, then $\bar{R}_\Bbbk$ is a prime ring.*

Proof. Since $\mathcal{Q}(\bar{R}_\Bbbk) = \mathcal{C}(Z(\bar{R}_\Bbbk))^{-1}\bar{R}_\Bbbk$ and the ring $\mathcal{Q}(\mathbf{A}_d(\Bbbk))$ is prime, this is an immediate consequence of Proposition 4.3. □

Conjecture 4.1. We conjecture that under the above assumptions on A the ring $\bar{R}_\Bbbk$ is prime for any finite dimensional simple Lie algebra $\mathfrak{g}$ and any primitive ideal $I = I_M$.

As Corollary 4.2 shows, this conjecture is weaker than Joseph's conjecture on the Goldie fields of the primitive quotients of $U(\mathfrak{g})$.

4.4. Write $\bar{I}_j$ for the image the minimal prime I_j of $R_\Bbbk$ in $\bar{R}_\Bbbk = R_\Bbbk/N_\Bbbk$. Since each quotient $\bar{R}_\Bbbk/\bar{I}_j$ is a prime ring, its central subalgebra $Z_p(\bar{R}_\Bbbk)/\bar{I}_j \cap Z_p(\bar{R}_\Bbbk)$ is a domain. Since $\bar{I}_i \cdot \bar{I}_j \subseteq 0$ for $i \neq j$ and $\cap_{j=1}^{\nu}(\bar{I}_j \cap Z_p(\bar{R}_\Bbbk)) = 0$, every $\bar{I}_j \cap Z_p(\bar{R}_\Bbbk)$ is a minimal prime of $Z_p(\bar{R}_\Bbbk)$ and every minimal prime of $Z_p(\bar{R}_\Bbbk)$ is one of the $\bar{I}_j \cap Z_p(\bar{R}_\Bbbk)$'s. It follows that there is $\ell \in \{1, \cdots, \nu\}$ such that $\dim\, Z_p(\bar{R}_\Bbbk) = \dim\, Z_p(\bar{R}_\Bbbk)/\bar{I}_\ell \cap Z_p(\bar{R}_\Bbbk)$. We now define $\mathcal{R} := \bar{R}_\Bbbk/\bar{I}_\ell$ and $Z_p(\mathcal{R}) := Z_p(\bar{R}_\Bbbk)/\bar{I}_\ell \cap Z_p(\bar{R}_\Bbbk)$. Then $\mathcal{R}$ is a prime Noetherian ring which is finitely generated as a $Z_p(\mathcal{R})$-module.

Since $G_\Bbbk$ is a connected group, every minimal prime $\bar{I}_j$ of $\bar{R}_\Bbbk$ is $G_\Bbbk$-stable. Therefore, $G_\Bbbk$ acts on $\mathcal{R}$ as algebra automorphisms. Recall from 4.2 the Zariski closed set $\mathcal{V}_\mathfrak{P}(M) \subset \mathfrak{g}_\Bbbk^*$ which we have identified with the maximal spectrum of $Z_p(\bar{R}_\Bbbk)$. As explained in the proof of Proposition 4.2, one of the components of $\mathcal{V}_\mathfrak{P}(M)$ contains a linear function $\Psi \in \chi + \mathfrak{m}_\Bbbk^\perp$ and $\dim\,(\mathrm{Ad}^*\, G)\,\Psi = 2d$.

By construction, the zero locus of $\bar{I}_\ell \cap Z_p(\bar{R}_\Bbbk)$ in $\mathcal{V}_\mathfrak{P}(M)$ is an irreducible component of maximal dimension in $\mathcal{V}_\mathfrak{P}(M)$. Since $\dim\, Z_p(\bar{R}_\Bbbk) = 2d$ by Proposition 4.2 and all irreducible components of $\mathcal{V}_\mathfrak{P}(M)$ are $G_\Bbbk$-stable, we see that Ψ, too, lies in an irreducible component of maximal dimension of $\mathcal{V}_\mathfrak{P}(M)$. But then the above discussion shows that we can choose $\ell \in \{1, \dots, \nu\}$ such that the zero locus of $\bar{I}_\ell \cap Z_p(\bar{R}_\Bbbk)$ in $\mathcal{V}_\mathfrak{P}(M)$ coincides with the Zariski closure of $(\mathrm{Ad}^*\, G)\,\Psi$ in $\mathfrak{g}_\Bbbk^*$. Therefore, no generality will be lost by assuming that $(\mathrm{Ad}^*\, G)\,\Psi$ is the unique open dense orbit of maximal spectrum $\mathrm{Specm}\, Z_p(\mathcal{R}) \subset \mathfrak{g}_\Bbbk^*$.

Since $\mathcal{R}$ is a Noetherian $Z_p(\mathcal{R})$-module, the centre $Z(\mathcal{R})$ is finitely generated and integral over $Z_p(\mathcal{R})$. Hence $Z_p(\mathcal{R})$ is an affine algebra over $\Bbbk$ and the morphism

$$\mu\colon\ \mathrm{Specm}\, Z(\mathcal{R}) \twoheadrightarrow \mathrm{Specm}\, Z_p(\mathcal{R})$$

induced by inclusion $Z_p(\mathcal{R}) \hookrightarrow Z(\mathcal{R})$ is finite. In particular, $\dim\, Z(\mathcal{R}) = \dim\, Z_p(\mathcal{R}) = 2d$. As the ring $\mathcal{R}$ is prime, the centre $Z(\mathcal{R})$ is a domain and hence the affine variety $\mathcal{V}(\mathcal{R}) := \mathrm{Specm}\, Z(\mathcal{R})$ is irreducible. By our choice of A, the rational action of the group $G_\Bbbk$ on $U(\mathfrak{g}_\Bbbk)$ induces that on $Z(\mathcal{R})$. Thus, $\mathcal{V}(\mathcal{R})$ is an irreducible affine $G_\Bbbk$-variety.

Proposition 4.4. *The following are true:*

(i) *The finite morphism* $\mu\colon \mathcal{V}(\mathcal{R}) \twoheadrightarrow \operatorname{Specm} Z_p(\mathcal{R})$ *is* $G_{\Bbbk}$*-equivariant and the inverse image of* $(\operatorname{Ad}^* G)\,\Psi \subset \operatorname{Specm} Z_p(\mathcal{R})$ *under* μ *is a unique open dense* $G_{\Bbbk}$*-orbit of* $\mathcal{V}(\mathcal{R})$.

(ii) *The stabiliser* $(G_{\Bbbk})_c = \{g \in G_{\Bbbk} \mid g \cdot c = c\}$ *of any* $c \in \mu^{-1}(\Psi)$ *has the property that* $Z_{G_{\Bbbk}}(\Psi)^{\circ} \subseteq (G_{\Bbbk})_c \subseteq Z_{G_{\Bbbk}}(\Psi)$.

(iii) *The coadjoint stabiliser* $Z_{G_{\Bbbk}}(\Psi)$ *acts transitively on the fibre* $\mu^{-1}(\Psi)$.

Proof. It is clear from our earlier remarks that μ is a finite morphism equivariant under the action of $G_{\Bbbk}$. Let $\mathcal{V}(\mathcal{R})_{\text{reg}}$ denote the inverse image of $(\operatorname{Ad}^* G)\,\Psi$ under μ. Since the map μ is $G_{\Bbbk}$-equivariant, we have that $\mathcal{V}(\mathcal{R})_{\text{reg}} = \bigcup_{c\in\mu^{-1}(\Psi)} G_{\Bbbk}\cdot c$. As the morphism μ is finite, $\mu^{-1}(\Psi)$ is a finite set and $\dim \mathcal{V}(\mathcal{R}) = 2d = (\operatorname{Ad}^* G)\,\Psi$. From this it is immediate that each orbit $G_{\Bbbk}\cdot c$ with $c \in \mu^{-1}(\Psi)$ is Zariski open in $\mathcal{V}(\mathcal{R})$. As the variety $\mathcal{V}(\mathcal{R})$ is irreducible, we see that $G_{\Bbbk}\cdot c \cap G_{\Bbbk}\cdot c' \neq \emptyset$ for any two $c, c' \in \mu^{-1}(\Psi)$. This forces $G_{\Bbbk}\cdot c = G_{\Bbbk}\cdot c'$ for all $c, c' \in \mu^{-1}(\Psi)$, implying that $\mu^{-1}\big(\mathcal{V}(\mathcal{R})_{\text{reg}}\big) = G_{\Bbbk}\cdot c$ for any $c \in \mu^{-1}(\Psi)$. This proves statement (i).

If $c \in \mu^{-1}(\Psi)$ and $g \in (G_{\Bbbk})_c$, then

$$\Psi = \mu(c) = \mu(g\cdot c) = (\operatorname{Ad}^* g)\,\mu(c) = (\operatorname{Ad}^* g)\,\Psi.$$

Therefore, $(G_{\Bbbk})_c \subseteq Z_{G_{\Bbbk}}(\Psi)$. On the other hand, the finite set $\mu^{-1}(c)$ is stable under the action of $Z_{G_{\Bbbk}}(\Psi)$. As $G_{\Bbbk}$ acts regularly on the affine algebraic variety $\mathcal{V}(\mathcal{R})$, it follows that the stabiliser $(G_{\Bbbk})_c$ of any $c \in \mu^{-1}(\Psi)$ is a Zariski closed subgroup of finite index in $Z_{G_{\Bbbk}}(\Psi)$. Therefore, it must contain the connected component of identity in $Z_{G_{\Bbbk}}(\Psi)$, and statement (ii) follows.

If $c, g(c) \in \mu^{-1}(\Psi)$ for some $g \in (G_{\Bbbk})_c$, then $\Psi = \mu(g(c)) = g(\mu(c)) = g(\Psi)$, forcing $g \in Z_{G_{\Bbbk}}(\Psi)$. Thus, statement (iii) is an immediate consequence of statement (i). □

Remark 4.2. If $\mathcal{D}_M \cong \mathcal{Q}(\mathbf{A}_d(\mathbb{C}))$, then $\bar{R}_{\Bbbk}$ is a prime ring by Corollary 4.2. So in this case we have that $\bar{R}_{\Bbbk} = \mathcal{R}$.

4.5. Recall from [2], [26], [1] that any prime PI-ring $\mathcal{A}$ has a simple Artinian ring of fractions $\mathcal{Q}(\mathcal{A})$ which satisfies the same identities as $\mathcal{A}$ and is spanned by $\mathcal{A}$ over its centre, K, which coincides with $\mathcal{Q}(Z(\mathcal{A}))$. Moreover, $\dim_K \mathcal{Q}(\mathcal{A}) = d^2$, and after tensoring by a suitable algebraic field extension $\tilde{K}$ of K, the ring $\mathcal{Q}(\mathcal{A})$ becomes the matrix algebra $\operatorname{Mat}_d(\tilde{K})$. Both $\mathcal{A}$ and $\mathcal{Q}(\mathcal{A})$ satisfy all the polynomial identities of $d \times d$ matrices over a commutative ring, but not those of smaller matrices, and d can be characterized as the least positive integer such that $S_{2d}(X_1, \ldots, X_{2d}) = 0$ for all $X_1, \ldots, X_{2d} \in \mathcal{A}$, where

$$S_{2d}(X_1, \ldots, X_{2d}) := \sum_{\sigma\in\mathfrak{S}_{2d}} (\operatorname{sgn}\sigma) X_{\sigma(1)} \cdots X_{\sigma(2d)}.$$

Definition 4.1. The *PI-degree* of of a prime PI-ring $\mathcal{A}$, denoted PI-deg($\mathcal{A}$), is defined as the least positive integer d such that $\mathcal{A}$ satisfies the *standard identity* $S_{2d} \equiv 0$.

Definition 4.2. We say that $\mathcal{A}$ is an *Azumaya algebra* over its centre $Z(\mathcal{A})$ if $\mathcal{A}$ is a finitely generated projective $Z(\mathcal{A})$-module and the natural map $\mathcal{A} \otimes_{Z(\mathcal{A})} \mathcal{A}^{\text{op}} \to \operatorname{End}_{Z(\mathcal{A})} \mathcal{A}$ is an isomorphism.

Now suppose that our PI-ring $\mathcal{A}$ is finitely generated over its centre $Z(\mathcal{A})$ which, in turn, is an affine algebra over $\Bbbk$. In this situation, it is known that PI-deg$(\mathcal{A}) = d(\mathcal{A})$, where $d(\mathcal{A})$ stands for the maximum $\Bbbk$-dimension of irreducible $\mathcal{A}$-modules; see [4], for example. Let V be an irreducible $\mathcal{A}$-module, $P = \operatorname{Ann}_{\mathcal{A}} V$ and $\mathfrak{c} = P \cap Z(\mathcal{A})$, a maximal ideal of $Z(\mathcal{A})$. It follows from the Artin–Procesi theorem [18, Thm. 13.7.4] that the equality $\dim V = \text{PI-deg}(\mathcal{A})$ holds if and only if $\mathcal{A}_\mathfrak{c} = \mathcal{A} \otimes_{Z(\mathcal{A})} Z(\mathcal{A})_\mathfrak{c}$ is an Azumaya algebra over the local ring $Z(\mathcal{A})_\mathfrak{c}$; see [4] for more detail.

The *Azumaya locus* of $\mathcal{A}$, denoted Az$(\mathcal{A})$, is defined as

$$\operatorname{Az}(\mathcal{A}) := \{\mathfrak{c} \in \operatorname{Specm} Z(\mathcal{A}) \mid \mathcal{A}_\mathfrak{c} \text{ is an Azumaya algebra}\}.$$

The above discussion shows that Az$(\mathcal{A})$ consists of all $\mathfrak{c} \in \operatorname{Specm} Z(\mathcal{A})$ with $\mathcal{A}_\mathfrak{c}/\mathfrak{c}\mathcal{A}_\mathfrak{c} \cong \operatorname{Mat}_{d(\mathcal{A})}(\Bbbk)$, whilst the Artin–Procesi theorem yields that Az$(\mathcal{A})$ is a nonempty Zariski open subset of $\operatorname{Specm} Z(\mathcal{A})$; see [18, Thm. 13.7.14(iii)].

4.6. In this subsection, we shall prove Theorems A and B. First suppose that $\mathcal{D}_M \cong \mathcal{Q}(\mathbf{A}_d(\mathbb{C}))$. Then Corollary 4.2 says that $\bar{R}_\Bbbk = \mathcal{R}$ is a prime ring. It follows from Proposition 4.3 that there exists a finite algebraic extension $\widetilde{K} \cong \Bbbk(X_1, \ldots, X_{2d})$ of the centre K of $\mathcal{Q}(\mathcal{R})$ (identified with $\mathcal{Q}(Z_d(\Bbbk))$, the centre of $\mathcal{Q}(\mathbf{A}_d(\Bbbk))$) such that

$$\mathcal{Q}(\mathcal{R}) \otimes_K \widetilde{K} \;\cong\; \operatorname{Mat}_{l'}\big(\mathcal{Q}(\mathbf{A}_d(\Bbbk)) \otimes_K \widetilde{K}\big) \;\cong\; \operatorname{Mat}_{l'p^d}(\widetilde{K}).$$

It follows that PI-deg$(\mathcal{R}) = l'p^d$. On the other hand, since the Azumaya locus of $\mathcal{R}$ is $G_\Bbbk$-stable and the dominant morphism $\mu\colon \operatorname{Specm} Z(\mathcal{R}) = \mathcal{V}(\mathcal{R}) \twoheadrightarrow \operatorname{Specm} Z_p(\mathcal{R})$ from 4.4 is $G_\Bbbk$-equivariant, it must be that $\Psi \in \mu(\operatorname{Az}(\mathcal{R}))$. But then $\mu^{-1}(\Psi) \cap \operatorname{Az}(\mathcal{R}) \neq \emptyset$. Applying Proposition 4.4 now yields $\mu^{-1}(\Psi) \subset \operatorname{Az}(\mathcal{R})$.

Let $\mathfrak{c}(\Psi)$ denote the annihilator in $Z(\mathcal{R})$ of the the irreducible $\mathcal{R}$-module $\widetilde{M_{\Bbbk,\Psi}}$ introduced in 2.6. Since $\mathfrak{c}(\Psi) \in \mu^{-1}(\Psi)$, the preceding remark shows that $\mathcal{R}_{\mathfrak{c}(\Psi)}$ is an Azumaya algebra. As $Z(\mathcal{R})_{\mathfrak{c}(\Psi)}$ is a local ring, our discussion in 4.5 now yields that $\widetilde{M_{\Bbbk,\Psi}}$ is the only irreducible $\mathcal{R}_{\mathfrak{c}(\Psi)}$-module (up to isomorphism) and it has dimension equal to $d(\mathcal{R}) = \text{PI-deg}(\mathcal{R})$. Therefore,

$$l'p^d = \text{PI-deg}(\mathcal{R}) = \dim_\Bbbk \widetilde{M_{\Bbbk,\Psi}} = lp^d = (\dim_\mathbb{C} M)p^d.$$

Since $l' = \operatorname{rk}\big(U(\mathfrak{g})/I_M\big)$, Theorem B follows.

It remains to prove Theorem A. Applying Proposition 4.4 and arguing as before we obtain the inclusion $\mu^{-1}(\Psi) \subset \operatorname{Az}(\mathcal{R})$ and hence the equality PI-deg$(\mathcal{R}) = lp^d$. On the other hand, Proposition 4.1 says that $\mathcal{Q}(\bar{R}_\Bbbk) \cong \mathfrak{C}_\Bbbk \otimes \mathfrak{D}_\Bbbk$, where $\mathfrak{D}_\Bbbk$ is the centraliser of $\mathfrak{C}_\Bbbk \cong \operatorname{Mat}_{l'}(\Bbbk)$ in $\mathcal{Q}(\bar{R}_\Bbbk)$. Since $\mathcal{Q}(\bar{R}_\Bbbk)$ is a semiprime Artinian ring, so is $\mathfrak{D}_\Bbbk$. Therefore, $\mathfrak{D}_\Bbbk \cong \bigoplus_{j=1}^{\nu'} \mathfrak{D}_{\Bbbk,j}$ for some simple Artinian rings $\mathfrak{D}_{\Bbbk,j}$. But we know that $\mathcal{Q}(\bar{R}_\Bbbk) = \bigoplus_{j=1}^{\nu} \mathcal{Q}(\bar{R}_\Bbbk/\bar{I}_j)$ and each $\mathcal{Q}(\bar{R}_\Bbbk/\bar{I}_j)$ is a simple Artinian ring; see our discussion in 4.4. Since $\mathcal{Q}(\bar{R}_\Bbbk) \cong \bigoplus_{j=1}^{\nu'} \big(\mathfrak{C}_\Bbbk \otimes \mathfrak{D}_{\Bbbk,j}\big)$ and each $\mathfrak{C}_\Bbbk \otimes \mathfrak{D}_{\Bbbk,j}$ is a simple Artinian ring, we now deduce that $\nu = \nu'$ and

$$\mathcal{Q}(\mathcal{R}) \;=\; \mathcal{Q}(\bar{R}_\Bbbk/\bar{I}_\ell) \;\cong\; \mathfrak{C}_\Bbbk \otimes \mathfrak{D}_{\Bbbk,\ell'}$$

for some $\ell' \leq \nu$. As $\mathfrak{C}_\Bbbk \cong \operatorname{Mat}_{l'}(\Bbbk)$, our discussion in 4.5 then shows that l' divides PI-deg$(\mathcal{Q}(\mathcal{R}))$ = PI-deg$(\mathcal{R}) = lp^d$. As $\Pi(A)$ contains almost all primes in $\mathbb{N}$, we can find $\mathfrak{P} \in \operatorname{Specm} A$ such that l' is coprime to $p = \operatorname{char} A/\mathfrak{P}$. Then we see that $l' = \operatorname{rk}\big(U(\mathfrak{g})/I_M\big)$ must divide $l = \dim_\mathbb{C} M$, which completes the proof of Theorem A.

4.7. Let M and M' be two generalised Gelfand–Graev models of a primitive ideal $\mathcal{I} \in \mathcal{X}_{\mathcal{O}}$, so that $\mathcal{I} = I_M = I_{M'}$. As we already mentioned in the Introduction, it was conjectured by the author and proved by Losev in [17] that $[M'] = {}^{\gamma}[M]$ for some $\gamma \in \Gamma(e)$. We would like to conclude this paper by showing that Conjecture 4.1 implies Losev's result.

Suppose $\bar{R}_{\Bbbk}$ is a prime ring and let $l = \dim V$, $l' = \dim V'$. Let Γ be a subset of $C(e) = G_e \cap G_f$ which maps bijectively onto $\Gamma(e)$ under the canonical homomorphisms $C(e) \to \Gamma(e) = C(e)/C(e)^{\circ}$. Let us assume for a contradiction that $M' \not\cong {}^{\gamma}M$ for any $\gamma \in \Gamma$. Arguing as in 2.6 we can find an admissible ring $A \subset \mathbb{C}$ and free A-submodules M_A and M'_A of M and M', respectively, stable under $U(\mathfrak{g}_A, e)$ and such that $M \cong M_A \otimes_A \mathbb{C}$ and $M' \cong M'_A \otimes_A \mathbb{C}$. For every $p \in \Pi(A)$ we then get $U(\mathfrak{g}_{\Bbbk}, e)$-modules $M_{\Bbbk} = M_A \otimes_A \Bbbk$ and $M'_{\Bbbk} = M'_A \otimes_A \Bbbk$, where $\Bbbk = \overline{\mathbb{F}}_p$. As in 2.6 we localise further to reduce to the case where $M_{\Bbbk}$ and $M'_{\Bbbk}$ are irreducible $U(\mathfrak{g}_{\Bbbk}, e)$-modules for all $p \in \Pi(A)$. Associated with $M_{\Bbbk}$ and $M'_{\Bbbk}$ are $\bar{R}_{\Bbbk}$-modules $\widetilde{M}_{\Bbbk,\Psi}$ and $\widetilde{M}'_{\Bbbk,\Psi'}$ be $\bar{R}_{\Bbbk}$-modules, where $\Psi, \Psi' \in \chi + \mathfrak{m}_{\Bbbk}^{\perp}$; see 2.6 for more detail.

Recall from 2.3 that $U(\mathfrak{g}_A, e)$ is a free A-module with basis consisting of the PBW monomials in $\Theta_1, \ldots, \Theta_r$. Since Γ is a finite set, we may assume (after extending A if necessary) that the A-form $U(\mathfrak{g}_A, e)$ of $U(\mathfrak{g}, e)$ is stable under the action of the subgroup of $C(e)$ generated by Γ. Then each ${}^{\gamma}M_A$ with $\gamma \in \Gamma$ can be regarded as a $U(\mathfrak{g}_A, e)$-module. For $\gamma \in \Gamma$, the equality $\operatorname{Hom}_{U(\mathfrak{g},e)}({}^{\gamma}M, M') = 0$ comes down to the fact that a certain homogeneous system of linear equations in ll' unknowns with coefficients in A has no nonzero solutions. After inverting in A one of the nonzero $ll' \times ll'$ minors of the matrix of this homogeneous system we may assume that $\operatorname{Hom}_{U(\mathfrak{g}_{\Bbbk},e)}({}^{\gamma}M_{\Bbbk}, M'_{\Bbbk}) = 0$ for all $p \in \Pi(A)$.

Recall from [23] and [25] the subset $\pi(A)$ of $\Pi(A)$; it consists of all primes $p \in \mathbb{N}$ such that $A/\mathfrak{P} \cong \mathbb{F}_p$ for some $\mathfrak{P} \in \operatorname{Specm} A$. By [25, Lemma 4.4], the set $\pi(A)$ is infinite. The preceding remark then shows that no generality will be lost by assuming that $p \in \pi(A)$ and ${}^{\gamma}M_{\Bbbk} \not\cong M'_{\Bbbk}$ as $U(\mathfrak{g}_{\Bbbk}, e)$-modules for all $\gamma \in \Gamma$. Enlarging A further if need be we may also assume that $\mathcal{I}_A = \operatorname{Ann}_{U(\mathfrak{g})} L_A(\lambda)$ for some irreducible highest weight module $L(\lambda)$ and that A satisfies all the requirements of [25, Sect. 4]. Since $\pi(A)$ is an infinite set, we may also assume that the base change $A \to A/\mathfrak{P} \hookrightarrow \Bbbk$ identifies $\Gamma \subset G(A)$ with a subset of $Z_{G_{\Bbbk}}(\chi)$ which maps *onto* the component group of $Z_{G_{\Bbbk}}(\chi)$ under the canonical homomorphism $Z_{G_{\Bbbk}}(\chi) \to Z_{G_{\Bbbk}}(\chi)/Z_{G_{\Bbbk}}(\chi)^{\circ}$.

Let $\mathfrak{P} \in \operatorname{Specm} A$ be such that $A/\mathfrak{P} \cong \mathbb{F}_p$. As explained in [25, 4.5] the $R_{\Bbbk}$-module $L_{\mathfrak{P}}(\lambda) = L_A(\lambda) \otimes_A \Bbbk_{\mathfrak{P}}$ has a composition factor, $L^{\eta}_{\mathfrak{P}}(\lambda)$ with p-character $\eta \in (\operatorname{Ad}^* G_{\Bbbk})\,\chi$. As the ideal $N_{\Bbbk}$ is nilpotent, $L^{\eta}_{\mathfrak{P}}(\lambda)$ is an irreducible $\bar{R}_{\Bbbk}$-module. Since we assume that the algebra $\bar{R}_{\Bbbk}$ is prime, the variety $\operatorname{Specm} Z_p(\bar{R}_{\Bbbk}) \subset \mathfrak{g}^*_{\Bbbk}$ is irreducible and $(\operatorname{Ad}^* G_{\Bbbk})$-stable. By Proposition 4.2, it has dimension $2d$ which forces $\operatorname{Specm} Z_p(\bar{R}_{\Bbbk}) = \overline{(\operatorname{Ad}^* G_{\Bbbk})\,\chi}$. But then both Ψ and Ψ' are $(\operatorname{Ad}^* G_{\Bbbk})$-conjugate to χ. As explained in Remark 2.1 we can replace Ψ and Ψ' by their $(\operatorname{Ad}^* \mathcal{M}_{\Bbbk})$-conjugates. In view of [25, Lemma 3.2] and standard properties of Slodowy slices, we therefore may assume further that $\Psi = \Psi' = \chi$.

Denote by $\mathfrak{c}$ and $\mathfrak{c}'$ the annihilators in $Z(\bar{R}_{\Bbbk})$ of $\widetilde{M}_{\Bbbk,\Psi}$ and $\widetilde{M}'_{\Bbbk,\Psi'}$, respectively. As $\mu(\mathfrak{c}) = \mu(\mathfrak{c}') = \chi$, Proposition 4.4 shows that $\mathfrak{c}' = \gamma_0(\mathfrak{c})$ for some $\gamma_0 \in \Gamma$. On the other hand, arguing as in 4.6 it is straightforward to see that $\mathfrak{c}, \mathfrak{c}' \in \operatorname{Az}(\bar{R}_{\Bbbk})$. From this it follows that $\widetilde{M'_{\Bbbk,\,\Psi'}} \cong {}^{\gamma_0}\widetilde{M_{\Bbbk,\,\Psi}}$ as $\bar{R}_{\Bbbk}$-modules and hence as $U_\chi(\mathfrak{g}_{\Bbbk})$-modules. But then $\mathrm{Wh}_\chi\,\widetilde{M'_{\Bbbk,\,\Psi'}} \cong \mathrm{Wh}_\chi\,{}^{\gamma_0}\widetilde{M_{\Bbbk,\,\Psi}}$ as $U(\mathfrak{g}_{\Bbbk}, e)$-modules. In view of the Morita equivalence mentioned in 2.5 this implies that $M'_{\Bbbk} \cong {}^{\gamma_0}M_{\Bbbk}$ as $U(\mathfrak{g}_{\Bbbk}, e)$-modules (one should keep in mind here that γ_0^{-1} acts on $\widetilde{M_{\Bbbk,\Psi}}$ and maps the subspace $\mathrm{Wh}_\chi\,{}^{\gamma_0}\widetilde{M_{\Bbbk,\,\Psi}}$ isomorphically onto the subspace of all common $\mathfrak{m}_{\Bbbk}$-eigenvectors of $\widetilde{M_{\Bbbk,\,\Psi}}$).

We have reached a contradiction thereby showing that $M' \cong {}^{\gamma}M$ for some $\gamma \in \Gamma$.

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