THE WORD PROBLEM AND THE AHARONI-BERGER-ZIV CONJECTURE ON THE CONNECTIVITY OF INDEPENDENCE COMPLEXES

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Abstract. For each finite simple graph G, Aharoni, Berger and Ziv consider a recursively defined number $\psi(G) \in \mathbb{Z} \cup \{+\infty\}$ which gives a lower bound for the topological connectivity of the independence complex I_G. They conjecture that this bound is optimal for every graph. We use a result of recursion theory to give a short disproof of this claim.

The map ψ is defined as follows: $\psi(\emptyset) = -2$; if G is a non-empty discrete graph, $\psi(G) = +\infty$; if G is non-discrete with edge set E, $\psi(G) = max\{min\{\psi(G - e), \psi(G \smallsetminus e) + 1\} \mid e \in E\}$. Here $G - e$ denotes the subgraph of G obtained by removing the edge e and $G \smallsetminus e$ denotes the subgraph of G induced by the vertices which are not adjacent to any of the vertices of e.

The *independence complex* I_G of a finite simple graph G is the simplicial complex whose simplices are the non-empty independent subsets of vertices of G. From an exact sequence of [6] (Claim 3.1) and from Van-Kampen and Hurewicz Theorems it is easy to deduce that I_G is $\psi(G)$-connected [2, Theorem 2.3]. It is conjectured in [2, Conjecture 2.4] that I_G is not $(\psi(G) + 1)$-connected, unless it is contractible. This was proved to be true in the particular case of chordal graphs [5]. However we will see that the conjecture is false in general, although we will not exhibit an explicit example. The following well-known result ([3, Corollary 3.9]) is a consequence of the non-existence of an effective way for determining whether a group Γ given by a finite presentation is trivial or not [1, 7] and a construction that associates to each presentation of Γ a 2-dimensional complex with fundamental group isomorphic to Γ (see [4] for example).

Theorem $*$. *There exists no algorithm that can decide whether a finite simplicial complex is simply connected or not.*

The truthfulness of the Aharoni-Berger-Ziv Conjecture would provide an algorithm (Turing machine) capable of determining if I_G is simply connected for every finite simple graph G (just computing $\psi(G)$ and checking if it is positive). On the other hand, given a finite simplicial complex K, there is a graph G such that I_G is isomorphic to the first barycentric subdivision of K. The vertices of G are the simplices of K and its edges are the pairs of simplices such that none of them is a face of the other. In particular, the conjecture contradicts Theorem $*$.

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