

VECTOR BUNDLES ON ELLIPTIC CURVES AND FACTORS OF AUTOMORPHY

OLEKSANDR IENA

ABSTRACT. We translate the Atiyah's results on classification of vector bundles on elliptic curves to the language of factors of automorphy.

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1. INTRODUCTION

1.1. **Motivation.** The problem of classification of vector bundles over an elliptic curve was considered and completely solved by Atiyah in [1].

For a group Γ acting on a complex manifold Y , an r -dimensional factor of automorphy is a holomorphic function $f : \Gamma \times Y \rightarrow \mathrm{GL}_r(\mathbb{C})$ satisfying $f(\lambda\mu, y) = f(\lambda, \mu y)f(\mu, y)$. Two factors of automorphy f and f' are equivalent if there exists a holomorphic function $h : Y \rightarrow \mathrm{GL}_r(\mathbb{C})$ such that $h(\lambda y)f(\lambda, y) = f'(\lambda, y)h(y)$.

Given a complex manifold X and the universal covering $Y \xrightarrow{p} X$, let Γ be the fundamental group of X acting naturally on Y by deck transformations. Then there is a one-to-one correspondence between equivalence classes of r -dimensional factors of automorphy and isomorphism classes of vector bundles on X with trivial pull-back along p . In particular, if Y does not possess any non-trivial vector bundles, one obtains a one-to-one correspondence between equivalence classes of r -dimensional factors of automorphy and isomorphism classes of vector bundles on X . In particular this is the case for complex tori.

Since it is known that one-dimensional complex tori correspond to elliptic curves and since the classification of holomorphic vector bundles on a projective variety over \mathbb{C} is equivalent to the classification of algebraic vector bundles (cf. [11]), it is possible to formulate the Atiyah's results in the language of factors of automorphy.

This paper is a shortened version of the diploma thesis [4] and aims to give an accessible reference to the proofs of some results definitely known to the experts but still unpublished or difficult to find. The statement of the main result of this manuscript, Theorem 5.23, coincides with the statement of Proposition 1 from [10], which was given without any proof.

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1.2. Structure of the paper. In Section 2 we establish a correspondence between vector bundles and factors of automorphy. Section 3 deals with properties of factors of automorphy, in particular we discuss a correspondence between operations on vector bundles and operations on factors of automorphy. From Section 4 on we restrict ourselves to the case of vector bundles on complex tori. It is shown in Theorem 4.10 that to define a vector bundle of rank r on a complex one-dimensional torus is the same as to fix a holomorphic function $\mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C})$. In Section 5 we first present in Theorem 5.13 a classification of indecomposable vector bundles of degree zero, using this we give then in Theorem 5.23 a complete classification of indecomposable vector bundles of fixed rank and degree in terms of factors of automorphy.

1.3. Notations and conventions. Following Atiyah's paper [1] we denote by $\mathcal{E}(r, d) = \mathcal{E}_X(r, d)$ the set of isomorphism classes of indecomposable vector bundles over X of rank r and degree d . For a vector bundle E we usually denote the corresponding locally free sheaf of its sections by \mathcal{E} . By **Vect** we denote the category of finite dimensional vector spaces. For a divisor D we denote by $[D]$ the corresponding line bundle.

2. CORRESPONDENCE BETWEEN VECTOR BUNDLES AND FACTORS OF AUTOMORPHY

Let X be a complex manifold and let $p : Y \rightarrow X$ be a covering of X . Let $\Gamma \subset \mathrm{Deck}(Y/X)$ be a subgroup in the group of deck transformations $\mathrm{Deck}(Y/X)$ such that for any two points y_1 and y_2 with $p(y_1) = p(y_2)$ there exists an element $\gamma \in \Gamma$ such that $\gamma(y_1) = y_2$. In other words, Γ acts transitively in each fiber. We call this property **(T)**.

Remark 2.1. Note that for any two points y_1 and y_2 there can be only one $\gamma \in \mathrm{Deck}(Y/X)$ with $\gamma(y_1) = y_2$ (see [2], Satz 4.8). Therefore, $\Gamma = \mathrm{Deck}(Y/X)$ and the property **(T)** simply means that $p : Y \rightarrow X$ is a normal (Galois) covering.

We have an action of Γ on Y :

$$\Gamma \times Y \rightarrow Y, \quad y \mapsto \gamma(y) =: \gamma y.$$

Definition 2.2. A holomorphic function $f : \Gamma \times Y \rightarrow \mathrm{GL}_r(\mathbb{C})$, $r \in \mathbb{N}$ is called an r -dimensional factor of automorphy if it satisfies the relation

$$f(\lambda\mu, y) = f(\lambda, \mu y)f(\mu, y).$$

Denote by $Z^1(\Gamma, r)$ the set of all r -dimensional factors of automorphy.

We introduce the relation \sim on $Z^1(\Gamma, r)$. We say that f is equivalent to f' if there exists a holomorphic function $h : Y \rightarrow \mathrm{GL}_r(\mathbb{C})$ such that

$$h(\lambda y)f(\lambda, y) = f'(\lambda, y)h(y).$$

We write in this case $f \sim f'$.

Claim. *The relation \sim is an equivalence relation on $Z^1(\Gamma, r)$.*

Proof. Straightforward verifications. □

We denote the set of equivalence classes of $Z^1(\Gamma, r)$ with respect to \sim by $H^1(\Gamma, r)$.

Consider $f \in Z^1(\Gamma, r)$ and a trivial vector bundle $Y \times \mathbb{C}^r \rightarrow Y$. Define a holomorphic action of Γ on $Y \times \mathbb{C}^r$:

$$\Gamma \times Y \times \mathbb{C}^r \rightarrow Y \times \mathbb{C}^r, \quad (\lambda, y, v) \mapsto (\lambda y, f(\lambda, y)v) =: \lambda(y, v).$$

Denote $E(f) = Y \times \mathbb{C}^r / \Gamma$ and note that for two equivalent points $(y, v) \sim_{\Gamma} (y', v')$ with respect to the action of Γ on $Y \times \mathbb{C}^r$ it follows that $p(y) = p(y')$. In fact, $(y, v) \sim_{\Gamma} (y', v')$ implies in particular that $y = \gamma y'$ for some $\gamma \in \Gamma$ and by the definition of deck transformations $p(y) = p(\gamma y') = p(y')$. Hence the projection $Y \times \mathbb{C}^r \rightarrow Y$ induces the map

$$\pi : E(f) \rightarrow X, \quad [y, v] \mapsto p(y).$$

We equip $E(f)$ with the quotient topology.

Theorem 2.3. *$E(f)$ inherits a complex structure from $Y \times \mathbb{C}^r$ and the map $\pi : E(f) \rightarrow X$ is a holomorphic vector bundle on X .*

Proof. First we prove that π is a topological vector bundle. Clearly π is a continuous map. Consider the commutative diagram

$$\begin{array}{ccc} Y \times \mathbb{C}^r & \longrightarrow & E(f) \\ \downarrow & & \downarrow \pi \\ Y & \xrightarrow{p} & X. \end{array}$$

Let x be a point of X . Since p is a covering, one can choose an open neighbourhood U of x such that its preimage is a disjoint union of open sets biholomorphic to U , i. e., $p^{-1}(U) = \bigsqcup_{i \in \mathcal{I}} V_i$, $p_i := p|_{V_i} : V_i \rightarrow U$ is a biholomorphism for each $i \in \mathcal{I}$. For each pair $(i, j) \in \mathcal{I} \times \mathcal{I}$ there exists a unique $\lambda_{ij} \in \Gamma$ such that $\lambda_{ij} p_j^{-1}(x) = p_i^{-1}(x)$ for all $x \in U$. This follows from the property **(T)**.

We have $\pi^{-1}(U) = ((\bigsqcup_{i \in \mathcal{I}} V_i) \times \mathbb{C}^r) / \Gamma$.

Choose some $i_U \in \mathcal{I}$. Consider the holomorphic map

$$\varphi'_U : \left(\bigsqcup_{i \in \mathcal{I}} V_i \right) \times \mathbb{C}^r \rightarrow U \times \mathbb{C}^r, \quad (y_i, v) \mapsto (p(y_i), f(\lambda_{i_U i}, y_i)v), \quad y_i \in V_i.$$

Suppose that $(y_i, v') \sim_{\Gamma} (y_j, v)$. This means

$$(y_i, v') = \lambda_{ij}(y_j, v) = (\lambda_{ij}y_j, f(\lambda_{ij}, y_j)v).$$

Therefore,

$$\begin{aligned} \varphi'_U(y_i, v') &= (p(y_i), f(\lambda_{i_U i}, y_i)v') = (p(\lambda_{ij}y_j), f(\lambda_{i_U i}, \lambda_{ij}y_j)f(\lambda_{ij}, y_j)v) = \\ &= (p(y_j), f(\lambda_{i_U j}, y_j)v) = \varphi'_U(y_j, v). \end{aligned}$$

Thus φ'_U factorizes through $((\bigsqcup_{i \in \mathcal{I}} V_i) \times \mathbb{C}^r) / \Gamma$, i. e., the map

$$\varphi_U : \left(\left(\bigsqcup_{i \in \mathcal{I}} V_i \right) \times \mathbb{C}^r \right) / \Gamma \rightarrow U \times \mathbb{C}^r, \quad [(y_i, v)] \mapsto (p(y_i), f(\lambda_{i_U i}, y_i)v), \quad y_i \in V_i$$

is well-defined and continuous. We claim that φ_U is bijective.

Suppose $\varphi_U([(y_i, v')]) = \varphi_U([(y_j, v)])$, where $y_i \in V_i$, $y_j \in V_j$. By definition this is equivalent to $(p(y_i), f(\lambda_{i_U i}, y_i)v') = (p(y_j), f(\lambda_{i_U j}, y_j)v)$, which means $y_i = \lambda_{ij}y_j$ and

$$\begin{aligned} f(\lambda_{i_U i}, \lambda_{ij}y_j)v' &= f(\lambda_{i_U i}, y_i)v' = f(\lambda_{i_U j}, y_j)v = \\ &= f(\lambda_{i_U i}\lambda_{ij}, y_j)v = f(\lambda_{i_U i}, \lambda_{ij}y_j)f(\lambda_{ij}, y_j)v. \end{aligned}$$

We conclude $v' = f(\lambda_{ij}, y_j)v$ and $[(y_i, v')] = [(y_j, v)]$, which means injectivity of φ_U .

At the same time for each element $(y, v) \in U \times \mathbb{C}^r$ one has

$$\begin{aligned} \varphi_U([(p_i^{-1}(y), f(\lambda_{i_U i}, p_i^{-1}(y))^{-1}v)]) &= \\ (pp_i^{-1}(y), f(\lambda_{i_U i}, p_i^{-1}(y))f(\lambda_{i_U i}, p_i^{-1}(y))^{-1}v) &= (y, v), \end{aligned}$$

i. e., φ_U is surjective and we obtain that φ_U is a bijective map.

This means, that φ_U is a trivialization for U and that $\pi : E(f) \rightarrow X$ is a (continuous) vector bundle. If U and V are two neighbourhoods of X defined as above for which $E(f)|_U, E(f)|_V$ are trivial, then the corresponding transition function is

$$\varphi_U \varphi_V^{-1} : (U \cap V) \times \mathbb{C}^r \rightarrow (U \cap V) \times \mathbb{C}^r, \quad (x, v) \mapsto (x, g_{UV}(x)v),$$

where $g_{UV} : U \cap V \rightarrow \mathrm{GL}_r(\mathbb{C})$ is a cocycle defining $E(f)$. But from the construction of φ_U it follows that

$$g_{UV}(x) = f(\lambda_{i_U i_V}, p_{i_V}^{-1}(x)).$$

Therefore, g_{UV} is a holomorphic map, hence $\varphi_U \varphi_V^{-1}$ is also a holomorphic map. Thus the maps φ_U give $E(f)$ a complex structure. Since π is locally a projection, one sees that π is a holomorphic map. \square

Remark 2.4. Note that $p^*E(f)$ is isomorphic to $Y \times \mathbb{C}^r$. An isomorphism can be given by the map

$$p^*E(f) \rightarrow Y \times \mathbb{C}^r, \quad (y, [\tilde{y}, v]) \mapsto (y, f(\lambda, \tilde{y})v), \quad \lambda \tilde{y} = y.$$

Now we have the map from $Z^1(\Gamma, r)$ to the set $K_r = \{[E] \mid p^*(E) \simeq Y \times \mathbb{C}^r\}$ of isomorphism classes of vector bundles of rank r over X with trivial pull back with respect to p .

$$\phi' : Z^1(\Gamma, r) \rightarrow K_r; \quad f \mapsto [E(f)].$$

Theorem 2.5. *Let K_r denote the set of isomorphism classes of vector bundles of rank r on X with trivial pull back with respect to p . Then the map*

$$H^1(\Gamma, r) \rightarrow K_r, \quad [f] \mapsto [E(f)]$$

is a bijection.

Proof. This proof generalizes the proof from [5] given only for line bundles.

Consider the map $\phi' : Z^1(\Gamma, r) \rightarrow K_r$ and let f and f' be two equivalent r -dimensional factors of automorphy. It means that there exists a holomorphic function $h : Y \rightarrow \mathrm{GL}_r(\mathbb{C})$ such that

$$f'(\lambda, y) = h(\lambda y) f(\lambda, y) h(y)^{-1}.$$

Therefore, for two neighbourhoods U, V constructed as above we have the following relation for cocycles corresponding to f and f' .

$$\begin{aligned} g'_{UV}(x) &= f'(\lambda_{UV}, p_{i_V}^{-1}(x)) = h(\lambda_{UV} p_{i_V}^{-1}(x)) f(\lambda_{UV}, p_{i_V}^{-1}(x)) h(p_{i_V}^{-1}(x))^{-1} = \\ &= h(p_{i_U}^{-1}(x)) g_{UV}(x) h(p_{i_V}^{-1}(x))^{-1} = h_U(x) g_{UV}(x) h_V(x)^{-1}, \end{aligned}$$

where $\lambda_{UV} = \lambda_{i_U i_V}$, $h_U(x) = h(p_{i_U}^{-1}(x))$ and $h_V(x) = h(p_{i_V}^{-1}(x))$. We obtained

$$g'_{UV} = h_U g_{UV} h_V^{-1},$$

which is exactly the condition for two cocycles to define isomorphic vector bundles. Therefore, $E(f) \simeq E(f')$ and it means that ϕ' factorizes through $H^1(\Gamma, r)$, i. e., the map

$$\phi : H^1(\Gamma, r) \rightarrow K_r; \quad [f] \mapsto [E(f)]$$

is well-defined.

It remains to construct the inverse map. Suppose $E \in K_r$, in other words $p^*(E)$ is the trivial bundle of rank r over Y . Let $\alpha : p^*E \rightarrow Y \times \mathbb{C}^r$ be a trivialization. The action of Γ on Y induces a holomorphic action of Γ on p^*E :

$$\lambda(y, e) := (\lambda y, e) \text{ for } (y, e) \in p^*E = Y \times_X E.$$

Via α we get for every $\lambda \in \Gamma$ an automorphism ψ_λ of the trivial bundle $Y \times \mathbb{C}^r$. Clearly ψ_λ should be of the form

$$\psi_\lambda(y, v) = (\lambda y, f(\lambda, y)v),$$

where $f : \Gamma \times Y \rightarrow \mathrm{GL}_r(\mathbb{C})$ is a holomorphic map. The equation for the action $\psi_{\lambda\mu} = \psi_\lambda\psi_\mu$ implies that f should be an r -dimensional factor of automorphy.

Suppose α' is another trivialization of p^*E . Then there exists a holomorphic map $h : Y \rightarrow \mathrm{GL}_r(\mathbb{C})$ such that $\alpha'\alpha^{-1}(y, v) = (y, h(y)v)$. Let f' be a factor of automorphy corresponding to α' . From

$$\begin{aligned} (\lambda y, f'(\lambda, y)v) &= \psi'_\lambda(y, v) = \alpha'\lambda\alpha'^{-1}(y, v) = \alpha'\alpha^{-1}\alpha\lambda\alpha^{-1}\alpha\alpha'^{-1}(y, v) = \\ &= \alpha'\alpha^{-1}\psi_\lambda(\alpha'\alpha^{-1})^{-1}(y, v) = \alpha'\alpha^{-1}\psi_\lambda(y, h(y)^{-1}v) = \\ &= \alpha'\alpha^{-1}(\lambda y, f(\lambda, y)h(y)^{-1}v) = (\lambda y, h(\lambda y)f(\lambda, y)h(y)^{-1}v), \end{aligned}$$

we obtain $f'(\lambda, y) = h(\lambda y)f(\lambda, y)h(y)^{-1}$. The last means that $[f] = [f']$, in other words, the class of a factor of automorphy in $H^1(\Gamma, r)$ does not depend on the trivialization and we get a map $K_r \rightarrow H^1(\Gamma, r)$. This map is the inverse of ϕ . \square

Let X be a connected complex manifold, let $p : \tilde{X} \rightarrow X$ be a universal covering of X , and let $\Gamma = \mathrm{Deck}(Y/X)$. Since universal coverings are normal coverings, Γ satisfies the property **(T)** (see [2, Satz 5.6]). Moreover, Γ is isomorphic to the fundamental group $\pi_1(X)$ of X (see [2, Satz 5.6]). An isomorphism is given as follows.

Fix $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ with $p(\tilde{x}_0) = x_0$. We define a map

$$\Phi : \mathrm{Deck}(\tilde{X}/X) \rightarrow \pi_1(X, x_0)$$

as follows. Let $\sigma \in \mathrm{Deck}(\tilde{X}/X)$ and $v : [0; 1] \rightarrow \tilde{X}$ be a curve with $v(0) = \tilde{x}_0$ and $v(1) = \sigma(\tilde{x}_0)$. Then a curve

$$pv : [0; 1] \rightarrow X, \quad t \mapsto pv(t)$$

is such that $pv(0) = pv(1) = x_0$. Define $\Phi(\sigma) := [pv]$, where $[pv]$ denotes a homotopy class of pv . The map Φ is well defined and is an isomorphism of groups.

So we can identify Γ with $\pi_1(X)$. Therefore, we have an action of $\pi_1(X)$ on \tilde{X} by deck transformations.

Consider an element $[w] \in \pi_1(X, x_0)$ represented by a path $w : [0; 1] \rightarrow X$. We denote $\sigma = \Phi^{-1}([w])$. Consider any $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = w(0) = w(1)$, then the path w can be uniquely lifted to the path

$$v : [0; 1] \rightarrow \tilde{X}$$

with $v(0) = \tilde{x}_0$ (see [2], Satz 4.14). Denote $\tilde{x}_1 = v(1)$. Then σ is a unique element in $\mathrm{Deck}(\tilde{X}/X)$ such that $\sigma(\tilde{x}_0) = \tilde{x}_1$. This gives a description of the action of $\pi_1(X, x_0)$ on \tilde{X} .

Now we have a corollary to Theorem 2.5.

Corollary 2.6. *Let X be a connected complex manifold, let $p : \tilde{X} \rightarrow X$ be the universal covering, let Γ be the fundamental group of X naturally acting on \tilde{X} by deck transformations. As above, $H^1(\Gamma, r)$ denotes the set of equivalence classes of r -dimensional factors of automorphy*

$$\Gamma \times \tilde{X} \rightarrow \mathrm{GL}_r(\mathbb{C}).$$

Then there is a bijection

$$H^1(\Gamma, r) \rightarrow K_r, \quad [f] \mapsto E(f),$$

where K_r denotes the set of isomorphism classes of vector bundles of rank r on X with trivial pull back with respect to p .

3. PROPERTIES OF FACTORS OF AUTOMORPHY

Definition 3.1. Let $f : \Gamma \times Y \rightarrow \mathrm{GL}_r(\mathbb{C})$ be an r -dimensional factor of automorphy. A holomorphic function $s : Y \rightarrow \mathbb{C}^r$ is called an f -theta function if it satisfies

$$s(\gamma y) = f(\gamma, y)s(y) \text{ for all } \gamma \in \Gamma, y \in Y.$$

Theorem 3.2. Let $f : \Gamma \times Y \rightarrow \mathrm{GL}_r(\mathbb{C})$ be an r -dimensional factor of automorphy. Then there is a one-to-one correspondence between sections of $E(f)$ and f -theta functions.

Proof. Let $\{V_i\}_{i \in \mathcal{I}}$ be a covering of Y such that p restricted to V_i is a homeomorphism. Denote $\varphi_i := (p|_{V_i})^{-1}$, $U_i := p(V_i)$. Then $\{U_i\}$ is a covering of X such that $E(f)$ is trivial over each U_i .

Consider a section of $E(f)$ given by functions $s_i : U_i \rightarrow \mathbb{C}^r$ satisfying

$$s_i(x) = g_{ij}(x)s_j(x) \text{ for } x \in U_i \cap U_j,$$

where

$$g_{ij}(x) = f(\lambda_{U_i U_j}, \varphi_j(x)), \quad x \in U_i \cap U_j$$

is a cocycle defining $E(f)$ (see the proof of Theorem 2.5). Define $s : Y \rightarrow \mathbb{C}^r$ by $s(\varphi_i(x)) := s_i(x)$. To prove that this is well-defined we need to show that $s_i(x) = s_j(x)$ when $\varphi_i(x) = \varphi_j(x)$. But since $\varphi_i(x) = \varphi_j(x)$ we obtain $\lambda_{U_i U_j} = 1$. Therefore,

$$s_i(x) = g_{ij}(x)s_j(x) = f(\lambda_{U_i U_j}, \varphi_j(x))s_j(x) = f(1, \varphi_j(x))s_j(x) = s_j(x).$$

For any $\gamma \in \Gamma$ for any point $y \in Y$ take $i, j \in \mathcal{I}$ and $x \in X$ such that $y = \varphi_j(x)$ and $\gamma y = \gamma \varphi_j(x) = \varphi_i(x)$. Thus $\gamma = \lambda_{U_i U_j}$ and one obtains

$$\begin{aligned} s(\gamma y) &= s(\varphi_i(x)) = s_i(x) = g_{ij}(x)s_j(x) = \\ &= f(\lambda_{U_i U_j}, \varphi_j(x))s_j(x) = f(\gamma, y)s(\varphi_j(x)) = f(\gamma, z)s(y). \end{aligned}$$

In other words, s is an f -theta function.

Vice versa, let $s : Y \rightarrow \mathbb{C}^r$ be an f -theta function. We define $s_i : U_i \rightarrow \mathbb{C}^r$ by $s_i(x) := s(\varphi_i(x))$. Then for a point $x \in U_i \cap U_j$ we have

$$\begin{aligned} s_i(x) &= s(\varphi_i(x)) = s(\lambda_{U_i U_j} \varphi_j(x)) = \\ &= f(\lambda_{U_i U_j}, \varphi_j(x))s(\varphi_j(x)) = g_{ij}(x)s_j(x), \end{aligned}$$

which means that the functions s_i define a section of $E(f)$. The described correspondences are clearly inverse to each other. \square

The following statement will be useful in the sequel.

Theorem 3.3. Let $f(\lambda, y) = \begin{pmatrix} f'(\lambda, y) & \tilde{f}(\lambda, y) \\ 0 & f''(\lambda, y) \end{pmatrix}$ be an $r' + r''$ -dimensional factor of automorphy, where $f'(\lambda, y) \in \mathrm{GL}_{r'}(\mathbb{C})$, $f''(\lambda, y) \in \mathrm{GL}_{r''}(\mathbb{C})$. Then

(a) $f' : \Gamma \times Y \rightarrow \mathrm{GL}_{r'}(\mathbb{C})$ and $f'' : \Gamma \times Y \rightarrow \mathrm{GL}_{r''}(\mathbb{C})$ are r' and r'' -dimensional factors of automorphy respectively;

(b) there is an extension of vector bundles

$$0 \longrightarrow E(f') \xrightarrow{i} E(f) \xrightarrow{\pi} E(f'') \longrightarrow 0.$$

Proof. The statement of (a) follows from straightforward verification. To prove (b) we define maps i and π as follows.

$$\begin{aligned} i : E(f') &\rightarrow E(f), & [y, v] &\mapsto [y, \begin{pmatrix} v \\ 0 \end{pmatrix}], & v &\in \mathbb{C}^{r'}, & \begin{pmatrix} v \\ 0 \end{pmatrix} &\in \mathbb{C}^{r'+r''} \\ \pi : E(f) &\rightarrow E(f''), & [y, \begin{pmatrix} v \\ w \end{pmatrix}] &\rightarrow [y, w], & v &\in \mathbb{C}^{r'}, & w &\in \mathbb{C}^{r''} \end{aligned}$$

Since $[\lambda y, f'(\lambda, y)v]$ is mapped via i to $[\lambda y, \begin{pmatrix} f'(\lambda, y)v \\ 0 \end{pmatrix}] = [\lambda y, f(\lambda, y) \begin{pmatrix} v \\ 0 \end{pmatrix}]$, one concludes that i is well-defined. Analogously, since $[\lambda y, f''(\lambda, y)w] = [y, w]$ one sees that π is well-defined. Using the charts from the proof of (2.3) one easily sees that the defined maps are holomorphic.

Notice that i and π respect fibers, i is injective and π is surjective in each fiber. This proves the statement. \square

Now we recall one standard construction from linear algebra. Let A be an $m \times n$ matrix. It represents some morphism $\mathbb{C}^n \rightarrow \mathbb{C}^m$ for fixed standard bases in \mathbb{C}^n and \mathbb{C}^m .

Let $\mathcal{F} : \mathbf{Vect}^p \rightarrow \mathbf{Vect}$ be a covariant functor. Let A_1, \dots, A_p be the matrices representing morphisms $\mathbb{C}_1^n \xrightarrow{f_1} \mathbb{C}_1^m, \dots, \mathbb{C}_p^n \xrightarrow{f_p} \mathbb{C}_p^m$ in standard bases.

If for each object $\mathcal{F}(\mathbb{C}^m)$ we fix some basis, then the matrix corresponding to the morphism $\mathcal{F}(f_1, \dots, f_p)$ is denoted by $\mathcal{F}(A_1, \dots, A_p)$. Clearly it satisfies

$$\mathcal{F}(A_1 B_1, \dots, A_p B_p) = \mathcal{F}(A_1, \dots, A_p) \mathcal{F}(B_1, \dots, B_p).$$

In this way $A \otimes B$, $S^q(A)$, $\Lambda^q(A)$ can be defined. As \mathcal{F} one considers the functors

$$- \otimes - : \mathbf{Vect}^2 \rightarrow \mathbf{Vect}, \quad S^n : \mathbf{Vect} \rightarrow \mathbf{Vect}, \quad \Lambda : \mathbf{Vect} \rightarrow \mathbf{Vect}$$

respectively.

Recall that every holomorphic functor $\mathcal{F} : \mathbf{Vect}^n \rightarrow \mathbf{Vect}$ can be canonically extended to the category of vector bundles of finite rank over X . By abuse of notation we will denote the extended functor by \mathcal{F} as well.

Theorem 3.4. *Let $\mathcal{F} : \mathbf{Vect}^n \rightarrow \mathbf{Vect}$ be a covariant holomorphic functor. Let f_1, \dots, f_n be r_i -dimensional factors of automorphy. Then $f = \mathcal{F}(f_1, \dots, f_n)$ is a factor of automorphy defining $\mathcal{F}(E(f_1), \dots, E(f_n))$.*

Proof. One clearly has

$$\begin{aligned} \mathcal{F}(f_1, \dots, f_n)(\lambda \mu, y) &= \mathcal{F}(f_1(\lambda \mu, y), \dots, f_n(\lambda \mu, y)) = \\ &= \mathcal{F}(f_1(\lambda, \mu y) f_1(\mu, y), \dots, f_n(\lambda, \mu y) f_n(\mu, y)) = \\ &= \mathcal{F}(f_1(\lambda, \mu y), \dots, f_n(\lambda, \mu y)) \mathcal{F}(f_1(\mu, y), \dots, f_n(\mu, y)) = \\ &= \mathcal{F}(f_1, \dots, f_n)(\lambda, \mu y) \mathcal{F}(f_1, \dots, f_n)(\mu, y). \end{aligned}$$

Since (f_1, \dots, f_n) represents an isomorphism in \mathbf{Vect}^n , $\mathcal{F}(f_1, \dots, f_n)$ also represents an isomorphism $\mathbb{C}^r \rightarrow \mathbb{C}^r$ for some $r \in \mathbb{N}$. Therefore, f is an r -dimensional factor of automorphy.

Since $f = \mathcal{F}(f_1, \dots, f_n)$, the same holds for cocycles defining the corresponding vector bundles, i. e., $g_{U_1 U_2} = \mathcal{F}(g_{1U_1 U_2}, \dots, g_{nU_1 U_2})$, where $g_{iU_1 U_2}$ is a cocycle defining $E(f_i)$. This is exactly the condition $E(f) = \mathcal{F}(E(f_1), \dots, E(f_n))$. \square

For example for $\mathcal{F} = - \otimes - : \mathbf{Vect}^2 \rightarrow \mathbf{Vect}$ we get the following obvious corollary.

Corollary 3.5. *Let $f' : \Gamma \times Y \rightarrow \mathrm{GL}_{r'}(\mathbb{C})$ and $f'' : \Gamma \times Y \rightarrow \mathrm{GL}_{r''}(\mathbb{C})$ be two factors of automorphy. Then $f = f' \otimes f'' : \Gamma \times Y \rightarrow \mathrm{GL}_{r'r''}(\mathbb{C})$ is also a factor of automorphy. Moreover, $E(f) \simeq E(f') \otimes E(f'')$.*

It is not essential that the functor in Theorem 3.4 is covariant. The following theorem is a generalization of Theorem 3.4.

Theorem 3.6. *Let $\mathcal{F} : \mathrm{Vect}^n \rightarrow \mathrm{Vect}$ be a holomorphic functor. Let \mathcal{F} be covariant in k first variables and contravariant in $n - k$ last variables. Let f_1, \dots, f_n be r_i -dimensional factors of automorphy. Then $f = \mathcal{F}(f_1, \dots, f_k, f_{k+1}^{-1}, \dots, f_n^{-1})$ is a factor of automorphy defining $\mathcal{F}(E(f_1), \dots, E(f_n))$.*

Proof. The proof is analogous to the proof of Theorem 3.4. □

4. VECTOR BUNDLES ON COMPLEX TORI

4.1. One dimensional complex tori. Let X be a complex torus, i. e., $X = \mathbb{C}/\Gamma$, $\Gamma = \mathbb{Z}\tau + \mathbb{Z}$, $\mathrm{Im}\tau > 0$. Then the universal covering is $\tilde{X} = \mathbb{C}$, namely

$$\mathrm{pr} : \mathbb{C} \rightarrow \mathbb{C}/\Gamma, \quad x \mapsto [x].$$

We have an action of Γ on \mathbb{C} :

$$\Gamma \times \mathbb{C} \rightarrow \mathbb{C}, \quad (\gamma, y) \mapsto \gamma + y.$$

Clearly Γ acts on \mathbb{C} by deck transformations and satisfies the property **(T)**.

Since \mathbb{C} is a non-compact Riemann surface, by [2, Theorem 30.4, p. 204], there are only trivial bundles on \mathbb{C} . Therefore, we have a one-to-one correspondence between classes of isomorphism of vector bundles of rank r on X and equivalence classes of factors of automorphy

$$f : \Gamma \times \mathbb{C} \rightarrow \mathrm{GL}_r(\mathbb{C}).$$

As usually, V_a denotes the standard parallelogram constructed at point a , U_a is the image of V_a under the projection, $\varphi_a : U_a \rightarrow V_a$ is the local inverse of the projection.

Remark 4.1. Let f be an r -dimensional factor of automorphy. Then

$$g_{ab}(x) = f(\varphi_a(x) - \varphi_b(x), \varphi_b(x))$$

is a cocycle defining $E(f)$. This follows from the construction of the cocycle in the proof of Theorem 2.5.

Example. There are factors of automorphy corresponding to classical theta functions. For any theta-characteristic $\xi = a\tau + b$, where $a, b \in \mathbb{R}$, there is a holomorphic function $\theta_\xi : \mathbb{C} \rightarrow \mathbb{C}$ defined by

$$\theta_\xi(z) = \sum_{n \in \mathbb{Z}} \exp(\pi i(n+a)^2\tau) \exp(2\pi i(n+a)(z+b)),$$

which satisfies

$$\theta_\xi(\gamma + z) = \exp(2\pi ia\gamma - \pi ip^2\tau - 2\pi ip(z + \xi)) \theta_\xi(z) = e_\xi(\gamma, z) \theta_\xi(z),$$

where $\gamma = p\tau + q$ and $e_\xi(\gamma, z) = \exp(2\pi ia\gamma - \pi ip^2\tau - 2\pi ip(z + \xi))$. Since

$$e_\xi(\gamma_1 + \gamma_2, z) = e_\xi(\gamma_1, \gamma_2 + z) e_\xi(\gamma_2, z),$$

we conclude that $e_\xi(\gamma, z)$ is a factor of automorphy.

By Theorem 3.2 $\theta_\xi(z)$ defines a section of $E(e_\xi(\gamma, z))$.

For more information on classical theta functions see [6, 7, 8].

Theorem 4.2. $\deg E(e_\xi) = 1$.

Proof. We know that sections of $E(e_\xi)$ correspond to e_ξ - theta functions. The classical e_ξ -theta function $\theta_\xi(z)$ defines a section s_ξ of $E(e_\xi)$. Since θ_ξ has only simple zeros and the set of zeros of $\theta_\xi(z)$ is $\frac{1}{2} + \frac{\tau}{2} + \xi + \Gamma$, we conclude that s_ξ has exactly one zero at point $p = [\frac{1}{2} + \frac{\tau}{2} + \xi] \in X$. Hence by [3, p. 136] we get $E(e_\xi) \simeq [p]$ and thus $\deg E(e_\xi) = 1$. \square

Theorem 4.3. *Let ξ and η be two theta-characteristics. Then*

$$E(e_\xi) \simeq t_{[\eta-\xi]}^* E(e_\eta),$$

where $t_{[\eta-\xi]} : X \rightarrow X$, $x \mapsto x + [\eta - \xi]$ is the translation by $[\eta - \xi]$.

Proof. As in the proof of Theorem 4.2 $E(e_\xi) \simeq [p]$ and $E(e_\eta) = [q]$ for $p = [\frac{1}{2} + \frac{\tau}{2} + \xi]$ and $q = [\frac{1}{2} + \frac{\tau}{2} + \eta]$. Since $t_{[\eta-\xi]}p = q$, we get

$$E(e_\xi) \simeq [p] \simeq t_{[\eta-\xi]}^*[q] \simeq t_{[\eta-\xi]}^* E(e_\eta),$$

which completes the proof. \square

Now we are going to investigate the extensions of the type

$$0 \rightarrow X \times \mathbb{C} \rightarrow E \rightarrow X \times \mathbb{C} \rightarrow 0.$$

In this case the transition functions are given by matrices of the type

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

and E is isomorphic to $E(f)$ for some factor of automorphy f of the form $f(\lambda, \tilde{x}) = \begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix}$. Note that the condition for f to be a factor of automorphy in this case is equivalent to the condition

$$\mu(\lambda + \lambda', \tilde{x}) = \mu(\lambda, \lambda' + \tilde{x}) + \mu(\lambda', \tilde{x}),$$

where we use the additive notation for the group operation since Γ is commutative.

Theorem 4.4. *f defines trivial bundle if and only if $\mu(\lambda, \tilde{x}) = \xi(\lambda\tilde{x}) - \xi(\tilde{x})$ for some holomorphic function $\xi : \mathbb{C} \rightarrow \mathbb{C}$.*

Proof. We know that E is trivial if and only if $h(\lambda\tilde{x}) = f(\lambda, \tilde{x})h(\tilde{x})$ for some holomorphic function $h : \tilde{X} \rightarrow \text{GL}_2(\mathbb{C})$. Let $h = \begin{pmatrix} a(\tilde{x}) & b(\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix}$, then the last condition is

$$\begin{pmatrix} a(\lambda\tilde{x}) & b(\lambda\tilde{x}) \\ c(\lambda\tilde{x}) & d(\lambda\tilde{x}) \end{pmatrix} = \begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a(\tilde{x}) & b(\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix} = \begin{pmatrix} a(\tilde{x}) + c(\tilde{x})\mu(\lambda, \tilde{x}) & b(\tilde{x}) + d(\tilde{x})\mu(\lambda, \tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix}.$$

In particular it means $c(\lambda\tilde{x}) = c(\tilde{x})$ and $d(\lambda\tilde{x}) = d(\tilde{x})$, i. e., c and d are doubly periodic functions on $\tilde{X} = \mathbb{C}$, so they should be constant, i. e., $c(\lambda, \tilde{x}) = c \in \mathbb{C}$, $d(\lambda, \tilde{x}) = d \in \mathbb{C}$.

Now we have

$$\begin{aligned} a(\tilde{x}) + c\mu(\lambda, \tilde{x}) &= a(\lambda\tilde{x}) \\ b(\tilde{x}) + d\mu(\lambda, \tilde{x}) &= b(\lambda\tilde{x}) \end{aligned}$$

which implies

$$\begin{aligned} c\mu(\lambda, \tilde{x}) &= a(\lambda\tilde{x}) - a(\tilde{x}) \\ d\mu(\lambda, \tilde{x}) &= b(\lambda\tilde{x}) - b(\tilde{x}). \end{aligned}$$

Since $\det h(\tilde{x}) \neq 0$ for all $\tilde{x} \in \tilde{X} = \mathbb{C}$ one of the numbers c and d is not equal to zero. Therefore, one concludes that $\mu(\lambda, \tilde{x}) = \xi(\lambda\tilde{x}) - \xi(\tilde{x})$ for some holomorphic function $\xi : \tilde{X} = \mathbb{C} \rightarrow \mathbb{C}$.

Now suppose $\mu(\lambda, \tilde{x}) = \xi(\lambda\tilde{x}) - \xi(\tilde{x})$ for some holomorphic function $\xi : \mathbb{C} \rightarrow \mathbb{C}$. Clearly for $h(\tilde{x}) = \begin{pmatrix} 1 & \xi(\tilde{x}) \\ 0 & 1 \end{pmatrix}$ one has that $\det h(\tilde{x}) = 1 \neq 0$ and

$$\begin{aligned} f(\lambda, \tilde{x})h(\tilde{x}) &= \begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \xi(\tilde{x}) \\ 0 & 1 \end{pmatrix} = \\ &= \begin{pmatrix} 1 & \xi(\tilde{x}) + \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \xi(\lambda\tilde{x}) \\ 0 & 1 \end{pmatrix} = h(\lambda\tilde{x}). \end{aligned}$$

We have shown, that f defines the trivial bundle. This proves the statement of the theorem. \square

Theorem 4.5. *Two factors of automorphy $f(\lambda, \tilde{x}) = \begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix}$ and $f'(\lambda, \tilde{x}) = \begin{pmatrix} 1 & \nu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix}$ defining non-trivial bundles are equivalent if and only if*

$$\mu(\lambda, \tilde{x}) - k\nu(\lambda, \tilde{x}) = \xi(\lambda\tilde{x}) - \xi(\tilde{x}), \quad k \in \mathbb{C}, \quad k \neq 0$$

for some holomorphic function $\xi : \mathbb{C} = \tilde{X} \rightarrow \mathbb{C}$.

Proof. Suppose the factors of automorphy $f(\lambda, \tilde{x}) = \begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix}$ and $f'(\lambda, \tilde{x}) = \begin{pmatrix} 1 & \nu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix}$ are equivalent. This means that $f(\lambda, \tilde{x})h(\tilde{x}) = h(\lambda\tilde{x})f(\lambda, \tilde{x})$ for some holomorphic function $h : \mathbb{C} = \tilde{X} \rightarrow \text{GL}_2(\mathbb{C})$. Let

$$h(\tilde{x}) = \begin{pmatrix} a(\tilde{x}) & b(\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix}.$$

The condition for equivalence of f and f' can be rewritten in the following way:

$$\begin{aligned} \begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a(\tilde{x}) & b(\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix} &= \begin{pmatrix} a(\lambda\tilde{x}) & b(\lambda\tilde{x}) \\ c(\lambda\tilde{x}) & d(\lambda\tilde{x}) \end{pmatrix} \begin{pmatrix} 1 & \nu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} a(\tilde{x}) + c(\tilde{x})\mu(\lambda, \tilde{x}) & b(\tilde{x}) + d(\tilde{x})\mu(\lambda, \tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix} &= \begin{pmatrix} a(\lambda\tilde{x}) & a(\lambda\tilde{x})\nu(\lambda, \tilde{x}) + b(\lambda\tilde{x}) \\ c(\lambda\tilde{x}) & c(\lambda\tilde{x})\nu(\lambda, \tilde{x}) + d(\lambda\tilde{x}) \end{pmatrix}. \end{aligned}$$

This leads to the system of equations

$$\begin{cases} a(\tilde{x}) + c(\tilde{x})\mu(\lambda, \tilde{x}) = a(\lambda\tilde{x}) \\ b(\tilde{x}) + d(\tilde{x})\mu(\lambda, \tilde{x}) = a(\lambda\tilde{x})\nu(\lambda, \tilde{x}) + b(\lambda\tilde{x}) \\ c(\tilde{x}) = c(\lambda\tilde{x}) \\ d(\tilde{x}) = c(\lambda\tilde{x})\nu(\lambda, \tilde{x}) + d(\lambda\tilde{x}). \end{cases}$$

The third equation means that c is a double periodic function. Therefore, c should be a constant function.

If $c \neq 0$ from the first and the last equations using Theorem 4.4 one concludes that f and f' define the trivial bundle.

In the case $c = 0$ one has

$$\begin{cases} a(\tilde{x}) = a(\lambda\tilde{x}) \\ b(\tilde{x}) + d(\tilde{x})\mu(\lambda, \tilde{x}) = a(\lambda\tilde{x})\nu(\lambda, \tilde{x}) + b(\lambda\tilde{x}) \\ d(\tilde{x}) = d(\lambda\tilde{x}), \end{cases}$$

i. e., as above, a and d are constant and both not equal to zero since $\det(h) \neq 0$. Finally one concludes that

$$(1) \quad d\mu(\lambda, \tilde{x}) - a\nu(\lambda, \tilde{x}) = b(\lambda\tilde{x}) - b(\tilde{x}), \quad a, d \in \mathbb{C}, \quad ad \neq 0$$

Vice versa, if μ and ν satisfy (1) for $h(\tilde{x}) = \begin{pmatrix} a & b(\tilde{x}) \\ 0 & d \end{pmatrix}$ we have

$$\begin{aligned} f(\lambda, \tilde{x})h(\tilde{x}) &= \begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b(\tilde{x}) \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b(\tilde{x}) + d\mu(\lambda, \tilde{x}) \\ 0 & d \end{pmatrix} = \\ &= \begin{pmatrix} a & b(\lambda\tilde{x}) + a\nu(\lambda, \tilde{x}) \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b(\lambda\tilde{x}) \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & \nu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix} = h(\lambda\tilde{x})f(\lambda, \tilde{x}). \end{aligned}$$

This means that f and f' are equivalent. \square

4.2. Higher dimensional complex tori. One can also consider higher dimensional complex tori. Let $\Gamma \subset \mathbb{C}^g$ be a lattice,

$$\Gamma = \Gamma_1 \times \cdots \times \Gamma_g, \quad \Gamma_i = \mathbb{Z} + \mathbb{Z}\tau_i, \quad \text{Im } \tau > 0.$$

Then as for one dimensional complex tori we obtain that $X = \mathbb{C}^g/\Gamma$ is a complex manifold. Clearly the map

$$\mathbb{C}^g \rightarrow \mathbb{C}^g/\Gamma = X, \quad x \mapsto [x]$$

is the universal covering of X . Since all vector bundles on C^g are trivial, we obtain a one-to-one correspondence between equivalence classes of r -dimensional factors of automorphy

$$f : \Gamma \times \mathbb{C}^g \rightarrow \text{GL}_r(\mathbb{C})$$

and vector bundles of rank r on X .

Let $\Gamma = \mathbb{Z}^g + \Omega\mathbb{Z}^g$, where Ω is a symmetric complex $g \times g$ matrix with positive definite real part. Note that Ω is a generalization of τ from one dimensional case.

For any theta-characteristic $\xi = \Omega a + b$, where $a \in \mathbb{R}^g$, $b \in \mathbb{R}^g$ there is a holomorphic function $\theta_\xi : \mathbb{C}^g \rightarrow \mathbb{C}$ defined by

$$\theta_\xi(z) = \theta_b^a(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i(n+a)^t \Omega(n+a)\tau) \exp(2\pi i(n+a)^t \Omega(z+b)),$$

which satisfies

$$\theta_\xi(\gamma + z) = \exp(2\pi i a^t \gamma - \pi i p^t \Omega p - 2\pi i p^t (z + \xi)) \theta_\xi(z) = e_\xi(\gamma, z) \theta_\xi(z),$$

where $\gamma = \Omega p + q$ and $e_\xi(\gamma, z) = \exp(2\pi i a^t \gamma - \pi i p^t \Omega p - 2\pi i p^t (z + \xi))$. Since

$$e_\xi(\gamma_1 + \gamma_2, z) = e_\xi(\gamma_1, \gamma_2 + z) e_\xi(\gamma_2, z),$$

we conclude that $e_\xi(\gamma, z)$ is a factor of automorphy.

As above $\theta_\xi(z)$ defines a section of $E(e_\xi(\gamma, z))$.

For more detailed information on higher dimensional theta functions see [6, 7, 8].

4.3. Factors of automorphy depending only on the τ -direction of the lattice Γ . Here X is a complex torus, $X = \mathbb{C}/\Gamma$, $\Gamma = \mathbb{Z}\tau + \mathbb{Z}$, $\text{Im } \tau > 0$. Denote $q = e^{2\pi i\tau}$. Consider the canonical projection

$$\text{pr} : \mathbb{C}^* \rightarrow \mathbb{C}^* / \langle q \rangle, \quad u \mapsto [u] = u \langle q \rangle.$$

Clearly one can equip $\mathbb{C}^* / \langle q \rangle$ with the quotient topology. Therefore, there is a natural complex structure on $\mathbb{C}^* / \langle q \rangle$.

Consider the homomorphism

$$\mathbb{C} \xrightarrow{\text{exp}} \mathbb{C}^* \xrightarrow{\text{pr}} \mathbb{C}^* / \langle q \rangle, \quad z \mapsto e^{2\pi iz} \mapsto [e^{2\pi iz}].$$

It is clearly surjective. An element $z \in \mathbb{C}$ is in the kernel of this homomorphism if and only if $e^{2\pi iz} = q^k = e^{2\pi ik\tau}$ for some integer k . But this holds if and only if $z - k\tau \in \mathbb{Z}$ or, in other words, if $z \in \Gamma$. Therefore, the kernel of the map is exactly Γ , and we obtain an isomorphism of groups

$$\text{iso} : \mathbb{C}/\Gamma \rightarrow \mathbb{C}^* / \langle q \rangle = \mathbb{C}^* / \mathbb{Z}, \quad [z] \mapsto [e^{2\pi iz}].$$

Since the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\text{pr}} & \mathbb{C}/\Gamma \\ \text{exp} \downarrow & & \downarrow \text{iso} \\ \mathbb{C}^* & \xrightarrow{\text{pr}} & \mathbb{C}^* / \mathbb{Z} = \mathbb{C}^* / \langle q \rangle \end{array}$$

is commutative, we conclude that the complex structure on $\mathbb{C}^* / \langle q \rangle$ inherited from \mathbb{C}/Γ by the isomorphism iso coincides with the natural complex structure on $\mathbb{C}^* / \langle q \rangle$. Therefore, iso is an isomorphism of complex manifolds. Thus complex tori can be represented as $\mathbb{C}^* / \langle q \rangle$, where $q = e^{2\pi i\tau}$, $\tau \in \mathbb{C}$, $\text{Im } \tau > 0$.

So for any complex torus $X = \mathbb{C}^* / \langle q \rangle$ we have a natural surjective holomorphic map

$$\mathbb{C}^* \rightarrow \mathbb{C}^* / \langle q \rangle = X, \quad u \mapsto [u].$$

This map is moreover a covering of X . Consider the group \mathbb{Z} . It acts holomorphically on $X = \mathbb{C}^*$:

$$\mathbb{Z} \times \mathbb{C}^* \rightarrow \mathbb{C}^*, \quad (n, u) \mapsto q^n u.$$

Moreover, since $\text{pr}(q^n u) = \text{pr}(u)$, \mathbb{Z} is naturally identified with a subgroup in the group of deck transformations $\text{Deck}(X/\mathbb{C}^*)$. It is easy to see that \mathbb{Z} satisfies the property **(T)**. We obtain that there is a one-to-one correspondence between classes of isomorphism of vector bundles over X and classes of equivalence of factors of automorphy

$$f : \mathbb{Z} \times \mathbb{C} \rightarrow \text{GL}_r(\mathbb{C}).$$

Consider the following action of Γ on \mathbb{C}^* :

$$\Gamma \times \mathbb{C}^* \rightarrow \mathbb{C}^*; \quad (\lambda, u) \mapsto \lambda u =: e^{2\pi i\lambda} u$$

Let $A : \Gamma \times \mathbb{C}^* \rightarrow \text{GL}_r(\mathbb{C})$ be a holomorphic function satisfying

$$A(\lambda + \lambda', u) = A(\lambda, \lambda' u) A(\lambda', u) \quad (*)$$

for all $\lambda, \lambda' \in \Gamma$. We call such functions \mathbb{C}^* -factors of automorphy. Consider the map

$$\text{id}_\Gamma \times \text{exp} : \Gamma \times \mathbb{C} \rightarrow \Gamma \times \mathbb{C}^*, \quad (\lambda, x) \mapsto (\lambda, e^{2\pi ix})$$

Then the function

$$f_A = A \circ (\text{id}_\Gamma \times \text{exp}) : \Gamma \times \mathbb{C} \rightarrow \text{GL}_r(\mathbb{C})$$

is an r -dimensional factor of automorphy, because

$$\begin{aligned} f_A(\lambda + \lambda', x) &= A(\lambda + \lambda', e^{2\pi i x}) = A(\lambda, e^{2\pi i \lambda'} e^{2\pi i x}) A(\lambda', e^{2\pi i x}) = \\ &= A(\lambda, e^{2\pi i(\lambda' + x)}) A(\lambda', e^{2\pi i x}) = f_A(\lambda, \lambda' + x) f_A(\lambda', x). \end{aligned}$$

So, factors of automorphy on \mathbb{C}^* define factors of automorphy on \mathbb{C} .

We restrict ourselves to factors of automorphy $f : \Gamma \times \mathbb{C} \rightarrow \mathrm{GL}_r(\mathbb{C})$ with the property

$$(2) \quad f(m\tau + n, x) = f(m\tau, x), \quad m, n \in \mathbb{Z}.$$

It follows from this property that $f(n, x) = f(0, x) = \mathrm{id}_{\mathbb{C}^r}$. Therefore,

$$f(\lambda + k, x) = f(\lambda, k + x) f(k, x) = f(\lambda, k + x) \text{ for all } \lambda \in \Gamma, k \in \mathbb{Z}$$

and it is possible to define the function

$$A_f : \Gamma \times \mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C}), \quad (\lambda, e^{2\pi i x}) \mapsto f(\lambda, x),$$

which is well-defined because from $e^{2\pi i x_1} = e^{2\pi i x_2}$ follows $x_1 = x_2 + k$ for some $k \in \mathbb{Z}$ and $f(\lambda, x_1) = f(\lambda, x_2 + k) = f(\lambda, x_2)$.

Consider A with the property $A(m\tau + n, u) = A(m\tau, u) =: A(m, u)$. Then clearly $f_A(m\tau + n, u) = f_A(m\tau, u)$. So for any \mathbb{C}^* -factor of automorphy $A : \Gamma \times \mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C})$ with the property $A(m\tau + n, u) = A(m\tau, u)$ one obtains the factor of automorphy f_A satisfying (2). We proved the following

Theorem 4.6. *Factors of automorphy $f : \Gamma \times \mathbb{C} \rightarrow \mathrm{GL}_r(\mathbb{C})$ with the property (2) are in a one-to-one correspondence with \mathbb{C}^* -factors of automorphy with property $A(m\tau + n, u) = A(m\tau, u)$.*

Now we want to translate the conditions for factors of automorphy with the property (2) to be equivalent in the language of \mathbb{C}^* -factors of automorphy with the same property.

Theorem 4.7. *Let f, f' be r -factors of automorphy with the property (2). Then $f \sim f'$ if and only if there exists a holomorphic function $B : \mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C})$ such that*

$$A_f(m, u) B(u) = B(q^m u) A_{f'}(m, u)$$

for $q := e^{2\pi i \tau}$, where $A(m, u) := A(m\tau, u)$. In this case we also say A_f is equivalent to $A_{f'}$ and write $A_f \sim A_{f'}$.

Proof. Let $f \sim f'$. By definition it means that there exists a holomorphic function $h : \mathbb{C} \rightarrow \mathrm{GL}_r(\mathbb{C})$ such that $f(\lambda, x) h(x) = h(\lambda x) f'(\lambda, x)$. Therefore, from $f(n, x) h(x) = h(n + x) f'(n, x)$ and $f(n, x) = f'(n, x) = \mathrm{id}_{\mathbb{C}^r}$ it follows $h(x) = h(n + x)$ for all $n \in \mathbb{Z}$. Therefore, the function

$$B : \mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C}), \quad e^{2\pi i x} \mapsto h(x)$$

is well-defined. We have

$$\begin{aligned} A_f(m, e^{2\pi i x}) B(e^{2\pi i x}) &= f(m\tau, x) h(x) = h(m\tau + x) f'(m\tau, x) = \\ &= B(e^{2\pi i(m\tau + x)}) f'(m, e^{2\pi i x}) = B(q^m e^{2\pi i x}) A_{f'}(m, e^{2\pi i x}). \end{aligned}$$

Vice versa, let B be such that $A_f(m, u) B(u) = B(q^m u) A_{f'}(m, u)$. Define $h = B \circ \exp$. We obtain

$$\begin{aligned} f(m\tau + n, x) h(x) &= A_f(m\tau + n, e^{2\pi i x}) B(e^{2\pi i x}) = \\ &= B(q^m e^{2\pi i x}) A_{f'}(m\tau + n, e^{2\pi i x}) = B(e^{2\pi i(m\tau + x)}) A_{f'}(m\tau + n, e^{2\pi i x}) = \\ &= B(e^{2\pi i(m\tau + n + x)}) A_{f'}(m\tau + n, e^{2\pi i x}) = h(m\tau + n + x) f'(m\tau + n, x), \end{aligned}$$

which means that $f \sim f'$ and completes the proof. \square

Remark 4.8. The last two theorems allow us to embed the set $Z^1(\mathbb{Z}, r)$ of factors of automorphy $\mathbb{Z} \times X \rightarrow \mathrm{GL}_r(\mathbb{C})$ to the set $Z^1(\Gamma, r)$. The embedding is

$$\Psi : Z^1(\mathbb{Z}, r) \rightarrow Z^1(\Gamma, r), \quad f \mapsto g, \quad g(n\tau + m, x) := f(n, x).$$

Two factors of automorphy from $Z^1(\mathbb{Z}, r)$ are equivalent if and only if their images under Ψ are equivalent in $Z^1(\Gamma, r)$. That is why it is enough to consider only factors of automorphy

$$\Gamma \times \mathbb{C} \rightarrow \mathrm{GL}_r(\mathbb{C})$$

satisfying (2).

Corollary 4.9. *A factor of automorphy f with property (2) is trivial if and only if $A_f(m, u) = B(q^m u)B(u)^{-1}$ for some holomorphic function $B : \mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C})$.*

Theorem 4.10. *Let A be a \mathbb{C}^* -factor of automorphy. $A(m, u)$ is uniquely determined by $A(u) := A(1, u)$.*

$$(3) \quad A(m, u) = A(q^{m-1}u) \dots A(qu)A(u), \quad m > 0$$

$$(4) \quad A(-m, u) = A(q^{-m}u)^{-1} \dots A(q^{-1}u)^{-1}, \quad m > 0.$$

$A(m, u)$ is equivalent to $A'(m, u)$ if and only if

$$(5) \quad A(u)B(u) = B(qu)A'(u)$$

for some holomorphic function $B : \mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C})$. In particular $A(m, u)$ is trivial iff $A(u) = B(qu)B(u)^{-1}$.

Proof. Since $A(1, u) = A(u)$ the first formula holds for $m = 1$. Therefore,

$$A(m+1, u) = A(1, q^m u)A(m, u) = A(q^m)A(m, u)$$

and we prove the first formula by induction.

Now $\mathrm{id} = A(0, u) = A(m-m, u) = A(m, q^{-m}u)A(-m, u)$ and hence

$$\begin{aligned} A(-m, u) &= A(m, q^{-m}u)^{-1} = (A(q^{m-1}q^{-m}u) \dots A(qq^{-m}u)A(q^{-m}u))^{-1} = \\ &= A(-m, u) = A(q^{-m}u)^{-1} \dots A(q^{-1}u)^{-1} \end{aligned}$$

which proves the second formula.

If $A(m, u) \sim A'(m, u)$ then clearly (5) holds.

Vice versa, suppose $A(u)B(u) = B(qu)A'(u)$. Then

$$\begin{aligned} A(m, u)B(u) &= A(q^{m-1}u) \dots A(qu)A(u)B(u) = \\ &= A(q^{m-1}u) \dots A(qu)B(qu)A'(u) = \dots = B(q^m u)A'(q^{m-1}u) \dots A'(qu)A'(u) = \\ &= B(q^m u)A'(m, u) \end{aligned}$$

for $m > 0$.

Since $A(-m, u) = A(m, q^{-m}u)^{-1}$ we have

$$\begin{aligned} A(-m, u)B(u) &= A(m, q^{-m}u)^{-1}B(u) = (B(u)^{-1}A(m, q^{-m}u))^{-1} = \\ &= (B(u)^{-1}A(m, q^{-m}u)B(q^{-m}u)B(q^{-m}u)^{-1})^{-1} = \\ &= (B(u)^{-1}B(q^m q^{-m}u)A'(m, q^{-m}u)B(q^{-m}u)^{-1})^{-1} = \\ &= (B(u)^{-1}B(u)A'(m, q^{-m}u)B(q^{-m}u)^{-1})^{-1} = \\ &= B(q^{-m}u)A'(m, q^{-m}u)^{-1} = B(q^{-m}u)A'(-m, u), \end{aligned}$$

which completes the proof. \square

Remark 4.11. Theorem 4.10 means that all the information about a vector bundle of rank r on a complex torus can be encoded by a holomorphic function $\mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C})$.

For a holomorphic function $A : \mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C})$, let us denote by $E(A)$ the corresponding vector bundle on X .

Theorem 4.12. *Let $A : \mathbb{C}^* \rightarrow \mathrm{GL}_n(\mathbb{C})$, $B : \mathbb{C}^* \rightarrow \mathrm{GL}_m(\mathbb{C})$ be two holomorphic maps. Then $E(A) \otimes E(B) \simeq E(A \otimes B)$.*

Proof. By theorem 3.5 we have

$$E(A) \otimes E(B) \simeq E(A(n, u)) \otimes E(B(n, u)) \simeq E(A(n, u) \otimes B(n, u)).$$

Since $A(1, u) \otimes B(1, u) = A(u) \otimes B(u)$, we obtain $E(A) \otimes E(B) \simeq E(A \otimes B)$. \square

5. CLASSIFICATION OF VECTOR BUNDLES OVER A COMPLEX TORUS

Here we work with factors of automorphy depending only on τ , i. e., with holomorphic functions $\mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C})$.

5.1. Vector bundles of degree zero. We return to extensions of the type $0 \rightarrow I_1 \rightarrow E \rightarrow I_1 \rightarrow 0$, where I_1 denotes the trivial vector bundle of rank 1.

Theorem 4.4 can be rewritten as follows.

Theorem 5.1. *A function $A(u) = \begin{pmatrix} 1 & a(u) \\ 0 & 1 \end{pmatrix}$ defines the trivial bundle if and only if $a(u) = b(qu) - b(u)$ for some holomorphic function $b : \mathbb{C}^* \rightarrow \mathbb{C}$.*

Corollary 5.2. *$A(u) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ defines a non-trivial vector bundle.*

Proof. Suppose A defines the trivial bundle. Then $1 = b(qu) - b(u)$ for some holomorphic function $b : \mathbb{C}^* \rightarrow \mathbb{C}$. Considering the Laurent series expansion $\sum_{-\infty}^{+\infty} b_k u^k$ of b we obtain $1 = b_0 - b_0 = 0$ which shows that our assumption was false. \square

Let $a : \mathbb{C}^* \rightarrow \mathbb{C}$ be a holomorphic function such that $A_2(u) = \begin{pmatrix} 1 & a(u) \\ 0 & 1 \end{pmatrix}$ defines non-trivial bundle, i. e., by Theorem 5.1, there is no holomorphic function $b : \mathbb{C}^* \rightarrow \mathbb{C}$ such that

$$a(u) = b(qu) - b(u).$$

Let F_2 be the bundle defined by A_2 . Then by Theorem 3.3 there exists an exact sequence

$$0 \rightarrow I_1 \rightarrow F_2 \rightarrow I_1 \rightarrow 0.$$

For $n \geq 3$ we define $A_n : \mathbb{C}^* \rightarrow \mathrm{GL}_n(\mathbb{C})$,

$$A_n = \begin{pmatrix} 1 & a & & & \\ & \ddots & \ddots & & \\ & & 1 & a & \\ & & & & 1 \end{pmatrix},$$

where empty entries stay for zeros.

Let F_n be the bundle defined by A_n . By (3.3) one sees that A_n defines the extension

$$0 \rightarrow I_1 \rightarrow F_n \rightarrow F_{n-1} \rightarrow 0.$$

Theorem 5.3. F_n is not the trivial bundle. The extension

$$0 \rightarrow I_1 \rightarrow F_n \rightarrow F_{n-1} \rightarrow 0.$$

is non-trivial for all $n \geq 2$.

Proof. Suppose F_n is trivial. Then $A_n(u)B(u) = B(qu)$ for some $B = (b_{ij})_{ij}^n$. In particular it means $b_{ni}(u) = b_{ni}(qu)$ for $i = \overline{1, n}$. Let $b_{ni} = \sum_{k=-\infty}^{+\infty} b_k^{(ni)} u^k$ be the expansion of b_{ni} in Laurent series. Then $b_{ni}(u) = b_{ni}(qu)$ implies $b_k^{(ni)} = q^k b_k^{(ni)}$ for all k .

Note that $|q| < 1$ because $\tau = \xi + i\eta$, $\eta > 0$ and

$$|q| = |e^{2\pi i \tau}| = |e^{2\pi i(\xi + i\eta)}| = |e^{2\pi i \xi} e^{-2\pi \eta}| = e^{-2\pi \eta} < 1.$$

Therefore, $b_k^{(ni)} = 0$ for $k \neq 0$ and we conclude that b_{ni} should be constant functions.

We also have

$$b_{n-1i}(u) + b_{ni}a(u) = b_{n-1i}(qu).$$

Since at least one of b_{ni} is not equal to zero because of invertibility of B , we obtain

$$a(u) = \frac{1}{b_{ni}}(b_{n-1i}(qu) - b_{n-1i}(u))$$

for some i , which contradicts the choice of a . Therefore, F_n is not trivial.

Assume now, that for some $n > 2$ the extension

$$0 \rightarrow I_1 \rightarrow F_n \rightarrow F_{n-1} \rightarrow 0$$

is trivial (for $n = 2$ it is not trivial since F_2 is not a trivial vector bundle). This means $A_n \sim \begin{pmatrix} 1 & 0 \\ 0 & A_{n-1} \end{pmatrix}$, i. e., there exists a holomorphic function $B : \mathbb{C}^* \rightarrow \text{GL}_n(\mathbb{C})$, $B = (b_{ij})_{i,j}^n$ such that

$$A_n(u)B(u) = B(qu) \begin{pmatrix} 1 & 0 \\ 0 & A_{n-1} \end{pmatrix}.$$

Considering the elements of the first and second columns we obtain for the first column

$$\begin{aligned} b_{n1}(u) &= b_{n1}(qu), \\ b_{i1}(u) + b_{i+11}(u)a(u) &= b_{i1}(qu), \quad i < n \end{aligned}$$

and for the second column

$$\begin{aligned} b_{n2}(u) &= b_{n2}(qu), \\ b_{i2}(u) + b_{i+12}(u)a(u) &= b_{i2}(qu), \quad i < n. \end{aligned}$$

For the first column as above considering Laurent series we have that b_{n1} should be a constant function. If $b_{n1} \neq 0$ it follows

$$a(u) = \frac{1}{b_{n1}}(b_{n-11}(qu) - b_{n-11}(u)),$$

which contradicts the choice of a . Therefore, $b_{n1} = 0$ and $b_{n-11}(qu) = b_{n-11}(u)$, in other words b_{n-11} is a constant function. Proceeding by induction one obtains that b_{11} is a constant function and $b_{i1} = 0$ for $i > 1$.

For the second column absolutely analogously we obtain a similar result: b_{12} is constant, $b_{i2} = 0$ for $i > 1$. This contradicts the invertibility of $B(u)$ and proves the statement. \square

Corollary 5.4. *The vector bundle F_n is the only indecomposable vector bundle of rank n and degree 0 that has non-trivial sections.*

Proof. This follows from [1, Theorem 5]. □

So we have that the vector bundles $F_n = E(A_n)$ are exactly F_n 's defined by Atiyah in [1].

Remark 5.5. Note that constant matrices A and B having the same Jordan normal form are equivalent. This is clear because $A = SBS^{-1}$ for some constant invertible matrix S , which means that A and B are equivalent.

Consider an upper triangular matrix $B = (b_{ij})_1^n$ of the following type:

$$(6) \quad b_{ii} = 1, \quad b_{ii+1} \neq 0.$$

It is easy to see that this matrix is equivalent to the upper triangular matrix A ,

$$(7) \quad a_{ii} = a_{ii+1} = 1, \quad a_{ij} = 0, \quad j \neq i+1, \quad j \neq i.$$

In fact, these matrices have the same characteristic polynomial $(t-1)^n$ and the dimension of the eigenspace corresponding to the eigenvalue 1 is equal to 1 for both matrices. Therefore, A and B have the same Jordan form. By Remark above we obtain that A and B are equivalent. We proved the following:

Lemma 5.6. *A matrix satisfying (6) is equivalent to the matrix defined by (7). Moreover, two matrices of the type (6) are equivalent, i. e., they define two isomorphic vector bundles.*

Theorem 5.7. $F_n \simeq S^{n-1}(F_2)$.

Proof. We know that F_2 is defined by the constant matrix $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We know by Theorem 3.4 that $S^n(F_2)$ is defined by $S^n(A_2)$. We calculate $S^n(f_2)$ for $n \in \mathbb{N}_0$. Since f_2 is a constant matrix, $S^n(f_2)$ is also a constant matrix defining a map $S^n(\mathbb{C}^2) \rightarrow S^n(\mathbb{C}^2)$. Let e_1, e_2 be the standard basis of \mathbb{C}^2 , then $S^n(\mathbb{C})$ has a basis

$$\{e_1^k e_2^{n-k} \mid k = n, n-1, \dots, 0\}.$$

Since $A_2(e_1) = e_1$ and $A_2(e_2) = e_1 + e_2$, we conclude that $e_1^k e_2^{n-k}$ is mapped to

$$A_2(e_1)^k A_2(e_2)^{n-k} = e_1^k (e_1 + e_2)^{n-k} = e_1^k \sum_{i=0}^{n-k} \binom{n-k}{i} e_1^{n-k-i} e_2^i = \sum_{i=0}^{n-k} \binom{n-k}{i} e_1^{n-i} e_2^i.$$

Therefore,

$$S^n(A_2) = \begin{pmatrix} 1 & 1 & 1 & \dots & \binom{n}{0} \\ & 1 & 2 & \dots & \binom{n}{1} \\ & & 1 & \dots & \binom{n}{2} \\ & & & \ddots & \vdots \\ & & & & \binom{n}{n} \end{pmatrix},$$

where empty entries stay for zero. In other words, the columns of $S^n(A_2)$ are columns of binomial coefficients. By Lemma 5.6 we conclude that $S^n(A_2)$ is equivalent to A_{n+1} . This proves the statement of the theorem. □

Let E be a 2-dimensional vector bundle over a topological space X . Then there exists an isomorphism

$$S^p(E) \otimes S^q(E) \simeq S^{p+q}(E) \oplus (\det E \otimes S^{p-1}(E) \otimes S^{q-1}(E)).$$

This is the Clebsch-Gordan formula. If $\det E$ is the trivial line bundle, then we have $S^p(E) \otimes S^q(E) \simeq S^{p+q}(E) \oplus S^{p-1}(E) \otimes S^{q-1}(E)$, and by iterating one gets

$$(8) \quad S^p(E) \otimes S^q(E) \simeq S^{p+q}(E) \oplus S^{p+q-2}(E) \oplus \cdots \oplus S^{p-q}(E), \quad p \geq q.$$

Theorem 5.8. $F_p \otimes F_q \simeq F_{p+q-1} \oplus F_{p+q-3} \oplus \cdots \oplus F_{p-q+1}$ for $p \geq q$.

Proof. Using Theorem 5.7 and (8) we obtain

$$F_p \otimes F_q \simeq S^{p-1}(F_2) \otimes S^{q-1}(F_2) \simeq S^{p+q-2}(F_2) \oplus S^{p+q-4}(F_2) \oplus \cdots \oplus S^{p-q}(F_2) \simeq F_{p+q-1} \oplus F_{p+q-3} \oplus \cdots \oplus F_{p-q+1}.$$

This completes the proof. \square

Remark 5.9. The possibility of proving the last theorem using Theorem 5.7 is exactly what Atiyah states in remark (1) after Theorem 9 (see [1, p. 439]).

We have already given (Corollary 5.4) a description of vector bundles of degree zero with non-trivial sections. We give now a description of all vector bundles of degree zero.

Consider the function $\varphi_0(z) = \exp(-\pi i \tau - 2\pi i z) = q^{-1/2} u^{-1} = \varphi(u)$, where $u = e^{2\pi i z}$. It defines the factor of automorphy

$$e_0(p\tau + q, z) = \exp(-\pi i p^2 \tau - 2\pi i z p) = q^{-\frac{p^2}{2}} u^{-p}$$

corresponding to the theta-characteristic $\xi = 0$.

Theorem 5.10. $\deg E(\varphi_0) = 1$, where as above $\varphi_0(z) = \exp(-\pi i \tau - 2\pi i z) = q^{-1/2} u^{-1} = \varphi(u)$.

Proof. Follows from Theorem 4.2 for $\xi = 0$. \square

Theorem 5.11. Let $L' \in \mathcal{E}(1, d)$. Then there exists $x \in X$ such that $L' \simeq t_x^* E(\varphi_0) \otimes E(\varphi_0)^{d-1}$.

Proof. Since $E(\varphi_0)^d$ has degree d , we obtain that there exists $\tilde{L} \in \mathcal{E}(1, 0)$ such that $L' \simeq E(\varphi_0)^d \otimes \tilde{L}$. We also know that $\tilde{L} \simeq t_x^* E(\varphi_0) \otimes E(\varphi_0)^{-1}$ (cf. proof of Theorem 4.2 and Theorem 4.3) for some $x \in X$. Combining these one obtains

$$L' \simeq E(\varphi_0)^d \otimes t_x^* E(\varphi_0) \otimes E(\varphi_0)^{-1} \simeq t_x^* E(\varphi_0) \otimes E(\varphi_0)^{d-1}.$$

This proves the required statement. \square

Theorem 5.12. The map

$$\mathbb{C}^* / \langle q \rangle \rightarrow \text{Pic}^0(X), \quad \bar{a} \mapsto E(a).$$

is well-defined and is an isomorphism of groups.

Proof. Let $\varphi_0(z) = \exp(-\pi i \tau - 2\pi i z)$ as above. For $x \in X$ consider $t_x^* E(\varphi_0)$, where the map

$$t_x : X \rightarrow X, \quad y \mapsto y + x$$

is the translation by x . Let $\xi \in \mathbb{C}$ be a representative of x . Clearly, $t_x^* E(\varphi_0)$ is defined by

$$\varphi_{0\xi}(z) = t_\xi \varphi_0(z) = \varphi_0(z + \xi) = \exp(-\pi i \tau - 2\pi i z - 2\pi i \xi) = \varphi_0(z) \exp(-2\pi i \xi).$$

(Note that if η is another representative of x , then $\varphi_{0\xi}$ and $\varphi_{0\eta}$ are equivalent.) Therefore, the bundle $t_x^* E(\varphi_0) \otimes E(\varphi_0)^{-1}$ is defined by

$$(\varphi_{0\xi} \varphi_0^{-1})(z) = \varphi_0(z) \exp(-2\pi i \xi) \varphi_0^{-1}(z) = \exp(-2\pi i \xi).$$

Since for any $L \in \mathcal{E}(1, 0)$ there exists $x \in X$ such that $L \simeq t_x^* E(\varphi_0) \otimes E(\varphi_0)^{-1}$, we obtain $L \simeq E(a)$ for $a = \exp(-2\pi i \xi) \in \mathbb{C}^*$, where $\xi \in \mathbb{C}$ is a representative of x . We proved that any line bundle of degree zero is defined by a constant function $a \in \mathbb{C}^*$.

Vice versa, let $L = E(a)$ for $a \in \mathbb{C}^*$. Clearly, there exists $\xi \in \mathbb{C}$ such that $a = \exp(-2\pi i \xi)$. Therefore,

$$L \simeq E(a) \simeq L(\varphi_{0\xi} \varphi_0^{-1}) \simeq t_x^* E(\varphi_0) \otimes E(\varphi_0)^{-1},$$

where x is the class of ξ in X , which implies that $E(a)$ has degree zero. So we obtained that the line bundles of degree zero are exactly the line bundles defined by constant functions.

We have the map

$$\phi : \mathbb{C}^* \rightarrow \text{Pic}^0(X), \quad a \mapsto E(a),$$

which is surjective. By Theorem 4.12 it is moreover a homomorphism of groups. We are looking now for the kernel of this map.

Suppose $E(a)$ is a trivial bundle. Then there exists a holomorphic function $f : \mathbb{C}^* \rightarrow \mathbb{C}^*$ such that $f(qu) = af(u)$. Let $f = \sum f_\nu a^\nu$ be the Laurent series expansion of f . Then from $f(qu) = af(u)$ one obtains

$$af_\nu = f_\nu q^\nu \text{ for all } \nu \in \mathbb{Z}.$$

Therefore, $f_\nu(a - q^\nu) = 0$ for all $\nu \in \mathbb{Z}$.

Since $f \not\equiv 0$, we obtain that there exists $\nu \in \mathbb{Z}$ with $f_\nu \neq 0$. Hence $a = q^\nu$ for some $\nu \in \mathbb{Z}$.

Vice versa, if $a = q^\nu$, for $f(u) = u^\nu$ we get

$$f(qu) = q^\nu u^\nu = af(u).$$

This means that $E(a)$ is the trivial bundle, which proves $\text{Ker } \phi = \langle q \rangle$. We obtain the required isomorphism

$$\mathbb{C}^* / \langle q \rangle \rightarrow \text{Pic}^0(X), \quad \bar{a} \mapsto E(a).$$

This completes the proof. \square

Theorem 5.13. *For any $F \in \mathcal{E}(r, 0)$ there exists a unique $\bar{a} \in \mathbb{C}^* / \langle q \rangle$ such that $F \simeq E(A_r(a))$, where*

$$A_r(a) = \begin{pmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & a & 1 \\ & & & a \end{pmatrix}.$$

Proof. By [1, Theorem 5] $F \simeq F_r \otimes L$ for a unique $L \in \mathcal{E}(1, 0)$. Since $F_r \simeq E(A_r)$ and $L \simeq E(a)$ for a unique $\bar{a} \in \mathbb{C}^* / \langle q \rangle$ we get $F \simeq E(A_r \otimes a)$. So F is defined by the matrix

$$\begin{pmatrix} a & a & & \\ & \ddots & \ddots & \\ & & a & a \\ & & & a \end{pmatrix},$$

where empty entries stay for zeros. It is easy to see that the Jordan normal form of this matrix is

$$\begin{pmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & a & 1 \\ & & & a \end{pmatrix}.$$

This proves the statement of the theorem. \square

5.2. Vector bundles of arbitrary degree. Denote by $E_\tau = \mathbb{C}/\Gamma_\tau$, where $\Gamma_\tau = \mathbb{Z}\tau + \mathbb{Z}$. Consider the r -covering

$$\pi_r : E_{r\tau} \rightarrow E_\tau, \quad [x] \mapsto [x].$$

Theorem 5.14. *Let F be a vector bundle of rank n on E_τ defined by $A(u) = A(1, u) = A(\tau, u)$. Then $\pi_r^*(F)$ is defined by*

$$\tilde{A}(r\tau, u) = \tilde{A}(u) = \tilde{A}(1, u) := A(r\tau, u) = A(q^{r-1}u) \dots A(qu)A(u).$$

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc} & \mathbb{C} & \\ p_{r\tau} \swarrow & & \searrow p_\tau \\ E_{r\tau} & \xrightarrow{\pi_r} & E_\tau \end{array}$$

Consider the map

$$\begin{aligned} \mathbb{C} \times \mathbb{C}^n / \tilde{A} = E(\tilde{A}) &\rightarrow \pi_r^*(E(A)) = E_{r\tau} \times_{E_\tau} E(A) = \{([z]_{r\tau}, [z, v]_\tau) \in E_{r\tau} \times E(A)\} \\ [z, v]_{r\tau} &\mapsto ([z]_{r\tau}, [z, v]_\tau). \end{aligned}$$

It is clearly bijective. It remains to prove that it is biholomorphic. From the construction of $E(A)$ and $E(\tilde{A})$ it follows that the diagram

$$\begin{array}{ccc} E(\tilde{A}) & & \\ \searrow & & \\ \pi_r^*(E(A)) & \longrightarrow & E(A) \\ \downarrow & & \downarrow \\ E_{r\tau} & \longrightarrow & E_\tau \end{array}$$

locally looks as follows:

$$\begin{array}{ccc} U \times \mathbb{C}^n & & (z, v) \\ \searrow & & \searrow \\ \Delta(U \times U) \times \mathbb{C}^n & \longrightarrow & U \times \mathbb{C}^n, & & ((z, z), v) & \longrightarrow & (z, v). \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ U & \xlongequal{\quad} & U & & z & \longrightarrow & z \end{array}$$

This proves the required statement. \square

Theorem 5.15. *Let F be a vector bundle of rank n on $E_{r\tau}$ defined by $\tilde{A}(u) = \tilde{A}(r\tau, u)$. Then $\pi_{r*}(F)$ is defined by $A(u) = \begin{pmatrix} 0 & I_{(r-1)n} \\ \tilde{A}(u) & 0 \end{pmatrix}$.*

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc} & \mathbb{C} & \\ p_{r\tau} \swarrow & & \searrow p_\tau \\ E_{r\tau} & \xrightarrow{\pi_r} & E_\tau \end{array}$$

Let $z \in \mathbb{C}$. Consider $y = p_{r\tau}(z) \in E_{r\tau}$ and $x = p_\tau(z) = \pi_r p_{r\tau}(z) \in E_\tau$.

Choose a point $b \in \mathbb{C}$ such that $z \in V_b$, where V_b is the standard parallelogram at point b . Clearly $x \in U_b = p_r(V_b)$ and we have the isomorphism $\varphi_b : U_b \rightarrow V_b$ with $\varphi_b(x) = z$.

Consider $\pi_r^{-1}(U_b) = W_b \sqcup \cdots \sqcup W_{b+(r-1)\tau}$, where $y \in W_b$ and $\pi_r|_{W_{b+i\tau}} : W_{b+i\tau} \rightarrow U_b$ is an isomorphism for each $0 \leq i < r$.

We have

$$\begin{aligned} \pi_{r*}(\mathcal{E}(\tilde{A}))(U_b) &= \mathcal{E}(\tilde{A})(\pi_r^{-1}(U_b)) = \mathcal{E}(\tilde{A})(W_b \sqcup \cdots \sqcup W_{b+(r-1)\tau}) = \\ &= \mathcal{E}(\tilde{A})(W_b) \oplus \cdots \oplus \mathcal{E}(\tilde{A})(W_{b+(r-1)\tau}), \end{aligned}$$

where $\mathcal{E}(\tilde{A})$ is the sheaf of sections of $E(\tilde{A})$.

Choose $a \in \mathbb{C}$ such that $z \notin V_a$, $z \in V_{a+\tau}$. We have $\varphi_a(x) = z + \tau$. As above, $\pi_r^{-1}(U_a) = W_a \sqcup \cdots \sqcup W_{a+(r-1)\tau}$ and

$$\begin{aligned} \pi_{r*}(\mathcal{E}(\tilde{A}))(U_a) &= \mathcal{E}(\tilde{A})(\pi_r^{-1}(U_a)) = \mathcal{E}(\tilde{A})(W_a \sqcup \cdots \sqcup W_{a+(r-1)\tau}) = \\ &= \mathcal{E}(\tilde{A})(W_a) \oplus \cdots \oplus \mathcal{E}(\tilde{A})(W_{a+(r-1)\tau}). \end{aligned}$$

Since $g_{ab}(x) = A(\varphi_a(x) - \varphi_b(x), \varphi_b(x))$, we obtain

$$g_{ab}(x) = A(\varphi_a(x) - \varphi_b(x), \varphi_b(x)) = A(z + \tau - z, z) = A(\tau, z).$$

Therefore, to obtain $A(\tau, z)$ it is enough to compute $g_{ab}(x)$.

Note that $\pi_{r*}(\mathcal{E}(\tilde{A}))_x = \mathcal{E}(\tilde{A})_y \oplus \cdots \oplus \mathcal{E}(\tilde{A})_{y+(r-1)\tau}$. Note also that g_{ab} is a map from $\pi_{r*}(\mathcal{E}(\tilde{A}))(U_b) = \mathcal{E}(\tilde{A})(W_b) \oplus \cdots \oplus \mathcal{E}(\tilde{A})(W_{b+(r-1)\tau})$ to $\pi_{r*}(\mathcal{E}(\tilde{A}))(U_a) = \mathcal{E}(\tilde{A})(W_a) \oplus \cdots \oplus \mathcal{E}(\tilde{A})(W_{a+(r-1)\tau})$.

One easily sees that $y \in W_b$, $y \in W_{a+(r-1)\tau}$ and $y + i\tau \in W_{b+i\tau}$, $y + i\tau \in W_{a+(i-1)\tau}$ for $0 < i < r$. Therefore,

$$g_{ab}(x) = \begin{pmatrix} 0 & \tilde{g}_{a \ b+\tau}(y + \tau) & & \\ \vdots & & \ddots & \\ 0 & & & \tilde{g}_{a+(r-2)\tau \ b+(r-a)\tau}(y + (r-1)\tau) \\ \tilde{g}_{a+(r-1)\tau \ b}(y) & 0 & \cdots & 0 \end{pmatrix}.$$

It remains to compute the entries of this matrix. Since

$$\begin{aligned} \tilde{g}_{a+(r-1)\tau \ b}(y) &= \tilde{A}(\tilde{\varphi}_{a+(r-1)\tau}(y) - \tilde{\varphi}_b(y), \tilde{\varphi}_b(y)) = \tilde{A}(z + r\tau - z, z) = \tilde{A}(r\tau, z) \text{ and} \\ \tilde{g}_{a+(i-1)\tau \ b+i\tau}(y + i\tau) &= \tilde{A}(\tilde{\varphi}_{a+(i-1)\tau}(y + i\tau) - \tilde{\varphi}_{b+i\tau}(y + i\tau), \tilde{\varphi}_{b+i\tau}(y + i\tau)) = \\ &= \tilde{A}(z + i\tau - (z + i\tau), z + i\tau) = \tilde{A}(0, z + i\tau) = I_n, \end{aligned}$$

one obtains

$$g_{ab}(x) = \begin{pmatrix} 0 & I_n & & \\ \vdots & & \ddots & \\ 0 & & & I_n \\ \tilde{A}(z) & 0 & \cdots & 0 \end{pmatrix}.$$

Therefore, $A(z) = \begin{pmatrix} 0 & I_n & & \\ \vdots & & \ddots & \\ 0 & & & I_n \\ \tilde{A}(z) & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_{(r-1)n} \\ \tilde{A}(z) & 0 \end{pmatrix}$. This proves the required statement. \square

Lemma 5.16. *Let $A_i \in \text{GL}_n(\mathbb{R})$, $i = 1, \dots, n$. Then*

$$\prod_{i=1}^r \begin{pmatrix} 0 & I_{(r-1)n} \\ A_i & 0 \end{pmatrix} = \text{diag}(A_r, \dots, A_1)$$

Proof. Straightforward calculation. \square

From Theorem 5.14 and Theorem 5.15 one obtains the following:

Corollary 5.17. *Let $E(A)$ be a vector bundle of rank n on $E_{r\tau}$, where $A : \mathbb{C}^* \rightarrow \mathrm{GL}_n(\mathbb{C})$ is a holomorphic function. Then $\pi_r^* \pi_{r*} E(A)$ is defined by*

$$\mathrm{diag}(A(q^{r-1}u), \dots, A(qu), A(u)).$$

In other words $\pi_r^ \pi_{r*} E(A)$ is isomorphic to the direct sum*

$$\bigoplus_{i=0}^{r-1} E(A(q^i u)).$$

Proof. We know that $\pi_r^* \pi_{r*} E(A)$ is defined by $B(r, u)$, where

$$B(1, u) = \begin{pmatrix} 0 & I_{(r-1)n} \\ A & 0 \end{pmatrix}.$$

Therefore, using Lemma 5.16, one obtains

$$B(r, u) = \begin{pmatrix} 0 & I_{(r-1)n} \\ A(q^{r-1}u) & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & I_{(r-1)n} \\ A(qu) & 0 \end{pmatrix} \begin{pmatrix} 0 & I_{(r-1)n} \\ A(u) & 0 \end{pmatrix} = \mathrm{diag}(A(q^{r-1}u), \dots, A(qu), A(u)),$$

which completes the proof. \square

Corollary 5.18. *Let $L \in \mathcal{E}(r, 0)$, then $\pi_r^* \pi_{r*} L = \bigoplus_1^r L$.*

Proof. Clear, since $L = E(A)$ for a constant matrix A by Theorem 5.13. \square

Note that for a covering $\pi_r : E_{r\tau} \rightarrow E_\tau$ the group of deck transformations $\mathrm{Deck}(E_{r\tau}/E_\tau)$ can be identified with the kernel $\mathrm{Ker}(\pi_r)$. But $\mathrm{Ker} \pi_r$ is cyclic and equals $\{1, [q], \dots, [q]^{r-1}\}$, where $[q]$ is a class of $q = e^{2\pi i\tau}$ in $E_{r\tau}$. Clearly

$$[q]^*(E(A(u))) = E(A(qu)).$$

Therefore, we get one more corollary.

Corollary 5.19. *Let ϵ be a generator of $\mathrm{Deck}(E_{r\tau}/E_\tau)$. Then for a vector bundle E on $E_{r\tau}$ we have*

$$\pi_r^* \pi_{r*} E = E \oplus \epsilon^* E \oplus \cdots \oplus (\epsilon^{r-1})^* E.$$

To proceed we need the following result from [9](Theorem 1.2, (i)):

Theorem. *Let $\varphi : Y \rightarrow X$ be an isogeny of g -dimensional abelian varieties over a field k , and let L be a line bundle on Y such that the restriction of the map*

$$\Lambda(L) : Y \rightarrow \mathrm{Pic}^0(Y), \quad y \mapsto t_y^* L \otimes L^{-1},$$

to the kernel of φ is an isomorphism. Then $\mathrm{End}(\varphi_ L) = k$ and $\varphi_* L$ is an indecomposable vector bundle on X .*

Theorem 5.20. *Let $L \in \mathcal{E}(1, d)$ and let $(r, d) = 1$. Then $\pi_{r*}(L) \in \mathcal{E}(r, d)$.*

Proof. It is clear that $\pi_{r*} L$ has rank r and degree d . It remains to prove that $\pi_{r*} L$ is indecomposable.

We have the isogeny $\pi_r : E_{r\tau} \rightarrow E_r$. Since $Y = E_{r\tau}$ is a complex torus (elliptic curve), $Y \simeq \mathrm{Pic}^0(Y)$ with the identification $y \leftrightarrow t_y^* E(\varphi_0) \otimes E(\varphi_0)^{-1}$. We know that

$L = E(\varphi_0)^d \otimes \tilde{L}$ for some $\tilde{L} = E(a) \in \mathcal{E}(1, 0)$, $a \in \mathbb{C}^*$. Since $t_y^*(\tilde{L}) = t_y^*(E(a)) = E(a) = \tilde{L}$, as in the proof of Theorem 5.12 one gets

$$\begin{aligned} \Lambda(L)(y) &= t_y^*(L) \otimes L^{-1} = t_y^*(E(\varphi_0)^d \otimes \tilde{L}) \otimes (E(\varphi_0)^d \otimes \tilde{L})^{-1} = \\ &= t_y^*(E(\varphi_0)^d) \otimes t_y^*(\tilde{L}) \otimes E(\varphi_0)^{-d} \otimes \tilde{L}^{-1} = t_y^*(E(\varphi_0)^d) \otimes E(\varphi_0)^{-d} = \\ &= t_y^*(E(\varphi_0^d)(z)) \otimes E(\varphi_0^{-d}) = E(\varphi_0^d(z + \eta)) \otimes E(\varphi_0^{-d}) = \\ &= E(\varphi_0^d(z + \eta)\varphi_0^{-d}(z)) = E(\exp(-2\pi i d \eta)) = t_{dy}^*(E(\varphi_0)) \otimes E(\varphi_0)^{-1}, \end{aligned}$$

where $\eta \in \mathbb{C}$ is a representative of y . This means that the map $\Lambda(L)$ corresponds to the map

$$d_Y : E_{r\tau} \rightarrow E_{r\tau}, \quad y \mapsto dy.$$

Since $\text{Ker } \pi_r$ is isomorphic to $\mathbb{Z}/r\mathbb{Z}$, we conclude that the restriction of d_Y to $\text{Ker } \pi_r$ is an isomorphism if and only if $(r, d) = 1$. Therefore, using Theorem mentioned above, we prove the required statement. \square

Now we are able to prove the following main theorem:

Theorem 5.21. (i) Every indecomposable vector bundle $F \in \mathcal{E}_{E_\tau}(r, d)$ is of the form $\pi_{r'*}(L' \otimes F_h)$, where $(r, d) = h$, $r = r'h$, $d = d'h$, $L' \in \mathcal{E}_{E_{r'\tau}}(1, d')$.

(ii) Every vector bundle of the form $\pi_{r'*}(L' \otimes F_h)$, where L' and r' are as above, is an element of $\mathcal{E}_{E_\tau}(r, d)$.

Proof. ¹ (i) By [1, Lemma 26] we obtain $F \simeq E_A(r, d) \otimes L$ for some line bundle $L \in \mathcal{E}(1, 0)$. By [1, Lemma 24] we have $E_A(r, d) \simeq E_A(r', d') \otimes F_h$, hence $F \simeq E_A(r', d') \otimes F_h \otimes L$.

Consider any line bundle $\tilde{L} \in \mathcal{E}_{E_{r'\tau}}(1, d')$. Since by Theorem 5.20 $\pi_{r'*}(\tilde{L}) \in \mathcal{E}(r', d')$, it follows from [1, Lemma 26] that there exists a line bundle L'' such that $E_A(r', d') \otimes L \simeq \pi_{r'*}(\tilde{L}) \otimes L''$.

Using the projection formula, we get

$$F \simeq \pi_{r'*}(\tilde{L}) \otimes L'' \otimes F_h \simeq \pi_{r'*}(\tilde{L} \otimes \pi_{r'}^*(L'')) \otimes \pi_{r'}^*(F_h) \simeq \pi_{r'*}(L' \otimes \pi_{r'}^*(F_h))$$

for $L' = \tilde{L} \otimes \pi_{r'}^*(L'')$.

Since F_h is defined by a constant matrix we obtain by Theorem 5.14 that $\pi_{r'}^*(F_h)$ is defined by $f_h^{r'}$, which has the same Jordan normal form as f_h . Therefore, $\pi_{r'}^*(F_h) \simeq F_h$ and finally one gets $F \simeq \pi_{r'*}(L' \otimes F_h)$.

(ii) Consider $F = \pi_{r'*}(L' \otimes F_h)$. As above $F_h = \pi_{r'}^*(F_h)$. Using the projection formula we get

$$F = \pi_{r'*}(L' \otimes F_h) = \pi_{r'*}(L' \otimes \pi_{r'}^*(F_h)) = \pi_{r'*}(L') \otimes F_h.$$

By Theorem 5.20 $\pi_{r'*}(L')$ is an element from $\mathcal{E}_{E_\tau}(r', d')$. Therefore, $\pi_{r'*}(L') = E_A(r', d') \otimes L$ for some line bundle $L \in \mathcal{E}_{E_\tau}(1, 0)$. Finally we obtain

$$F = \pi_{r'*}(L') \otimes F_h = E_A(r', d') \otimes L \otimes F_h = E_A(r'h, d'h) \otimes L = E_A(r, d) \otimes L,$$

which means that F is an element of $\mathcal{E}_{E_\tau}(r, d)$. \square

Remark 5.22. Since any line bundle of degree d' is of the form $t_x^*E(\varphi_0) \otimes E(\varphi_0)^{d'-1}$, Theorem 5.21(i) takes exactly the form of Proposition 1 from [10].

Any line bundle of degree d' over $E_{r\tau}$ is of the form $E(a) \otimes E(\varphi^{d'})$, where $a \in \mathbb{C}^*$. Therefore, $L' \otimes F_h = E(a) \otimes E(\varphi_0^{d'}) \otimes E(A_h) = E(\varphi_0^{d'} A_h(a))$. Using Theorem 5.15 we obtain the following:

¹The proof of this theorem uses the ideas from lectures presented by Bernd Kreußler at the University of Kaiserslautern.

Theorem 5.23. *Indecomposable vector bundles of rank r and degree d on E_τ are exactly those defined by the matrices*

$$\begin{pmatrix} 0 & I_{(r'-1)h} \\ \varphi_0^{d'} A_h(a) & 0 \end{pmatrix},$$

where $(r, d) = h$, $r' = r/h$, $d' = d/h$, $\varphi_0(u) = q^{-\frac{r}{2}} u^{-1}$, $q = e^{2\pi i \tau}$, $a \in \mathbb{C}^*$, and

$$A_h(a) = \begin{pmatrix} a & 1 & & & \\ & \ddots & \ddots & & \\ & & & a & 1 \\ & & & & a \end{pmatrix} \in \mathrm{GL}_h(\mathbb{C}).$$

Note that if $d = 0$, we get $h = r$, $r' = 1$, and $d' = 0$. In this case the statement of Theorem 5.23 is exactly Theorem 5.13.

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SISSA, VIA BONOMEA, 265, 34136 TRIESTE, ITALY
E-mail address: iena@sisssa.it