UNITARY SK_1 OF SEMIRAMIFIED GRADED AND VALUED DIVISION ALGEBRAS

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1. Introduction

Let D be a division algebra finite-dimensional over its center K. Then,

$$\mathrm{SK}_1(D) \;=\; \{d \in D^* \mid \mathrm{Nrd}_D(d) = 1\} \,/\, [D^*, D^*],$$

where Nrd_D denotes the reduced norm and $[D^*, D^*]$ is the commutator group of the group of units D^* of D. If D has a unitary involution τ (i.e., an involution τ on D with $\tau|_K \neq \mathrm{id}$), then the unitary SK_1 for τ on D is

$$\mathrm{SK}_1(D, \tau) \;=\; \Sigma'_\tau(D) \,/\, \Sigma_\tau(D), \tag{1.1}$$

where

$$\Sigma'_\tau(D) \;=\; \{d \in D^* \mid \mathrm{Nrd}_D(d) = \tau(\mathrm{Nrd}_D(d))\} \qquad \text{and} \qquad \Sigma_\tau(D) \;=\; \big\langle \{d \in D^* \mid d = \tau(d)\} \big\rangle.$$

The groups $\mathrm{SK}_1(D)$ and $\mathrm{SK}_1(D, \tau)$ are of considerable interest as subtle invariants of D, and as reduced Whitehead groups for certain algebraic groups (cf. [Ti], [P_6], [G]).

In this paper we will prove formulas for $\mathrm{SK}_1(\mathsf{E})$ and $\mathrm{SK}_1(\mathsf{E}, \tau)$ for E a semiramified graded division algebra E of finite rank over its center. In view of the isomorphisms in [HW_1, Th. 4.8] and [HW_2, Th. 3.5], the formulas for E imply analogous formulas for SK_1 and unitary SK_1 for a tame semiramified division algebra D over a Henselian valued field K. The formulas thus obtained in the Henselian case generalize ones given by Platonov for $\mathrm{SK}_1(D)$ and Yanchevskiĭ for $\mathrm{SK}_1(D, \tau)$ for bicyclic decomposably semiramified division algebras over iterated Laurent fields. Most of our work will be in the unitary setting, which is not as well developed as the nonunitary setting.

Ever since Platonov gave examples of division algebras with $\mathrm{SK}_1(D)$ nontrivial there has been ongoing interest in SK_1. Platonov showed in [P_5] that nontriviality of $\mathrm{SK}_1(D)$ implies that the algebraic group group $\mathrm{SL}_1(D)$ (with K-points $\{d \in D \mid \mathrm{Nrd}_D(d) = 1\}$) is not a rational variety. Also, Voskresenskiĭ showed in [V_1] and [V_2, Th., p. 186] that $\mathrm{SK}_1(D) \cong \mathrm{SL}_1(D)/R$, the group of R-equivalence classes of the variety $\mathrm{SL}_1(D)$. The corresponding unitary result, $\mathrm{SK}_1(D, \tau) \cong \mathrm{SU}_1(D, \tau)/R$ was given in [Y_5, Remark, p. 537] and [CM, Th. 5.4]. More recently, Suslin in [Su_1] and [Su_2] has related $\mathrm{SK}_1(D)$ to certain 4-th cohomology groups associated to D, and has conjectured that whenever the Schur index $\mathrm{ind}(D)$ is not square-free then $\mathrm{SK}_1(D \otimes_K L)$ is nontrivial for some field $L \supseteq K$. (This has been proved by Merkurjev in [M_1] and [M_4] if $4 | \mathrm{ind}(D)$, but remains open otherwise.) Nonetheless, explicit computable formulas for $\mathrm{SK}_1(D)$ and $\mathrm{SK}_1(D, \tau)$ have remained elusive, and are principally available, when $\mathrm{ind}(D) > 4$, only for algebras over Henselian fields (cf. [E_2] and [HW_1, Th. 3.4]) and quotients of iterated twisted polynomial algebras (cf. [HW_1, Th. 5.7]).

Platonov's original examples with nontrivial SK_1 in [P_1] and [P_2] were division algebras D over a twice iterated Laurent power series field $K = k(((x))((y))$, where k is a local or global field or an infinite algebraic extension of such a field. His K has a naturally associated rank 2 Henselian valuation which extends uniquely to a valuation on D. With respect to this valuation, his D is tame and "decomposably

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semiramified" and, in addition, its residue division algebra $\overline{D}$ is a field with $\overline{D} = L_1 \otimes_k L_2$, where each L_i is cyclic Galois over k. His basic formula for such D is:

$$\mathrm{SK}_1(D) \;\cong\; \mathrm{Br}(\overline{D}/k)\big/\big[\,\mathrm{Br}(L_1/k)\cdot \mathrm{Br}(L_2/k)\big], \tag{1.2}$$

where k is any field, $\mathrm{Br}(k)$ is the Brauer group of k, and for a field $M \supseteq k$, $\mathrm{Br}(M/k)$ denotes the relative Brauer group $\ker(\mathrm{Br}(k) \to \mathrm{Br}(M))$, a subgroup of $\mathrm{Br}(k)$. That D is tame and semiramified means that $[\overline{D}\!:\!\overline{K}] = |\Gamma_D\!:\!\Gamma_K| = \sqrt{[D\!:\!K]}$ and $\overline{D}$ is a field separable (hence abelian Galois) over $\overline{K}$, where Γ_D is the value group the valuation on D. We say that D is *decomposably semiramified* (abbreviated DSR) if D is a tensor product of cyclic tame and semiramified division algebras. Using (1.2) with k a global field or an algebraic extension of a global field, Platonov showed in [P$_4$] that every finite abelian group and some infinite abelian groups of bounded torsion appear as $\mathrm{SK}_1(D)$ for suitable D.

Shortly after Platonov's work, Yanchevskiĭ obtained in [Y$_2$], [Y$_3$], [Y$_4$] similar results for the unitary SK_1 for similar types of division algebras, namely D decomposably semiramified over $K = k((x))((y))$, with k any field, given that D has a unitary involution τ with fixed field $K^\tau = \ell((x))((y))$ for some field $\ell \subseteq k$ with $[k\!:\!\ell] = 2$. Yanchevskiĭ's key formula (when $\overline{D} = L_1 \otimes_k L_2$ as above) is:

$$\mathrm{SK}_1(D,\tau) \cong \mathrm{Br}(\overline{D}/k;\ell)\big/\big[\,\mathrm{Br}(L_1/k;\ell)\cdot \mathrm{Br}(L_2/k;\ell)\big], \tag{1.3}$$

where for a field $M \supseteq k$, $\mathrm{Br}(M/k;\ell) = \ker\big(\mathrm{cor}_{k\to\ell}\colon \mathrm{Br}(M/k) \to \mathrm{Br}(\ell)\big)$; this is the subgroup of $\mathrm{Br}(k)$ consisting of the classes of central simple k-algebras split by M and having a unitary involution τ with fixed field $k^\tau = \ell$. He used this in [Y$_4$] with k and ℓ global fields to show that any finite abelian group is realizable as $\mathrm{SK}_1(D,\tau)$. He obtained remarkably similar analogues for the unitary SK_1 to other results of Platonov for the nonunitary SK_1, but generally with substantially more difficult and intricate proofs.

Ershov showed in [E$_1$] and [E$_2$] that the natural setting for viewing Platonov's examples of nontrivial $\mathrm{SK}_1(D)$ is that of tame division algebras D over a Henselian valued field K. (Platonov considered his K in a somewhat cumbersome way as a field with complete discrete valuation with residue field which also has a complete discrete valuation.) The Henselian valuation on K has a unique extension to a valuation on D, and Ershov gave exact sequences that describe $\mathrm{SK}_1(D)$ in terms of various data related to the residue division ring $\overline{D}$. In particular he showed (combining [E$_2$, p. 69, (6) and Cor. (b)]) that if D is DSR (with K Henselian), then

$$\mathrm{SK}_1(D) \;\cong\; \widehat{H}^{-1}(\mathrm{Gal}(\overline{D}/\overline{K}), \overline{D}^*). \tag{1.4}$$

More recently, there has been work on associated graded rings of valued division algebras, see especially [HwW$_2$], [Mou], [TW]. The tenor of this work has been that for a tame division algebra D over a Henselian valued field, most of the structure of D is inherited by its associated graded ring $\mathsf{gr}(D)$, while $\mathsf{gr}(D)$ is often much easier to work with than D itself. This theme was applied quite recently by R. Hazrat and the author in [HW$_1$] and [HW$_2$] to calculations of SK_1 and unitary SK_1. It was shown in [HW$_1$, Th. 4.8] that if D is tame over K with respect to a Henselian valuation, then $\mathrm{SK}_1(D) \cong \mathrm{SK}_1(\mathsf{gr}(D))$; the corresponding result for unitary SK_1 was proved in [HW$_2$, Th. 3.5]. Calculations of SK_1 in the graded setting are significantly easier and more transparent than in the original ungraded setting, allowing almost effortless recovery of Ershov's exact sequences, with some worthwhile improvements. Notably, it was shown in [HW$_1$, Cor. 3.6(iii)] that if K is Henselian and D is tame and semiramified (but not necessarily DSR), then there is an exact sequence

$$H \wedge H \;\longrightarrow\; \widehat{H}^{-1}(H, \overline{D}^*) \;\longrightarrow\; \mathrm{SK}_1(D) \;\longrightarrow\; 1, \qquad \text{where} \qquad H \;=\; \mathrm{Gal}(\overline{D}/\overline{K}) \;\cong\; \Gamma_D/\Gamma_K. \tag{1.5}$$

When D is DSR, the image of $H \wedge H$ in $\widehat{H}^{-1}(H, \overline{D}^*)$ is trivial, yielding (1.4). Then, Platonov's formula (1.2) is obtained from (1.4) via the following isomorphism: For a field $M = L_1 \otimes_k L_2$ where each L_i is cyclic Galois over k,

$$\widehat{H}^{-1}(\mathrm{Gal}(M/k), M^*) \;\cong\; \mathrm{Br}(M/k)/\big[\,\mathrm{Br}(L_1/k)\cdot \mathrm{Br}(L_2/k)\big]. \tag{1.6}$$

See (3.6)–(3.9) below for a short proof of (1.6) using facts about abelian crossed products.

When D is semiramified but not DSR, the contribution of the first term in (1.5) can be better understood in terms of the $I \otimes N$ decomposition of D: Our semiramified D is equivalent in $\mathrm{Br}(K)$ to $I \otimes_K N$, where I is inertial (= unramified) over K and N is DSR, so $\overline{N} \cong \overline{D}$ and $\Gamma_N = \Gamma_D$. Thus, the $\widehat{H}^{-1}$ term in (1.5) coincides with $\mathrm{SK}_1(N)$. We will show in Cor. 3.8(i) below that the image of $H \wedge H$ in $\widehat{H}^{-1}(H, \overline{D}^*)$ is expressible in terms of parameters describing the residue algebra $\overline{I}$ of I, which is central simple over $\overline{K}$ and split by the field $\overline{D}$. This $\overline{I}$ does not show up within D or $\overline{D}$, but nonetheless has significant influence on the structure of D. (For example, it determines whether D can be a crossed product or nontrivially decomposable—see [JW, pp. 162–166, Remarks 5.16]. In [JW] DSR algebras were called "nicely semiramified," and abbreviated NSR. We prefer the more descriptive term decomposably semiramified.) Also, $\overline{I}$ is not uniquely determined by D, but determined only modulo the group $\mathrm{Dec}(\overline{D}/\overline{K})$ of simple $\overline{K}$-algebras which "decompose according to $\overline{D}$"—see §3 below for the definition of $\mathrm{Dec}(\overline{D}/\overline{K})$. In the bicyclic case where D is semiramified and K Henselian and $\overline{D} \cong L_1 \otimes_{\overline{K}} L_2$ with each L_i cyclic Galois over $\overline{K}$, we will show in Cor. 3.8(ii) that

$$\mathrm{SK}_1(D) \;\cong\; \mathrm{Br}(\overline{D}/\overline{K})\big/\big[\,\mathrm{Br}(L_1/\overline{K}) \cdot \mathrm{Br}(L_2/\overline{K}) \cdot \langle [\overline{I}] \rangle\big], \tag{1.7}$$

which is a natural generalization of Platonov's formula (1.2).

The principal aim of this paper is to prove unitary versions of the results described above for nonunitary SK_1, especially (1.4), (1.6), and (1.7). The unitary versions of these are, respectively, Th. 7.1(i), Prop. 6.2, and Th. 7.3(ii). Along the way, it will be necessary to develop a unitary version of the $I \otimes N$ decomposition for semiramified division algebras. This is given in Prop. 4.5. In the final section we will apply some of these formulas to give an example where the natural map $\mathrm{SK}_1(D, \tau) \to \mathrm{SK}_1(D)$ is not injective.

This paper is a sequel to [HW_2], which describes the equivalence of the graded setting and the Henselian valued setting for computing unitary SK_1, and has calculations of $\mathrm{SK}_1(D, \tau)$ for several cases other than the semiramified one considered here. However, the present paper can be read independently of [HW_2]. We will work here primarily with graded division algebras, where the calculations are more transparent than for valued algebras. Some basic background on the graded objects is given in §2. But we reiterate that by [HW_2, Th. 3.5] every result in the graded setting yields a corresponding result for tame division algebras over Henselian valued fields. While what is proved here is for a rather specialized type of algebra, we note that detailed knowledge of SK_1 in special cases sometimes has wider consequences. See, e.g., the paper [RTY] where Suslin's conjecture is reduced to the case of cyclic algebras. See also [W, Th. 4.11], where the proof of nontriviality of a cohomological invariant of Kahn uses a careful analysis of $\mathrm{SK}_1(D)$ for the D in Platonov's original example.

From the perspective of algebraic groups, it is perhaps unsurprising that there should be results for the unitary SK_1 similar to those in the nonunitary case. For, $\mathrm{SL}_1(D)$ is a group of inner type A_{n-1} where $n = \deg(D)$, and $\mathrm{SU}_1(D, \tau)$ is a group of outer type A_{n-1} (cf. [KMRT, Th. (26.9)]). Nonetheless, the similarities in formulas for $\mathrm{SK}_1(D, \tau)$ given in Yanchevskiĭ's work and in [HW_2] and here to those for $\mathrm{SK}_1(D)$ seem quite striking. Likewise, the results by Rost on $\mathrm{SK}_1(D)$ for biquaternion algebras (see [KMRT, §17A]) and by Merkurjev in [M_2] for arbitrary algebras of degree 4, have a unitary analogue proved by Merkurjev in [M_3]. This suggests that a further analysis of the unitary SK_1 would be worthwhile, notably to investigate whether there are unitary versions of the deep results by Suslin [Su_2] and Kahn [K] relating $\mathrm{SK}_1(D)$ to higher étale cohomology groups.

2. Graded division algebras and simple algebras

We will be working throughout with graded algebras graded by a torsion-free abelian group. We now set up the terminology for such algebras and recall some of the basic facts we will use frequently.

Let Γ be a torsion-free abelian group, and let R be a ring graded by Γ, i.e., $\mathsf{R} = \bigoplus_{\gamma\in\Gamma} \mathsf{R}_\gamma$, where each R_γ is an additive subgroup of R and $\mathsf{R}_\gamma \cdot \mathsf{R}_\delta \subseteq \mathsf{R}_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. The homogeneous elements of R are those lying in $\bigcup_{\gamma\in\Gamma} \mathsf{R}_\gamma$. If $r \in \mathsf{R}_\gamma$, $r \neq 0$, then we write $\deg(r) = \gamma$. The grade set of R is $\Gamma_\mathsf{R} = \{\gamma \in \Gamma \mid \mathsf{R}_\gamma \neq \{0\}\}$. (We work only with gradings by torsion-free abelian groups because we are interested in the associated graded rings determined by valuations on division algebras; for such rings the grading is indexed by the value group of the valuation, which is torsion-free abelian.) If $\mathsf{R}' = \bigoplus_{\gamma\in\Gamma} \mathsf{R}'_\gamma$ is another graded ring, a *graded ring homomorphism* $\varphi\colon \mathsf{R} \to \mathsf{R}'$ is a ring homomorphism such that $\varphi(\mathsf{R}_\gamma) \subseteq \mathsf{R}'_\gamma$ for all $\gamma \in \Gamma$. If φ is an isomorphism, we say that R and R' are graded ring isomorphic, and write $\mathsf{R} \cong_g \mathsf{R}'$. For example, if $a \in \mathsf{R}$ is homogeneous and $a \in \mathsf{R}^*$, the group of units of R, then the map $\operatorname{int}(a)\colon \mathsf{R} \to \mathsf{R}$ given by $r \mapsto ara^{-1}$ is a graded ring automorphism of R.

A graded ring $\mathsf{E} = \bigoplus_{\gamma\in\Gamma} \mathsf{E}_\gamma$ is said to be a *graded division ring* if every nonzero homogeneous element of E lies in the multiplicative group E^* of units of E. See [HwW$_2$] for background on graded division ring and proofs of the properties mentioned here. Notably (as Γ is torsion-free abelian), E has no zero divisors, E^* consists entirely of homogeneous elements, Γ_E is a subgroup of Γ, E_0 is a division ring, and each nonzero homogeneous component E_γ of E is a 1-dimensional left and right E_0-vector space. Furthermore, if M is any left graded E- module (i.e., an E-module such that $\mathsf{M} = \bigoplus_{\gamma\in\Gamma} \mathsf{M}_\gamma$ with $\mathsf{E}_\gamma{\cdot}\mathsf{M}_\delta \subseteq \mathsf{M}_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$), then M is a free E-module with a homogeneous base, and any two such bases have the same cardinality; this cardinality is called the dimension of M and denoted $\dim_\mathsf{E}(\mathsf{M})$. Any such M is therefore called a left graded E-vector space.

A commutative graded division ring $\mathsf{T} = \bigoplus_{\gamma\in\Gamma} \mathsf{T}_\gamma$ is called a *graded field*. Such a T is an integral domain; let $q(\mathsf{T})$ denote the quotient field of T. A graded ring A which is a T-algebra is called a *graded* T*-algebra* if the module action of T on A makes A into a graded T-module. When this occurs, T is graded isomorphic to a graded subring of the center of A, which is denoted $Z(\mathsf{A})$. *All graded* T*-algebras considered in this paper are assumed to be finite-dimensional graded* T*-vector spaces.* Note that if A is a graded T-algebra, then $\mathsf{A} \otimes_\mathsf{T} q(\mathsf{T})$ is a $q(\mathsf{T})$-algebra of the same dimension. That is, $[\mathsf{A}:\mathsf{T}] = [\mathsf{A} \otimes_\mathsf{T} q(\mathsf{T}):q(\mathsf{T})]$, where $[\mathsf{A}:\mathsf{T}]$ denotes $\dim_\mathsf{T}(\mathsf{A})$ and $[\mathsf{A} \otimes_\mathsf{T} q(\mathsf{T}):q(\mathsf{T})] = \dim_{q(\mathsf{T})}(\mathsf{A} \otimes_\mathsf{T} q(\mathsf{T}))$.

Note that if A and B are graded algebras over a graded field T then $\mathsf{A} \otimes_\mathsf{T} \mathsf{B}$ is also a graded T-algebra with $(\mathsf{A} \otimes_\mathsf{T} \mathsf{B})_\gamma = \sum_{\delta\in\Gamma} \mathsf{A}_\delta \otimes_{\mathsf{T}_0} \mathsf{B}_{\gamma-\delta}$ for all $\gamma \in \Gamma$. Clearly, $\Gamma_{\mathsf{A}\otimes_\mathsf{T}\mathsf{B}} = \Gamma_\mathsf{A} + \Gamma_\mathsf{B}$. Also, if C is a finite-dimensional T_0-algebra, then $C \otimes_{\mathsf{T}_0} \mathsf{A}$ is a graded T-algebra with $(C \otimes_{\mathsf{T}_0} \mathsf{A})_\gamma = C \otimes_{\mathsf{T}_0} \mathsf{A}_\gamma$ for all $\gamma \in \Gamma$, and $\Gamma_{C\otimes_{\mathsf{T}_0}\mathsf{A}} = \Gamma_\mathsf{A}$.

A graded T-algebra A is said to be *simple* if it has no homogeneous two-sided ideals except A and $\{0\}$. A is called a *central simple* T-algebra if in addition its center $Z(\mathsf{A})$ is T. The theory of simple graded algebras is analogous to the usual theory of finite-dimensional simple algebras. This is described in [HwW$_2$, §1], where proofs of the following facts can be found. There is a graded Wedderburn Theorem for simple graded algebras: Any such A is graded isomorphic to $\mathsf{End}_\mathsf{E}(\mathsf{M})$ for some finite-dimensional graded vector space M over a graded division algebra E, and E is unique up to graded isomorphism. Also, while A_0 need not be simple, it is always semisimple, and $\mathsf{A}_0 \cong \prod_{j=1}^s M_{\ell_j}(\mathsf{E}_0)$ for some $\ell_j \times \ell_j$ matrix rings over E_0 (see the proof of Lemma 2.2 below). We write $[\mathsf{A}]$ for the equivalence class of A under the equivalence relation $\sim_g$ given by: $\mathsf{A} \sim_g \mathsf{A}'$ iff $\mathsf{A} \cong_g \mathsf{End}_\mathsf{E}(\mathsf{M})$ and $\mathsf{A}' \cong_g \mathsf{End}_\mathsf{E}(\mathsf{M}')$ for the same graded division algebra E. The Brauer group (of graded algebras) for T is

$$\mathsf{Br}(\mathsf{T}) = \{[\mathsf{A}] \mid \mathsf{A} \text{ is a graded central simple } \mathsf{T}\text{-algebra}\},$$

with the well-defined group operation $[\mathsf{A}]\cdot[\mathsf{A}'] = [\mathsf{A}\otimes_{\mathsf{T}}\mathsf{A}']$. When $\mathsf{A}\cong_g \mathsf{End}_{\mathsf{E}}(\mathsf{M})$ as above, then $[\mathsf{A}] = [\mathsf{E}]$, and up to graded isomorphism E is the only graded division algebra with $\mathsf{A}\sim_g \mathsf{E}$. There is a graded version of the Double Centralizer Theorem, see [HwW$_2$, Prop. 1.5] and also the Skolem-Noether Theorem, see [HwW$_2$, Prop. 1.6]. We recall the latter, since it has an added condition not appearing in the ungraded version.

Proposition 2.1 ([HwW$_2$, Prop. 1.6(b),(c)]). *Let A be a central simple graded algebra over the graded field T, and let B and B' be simple graded T-subalgebras of A. Let $\mathsf{C} = C_{\mathsf{A}}(\mathsf{B})$, the centralizer of B in A, and let $\mathsf{Z} = Z(\mathsf{C}) = Z(\mathsf{B})$ and $\mathsf{C}' = C_{\mathsf{A}}(\mathsf{B}')$. Let $\alpha\colon \mathsf{B}\to\mathsf{B}'$ be a graded T-algebra isomorphism. Then there is a homogeneous $a\in A^*$ such that $\alpha(b) = aba^{-1}$ for all $b\in\mathsf{B}$ if and only if there is a graded T-algebra isomorphism $\gamma\colon \mathsf{C}\to\mathsf{C}'$ such that $\gamma|_Z = \alpha|_{\mathsf{Z}}$. Such a γ exists whenever C_0 is a division ring.*

If E is a graded division algebra over a graded field T, we write $[\mathsf{E}\,{:}\,\mathsf{T}]$ for $\dim_{\mathsf{T}}(\mathsf{E})$. A basic fact is the Fundamental Equality

$$[\mathsf{E}\,{:}\,\mathsf{T}] \;=\; [\mathsf{E}_0\,{:}\,\mathsf{T}_0]\,|\Gamma_{\mathsf{E}}\,{:}\,\Gamma_{\mathsf{T}}|, \tag{2.1}$$

where $|\Gamma_{\mathsf{E}}\,{:}\,\Gamma_{\mathsf{T}}|$ denotes the index in Γ_{E} of its subgroup Γ_{T}. Also, it is known that $Z(\mathsf{E}_0)$ is abelian Galois over T_0, and there is a well-defined group epimorphism

$$\Theta_{\mathsf{E}}\colon \Gamma_{\mathsf{E}}\to \mathrm{Gal}(Z(\mathsf{E}_0)/\mathsf{T}_0) \quad \text{ given by } \Theta_{\mathsf{E}}(\gamma)(z) = aza^{-1} \text{ for any } z\in Z(\mathsf{E}_0) \text{ and } a\in\mathsf{E}_\gamma\setminus\{0\}. \tag{2.2}$$

Clearly, $\Gamma_{\mathsf{T}}\subseteq\ker(\Theta_{\mathsf{E}})$, so Θ_{E} induces an epimorphism of finite groups $\overline{\Theta}_{\mathsf{E}}\colon \Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}\to\mathrm{Gal}(Z(\mathsf{E}_0)/\mathsf{E}_0)$.

The terminology for different cases in (2.1) is carried over from valuation theory: We say that a graded field $\mathsf{S}\supseteq\mathsf{T}$ is *inertial over* T if $[\mathsf{S}_0:\mathsf{T}_0] = [\mathsf{S}:\mathsf{T}] < \infty$ and the field S_0 is separable over T_0. When this occurs, $\Gamma_{\mathsf{S}} = \Gamma_{\mathsf{T}}$, and the graded monomorphism $\mathsf{S}_0\otimes_{\mathsf{T}_0}\mathsf{T}\to\mathsf{S}$ given by multiplication in S is surjective by dimension count; so $\mathsf{S}\cong_g\mathsf{S}_0\otimes_{\mathsf{T}_0}\mathsf{T}$. At the other extreme, we say that a graded field $\mathsf{J}\supseteq\mathsf{T}$ is *totally ramified over* T if $|\Gamma_{\mathsf{J}}\,{:}\,\Gamma_{\mathsf{T}}| = [\mathsf{J}\,{:}\,\mathsf{T}] < \infty$. When this occurs, $\mathsf{J}_0 = \mathsf{T}_0$ and, more generally, for any $\gamma\in\Gamma_{\mathsf{T}}$, we have $\mathsf{J}_\gamma = \mathsf{T}_\gamma$ since $\dim_{\mathsf{T}_0}(\mathsf{J}_\gamma) = \dim_{\mathsf{J}_0}(\mathsf{J}_\gamma) = 1 = \dim_{\mathsf{T}_0}(\mathsf{T}_\gamma)$.

There is an extensive theory of finite-degree graded field extensions; [HwW$_1$] is a good reference for what we need here. Notably, there is a version of Galois theory: For graded fields $\mathsf{T}\subseteq\mathsf{F}$, with $[\mathsf{F}\,{:}\,\mathsf{T}]<\infty$, the (graded) Galois group of F over T is defined to be:

$$\mathsf{Gal}(\mathsf{F}/\mathsf{T}) \;=\; \{\psi\colon \mathsf{F}\to\mathsf{F}\mid \psi \text{ is a graded field automorphism of } \mathsf{F} \text{ and } \ \psi|_{\mathsf{T}} = \mathrm{id}\}.$$

Galois theory for graded fields follows easily from the classical ungraded theory since for the quotient fields of F and T we have $q(\mathsf{F})\cong\mathsf{F}\otimes_{\mathsf{T}} q(\mathsf{T})$, so $[q(\mathsf{F}):q(\mathsf{T})] = [\mathsf{F}:\mathsf{T}]$, and there is a canonical isomorphism $\mathsf{Gal}(\mathsf{F}/\mathsf{T})\to\mathrm{Gal}(q(\mathsf{F})/q(\mathsf{T}))$ (the usual Galois group) given by $\psi\mapsto\psi\otimes\mathrm{id}_{q(\mathsf{T})}$ (see [HwW$_1$, Cor. 2.5(d), Th. 3.11]). Thus, F is Galois over T iff $q(\mathsf{F})$ is Galois over $q(\mathsf{T})$, iff $|\mathsf{Gal}(\mathsf{F}/\mathsf{T})| = [\mathsf{F}\,{:}\,\mathsf{T}]$, iff T is the fixed ring of $\mathsf{Gal}(\mathsf{F}/\mathsf{T})$. This will arise here primarily in the inertial case: Suppose S is a graded field which contains and is inertial over T, with $[\mathsf{S}\,{:}\,\mathsf{T}]<\infty$. For any $\psi\in\mathsf{Gal}(\mathsf{S}/\mathsf{T})$ clearly the restriction $\psi|_{\mathsf{S}_0}$ lies in $\mathrm{Gal}(\mathsf{S}_0/\mathsf{T}_0)$. Moreover, as $\mathsf{S}\cong_g\mathsf{S}_0\otimes_{\mathsf{T}_0}\mathsf{T}$, for any $\rho\in\mathrm{Gal}(\mathsf{S}_0/\mathsf{T}_0)$ we have $\rho\otimes\mathrm{id}_{\mathsf{T}}\in\mathsf{Gal}(\mathsf{S}/\mathsf{T})$. Thus, the restriction map $\psi\mapsto\psi|_{\mathsf{S}_0}$ yields a canonical isomorphism $\mathsf{Gal}(\mathsf{S}/\mathsf{T})\to\mathrm{Gal}(\mathsf{S}_0/\mathsf{T}_0)$. Hence, as $[\mathsf{S}\,{:}\,\mathsf{T}] = [\mathsf{S}_0\,{:}\,\mathsf{T}_0]$, S is Galois over T iff S_0 is Galois over T_0.

Just as in the ungraded case, we can use Galois graded field extensions to build central simple graded algebras. If F is a Galois graded field extension of T, set $G = \mathsf{Gal}(\mathsf{F}/\mathsf{T})$ and take any 2-cocycle $f\in Z^2(G,\mathsf{F}^*)$. Then we can build a crossed product graded algebra $\mathsf{B} = (\mathsf{F}/\mathsf{T},G,f) = \bigoplus_{\sigma\in G}\mathsf{F}x_\sigma$ with multiplication given by $(ax_\sigma)(bx_\rho) = a\sigma(b)f(\sigma,\rho)x_{\sigma\rho}$ for all $a,b\in\mathsf{F}$, $\sigma,\rho\in G$. The grading is given by viewing B as a left graded F-vector space with $(x_\sigma)_{\sigma\in G}$ as a homogeneous base with $\deg(x_\sigma) = \frac{1}{|G|}\sum_{\rho\in G}\deg(f(\sigma,\rho))$. A short calculation shows that $\deg(f(\sigma,\tau)\,x_{\sigma\tau}) = \deg(x_\sigma)+\deg(x_\tau)$ for all $\sigma,\tau\in G$; it follows easily that B is a graded T-algebra. Indeed, B is a simple graded algebra with $Z(\mathsf{B})\cong_g\mathsf{T}$. Conversely, if A is any central simple graded T-algebra containing F as a strictly maximal graded subfield (i.e., $[\mathsf{F}\,{:}\,\mathsf{T}] = \deg(\mathsf{A})\ (=\sqrt{\dim_{\mathsf{T}}(\mathsf{A})}\ )$, then by the graded Double Centralizer Theorem $C_{\mathsf{A}}(\mathsf{F}) = \mathsf{F} = Z(\mathsf{F})$; so the graded Skolem-Noether Theorem, Prop. 2.1 above, applies to the graded isomorphisms in G, which yields that $\mathsf{A} \cong_g (\mathsf{F}/\mathsf{T}, G, f)$ for some $f \in Z^2(G, \mathsf{F}^*)$. From this one deduces, as in the ungraded case, that $\mathsf{Br}(\mathsf{F}/\mathsf{T}) \cong H^2(G, \mathsf{F}^*)$, where $\mathsf{Br}(\mathsf{F}/\mathsf{T})$ denotes the kernel of the canonical map $\mathsf{Br}(\mathsf{T}) \to \mathsf{Br}(\mathsf{F})$ given by $[\mathsf{A}] \mapsto [\mathsf{A} \otimes_{\mathsf{T}} \mathsf{F}]$. In particular, if $\mathsf{Gal}(\mathsf{F}/\mathsf{T})$, is cyclic, say with generator σ, then for any $b \in \mathsf{T}^*$ we have the graded cyclic algebra $\mathsf{C} = (\mathsf{F}/\mathsf{T}, \sigma, b) = \bigoplus_{i=0}^{r-1} \mathsf{F} y^i$, in which $ya = \sigma(a)y$ for all $a \in \mathsf{F}$ and $y^r = b$, where $r = [\mathsf{F} : \mathsf{T}]$. For the grading, we view C as a left graded F-vector space with homogeneous base $(1, y, y^2, \ldots, y^{r-1})$ with $\deg(y^i) = \frac{i}{r} \deg(b)$. Then C is a central simple graded T-algebra.

There are also norm maps in the graded setting: If $\mathsf{T} \subseteq \mathsf{F}$ are graded fields with $[\mathsf{F} : \mathsf{T}] < \infty$, then because F is a free module the norm $N_{\mathsf{F}/\mathsf{T}} \colon \mathsf{F} \to \mathsf{T}$ can be defined by $c \mapsto \det(\lambda_c)$, where for $c \in \mathsf{F}$, $\lambda_c \in \mathrm{Hom}_{\mathsf{T}}(\mathsf{F}, \mathsf{F})$ is the map $a \mapsto ca$. Clearly, $N_{\mathsf{F}/\mathsf{T}}(c) = N_{q(\mathsf{F})/q(\mathsf{T})}(c)$, where $N_{q(\mathsf{F})/q(\mathsf{T})}$ is the usual norm for the quotient fields. Also, if $c \in \mathsf{F}$ is homogeneous, say $c \in \mathsf{F}_\gamma$, then $N_{\mathsf{F}/\mathsf{T}}(c) \in \mathsf{T}_{[\mathsf{F}:\mathsf{T}]\gamma}$. Likewise, if B is a central simple graded T-algebra, then it is known that B is an Azumaya algebra of constant rank $[\mathsf{B} : \mathsf{T}]$ over T; hence there is a reduced norm map $\mathrm{Nrd}_{\mathsf{B}} \colon \mathsf{B} \to \mathsf{T}$. It is easy to see that for the central ring of quotients $q(\mathsf{B}) = \mathsf{B} \otimes_{\mathsf{T}} q(\mathsf{T})$ of B, we have $q(\mathsf{B})$ is a central simple algebra over the field $q(\mathsf{T})$, and it is known (see [HW_1, proof of Prop. 3.2(i)]) that for any $b \in \mathsf{B}$, $\mathrm{Nrd}_{\mathsf{B}}(b) = \mathrm{Nrd}_{q(\mathsf{B})}(b)$, where $\mathrm{Nrd}_{q(\mathsf{B})} \colon q(\mathsf{B}) \to q(\mathsf{T})$ is the reduced norm for $q(\mathsf{B})$. As usual, $b \in \mathsf{B}^*$ iff $\mathrm{Nrd}_{\mathsf{B}}(b) \in \mathsf{T}^*$. Also, if $b \in \mathsf{B}_\gamma$, then $\mathrm{Nrd}_{\mathsf{B}}(b) \in \mathsf{T}_{\deg(\mathsf{B})\gamma}$. Now assume further that B is a graded division algebra, so that all its units are homogeneous. Then for the commutator group $[\mathsf{B}^*, \mathsf{B}^*]$ of B, we have $[\mathsf{B}^*, \mathsf{B}^*] \subseteq \{b \in \mathsf{B} \mid \mathrm{Nrd}_{\mathsf{B}}(b) = 1\} \subseteq \mathsf{B}_0^*$. We define

$$\mathrm{SK}_1(\mathsf{B}) \ = \ \{b \in \mathsf{B} \mid \mathrm{Nrd}_{\mathsf{B}}(b) = 1\} \,/\, [\mathsf{B}^*, \mathsf{B}^*]. \tag{2.3}$$

The fact that both terms in the right quotient lie in B_0^* often makes that calculation of $\mathrm{SK}_1(\mathsf{B})$ much more tractable in this graded setting than for ungraded division algebras.

We need terminology for some types of simple graded algebras and graded division algebras over a graded field T. A central simple graded T-algebra I is said to be *inertial* (or unramified) if $[\mathsf{I}_0 : \mathsf{T}_0] = [\mathsf{I} : \mathsf{T}]$. When this occurs, the injective graded T-algebra homomorphism $\mathsf{I}_0 \otimes_{\mathsf{T}_0} \mathsf{T} \to \mathsf{I}$ is surjective by dimension count. So, $\Gamma_{\mathsf{I}} = \Gamma_{\mathsf{T}}$ and $\mathsf{I} \cong_g \mathsf{I}_0 \otimes_{\mathsf{T}_0} \mathsf{T}$. Hence, I_0 must be a central simple T_0-algebra. Moreover, if we let D be the T_0-central division algebra with $\mathsf{I}_0 \cong M_\ell(D)$, then $D \otimes_{\mathsf{T}_0} \mathsf{T}$ is clearly a graded division algebra over T which is also inertial over T, and $D \otimes_{\mathsf{T}_0} \mathsf{T} \sim_g \mathsf{I}$ (see Lemma 2.2 below).

The principal focus of this paper is on calculating SK_1 and unitary SK_1 for semiramified graded division algebras. Let E be a central graded division algebra over a graded field T. This E is said to be *semiramified* if $[\mathsf{E}_0 : \mathsf{T}_0] = |\Gamma_{\mathsf{E}} : \Gamma_{\mathsf{T}}| = \deg(\mathsf{E})$ and E_0 is a field. Since $\mathsf{E}_0 = Z(\mathsf{E}_0)$, E_0 is abelian Galois over T_0 and the epimorphism $\overline{\Theta}_{\mathsf{E}} \colon \Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}} \to \mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0)$ (see (2.2)) must be an isomorphism as $|\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}| = [\mathsf{E}_0 : \mathsf{T}_0] = |\,\mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0)|$. Furthermore, E has the graded subfield $\mathsf{E}_0\mathsf{T} \cong_g \mathsf{E}_0 \otimes_{\mathsf{T}_0} \mathsf{T}$, which is inertial and Galois over T with $\mathsf{Gal}(\mathsf{E}_0\mathsf{T}/\mathsf{T}) \cong \mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0)$. Because $[\mathsf{E}_0\mathsf{T} : \mathsf{T}] = \deg(\mathsf{E})$, the graded Double Centralizer Theorem [HwW_2, Prop. 1.5] shows that $C_{\mathsf{E}}(\mathsf{E}_0\mathsf{T}) = \mathsf{E}_0\mathsf{T}$, and hence $\mathsf{E}_0\mathsf{T}$ is a maximal graded subfield of E; thus, E is a graded abelian crossed product, as will be discussed in §3.

There is a significant special class of semiramified graded division algebras which are building blocks for all semiramified algebras. We say that a T-central graded division algebra N is *decomposably semiramified* (abbreviated DSR) if N has a maximal graded subfield S which is inertial over T and another maximal graded subfield J which is totally ramified over T. The graded Double Centralizer Theorem yields that $[\mathsf{S} : \mathsf{T}] = [\mathsf{J} : \mathsf{T}] = \deg(\mathsf{N})$. We thus have

$$\deg(\mathsf{N}) \ = \ [\mathsf{J} : \mathsf{T}] \ = \ |\Gamma_{\mathsf{J}} : \Gamma_{\mathsf{T}}| \ \le \ |\Gamma_{\mathsf{N}} : \Gamma_{\mathsf{T}}| \quad \text{and} \quad \deg(\mathsf{N}) \ = \ [\mathsf{S} : \mathsf{T}] \ = \ [\mathsf{S}_0 : \mathsf{T}_0] \ \le \ [\mathsf{N}_0 : \mathsf{T}_0]. \tag{2.4}$$

Since $|\Gamma_{\mathsf{N}} : \Gamma_{\mathsf{T}}|\,[\mathsf{N}_0 : \mathsf{T}_0] = [\mathsf{N} : \mathsf{T}] = \deg(\mathsf{N})^2$, the inequalities in (2.4) must be equalities, showing that $\mathsf{N}_0 = \mathsf{S}_0$ and $\Gamma_{\mathsf{N}} = \Gamma_{\mathsf{J}}$, hence N is semiramified. We call such an N decomposably semiramified because it is always decomposable into a tensor product of cyclic semiramified graded division algebras (see Prop. 4.4 below for the unitary analogue to this). The older term for such algebras is nicely semiramified (NSR).

While our focus in this paper is on central graded division algebras we will often take tensor products of such algebras, obtaining simple graded algebras which may have zero divisors. The next lemma allows us to recover information about the graded division algebra Brauer equivalent to such a tensor product.

Lemma 2.2. *Let B be a central simple graded algebra over the graded field T. Let D be the graded division algebra Brauer equivalent to B. Suppose B_0 is a simple ring. Then,*

(i) $\mathsf{B} \cong_g M_\ell(\mathsf{D})$ *for some ℓ, where the matrix ring $M_\ell(\mathsf{D})$ is given the standard grading in which $\big(M_\ell(\mathsf{D})\big)_\gamma = M_\ell(\mathsf{D}_\gamma)$ for all $\gamma \in \Gamma_{\mathsf{D}}$. Hence, $\mathsf{B}_0 \cong M_\ell(\mathsf{D}_0)$, $\Gamma_{\mathsf{B}} = \Gamma'_{\mathsf{B}} = \Gamma_{\mathsf{D}}$, and $\Theta_{\mathsf{B}} = \Theta_{\mathsf{D}}$, where $\Gamma'_{\mathsf{B}} = \{\deg(b) \mid b \in B^* \text{ and } b \text{ is homogeneous}\}$, and*

$$\Theta_{\mathsf{B}} \colon \Gamma'_{\mathsf{B}} \to \mathrm{Gal}(Z(\mathsf{B}_0)/\mathsf{T}_0) \text{ is given by } \deg(b) \mapsto \mathrm{int}(b)|_{Z(\mathsf{B}_0)}, \text{ for any homogeneous } b \in \mathsf{B}^* \tag{2.5}$$

where $\mathrm{int}(b)$ denotes conjugation by b.

(ii) B *is a graded division algebra if and only if B_0 is a division ring.*

Proof. (i) By the graded Wedderburn Theorem [HwW$_2$, Prop. 1.3], $\mathsf{B} \cong_g \mathsf{End}_{\mathsf{D}}(\mathsf{V})$ for some right graded vector space V of D. The grading on $\mathsf{End}_{\mathsf{D}}(\mathsf{V})$ is given by

$$\big(\mathsf{End}_{\mathsf{D}}(\mathsf{V})\big)_\varepsilon \;=\; \{f \in \mathsf{End}_{\mathsf{D}}(\mathsf{V}) \mid f(\mathsf{V}_\delta) \subseteq \mathsf{V}_{\varepsilon+\delta} \text{ for all } \delta \in \Gamma_{\mathsf{V}}\}.$$

Take a homogeneous D-base $(v_1, \ldots, v_\ell)$ of V, and let $\gamma_i = \deg(v_i)$, for $1 \le i \le \ell$; then, $\Gamma_{\mathsf{V}} = \bigcup_{i=1}^{\ell} \gamma_i + \Gamma_{\mathsf{D}}$. Let $\delta_1 + \Gamma_{\mathsf{D}}, \ldots, \delta_s + \Gamma_{\mathsf{D}}$ be the distinct cosets of Γ_{D} appearing in Γ_{V}, and let t_j be the number of i with $\gamma_i \in \delta_j + \Gamma_{\mathsf{D}}$. So, $t_1 + \ldots + t_s = \ell$. By replacing each v_i by a D^*-multiple of it, we may assume that $\deg(v_i) = \delta_j$ whenever $\gamma_i \in \delta_j + \Gamma_{\mathsf{D}}$. Then, we can reindex $(v_1, \ldots, v_\ell) = (v_{11}, \ldots, v_{1t_1}, \ldots, v_{s1}, \ldots, v_{st_s})$ with $\deg(v_{jk}) = \delta_j$ for all j, k. Then, $\mathsf{V}_{\delta_j} = \mathsf{D}_0\text{-span}(v_{j1}, \ldots, v_{jt_j})$ for $j = 1, 2, \ldots, s$, and

$$\begin{aligned}\big(\mathsf{End}_{\mathsf{D}}(\mathsf{V})\big)_0 \;&=\; \big\{f \in \mathsf{End}_{\mathsf{D}}(\mathsf{V}) \mid f(\mathsf{V}_\varepsilon) \subseteq \mathsf{V}_\varepsilon \text{ for all } \varepsilon \in \Gamma_{\mathsf{V}}\big\} \\ &\cong\; \prod_{j=1}^{s} \mathrm{End}_{\mathsf{D}_0}(\mathsf{D}_0\text{-span}(v_{j1}, \ldots, v_{jt_j})) \;\cong\; \prod_{j=1}^{s} M_{t_j}(\mathsf{D}_0).\end{aligned}$$

This is a direct product of s simple algebras. Since we have assumed that B_0 is simple, we must have $s = 1$, i.e., all the v_i have degree δ_1. It is then clear that when we use the base $(v_1, \ldots, v_\ell)$ for the isomorphism $\mathsf{End}_{\mathsf{D}}(\mathsf{V}) \cong M_\ell(\mathsf{D})$, the grading on $M_\ell(\mathsf{D})$ induced by the isomorphism is the standard grading. Thus, $\mathsf{B} \cong_g M_\ell(\mathsf{D})$ and hence $\mathsf{B}_0 \cong M_\ell(\mathsf{D}_0)$ and $\Gamma_{\mathsf{B}} = \Gamma_{\mathsf{D}}$. Then, $\Gamma'_{\mathsf{B}} = {\Gamma'}_{M_\ell(\mathsf{D})} = \Gamma_{\mathsf{D}}$ and, when we identify $Z(\mathsf{B}_0)$ with $Z(M_\ell(\mathsf{D}_0))$ and with $Z(\mathsf{D}_0)$, clearly $\Theta_{\mathsf{B}} = \Theta_{M_\ell(\mathsf{D})} = \Theta_{\mathsf{D}}$.

(ii) If B is a graded division algebra, then every nonzero homogeneous element of B lies in B^*. In particular, $\mathsf{B}_0 \setminus \{0\} \subseteq \mathsf{B}^*$, so B_0 is a division ring. Conversely, suppose B_0 is a division ring. Since B_0 is then simple, part (i) applies, showing that for some graded division algebra D, we have $\mathsf{B} \cong_g M_\ell(\mathsf{D})$ where $\mathsf{B}_0 \cong M_\ell(\mathsf{D}_0)$. Necessarily $\ell = 1$, as B_0 is a division ring. □

Corollary 2.3. *Let I and E be central graded division algebras over a graded field T, with I inertial, and let D be the graded division algebra with $\mathsf{D} \sim_g \mathsf{I} \otimes_{\mathsf{T}} \mathsf{E}$. Then, $\mathsf{D}_0 \sim \mathsf{I}_0 \otimes_{\mathsf{T}_0} \mathsf{E}_0$, $Z(\mathsf{D}_0) \cong Z(\mathsf{E}_0)$, $\Gamma_{\mathsf{D}} = \Gamma_{\mathsf{E}}$, and $\Theta_{\mathsf{D}} = \Theta_{\mathsf{E}}$.*

Proof. Let $\mathsf{B} = \mathsf{I} \otimes_{\mathsf{T}} \mathsf{E}$. Since $\mathsf{I} \cong_g \mathsf{I}_0 \otimes_{\mathsf{T}_0} \mathsf{T}$, we have $\mathsf{B} \cong_g \mathsf{I}_0 \otimes_{\mathsf{T}_0} \mathsf{E}$. Hence, $\mathsf{B}_0 \cong \mathsf{I}_0 \otimes_{\mathsf{T}_0} \mathsf{E}_0$, $Z(\mathsf{B}_0) \cong Z(\mathsf{I}_0) \otimes_{\mathsf{T}_0} Z(\mathsf{E}_0) \cong Z(\mathsf{E}_0)$, and $\Gamma_{\mathsf{B}} = \Gamma_{\mathsf{E}}$. Moreover, B_0 is simple as I_0 is central simple over T_0, so Lemma 2.2 applies to B. In particular, $\Gamma'_{\mathsf{B}} = \Gamma_{\mathsf{B}}$ and $\Theta_{\mathsf{B}} = \Theta_{\mathsf{E}}$. Since D is the graded division algebra with $\mathsf{D} \sim_g \mathsf{B}$, the Lemma yields $\mathsf{D}_0 \sim \mathsf{B}_0 \cong \mathsf{I}_0 \otimes_{\mathsf{T}_0} \mathsf{E}_0$, so $Z(\mathsf{D}_0) \cong Z(\mathsf{B}_0) \cong Z(\mathsf{E}_0)$, and $\Gamma_{\mathsf{D}} = \Gamma_{\mathsf{B}} = \Gamma_{\mathsf{E}}$, and $\Theta_{\mathsf{D}} = \Theta_{\mathsf{B}} = \Theta_{\mathsf{E}}$. □

3. Abelian crossed products and nonunitary SK_1 for semiramified algebras

Let M be a finite degree abelian Galois extension of a field K, and let $H = \text{Gal}(M/K)$. Let $X(M/K) = \text{Hom}(H, \mathbb{Q}/\mathbb{Z})$, the character group of H. Take any cyclic decomposition $H = \langle\sigma_1\rangle \times \ldots \times \langle\sigma_k\rangle$, and let r_i be the order of σ_i in H. Let $(\chi_1, \ldots, \chi_k)$ be the base of $X(M/K)$ dual to $(\sigma_1, \ldots, \sigma_k)$; so $\chi_i(\sigma_j) = \delta_{ij}/r_i + \mathbb{Z}$, where $\delta_{ij} = 1$ if $j = i$ and $= 0$ if $j \neq i$. Let L_i be the fixed field of $\ker(\chi_i)$. So, $M = L_1 \otimes_K \ldots \otimes_K L_k$, and for each i, L_i is cyclic Galois over K with $[L_i \colon K] = r_i$ and $\text{Gal}(L_i/K) = \langle\sigma_i|_K\rangle$. Let A be any central simple K-algebra containing M as a strictly maximal subfield (i.e., M is a maximal subfield of A with $[M \colon K] = \deg(A)$). By the Double Centralizer Theorem, the centralizer $C_A(M)$ is M. Recall that every algebra class in $\text{Br}(M/K)$ is represented by a unique such A. By Skolem-Noether, for each i there is $z_i \in A^*$ with $\text{int}(z_i)|_M = \sigma_i$, where $\text{int}(z_i)$ denotes conjugation by z_i. Set

$$u_{ij} \;=\; z_i z_j z_i^{-1} z_j^{-1} \quad \text{and} \quad b_i \;=\; z_i^{r_i}.$$

Since $\text{int}(u_{ij})|_M = \sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1} = \text{id}_M$ and $\text{int}(b_i)|_M = \sigma^{r_i} = \text{id}_M$, all the u_{ij} and b_i lie in $C_A(M)^* = M^*$. Take the index set $\mathcal{I} = \prod_{i=1}^k \{0, 1, 2, \ldots, r_i - 1\} \subseteq \mathbb{Z}^k$. For $\mathbf{i} = (i_1, \ldots, i_k) \in \mathcal{I}$, set $\sigma^{\mathbf{i}} = \sigma_1^{i_1} \ldots \sigma_k^{i_k}$ and $z^{\mathbf{i}} = z_1^{i_1} \ldots z_k^{i_k}$. So, $\text{int}(z^{\mathbf{i}})|_M = \sigma^{\mathbf{i}}$ and, as the map $\mathbf{i} \mapsto \sigma^{\mathbf{i}}$ is a bijection $\mathcal{I} \to H$, we have the crossed product decomposition

$$A \;=\; \bigoplus_{\mathbf{i} \in \mathcal{I}} M z^{\mathbf{i}}.$$

For $\mathbf{i}, \mathbf{j} \in \mathcal{I}$, if we set $\mathbf{i}{*}\mathbf{j}$ to be the element of $\mathcal{I}$ congruent to $\mathbf{i} + \mathbf{j}$ mod $r_1\mathbb{Z} \times \ldots \times r_k\mathbb{Z}$ in $\mathbb{Z}^k$, and set

$$f(\sigma^{\mathbf{i}}, \sigma^{\mathbf{j}}) \;=\; z^{\mathbf{i}} z^{\mathbf{j}} (z^{\mathbf{i}*\mathbf{j}})^{-1} \;\in\; M^*,$$

then $f \in Z^2(H, M^*)$ and the multiplication in A is given by

$$az^{\mathbf{i}} \cdot cz^{\mathbf{j}} \;=\; a\sigma^{\mathbf{i}}(c) f(\sigma^{\mathbf{i}}, \sigma^{\mathbf{j}})\, z^{\mathbf{i}*\mathbf{j}}, \quad \text{for all } \; a, c \in M \; \text{ and } \mathbf{i}, \mathbf{j} \in \mathcal{I}.$$

Since each $f(\sigma^{\mathbf{i}}, \sigma^{\mathbf{j}})$ is expressible as a computable product of the u_{ij} and the b_i and their images under H, the multiplication for A is completely determined by M, H, and the u_{ij} and b_i. Thus, we write $A = A(M/K, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$, where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_k)$, $\mathbf{u} = (u_{ij})_{i=1,\, j=1}^{k \quad k}$, and $\mathbf{b} = (b_1, \ldots, b_k)$.

It is easy to check (cf. [AS, Lemma 1.2] or [T_2, p. 423]) that the u_{ij} and the b_i satisfy the following relations, for all i, j, ℓ,

$$u_{ii} \;=\; 1, \quad u_{ji} \;=\; u_{ij}^{-1}, \quad \sigma_i(u_{j\ell})\sigma_j(u_{\ell i})\sigma_\ell(u_{ij}) \;=\; u_{j\ell}u_{\ell i}u_{ij} \tag{3.1}$$

and

$$N_{M/M^{\langle\sigma_i\rangle}}(u_{ij}) \;=\; b_i/\sigma_j(b_i), \tag{3.2}$$

where $M^{\langle\sigma_i\rangle}$ is the fixed field of M under $\langle\sigma_i\rangle$. It is known (cf. [AS, Th. 1.3]) that for any family of u_{ij} and b_i in M^* satisfying (3.1) and (3.2) there is a central simple K-algebra $A(M/K, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$.

Lemma 3.1. *Let $A = A(M/K, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$ as above, and let $B = A(M/K, \boldsymbol{\sigma}, \mathbf{v}, \mathbf{c})$. Then, there is a well-defined abelian crossed product $A(M/K, \boldsymbol{\sigma}, \mathbf{w}, \mathbf{d})$ where $w_{ij} = u_{ij}v_{ij}$ and $d_i = b_ic_i$ for all i, j. Moreover, $A \otimes_K B \sim A(M/K, \boldsymbol{\sigma}, \mathbf{w}, \mathbf{d})$ (Brauer equivalent).*

Proof. Because the u_{ij} and b_i satisfy (3.1) and (3.2) as do the v_{ij} and c_i, and the σ_i and the norm maps are multiplicative, the w_{ij} and d_i also satisfy (3.1) and (3.2). Therefore $A(M/K, \boldsymbol{\sigma}, \mathbf{w}, \mathbf{d})$ is a well-defined abelian crossed product.

We have the 2-cycle $f \in Z^2(H, M^*)$ representing A defined as above by, $f(\sigma^{\mathbf{i}}, \sigma^{\mathbf{j}}) \;=\; z^{\mathbf{i}} z^{\mathbf{j}} (z^{\mathbf{i}*\mathbf{j}})^{-1}$. The relations $z_i^{r_i} = b_i$ and $[z_i, z_j] = u_{ij}$ are encoded in f by

$$f(\sigma_i^\ell, \sigma_i) = \begin{cases} 1, & \text{if } 0 \le \ell \le r_i - 2 \\ b_i, & \text{if } \ell = r_i - 1 \end{cases} \qquad \text{and} \qquad f(\sigma_i, \sigma_j) = \begin{cases} 1 & \text{if } i < j \\ u_{ij} & \text{if } i > j. \end{cases} \tag{3.3}$$

We likewise build a cocycle $g \in Z^2(H, M^*)$ for $B = A(M/K, \boldsymbol{\sigma}, \mathbf{v}, \mathbf{c})$. Then, the cocycle $f{\cdot}g$ satisfies conditions corresponding to those for f in (3.3), so $f{\cdot}g$ is a cocycle for $C = A(M/K, \boldsymbol{\sigma}, \mathbf{w}, \mathbf{d})$ where $w_{ij} = u_{ij}v_{ij}$ and $d_i = b_i c_i$. From the group isomorphism $H^2(H, M^*) \cong \mathrm{Br}(M/K)$ it follows that $A \otimes_K B \sim C$. □

In Tignol's terminology in [T2], a central simple K-algebra containing M as a strictly maximal subfield *decomposes according to M* if $A \cong (L_1/K, \sigma_1, b_1) \otimes_K \ldots \otimes_K (L_k/K, \sigma_k, b_k)$ for some $b_1, \ldots, b_k \in K^*$. Clearly then, $A \cong A(M/K, \boldsymbol{\sigma}, \mathbf{1}, \mathbf{b})$, i.e., each $u_{ij} = 1$. Conversely, for any algebra $A(M/K, \boldsymbol{\sigma}, \mathbf{1}, \mathbf{b})$ (i.e., the z_i commute with each other), each z_j centralizes $b_i = z_i^{r_i}$, so $b_i \in M^H = K$ and the algebra decomposes according to M. The collection of such algebras yields an important distinguished subgroup $\mathrm{Dec}(M/K)$ of $\mathrm{Br}(M/K)$, i.e.

$$\begin{aligned} \mathrm{Dec}(M/K) &= \{\, [A] \in \mathrm{Br}(M/K) \mid A \text{ decomposes according to } M \} \\ &= \{\, [A(M/K, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})] \mid \text{every } u_{ij} = 1 \text{ and every } b_i \in K^* \,\}. \end{aligned} \tag{3.4}$$

Since $\mathrm{Br}(L_i/K) = \{\, [(L_i/K, \sigma_i, b)] \mid b \in K^* \,\}$, we have also $\mathrm{Dec}(M/K) = \prod_{i=1}^k \mathrm{Br}(L_i/K) \subseteq \mathrm{Br}(M/K)$. Tignol also also points out in [T2, p. 426] a homological characterization: From the short exact sequence of trivial H-modules $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ the long exact cohomology sequence yields the connecting homomorphism $\delta\colon H^1(H, \mathbb{Q}/\mathbb{Z}) \to H^2(H, \mathbb{Z})$, which is an isomorphism since $H^i(H, \mathbb{Q}) = 1$ for $i \geq 1$ as $\mathbb{Q}$ is uniquely divisible. For any $\chi \in X(M/K) = H^1(H, \mathbb{Q}/\mathbb{Z})$ and any $c \in K^* = H^0(H, M^*)$ it is known (cf. [Se, p. 204, Prop. 2]) that under the cup product pairing $\cup : H^2(H, \mathbb{Z}) \times H^0(H, M^*) \to H^2(H, M^*) = \mathrm{Br}(M/K)$, we have $\delta(\chi) \cup c = [(N/K, \rho|_N, c)]$, where N is the fixed field of $\ker(\chi)$ and $\rho \in H$ is determined by $\chi(\rho) = (1/|\chi|) + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$. Thus, the algebra class $[(L_1/K, \sigma_1, b_1) \otimes_K \ldots \otimes_K (L_k/K, \sigma_k, b_k)]$ in $\mathrm{Br}(M/K)$ corresponds to $(\delta(\chi_1) \cup b_1) + \ldots + (\delta(\chi_k) \cup b_k)$ in $H^2(H, M^*)$. Since the cup product is bimultiplicative and $X(M/K) = \langle \chi_1, \ldots, \chi_k \rangle$, we have

$$\mathrm{Dec}(M/K) \;=\; \big\langle \operatorname{im}\big(\cup : H^2(H, \mathbb{Z}) \times H^0(H, M^*) \to H^2(H, M^*) \big) \big\rangle \;=\; \prod_{\substack{K \subseteq L \subseteq M \\ \mathrm{Gal}(L/K) \text{ cyclic}}} \mathrm{Br}(L/K), \tag{3.5}$$

showing that $\mathrm{Dec}(M/K)$ is independent of the choice of the σ_i and the L_i. (Actually, Tignol uses (3.5) as his definition of $\mathrm{Dec}(M/F)$, and proves in [T2, Cor. 1.4] that this is equivalent to the definition given here in (3.4).)

The case when H is bicyclic is of particular interest, i.e., $H = \langle \sigma_1 \rangle \times \langle \sigma_2 \rangle$ and $M = L_1 \otimes_K L_2$. Then, for any algebra $A = A(M/K, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$, if we set $u = u_{12}$, then u determines all the u_{ij} as $u_{21} = u_{12}^{-1}$ and $u_{11} = u_{22} = 1$. We write, for short, $A = A(u, b_1, b_2)$. The conditions in (3.2) can then be restated:

$$b_1 \in M^{\langle \sigma_1 \rangle} = L_2, \quad b_2 \in M^{\langle \sigma_2 \rangle} = L_1, \quad N_{M/L_2}(u) = b_1/\sigma_2(b_1), \quad N_{M/L_1}(u) = \sigma_1(b_2)/b_2. \tag{3.6}$$

Note that $N_{M/K}(u) = N_{L_2/K}(b_1/\sigma_2(b_1)) = 1$. An easy calculation (cf. [AS, Th. 1.4]) shows that

$$\begin{aligned} &A(u, b_1, b_2) \;\cong\; A(u', b_1', b_2') \text{ if and only if there exist } c_1, c_2 \in M^* \text{ such that} \\ &u' = \big[c_1/\sigma_2(c_1)\big]\big[\sigma_1(c_2)/c_2\big] u, \quad b_1' = N_{M/L_2}(c_1) b_1, \quad \text{and} \quad b_2' = N_{M/L_1}(c_2) b_2. \end{aligned} \tag{3.7}$$

These observations can be formulated homologically: Recall that $\widehat{H}^{-1}(H, M^*) = \ker(N_{M/K})/I_H(M^*)$, where $\ker(N_{M/K}) = \{ m \in M^* \mid N_{M/K}(m) = 1 \}$ and, as $H = \langle \sigma_1 \rangle \times \langle \sigma_2 \rangle$,

$$I_H(M^*) \;=\; \big\{\, [a/\sigma_1(a)]\,[b/\sigma_2(b)] \mid a, b \in M^* \big\}.$$

We define a map

$$\eta\colon \mathrm{Br}(M/K) \longrightarrow \widehat{H}^{-1}(H, M^*) \quad \text{given by} \quad \big[A(u, b_1, b_2)\big] \mapsto u I_H(M^*). \tag{3.8}$$

By (3.7) above η is well-defined, and Lemma 3.1 shows that η is a group homomorphism. Given any $u \in M^*$ with $N_{M/K}(u) = 1$, Hilbert 90 gives $b_1 \in L_2^*$ and $b_2 \in L_1^*$ so that the conditions in (3.6) are satisfied and the algebra $A(u, b_1, b_2)$ exists. Therefore η is surjective. By (3.7),

$$\ker(\eta) \;=\; \big\{[A(u, b_1, b_2)] \mid u = 1\big\} \;=\; \mathrm{Dec}(M/K),$$

so η yields an isomorphism

$$\mathrm{Br}(M/K)\big/\,\mathrm{Dec}(M/K) \;\cong\; \widehat{H}^{-1}(\mathrm{Gal}(M/K), M^*) \qquad \text{whenever } M \text{ is bicyclic over } K. \tag{3.9}$$

This isomorphism is known (see, e.g., [T_2, Remarque, pp. 427–428]); indeed, it follows by comparing Draxl's formula [D, Kor. 8, p. 133] for SK_1 of the division algebras considered by Platonov in [P_2] with Platonov's formula in [P_2, Th. 4.11, Th. 4.17]. I learned of this description of the isomorphism from Tignol. Its relevance for SK_1 calculations is shown in the next proposition, which is the graded version of (1.4) and (1.2) above.

Proposition 3.2. *Suppose* N *is a DSR central graded division algebra over the graded field* T. *Then,*

(i) $\mathrm{SK}_1(\mathsf{N}) \;\cong\; \widehat{H}^{-1}(H, \mathsf{N}_0^*)$ *where* $H = \mathrm{Gal}(\mathsf{N}_0/\mathsf{T}_0)$.

(ii) *If* $\mathsf{N}_0 \cong L_1 \otimes_{\mathsf{T}_0} L_2$ *with each* L_i *cyclic Galois over* T_0, *then*

$$\mathrm{SK}_1(\mathsf{N}) \;\cong\; \mathrm{Br}(\mathsf{N}_0/\mathsf{T}_0)\big/\,\mathrm{Dec}(\mathsf{N}_0/\mathsf{T}_0).$$

Proof. (i) was given in [HW_1, Cor. 3.6(iv)], and (ii) follows from (i) and (3.9) above. □

We will generalize Prop. 3.2 in Th. 3.7 below by giving formulas for $\mathrm{SK}_1(\mathsf{E})$ when E is semiramified but not necessarily DSR. For this we need, first, a graded version of the abelian crossed products described at the beginning of this section. Second, we need a graded version of the $I \otimes N$ decomposition for semiramified division algebras over a Henselian valued field. Here I is inertial and N is DSR. (See [JW, Lemma 5.14, Th. 5.15] for the valued $I \otimes N$ decomposition.)

Here is the graded version of abelian crossed products. Let B be a central simple graded algebra over a graded field T. Assume that B contains a maximal graded subfield S with $[\mathsf{S}:\mathsf{T}] = \deg(\mathsf{B})$ $\big(= \sqrt{[\mathsf{B}:\mathsf{T}]}\,\big)$ such that S is Galois over T and $H = \mathsf{Gal}(\mathsf{S}/\mathsf{T})$ is abelian. We have $C_{\mathsf{B}}(\mathsf{S}) = \mathsf{S}$ by the graded Double Centralizer Theorem. For any cyclic decomposition $H = \langle\sigma_1\rangle \times \ldots \times \langle\sigma_k\rangle$, the graded Skolem-Noether Theorem, Prop. 2.1, is available as $C_{\mathsf{B}}(\mathsf{S}) = \mathsf{S} = Z(\mathsf{S})$; it shows that for each i there is $y_i \in \mathsf{B}^*$ with y_i homogeneous and $\mathrm{int}(y_i)|_{\mathsf{S}} = \sigma_i$. Set $c_i = y_i^{r_i}$ where r_i is the order of σ_i in H, and set $v_{ij} = y_i y_j y_i^{-1} y_j^{-1}$. Then, each $c_i \in C_{\mathsf{B}}(\mathsf{S})^* = \mathsf{S}^*$ with $\deg(y_i) = \frac{1}{r_i}\deg(c_i)$, and each $v_{ij} \in \mathsf{S}_0^*$. For each $\mathbf{i} = (i_1, \ldots, i_k) \in \mathcal{I} = \prod_{j=1}^{k}\{0, 1, 2, \ldots, r_j - 1\}$, set $y^{\mathbf{i}} = y_1^{i_1} \ldots y_k^{i_k}$. Then, $\mathrm{int}(y^{\mathbf{i}})|_{\mathsf{S}} = \sigma^{\mathbf{i}}$, and we have

$$\mathsf{B} \;=\; \bigoplus_{\mathbf{i}\in\mathcal{I}} \mathsf{S}\, y^{\mathbf{i}}. \tag{3.10}$$

For, the sum in the equation is direct since $\mathsf{B} \otimes_{\mathsf{T}} q(\mathsf{T}) = \bigoplus_{\mathbf{i}\in\mathcal{I}} (\mathsf{S} \otimes_{\mathsf{T}} q(\mathsf{T}))\, y^{\mathbf{i}}$ by the ungraded case. Then equality holds in (3.10) by dimension count. Note that B is a left graded S-vector space with homogeneous base $\big(y^{\mathbf{i}}\big)_{\mathbf{i}\in\mathcal{I}}$, and

$$\deg(y^{\mathbf{i}}) \;=\; \sum_{j=1}^{k} \tfrac{i_j}{r_j}\deg(c_j). \tag{3.11}$$

So,

$$\Gamma_{\mathsf{B}} \;=\; \big\langle \tfrac{1}{r_1}\deg(c_1), \ldots, \tfrac{1}{r_k}\deg(c_k)\big\rangle + \Gamma_{\mathsf{S}} \qquad \text{and} \qquad \text{each } \ B_\delta \;=\; \bigoplus_{\mathbf{i}\in\mathcal{I}} S_{(\delta - \deg(y^{\mathbf{i}}))} y^{\mathbf{i}}. \tag{3.12}$$

Since B is determined as a graded T-algebra by S, the σ_i, the v_{ij}, and the c_i, we write $\mathsf{B} = \mathsf{A}(\mathsf{S}/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{v}, \mathbf{c})$, where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_k)$, $\mathbf{v} = (v_{ij})_{i=1,\,j=1}^{k\quad\,k}$, and $\mathbf{c} = (c_1, \ldots, c_k)$. Note that the v_{ij} and the c_i satisfy the identities corresponding to (3.1) and (3.2). Conversely, given any $v_{ij} \in \mathsf{S}_0^*$ and $c_i \in \mathsf{S}^*$ satisfying those identities there is a central simple graded T algebra $\mathsf{A}(\mathsf{S}/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{v}, \mathbf{c})$. This is obtainable as $\mathsf{B} = \bigoplus_{\mathbf{i}\in\mathcal{I}} \mathsf{S}\, y^{\mathbf{i}}$ within the ungraded abelian crossed product $A = A(q(\mathsf{S})/q(\mathsf{T}), \boldsymbol{\sigma}, \mathbf{v}, \mathbf{c})$, with the grading on B determined by that on S and $\deg(y_i) = \frac{1}{r_i}\deg(c_i)$, as described above. To see that B is a graded ring, one uses that each $\sigma \in H$ is a (degree-preserving) graded automorphism of S and that $\deg(y^{\mathbf{i}} \cdot y^{\mathbf{j}}) = \deg(y^{\mathbf{i}}) + \deg(y^{\mathbf{j}})$ for all $\mathbf{i}, \mathbf{j} \in \mathcal{I}$, since all the v_{ij} have degree 0. This B is graded simple, since any nontrivial proper homogeneous ideal would localize to a nontrivial proper ideal of the simple $q(\mathsf{T})$-algebra A.

Remark 3.3. The graded analogue to Lemma 3.1 holds, with the same proof, since for S Galois over T, we have $\mathsf{Br}(\mathsf{S}/\mathsf{T}) \cong H^2(\mathsf{Gal}(\mathsf{S}/\mathsf{T}), \mathsf{S}^*)$.

The graded abelian crossed products we work with here will have S inertial over T and will be semiramified, as described in the next lemma.

Lemma 3.4. *Let S be an inertial graded field extension of T with S abelian Galois over T. Let $H = \mathsf{Gal}(\mathsf{S}/\mathsf{T}) = \langle\sigma_1\rangle \times \ldots \times \langle\sigma_k\rangle$ as above with r_i the order of σ_i, and let $\mathsf{B} = \mathsf{A}(\mathsf{S}/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{v}, \mathbf{c})$ be a graded abelian crossed product. Let $\delta_i = \frac{1}{r_i}\deg(c_i) \in \Gamma_{\mathsf{B}}$ and $\overline{\delta_i} = \delta_i + \Gamma_{\mathsf{T}} \in \Gamma_{\mathsf{B}}/\Gamma_{\mathsf{T}}$. Then, B is a semiramified graded division algebra if and only if each $\overline{\delta_i}$ has order r_i and $\overline{\delta_1}, \ldots, \overline{\delta_k}$ are independent in $\Gamma_{\mathsf{B}}/\Gamma_{\mathsf{T}}$. When this occurs, $\mathsf{B}_0 = \mathsf{S}_0$ and $\Gamma_{\mathsf{B}}/\Gamma_{\mathsf{T}} = \langle\overline{\delta_1}\rangle \times \ldots \times \langle\overline{\delta_k}\rangle \cong H$.*

Proof. Since S is inertial and Galois over T, S_0 is Galois over T_0 with $\mathsf{Gal}(\mathsf{S}_0/\mathsf{T}_0) \cong \mathsf{Gal}(\mathsf{S}/\mathsf{T}) = H$. We identify H with $\mathrm{Gal}(\mathsf{S}_0/\mathsf{T}_0)$. We have $\mathsf{S}_0 \subseteq \mathsf{B}_0$ and $[\mathsf{S}_0 : \mathsf{T}_0] = [\mathsf{S} : \mathsf{T}] = \deg(\mathsf{B})$.

Suppose B is a semiramified graded division algebra. Then, $[\mathsf{B}_0 : \mathsf{T}_0] = \deg(\mathsf{B}) = [\mathsf{S}_0 : \mathsf{T}_0]$, so $\mathsf{B}_0 = \mathsf{S}_0$. Since B is semiramified, the epimorphism $\overline{\Theta}_{\mathsf{B}} : \Gamma_{\mathsf{B}}/\mathsf{T} \to H$ is an isomorphism, as noted in §2. When we represent $\mathsf{B} = \bigoplus_{\mathbf{i}\in\mathcal{I}} \mathsf{S}y^{\mathbf{i}}$ as above, since $\mathrm{int}(y_i) = \sigma_i$ and $\deg(y_i) = \frac{1}{r_i}\deg(c_i) = \delta_i$, we have $\overline{\Theta}_{\mathsf{B}}(\overline{\delta_i}) = \sigma_i$. Hence, $\overline{\delta_i}$ has the same order r_i as σ_i, and

$$\Gamma_{\mathsf{B}}/\Gamma_{\mathsf{T}} \ = \ \overline{\Theta}_{\mathsf{B}}^{-1}(H) \ = \ \overline{\Theta}_{\mathsf{B}}^{-1}(\langle\sigma_1\rangle) \times \ldots \times \overline{\Theta}_{\mathsf{B}}^{-1}(\langle\sigma_k\rangle) \ = \ \langle\overline{\delta_i}\rangle \times \ldots \times \langle\overline{\delta_k}\rangle,$$

so the $\overline{\delta_i}$ are independent in $\Gamma_{\mathsf{B}}/\Gamma_{\mathsf{T}}$.

Conversely, suppose each $\overline{\delta_i}$ has order r_i and the $\overline{\delta_i}$ are independent in $\Gamma_{\mathsf{B}}/\Gamma_{\mathsf{T}}$. Then,

$$|\Gamma_{\mathsf{B}} : \Gamma_{\mathsf{T}}| \ \geq \ \prod_{i=1}^{k} |\langle\overline{\delta_i}\rangle| \ = \ r_1 \ldots r_k \ = \ |H| \ = \ \deg(\mathsf{B}).$$

Hence,

$$[\mathsf{B}_0 : \mathsf{T}_0] \ = \ [\mathsf{B} : \mathsf{T}]\big/|\Gamma_{\mathsf{B}} : \Gamma_{\mathsf{T}}| \ \leq \ \deg(\mathsf{B})^2/\deg(\mathsf{B}) \ = \ [\mathsf{S}_0 : \mathsf{T}_0]. \tag{3.13}$$

Since $\mathsf{S}_0 \subseteq \mathsf{B}_0$, (3.13) shows that $\mathsf{B}_0 = \mathsf{S}_0$, so equality holds in (3.13). Since B_0 is a field, B is a graded division algebra by Lemma 2.2(ii), and it is semiramified by the equality in (3.13). □

Observe that if E is any semiramified graded T-central division algebra, then E is a graded abelian crossed product as described in Lemma 3.4. For, $\mathsf{E}_0\mathsf{T}$ is a maximal graded subfield of E which is inertial and Galois over T with $\mathsf{Gal}(\mathsf{E}_0\mathsf{T}/\mathsf{T}) \cong \mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0)$, which is abelian.

Proposition 3.5. *Let E be a semiramified central graded division algebra over the graded field T. Then,*

(i) *There exist graded T-central division algebras I and N such that I is inertial, N is DSR, and $\mathsf{E} \sim_g \mathsf{I} \otimes_{\mathsf{T}} \mathsf{N}$ in $\mathsf{Br}(\mathsf{T})$. When this occurs, $\mathsf{N}_0 \cong \mathsf{E}_0$, $\Gamma_N = \Gamma_E$, $\Theta_{\mathsf{N}} = \Theta_{\mathsf{E}}$, and E_0 splits I_0.*

(ii) *For any other decomposition $\mathsf{E} \sim_g \mathsf{I}' \otimes_{\mathsf{T}} \mathsf{N}'$ with I' inertial and N' DSR, we have $\mathsf{I}'_0 \equiv \mathsf{I}_0 \pmod{\mathrm{Dec}(\mathsf{E}_0/\mathsf{T}_0)}$.*

We do not give a proof of Prop. 3.5 because it is a simpler version of the proof of the analogous unitary result, which is Prop. 4.5 below. Also, Prop. 3.5 is the graded analogue of a known result for semiramified division algebras over Henselian valued fields, [JW, Lemma 5.14, Th. 5.15], and the graded result given here is deducible from the Henselian one.

Lemma 3.6. *For the semiramified graded division algebra* $\mathsf{E} = \mathsf{A}(\mathsf{E}_0\mathsf{T}/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{v}, \mathbf{c})$ *as above, write* $\mathsf{E} \sim_g \mathsf{I} \otimes_T \mathsf{N}$ *with* I *inertial and* N *DSR; so* $[\mathsf{I}_0] \in \mathrm{Br}(\mathsf{E}_0/\mathsf{T}_0)$*. If* $\mathsf{I}_0 \sim A(\mathsf{E}_0/\mathsf{T}_0, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$*, then by changing the chioce of the* $y_i \in \mathsf{E}^*$ *inducing* σ_i *on* $\mathsf{E}_0\mathsf{T}$ *we have* $\mathsf{E} = \mathsf{A}(\mathsf{E}_0\mathsf{T}/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{e})$ *with the same* $\mathbf{u}$ *as for* I_0*.*

Proof. Let J be a maximal graded subfield of N which is totally ramified over T, so $\Gamma_{\mathsf{N}} = \Gamma_{\mathsf{J}}$. Because N is semiramified, the map $\Theta_{\mathsf{N}} \colon \Gamma_{\mathsf{N}}/\Gamma_{\mathsf{T}} \to \mathrm{Gal}(\mathsf{N}_0/\mathsf{T}_0)$ is an isomorphism. But also $\mathsf{N}_0 = \mathsf{E}_0$. Thus, for each i, we can choose $x_i \in \mathsf{J}^*$ with $\Theta_{\mathsf{N}}(\deg(x_i)) = \sigma_i|_{\mathsf{E}_0}$. Let $d_i = x_i^{r_i} \in (\mathsf{N}_0\mathsf{T})^* = (\mathsf{E}_0\mathsf{T})^*$. Then, $\mathsf{N} \cong_g \mathsf{A}(\mathsf{E}_0\mathsf{T}/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{w}, \mathbf{d})$, where each $w_{ij} = x_i x_j x_i^{-1} x_j^{-1} = 1$, as all the x_i lie in the graded field J. Let $\mathsf{I}_0' = A(\mathsf{E}_0/\mathsf{T}_0, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$, which is Brauer equivalent to I_0. Then set $\mathsf{I}' = \mathsf{I}_0' \otimes_{\mathsf{T}_0} \mathsf{T}$, which is an inertial T-algebra with $\mathsf{I}' \sim_g \mathsf{I}$. Since $\mathsf{I}' \otimes_{\mathsf{T}} \mathsf{N} \sim_g \mathsf{I} \otimes_{\mathsf{T}} \mathsf{N} \sim_g \mathsf{E}$, we may without any loss replace I by I'. Then, as $\mathsf{I}_0 \cong A(\mathsf{E}_0/\mathsf{T}_0, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$, clearly $\mathsf{I} \cong_g \mathsf{I}_0 \otimes_{\mathsf{T}_0} \mathsf{T} \cong_g \mathsf{A}(\mathsf{E}_0\mathsf{T}/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$. Let $\mathsf{E}' = \mathsf{A}(\mathsf{E}_0\mathsf{T}/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{e})$, where each $e_i = b_i d_i$, and let $y_1', \dots, y_k'$ be the associated generators of E' over $\mathsf{E}_0\mathsf{T}$. Then, $\mathsf{E} \sim_g \mathsf{I} \otimes_{\mathsf{T}} \mathsf{N} \sim_g \mathsf{E}'$, by Remark 3.3, as $u_{ij} w_{ij} = u_{ij}$. Note that for each i, $\deg(e_i) = \deg(d_i)$, as $\deg(b_i) = 0$. Hence, $\Gamma_{\mathsf{E}'} = \Gamma_{\mathsf{N}}$ by (3.12). Furthermore, E' is a semiramified graded division algebra since N is, because Lemma 3.4 shows that this is determined by the $\deg(e_i)$, resp. $\deg(d_i)$. Because E' is a graded division algebra (not just a graded simple algebra), as is E, from $\mathsf{E} \sim_g \mathsf{E}'$ the uniqueness in the graded Wedderburn Theorem [HwW_2, Prop. 1.3] yields a graded T-isomorphism $\eta \colon \mathsf{E} \to \mathsf{E}'$. By the graded Skolem-Noether Theorem, Prop. 2.1, η can be chosen so that $\eta|_{\mathsf{E}_0\mathsf{T}} = \mathrm{id}$. Then replacing the y_i by $\eta^{-1}(y_i')$ in the presentation of E changes each v_{ij} to u_{ij}. □

Theorem 3.7. *Suppose* E *is a semiramified* T*-central graded division algebra, and take any decomposition* $\mathsf{E} \sim_g \mathsf{I} \otimes_{\mathsf{T}} \mathsf{N}$ *where* I *is an inertial graded* T*-algebra and* N *is DSR. Then,*

(i) *Since* $\mathsf{I}_0 \in \mathrm{Br}(\mathsf{E}_0/\mathsf{T}_0)$ *with* E_0 *abelian Galois over* T_0*, we can write* $\mathsf{I}_0 \sim A(\mathsf{E}_0/\mathsf{T}_0, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$ *in* $\mathrm{Br}(\mathsf{T}_0)$*. Then,*

$$\mathrm{SK}_1(\mathsf{E}) \;\cong\; \widehat{H}^{-1}(H, \mathsf{E}_0^*)\big/\big\langle \mathrm{im}\{u_{ij} \mid 1 \le i, j \le k\}\big\rangle, \quad \textit{where } H = \mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0).$$

(ii) *If* $\mathsf{E}_0 \cong L_1 \otimes_{\mathsf{T}_0} L_2$ *with each* L_i *cyclic Galois over* T_0*, then*

$$\mathrm{SK}_1(\mathsf{E}) \;\cong\; \mathrm{Br}(\mathsf{E}_0/\mathsf{T}_0)\big/\big[\,\mathrm{Dec}(\mathsf{E}_0/\mathsf{T}_0)\cdot\langle[\mathsf{I}_0]\rangle\big],$$

where $\mathrm{Dec}(\mathsf{E}_0/\mathsf{T}_0) = \mathrm{Br}(L_1/\mathsf{T}_0)\cdot \mathrm{Br}(L_2/\mathsf{T}_0)$*.*

Proof. The definition of SK_1 for graded division algebras is given in (2.3) above. (i) We have $H = \mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0) \cong \mathsf{Gal}(\mathsf{E}_0\mathsf{T}/\mathsf{T})$. Since E is semiramified, $H \cong \Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$ via $\overline{\Theta}_{\mathsf{E}}^{-1}$ (see §2). By [HW_1, Cor. 3.6(ii)] there is an exact sequence

$$0 \longrightarrow H \wedge H \xrightarrow{\;\Phi\;} \widehat{H}^{-1}(G, \mathsf{E}_0^*) \xrightarrow{\;\Psi\;} \mathrm{SK}_1(\mathsf{E}) \longrightarrow 0. \tag{3.14}$$

The maps in (3.14) are given as follows: Let $\ker(\mathrm{Nrd}_{\mathsf{E}}) = \{a \in \mathsf{E}^* \mid \mathrm{Nrd}_{\mathsf{E}}(a) = 1\} \subseteq \mathsf{E}_0^*$, and let $\ker(N_{\mathsf{E}_0/\mathsf{T}_0}) = \{a \in \mathsf{E}_0^* \mid N_{\mathsf{E}_0/\mathsf{T}_0}(a) = 1\}$. Because E is semiramified, by [HW_2, Remark 2.1(iii), Lemma 2.2], $\ker(\mathrm{Nrd}_{\mathsf{E}}) = \ker(N_{\mathsf{E}_0/\mathsf{T}_0})$. For every $\rho \in H$, choose any $y_\rho \in \mathsf{E}^*$ with $\mathrm{int}(y_\rho)|_{\mathsf{E}_0} = \rho$. The map Φ is given by: for $\rho, \pi \in H$,

$$\Phi(\rho \wedge \pi) \;=\; y_\rho y_\pi y_\rho^{-1} y_\pi^{-1} I_H(\mathsf{E}_0^*) \;\in \ker(N_{\mathsf{E}_0/\mathsf{T}_0})\big/I_H(\mathsf{E}_0^*) \;=\; \widehat{H}^{-1}(H, \mathsf{E}_0^*).$$

The map Ψ is given by: for $a \in \ker(N_{\mathsf{E}_0/\mathsf{T}_0})$,

$$\Psi\big(a\, I_H(\mathsf{E}_0)^*\big) \;=\; a\,[\mathsf{E}^*, \mathsf{E}^*] \;\in \ker(\mathrm{Nrd}_{\mathsf{E}})\big/[\mathsf{E}^*, \mathsf{E}^*] \;=\; \mathrm{SK}_1(\mathsf{E}).$$

By Lemma 3.6, we can assume $\mathsf{E} = \mathsf{A}(\mathsf{E}_0\mathsf{T}/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{c})$ (with the same u_{ij} as for I_0). Since $H \cong \mathsf{Gal}(\mathsf{E}_0\mathsf{T}/\mathsf{T}) = \langle\sigma_1\rangle \times \ldots \times \langle\sigma_k\rangle$, we have $H \wedge H = \langle \sigma_i \wedge \sigma_j \mid 1 \le i, j \le k\rangle$. There are $y_1, \dots, y_k \in \mathsf{E}^*$, with $\mathrm{int}(y_i)|_{\mathsf{E}_0} = \sigma_i$ and $y_i y_j y_i^{-1} y_j^{-1} = u_{ij}$. So we can take $y_{\sigma_i} = y_i, 1 \le i \le k$, yielding for the Φ in (3.14),

$\Phi(\sigma_i \wedge \sigma_j) = u_{ij} I_H(\mathsf{E}_0^*) \in \widehat{H}^{-1}(H, \mathsf{E}_0^*)$. Thus, $\mathrm{im}(\Phi) = \langle \mathrm{im}(u_{ij}) \mid 1 \le i,j \le k\rangle$, and part (i) follows from the exact sequence (3.14).

(ii) When $\mathsf{E}_0 = L_1 \otimes_{\mathsf{T}_0} L_2$, $H = \mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0)$ has rank 2, say $H = \langle \sigma_1 \rangle \times \langle \sigma_2 \rangle$. So, $H \wedge H = \langle \sigma_1 \wedge \sigma_2 \rangle$ and $\mathrm{im}(\Phi) = \langle u_{12} I_H(\mathsf{E}_0^*)\rangle$. As we saw in discussion of (3.9) above, the isomorphism

$$\mathrm{Br}(\mathsf{E}_0/\mathsf{T}_0)\big/ \mathrm{Dec}(\mathsf{E}_0/\mathsf{T}_0) \longrightarrow \widehat{H}^{-1}(H, \mathsf{E}_0^*),$$

maps $[\mathsf{I}_0] + \mathrm{Dec}(\mathsf{E}_0/\mathsf{T}_0)$ to $u_{12} I_H(\mathsf{E}_0^*)$. Thus using part (i),

$$\mathrm{SK}_1(\mathsf{E}) \;\cong\; \widehat{H}^{-1}(H, \mathsf{E}_0^*)\big/\big\langle \mathrm{im}(u_{12})\big\rangle \;\cong\; \mathrm{Br}(\mathsf{E}_0/\mathsf{T}_0)\big/\big[\mathrm{Dec}(\mathsf{E}_0/\mathsf{T}_0)\cdot\langle[\mathsf{I}_0]\rangle\big]. \qquad \square$$

For any division algebra D over a Henselian valued field F, the valuation on F extends uniquely to a valuation on D, and we write $\overline{D}$ for its residue division algebra and Γ_D for its value group. Recall the isomorphism $\mathrm{SK}_1(D) \cong \mathrm{SK}_1(\mathsf{gr}(D))$ for a tame such D, proved in [HW$_1$, Th. 4.8]. By using this isomorphism, Th. 3.7 yields the following:

Corollary 3.8. *Let F be field with Henselian valuation v, and let D be an F-central division algebra which (with respect to the unique extension of v to D) is tame and semiramified. Take any decomposition $D \sim I \otimes_F N$, where I and N are F-central division algebras with I inertial and N DSR.*

(i) *Since $\overline{I} \in \mathrm{Br}(\overline{D}/\overline{F})$ with $\overline{D}$ abelian Galois over $\overline{F}$, we can write $\overline{I} \sim A(\overline{D}/\overline{F}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$ in $\mathrm{Br}(\overline{F})$. Then,*

$$\mathrm{SK}_1(D) \;\cong\; \widehat{H}^{-1}(H, \overline{D}^*)\big/\big\langle \mathrm{im}\{u_{ij} \mid 1 \le i,j \le k\}\big\rangle, \quad \textit{where} \;\; H = \mathrm{Gal}(\overline{D}/\overline{F}).$$

(ii) *If $\overline{D} \cong L_1 \otimes_{\overline{F}} L_2$ with each L_i cyclic Galois over $\overline{F}$, then*

$$\mathrm{SK}_1(D) \;\cong\; \mathrm{Br}(\overline{D}/\overline{F})\big/\big[\mathrm{Dec}(\overline{D}/\overline{F})\cdot\langle[\overline{I}]\rangle\big],$$

where $\mathrm{Dec}(\overline{D}/\overline{F}) = \mathrm{Br}(L_1/\overline{F}) \cdot \mathrm{Br}(L_2/\overline{F})$.

Proof. (That D is tame and semiramified means $[\overline{D}:\overline{F}] = |\Gamma_D : \Gamma_F| = \sqrt{[D{:}F]}$ and $\overline{D}$ is a field separable over $\overline{F}$.) Let $\mathsf{T} = \mathsf{gr}(F)$, the associated graded ring of F with respect to the filtration on it induced by the valuation (cf. [HwW$_2$] or [HW$_1$]). Since F is a field, T is a graded field with $\mathsf{T}_0 = \overline{F}$ and $\Gamma_{\mathsf{T}} = \Gamma_F$. Since v is Henselian, it has unique extensions to valuations on D, I, and N; with respect to these valuations, let $\mathsf{E} = \mathsf{gr}(D)$, $\mathsf{I} = \mathsf{gr}(I)$, and $\mathsf{N} = \mathsf{gr}(N)$. These are graded division rings, with $\mathsf{E}_0 = \overline{D}$, $\mathsf{I}_0 = \overline{I} \sim A(\mathsf{E}_0/\mathsf{T}_0, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$, and $\mathsf{N}_0 = \overline{N} \cong \overline{D} = \mathsf{E}_0$. Moreover, as D, I, and N are each tame over F, it follows by [HwW$_2$, Prop. 4.3] that T is the center of E, I, and N, and $[\mathsf{E}{:}\mathsf{T}] = [D{:}F]$, $[\mathsf{I}{:}\mathsf{T}] = [I{:}F]$, and $[\mathsf{N}{:}\mathsf{T}] = [N{:}F]$. Since I is inertial over F, we have I is inertial over T. That N is DSR means (cf. [JW, p. 149], where the term NSR is used) that N has maximal subfields S and J with S inertial over F and J totally ramified of radical type over F. Then, $\mathsf{gr}(S)$ and $\mathsf{gr}(J)$ are maximal graded subfields of N with $\mathsf{gr}(S)$ inertial over T and $\mathsf{gr}(J)$ totally ramified over T. So, N is DSR. Similarly, E is semiramified since D is tame and semiramified. Let $\mathrm{Br}_t(F)$ be the tame part of the Brauer group $\mathrm{Br}(F)$. From the isomorphism $\mathrm{Br}_t(F) \cong \mathsf{Br}(\mathsf{T})$ given by [HwW$_2$, Th. 5.3], we obtain $\mathsf{E} \sim_g \mathsf{I} \otimes_{\mathsf{T}} \mathsf{N}$ from $D \sim I \otimes_F N$. Thus, Th. 3.7 applies to E with the decomposition $\mathsf{E} \sim_g \mathsf{I} \otimes_{\mathsf{T}} \mathsf{N}$, and the assertions of Cor. 3.8 follow immediately as $\mathrm{SK}_1(D) \cong \mathrm{SK}_1(\mathsf{E})$ by [HW$_1$, Th. 4.8]. $\square$

Example 3.9. Take any integer $n \ge 2$ and let K be any field containing a primitive n^2-root of unity ω. Let $\mathsf{T} = K[x, x^{-1}, y, y^{-1}]$, the Laurent polynomial ring, graded as usual by $\mathbb{Z} \times \mathbb{Z}$ with $\mathsf{T}_{(k,\ell)} = Kx^k y^\ell$; in particular, $\mathsf{T}_0 = K$. (So $\mathsf{T} \cong_g \mathsf{gr}\big(K((x))((y))\big)$ where the iterated Laurent power series ring $K((x))(y))$ is given its usual rank 2 Henselian valuation.) Take any $a, b \in K^*$ such that $\big[K(\sqrt[n]{a}, \sqrt[n]{b}\,){:}K\big] = n^2$, and let E be the graded symbol algebra $\mathsf{E} = (ax^n, by^n, \mathsf{T})_\omega$, of degree n^2. That is, E is the graded central simple T-algebra with homogenous generators i and j such that $i^{n^2} = ax^n$, $j^{n^2} = by^n$, and $ij = \omega ji$, and $\deg(i) = (\frac{1}{n}, 0)$, $\deg(j) = (0, \frac{1}{n})$. Then, $\Gamma_{\mathsf{E}} = (\frac{1}{n}\mathbb{Z}) \times (\frac{1}{n}\mathbb{Z})$, and $\mathsf{E}_0 = K(i^n x^{-1}, j^n y^{-1}) \cong K(\sqrt[n]{a}, \sqrt[n]{b}\,)$. Since E_0 is a field, by Lemma 2.2(ii) E is a graded division ring, which is clearly semiramified. We can write $\mathsf{E}_0 = L_1 \otimes_K L_2$ where $L_1 = K(\sqrt[n]{a})$ and $L_2 = K(\sqrt[n]{b})$, and $H = \operatorname{Gal}(\mathsf{E}_0/K) = \langle\sigma_1\rangle \times \langle\sigma_2\rangle$ where $\sigma_1(\sqrt[n]{a}) = \omega^n \sqrt[n]{a}$, $\sigma_1(\sqrt[n]{b}) = \sqrt[n]{b}$ and $\sigma_2(\sqrt[n]{b}) = \omega^n \sqrt[n]{b}$, $\sigma_2(\sqrt[n]{a}) = \sqrt[n]{a}$. Since $\operatorname{int}(j^{-1})|_{\mathsf{E}_0} = \sigma_1$ and $\operatorname{int}(i)|_{\mathsf{E}_0} = \sigma_2$, we can express E as a graded abelian crossed product with $y_1 = j^{-1}$ and $y_2 = i$, obtaining $\mathsf{E} = \mathsf{A}(\mathsf{T}(\sqrt[n]{a}, \sqrt[n]{b})/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{d})$, where $u_{11} = u_{22} = 1$, $u_{12} = \omega$, $u_{21} = \omega^{-1}$, and $d_1 = 1/(y\sqrt[n]{b})$, $d_2 = x\sqrt[n]{a}$. Graded symbol algebras satisfy the same multiplicative rules in the graded Brauer group as do the usual ungraded symbol algebras in the Brauer group. (This follows, e.g., by the injectivity of the scalar extension map $\mathsf{Br}(\mathsf{T}) \to \operatorname{Br}(q(\mathsf{T}))$, cf. [HwW$_2$, p. 90].) Thus, in $\operatorname{Br}(\mathsf{T})$, we have

$$\begin{aligned}\mathsf{E} &\sim_g (a,b,\mathsf{T})_\omega \otimes_\mathsf{T} (x^n,b,\mathsf{T})_\omega \otimes_\mathsf{T} (a,y^n,\mathsf{T})_\omega \otimes_\mathsf{T} (x^n,y^n,\mathsf{T})_\omega \\ &\sim_g (a,b,\mathsf{T})_\omega \otimes_\mathsf{T} (x,b,\mathsf{T})_{\omega^n} \otimes (a,y,\mathsf{T})_{\omega^n}.\end{aligned}$$

(The last two terms are symbol algebras of degree n.) Thus, $\mathsf{E} \sim_g \mathsf{I} \otimes_\mathsf{T} \mathsf{N}$ where $\mathsf{I} = (a,b,\mathsf{T})_\omega$ and $\mathsf{N} = (x,b,\mathsf{T})_{\omega^n} \otimes_\mathsf{T} (a,y,\mathsf{T})_{\omega^n}$. Then, $\mathsf{I} \cong_g \mathsf{I}_0 \otimes_{\mathsf{T}_0} \mathsf{T}$, where $\mathsf{I}_0 = (a,b,\mathsf{T}_0)_\omega = A(K(\sqrt[n]{a}, \sqrt[n]{b})/K, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$, with the same $\mathbf{u}$ as for E and $b_1 = 1/\sqrt[n]{b}$, $b_2 = \sqrt[n]{a}$. So, I is an inertial central simple graded T-algebra. We have N_0 is the field $K(\sqrt[n]{a}, \sqrt[n]{b})$, so N is a graded division algebra by Lemma 2.2(ii). N is DSR since it has the inertial maximal graded subfield $\mathsf{T}(\sqrt[n]{a}, \sqrt[n]{b}) = \mathsf{N}_0\mathsf{T}$ and the totally ramified maximal graded subfield $\mathsf{T}(\sqrt[n]{x}, \sqrt[n]{y})$. As a graded abelian crossed product, $\mathsf{N} \cong_g \mathsf{A}(\mathsf{T}(\sqrt[n]{a}, \sqrt[n]{b})/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{1}, \mathbf{c})$, where $c_1 = 1/y$, $c_2 = x$. Let $M = K(\sqrt[n]{a}, \sqrt[n]{b})$. By Prop. 3.2,

$$\operatorname{SK}_1(\mathsf{N}) \;\cong\; \widehat{H}^{-1}(H, M^*) \;\cong\; \operatorname{Br}(M/K)\big/\operatorname{Dec}(M/K),$$

where $H = \operatorname{Gal}(M/K)$ and $\operatorname{Dec}(M/K) = \operatorname{Br}(K(\sqrt[n]{a})/K) \cdot \operatorname{Br}(K(\sqrt[n]{b})/K)$; but, by Th. 3.7,

$$\operatorname{SK}_1(\mathsf{E}) \;\cong\; \widehat{H}^{-1}(H, M^*)/\langle \operatorname{im}(\omega)\rangle \;\cong\; \operatorname{Br}(M/K)\big/\big[\operatorname{Dec}(M/K) \cdot \langle[(a,b,K)_\omega]\rangle\big]. \tag{3.15}$$

This example is the graded version of Platonov's example in [P$_3$] and [P$_4$] of a cyclic algebra with nontrivial SK_1, where K is a suitably chosen global field. (Platonov worked with the Henselian valued ground field $K' = K((x))((y))$ in place of the graded field $\mathsf{T} = \mathsf{gr}(K')$ considered here.) In [P$_4$, Th. 2] the added term distingushing $\operatorname{SK}_1(\mathsf{E})$ from $\operatorname{SK}_1(\mathsf{N})$ is omitted. This error is corrected in [Y$_5$, p. 536, footnote 1] and in [E$_2$, p. 70], giving the first isomorphism of (3.15) but not the second.

4. Unitary graded $I \otimes N$ decomposition

The goal for §§4–7 is to give a unitary version of the formulas for SK_1 in Prop. 3.2 and Th. 3.7 for semiramified graded division algebras with graded unitary involution. In this section we consider abelian crossed products with unitary involution and prove a unitary analogue to the $I \otimes N$ decomposition of Prop. 3.5.

A *unitary involution* on a central simple algebra A over a field K is a ring antiautomorphism τ of A such that $\tau^2 = \mathrm{id}_A$ and $\tau|_K \neq \mathrm{id}$. (Such a τ is also called an involution on A of the second kind.) Let $F = K^\tau = \{c \in K \mid \tau(c) = c\}$, which is a subfield of K with $[K:F] = 2$ and K Galois over F with $\operatorname{Gal}(K/F) = \{\tau|_K, \mathrm{id}_K\}$. Our τ is also called a *unitary K/F-involution*. The unitary $\operatorname{SK}_1(A,\tau)$ is defined just as for $\operatorname{SK}_1(D,\tau)$ in (1.1). Recall (see [KMRT, Prop. (17.24)(2)]) that if τ' is another unitary K/F-involution on A, then $\operatorname{SK}_1(A,\tau') = \operatorname{SK}_1(A,\tau)$. Thus, we will freely pass from one unitary K/F-involution on A to another when convenient.

In the unitary setting generalized dihedral Galois groups often arise where abelian Galois groups appear in the nonunitary setting. A group G is said to be *generalized dihedral* with respect to a subgroup H if $|G:H| = 2$ and for some $\theta \in G \setminus H$, $\theta^2 = 1$ and $\theta h \theta^{-1} = h^{-1}$ for every $h \in H$. Equivalently, every element of $G \setminus H$ has order 2. See [HW$_2$, §2.4] for some remarks on such groups. Note that H is necessarily abelian. If H is cyclic, we say that G is *dihedral*. (This includes the trivial cases where $|H| = 1$ or 2.) For fields $F \subseteq K \subseteq M$, we say that M is *K/F-generalized dihedral* if $[M:F] < \infty$, M is Galois over F, and $G = \operatorname{Gal}(M/F)$ is generalized dihedral with respect to its subgroup $H = \operatorname{Gal}(M/K)$.

Lemma 4.1. *Let $F \subseteq K \subseteq M$ be fields, and suppose M is K/F-generalized dihedral. Let A be a central simple K-algebra containing M as a strictly maximal subfield. Let $G = \mathrm{Gal}(M/F)$ and $H = \mathrm{Gal}(M/K)$, and fix any $\theta \in G \setminus H$ (so $\theta^2 = \mathrm{id}_M$). Then, the following conditions are equivalent:*

(i) *A has a unitary K/F-involution.*
(ii) *A has a unitary K/F-involution τ such that $\tau|_M = \theta$.*
(iii) *$A \cong A(M/K, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$ where (in addition to conditions (3.1) and (3.2))*

$$u_{ij} \cdot \sigma_i\sigma_j\theta(u_{ij}) \;=\; 1 \quad \text{and} \quad b_i = \theta(b_i) \quad \text{for all } i, j. \tag{4.1}$$

The A in (iii), has a unitary K/F-involution τ with $\tau|_M = \theta$ and $\tau(z_i) = z_i$ for each of the standard generators z_i of A.

Proof. Note that as $\theta \notin H$ and K is Galois over F, we have $\theta(K) = K$ and $\mathrm{Gal}(K/F) = \{\mathrm{id}_K, \theta|_K\}$.

(i) $\Rightarrow$ (ii) This is a special case of a substantial result [KMRT, Th. 4.14] on simple subalgebras with compatible involutions. For the convenience of the reader we give a short direct proof. Let ρ be a unitary K/F-involution on A, so $\rho|_K = \theta|_K$. Since $\rho\theta$ is a K-linear homomorphism $M \to A$, by the Skolem-Noether Theorem, there is $y \in A^*$ with $\mathrm{int}(y)|_M = \rho\theta$. For any $a \in M$, as $\rho^2 = \theta^2 = \mathrm{id}|_M$, we have

$$\rho(y)a\rho(y)^{-1} \;=\; \rho(y^{-1}\rho(a)y) \;=\; \rho(\rho\theta)^{-1}\rho(a) \;=\; \rho\theta(a) \;=\; yay^{-1}.$$

Therefore, letting $c = y^{-1}\rho(y)$, we have $c \in C_A(M)^* = M^*$ and $\rho(y) = yc$. Hence,

$$y \;=\; \rho^2(y) \;=\; \rho(yc) \;=\; \rho(c)yc \;=\; \rho(c)\rho\theta(c)y \;=\; \rho(c\theta(c))y;$$

so, $c\theta(c) = 1$. Since $\theta^2 = \mathrm{id}|_M$, by Hilbert 90 applied to the quadratic extension M/M^θ there is $d \in M^*$ with $c = d\theta(d)^{-1}$. Let $z = yd$. Then, as $\theta(c) = \theta(d)d^{-1}$,

$$\rho(z) \;=\; \rho(d)yc \;=\; \rho(d)\rho\theta(c)\,y \;=\; \rho\theta(d)\,y \;=\; yd \;=\; z.$$

Let $\tau = \rho\circ\mathrm{int}(z)$, which is an involution on A, as $\rho(z) = z$. Then, $\tau|_M = \rho\,\mathrm{int}(z)|_M = \rho\,\mathrm{int}(y)|_M = \rho^2\theta = \theta$, as desired.

(ii) $\Rightarrow$ (iii) Let τ be a unitary K/F-involution on A such that $\tau|_M = \theta$. For any $\sigma \in H$, we claim that there is $z \in A^*$ with $\mathrm{int}(z)|_M = \sigma$ and $\tau(z) = z$. For this, first apply Skolem-Noether to obtain $y \in A^*$ with $\mathrm{int}(y)|_M = \sigma$. For any $a \in M$ we have, as $\tau\sigma^{-1}\tau = \sigma$ on M since $\tau\sigma^{-1}|_M \in G \setminus H$,

$$\tau(y)a\tau(y)^{-1} \;=\; \tau(y^{-1}\tau(a)y) \;=\; \tau\sigma^{-1}\tau(a) \;=\; \sigma(a) \;=\; yay^{-1}.$$

Hence, $\tau(y) = cy$, where $c \in C_A(M)^* = M^*$. Now,

$$y \;=\; \tau^2(y) \;=\; \tau(cy) \;=\; \tau(y)\tau(c) \;=\; cy\theta(c) \;=\; c\sigma\theta(c)\,y,$$

so $c\,\sigma\theta(c) = 1$. Since $\sigma\theta$ has order 2, Hilbert 90 applied to the quadratic extension $M/M^{\sigma\theta}$ shows that there is $d \in M^*$ with $c = d\,\sigma\theta(d)^{-1}$. Let $z = dy$. Then, $\mathrm{int}(z)|_M = \mathrm{int}(y)|_M = \sigma$ and

$$\tau(z) \;=\; cy\theta(d) \;=\; [d\,\sigma\theta(d)^{-1}]\,\sigma\theta(d)\,y \;=\; z,$$

proving the claim. Thus, with our cyclic decomposition $H = \langle\sigma_1\rangle \times \ldots \times \langle\sigma_k\rangle$, we can choose $z_1, \ldots, z_k \in A^*$ with $\mathrm{int}(z_i)|_M = \sigma_i$ and $\tau(z_i) = z_i$. Then, for $b_i = z_i^{r_i} \in M^*$, we have $\theta(b_i) = \tau(b_i) = \tau(z_i^{r_i}) = b_i$. Also, for $u_{ij} = z_iz_jz_i^{-1}z_j^{-1}$, we have

$$\sigma_i\sigma_j\theta(u_{ij}) \;=\; z_iz_j\,\tau(z_iz_jz_i^{-1}z_j^{-1})z_j^{-1}z_i^{-1} \;=\; z_iz_j(z_j^{-1}z_i^{-1}z_jz_i)z_j^{-1}z_i^{-1} \;=\; z_jz_iz_j^{-1}z_i^{-1} \;=\; u_{ij}^{-1},$$

so $u_{ij}\,\sigma_i\sigma_j\theta(u_{ij}) = 1$. Thus, $A \cong A(M/K, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$ with the u_{ij} and b_i satisfying the equations in (4.1).

(iii) $\Rightarrow$ (i) Assume $A = A(M/K, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$ where the u_{ij} and b_i satisfy the conditions in (4.1). Take $z_1, \ldots, z_k \in A^*$ with $\mathrm{int}(z_i)|_M = \sigma$, $z_i^{r_i} = b_i$ and $z_iz_jz_i^{-1}z_j^{-1} = u_{ij}$. We show that there is a unitary K/F-involution τ on A satisfying (and determined by) $\tau|_M = \theta$ and $\tau(z_i) = z_i$ for each i. Basically, this is a matter of checking that the τ just described is compatible with the defining relations of A. Here is a more complete argument, based on the description of $A(M/K, \sigma_i, u_{ij}, b_i)$ given in the proof of [AS, Th. 1.3]. First, take any ring B with an automorphism σ, and let $B[y;\sigma]$ be the twisted polynomial ring $\{\sum c_i y^i \mid c_i \in B\}$ with the multiplication determined by $yc = \sigma(c)y$ for all $c \in B$. It is easy to check that an involution ρ on B extends to an involution ρ' on $B[y;\sigma]$ with $\rho'(y) = y$ iff $\sigma\rho\sigma = \rho$. Also, for $d \in B^*$, an automorphism η of B extends to an automorphism η' of $B[y;\sigma]$ with $\eta'(y) = dy$ iff $\operatorname{int}(d)\sigma\eta = \eta\sigma$. Here, let $B_0 = M$, $B_1 = B_0[y_1;\sigma_1^*]$, $\dots$, $B_\ell = B_{\ell-1}[y_\ell;\sigma_\ell^*]$, $\dots$, $B_k = B_{k-1}[y_k;\sigma_k^*]$, where $\sigma_1^* = \sigma_1$ and for $\ell > 1$, the automorphism σ_ℓ^* of $B_{\ell-1}$ is defined by $\sigma_\ell^*|_M = \sigma_\ell$ and $\sigma_\ell^*(y_i) = u_{\ell i} y_i$ for $1 \le i < \ell$. (One checks inductively using the identities in (3.1) that for $1 \le i \le \ell - 1$, σ_ℓ^* satisfies $\operatorname{int}(u_{\ell i})\sigma_i^*\sigma_\ell^* = \sigma_\ell^*\sigma_i^*$ on B_{i-1}, hence σ_ℓ^* extends from B_{i-1} to B_i; thus, σ_ℓ^* is an automorphism of $B_{\ell-1}$.) Define inductively involutions τ_i on B_i by $\tau_0 = \theta$ and for $\ell > 0$, $\tau_\ell|_{B_{\ell-1}} = \tau_{\ell-1}$ and $\tau_\ell(y_\ell) = y_\ell$. Given $\tau_{\ell-1}$, the condition for the existence of τ_ℓ is that $\sigma_\ell^*\tau_{\ell-1}\sigma_\ell^* = \tau_{\ell-1}$. For this, note first that $\sigma_\ell^*\tau_{\ell-1}\sigma_\ell^*|_M = \sigma_\ell\theta\sigma_\ell = \theta = \tau_{\ell-1}|_M$ as G is generalized dihedral. Furthermore, for $1 \le i < \ell$,

$$\sigma_\ell^*\tau_{\ell-1}\sigma_\ell^*(y_i) \;=\; \sigma_\ell^*\tau_{\ell-1}(u_{\ell i}y_i) \;=\; \sigma_\ell^*\big(y_i\,\theta(u_{\ell i})\big) \;=\; \sigma_\ell^*[\sigma_i\theta(u_{\ell i})\,y_i] \;=\; [\sigma_\ell\sigma_i\theta(u_{\ell i})]u_{\ell i}\,y_i \;=\; y_i \;=\; \tau_{\ell-1}(y_i).$$

Thus, $\sigma_\ell^*\tau_{\ell-1}\sigma_\ell^*$ agrees with $\tau_{\ell-1}$ throughout $B_{\ell-1}$, as needed. By induction, we have the involution τ_k on B_k. As pointed out in [AS, p. 79], $A \cong B_k/I$, where I is the two-sided ideal of B_k generated by $\{y_i^{r_i} - b_i \mid 1 \le i \le k\}$. Since $\tau_k(b_i) = \theta(b_i) = b_i$, τ_k maps each generator of I to itself. Therefore, τ_k induces an involution τ on $A \cong B_k/I$ which clearly restricts to θ on M; so τ is a unitary K/F-involution on A. □

We write $\operatorname{Br}(M/K;F)$ for the subgroup of $\operatorname{Br}(M/K)$ of algebra classes $[A]$ such that A has a unitary K/F-involution. By Albert's theorem [KMRT, Th. 3.1(2)], $\operatorname{Br}(M/K;F)$ is the kernel of the corestriction map $\operatorname{cor}_{K\to F}\colon \operatorname{Br}(M/K) \to \operatorname{Br}(M/F)$. For M a K/F-generalized dihedral extension of F, as above, there is in addition a corresponding subgroup of $\operatorname{Dec}(M/K)$. For this, note first that for any field L with $K \subseteq L \subseteq M$ and L cyclic Galois over K, say $\operatorname{Gal}(L/K) = \langle\sigma\rangle$, L is K/F-dihedral, so Lemma 4.1 (with $k = 1$) implies that $\operatorname{Br}(L/K;F) = \{[(L/K,\sigma,b)] \mid b \in F^*\}$. For $H = \operatorname{Gal}(M/K) = \langle\sigma_1\rangle \times \ldots \times \langle\sigma_k\rangle$ and $(\chi_1,\dots,\chi_k)$ the base of $X(M/K)$ dual to $(\sigma_1,\dots,\sigma_k)$, and L_i the fixed field of $\ker(\chi_i)$, as at the beginning of §3, define

$$\operatorname{Dec}(M/K;F) \;=\; \{[(L_1/K,\sigma_1,b_1)\otimes_K \ldots \otimes_K (L_k/K,\sigma_k,b_k)] \mid \text{each } b_i \in F^*\} \;\subseteq\; \operatorname{Br}(M/K;F)\}. \tag{4.2}$$

Note that $\operatorname{Dec}(M/K;F)$ is generated as a group by the image under the cup product of $H^2(H,\mathbb{Z}) \times F^*$. Thus $\operatorname{Dec}(M/K;F)$ is independent of the choice of cyclic decomposition of H, and we have analogously to (3.5),

$$\operatorname{Dec}(M/K;F) \;=\; \prod_{i=1}^{k} \operatorname{Br}(L_i/K;F) \;=\; \prod_{\substack{K\subseteq L\subseteq M\\ \operatorname{Gal}(L/K)\text{ cyclic}}} \operatorname{Br}(L/K;F). \tag{4.3}$$

For the rest of this section we fix a graded field T and a graded subfield $\mathsf{R} \subseteq \mathsf{T}$ such that $[\mathsf{T}:\mathsf{R}] = 2$ and T is inertial and Galois over R. Let ψ be the nonidentity graded R-automorphism of T, and let ψ_0 be the restriction $\psi|_{\mathsf{T}_0}$. Thus, $\Gamma_{\mathsf{T}} = \Gamma_{\mathsf{R}}$, $[\mathsf{T}_0:\mathsf{R}_0] = 2$, $\mathsf{T} \cong_g \mathsf{T}_0 \otimes_{\mathsf{R}_0} \mathsf{R}$, T_0 is Galois over R_0, and ψ on T corresponds to $\psi_0 \otimes \mathrm{id}_{\mathsf{R}}$ on $\mathsf{T}_0 \otimes_{\mathsf{R}_0} \mathsf{R}$. We are interested in central simple graded T-algebras A with graded unitary T/R-involutions τ. This means that τ is a degree-preserving ring antiautomorphism of A with $\tau^2 = \mathrm{id}_{\mathsf{A}}$ and the ring of invariants $\mathsf{T}^\tau = \mathsf{R}$; the last condition is equivalent to $\tau|_{\mathsf{T}} = \psi$. Suppose now that A is a graded division algebra. Set $\tau_0 = \tau|_{\mathsf{A}_0}$, which is a unitary involution on A_0, as $\tau_0|_{\mathsf{T}_0} = \psi_0 \neq \mathrm{id}$ and $\mathsf{T}_0 \subseteq Z(\mathsf{A}_0)$. Just as for any graded division algebra, $Z(\mathsf{A}_0)$ is abelian Galois over T_0. But the presence of the involution τ implies further that $Z(\mathsf{A}_0)$ is actually $\mathsf{T}_0/\mathsf{R}_0$-generalized dihedral, by [HW_2, Lemma 4.6(ii)].

A central graded division algebra N over T is said to be *decomposably semiramified for* T/R (abbreviated DSR for T/R) if N has a unitary graded T/R-involution τ and a maximal graded subfield M inertial over T and another maximal graded subfield J with J totally ramified over T and $\tau(\mathsf{J}) = \mathsf{J}$. When this occurs, N is semiramified with $\mathsf{N}_0 = \mathsf{M}_0$, a field, which as just noted is $\mathsf{T}_0/\mathsf{R}_0$-generalized dihedral. Also, $\Gamma_{\mathsf{N}} = \Gamma_{\mathsf{J}}$ and Θ_{N} induces an isomorphism $\Gamma_{\mathsf{N}}/\Gamma_{\mathsf{T}} \cong \mathrm{Gal}(\mathsf{N}_0/\mathsf{T}_0)$. Furthermore, as $\mathsf{M} = \mathsf{M}_0\mathsf{T} = \mathsf{N}_0\mathsf{T}$, we have $\tau(\mathsf{M}) = \mathsf{M}$.

Example 4.2. Let L be any cyclic Galois field extension of T_0 with L dihedral over R_0. (That is, L is Galois over R_0 and there is $\theta \in \mathrm{Gal}(L/\mathsf{R}_0) \setminus \mathrm{Gal}(L/\mathsf{T}_0)$ with $\theta^2 = \mathrm{id}_{\mathsf{L}}$ and $\theta h \theta^{-1} = h^{-1}$ for every $h \in \mathrm{Gal}(L/\mathsf{T}_0)$. Thus, the group $\mathrm{Gal}(L/\mathsf{R}_0)$ is either dihedral or isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$.) Let $r = [L:\mathsf{T}_0]$, and take any $b \in \mathsf{R}^*$ with the image of $\deg(b)$ having order r in $\Gamma_{\mathsf{T}}/r\Gamma_{\mathsf{T}}$. Take any generator σ of $\mathrm{Gal}(L/\mathsf{T}_0)$, and let σ denote also its canonical extension $\sigma \otimes \mathrm{id}_{\mathsf{T}}$ in $\mathsf{Gal}((L \otimes_{\mathsf{T}_0} \mathsf{T})/\mathsf{T})$. Let

$$\mathsf{N} \;=\; ((L \otimes_{\mathsf{T}_0} \mathsf{T})/\mathsf{T}, \sigma, b), \text{ a cyclic graded algebra over } \mathsf{T}.$$

We show that N is a central graded division algebra over T of degree r, and N is DSR for T/R. For, letting $L\mathsf{T}$ denote $L \otimes_{\mathsf{T}_0} \mathsf{T}$, note that $L\mathsf{T}$ is a graded field which is inertial over T and is Galois over T with $\mathsf{Gal}(L\mathsf{T}/\mathsf{T}) = \langle\sigma\rangle$. Our N is $\bigoplus_{i=0}^{r-1} L\mathsf{T}z^i$, where $zcz^{-1} = \sigma(c)$ for all $c \in L\mathsf{T}$, and $z^r = b$, with the grading on N extending that on $L\mathsf{T}$ by setting $\deg(z) = \frac{1}{r}\deg(b)$. A graded cyclic T-algebra is always graded simple with center T. Note that for $j \in \mathbb{Z}$, if $j\deg(b)/r \in \Gamma_{L\mathsf{T}} = \Gamma_{\mathsf{T}}$, then $j\deg(b) \in r\Gamma_{\mathsf{T}}$, so by hypothesis $r \,|\, j$. Hence,

$$\mathsf{N}_0 \;=\; \sum_{i=0}^{r-1} (L\mathsf{T})_{-i\deg(b)/r}\, z^i \;=\; (L\mathsf{T})_0 \;=\; L.$$

Since N_0 is a division ring, the simple graded algebra N is a graded division ring, by Lemma 2.2(ii). Also, as $[L\mathsf{T}:\mathsf{T}] = [L:\mathsf{T}_0] = r = \deg(\mathsf{N})$, $L\mathsf{T}$ is a maximal graded subfield of N which is inertial over T. Take any $\theta \in \mathrm{Gal}(L/\mathsf{R}_0)$ with $\theta|_{\mathsf{T}_0} = \psi_0$, and let θ denote also its canonical extension $\theta \otimes \mathrm{id}\,|_{\mathsf{R}}$ to $\mathsf{Gal}(L\mathsf{T}/\mathsf{R})$. Define a map $\tau\colon \mathsf{N} \to \mathsf{N}$ by

$$\tau\Big(\sum_{i=0}^{r-1} c_i z^i\Big) \;=\; \sum_{i=0}^{r-1} z^i\theta(c_i) \;=\; \sum_{i=0}^{r-1} \sigma^i\theta(c_i) z^i.$$

Since $\theta|_{\mathsf{T}} = \psi$, $\theta^2 = \mathrm{id}$, and $\theta\sigma\theta^{-1} = \sigma^{-1}$ (as L is $\mathsf{T}_0/\mathsf{R}_0$-dihedral), it is easy to check that τ is a graded T/R-involution of N. Moreover, if we let $\mathsf{J} = \bigoplus_{i=0}^{r-1} \mathsf{T}z^i = \mathsf{T}[z]$, then J is a maximal graded subfield of N, and the hypothesis on $\deg(b)$ assures that J is totally ramified over T; also $\tau(\mathsf{J}) = \mathsf{J}$. This verifies that N is DSR for T/R. Note that $\mathsf{N}_0 = L$ and $\Gamma_{\mathsf{N}} = \langle \frac{1}{r}\deg(b)\rangle + \Gamma_{\mathsf{T}}$.

Lemma 4.3. *Let N and N' be graded division algebras which are each DSR for T/R. Suppose N_0 and N_0' are linearly disjoint over T_0 and $\Gamma_{\mathsf{N}} \cap \Gamma_{\mathsf{N}'} = \Gamma_{\mathsf{T}}$. Then, $\mathsf{N} \otimes_{\mathsf{T}} \mathsf{N}'$ is a graded division algebra which is DSR for T/R. Also, $(\mathsf{N} \otimes_{\mathsf{T}} \mathsf{N}')_0 \cong \mathsf{N}_0 \otimes_{\mathsf{T}_0} \mathsf{N}_0'$ and $\Gamma_{\mathsf{N}\otimes_{\mathsf{T}}\mathsf{N}'} = \Gamma_{\mathsf{N}} + \Gamma_{\mathsf{N}'}$.*

Proof. Let $\mathsf{B} = \mathsf{N} \otimes_{\mathsf{T}} \mathsf{N}'$, which is a central simple graded T-algebra, since this is true for N and N' by [HwW_2, Prop. 1.1]. For each $\gamma \in \Gamma_{\mathsf{T}}$ choose a nonzero $t_\gamma \in \mathsf{T}_\gamma$. Then,

$$\mathsf{B}_0 \;=\; \sum_{\gamma\in\Gamma_{\mathsf{N}}\cap\Gamma_{\mathsf{N}'}} \mathsf{N}_\gamma \otimes_{\mathsf{T}_0} \mathsf{N}'_{-\gamma} \;=\; \sum_{\gamma\in\Gamma_{\mathsf{T}}} \mathsf{N}_0\, t_\gamma \otimes_{\mathsf{T}_0} \mathsf{N}_0'\, t_\gamma^{-1} \;=\; \mathsf{N}_0 \otimes_{\mathsf{T}_0} \mathsf{N}_0'.$$

The linear disjointness hypothesis assures that B_0 is a field, and hence B is a graded division ring, by Lemma 2.2(ii). Moreover, by dimension count $\mathsf{B}_0\mathsf{T}$ is a graded maximal subfield of B which is inertial over T. Let τ be a graded T/R-involution of N, and let J be a graded maximal subfield of N with $\tau(\mathsf{J}) = \mathsf{J}$. Take τ' and J' correspondingly for N'. Then, $\mathsf{J}\mathsf{J}' = \mathsf{J} \otimes_{\mathsf{T}} \mathsf{J}'$ and $\tau \otimes \tau'$ is a graded T/R-involution on B with $(\tau \otimes \tau')(\mathsf{J}\mathsf{J}') = \mathsf{J}\mathsf{J}'$. Moreover, $\mathsf{J}\mathsf{J}'$ is a maximal graded subfield of B by dimension count, and, as $\Gamma_{\mathsf{J}} \cap \Gamma_{\mathsf{J}'} = \Gamma_{\mathsf{N}} \cap \Gamma_{\mathsf{N}'} = \Gamma_{\mathsf{T}}$, we have

$$|\Gamma_{\mathsf{J}\mathsf{J}'}:\Gamma_{\mathsf{T}}| \;\ge\; |\Gamma_{\mathsf{J}} + \Gamma_{\mathsf{J}'}:\Gamma_{\mathsf{T}}| \;=\; |\Gamma_{\mathsf{J}}:\Gamma_{\mathsf{T}}| \cdot |\Gamma_{\mathsf{J}'}:\Gamma_{\mathsf{T}}| \;=\; [\mathsf{J}:\mathsf{T}] \cdot [\mathsf{J}':\mathsf{T}] \;=\; [\mathsf{J}\mathsf{J}':\mathsf{T}].$$

Hence, $\mathsf{J}\mathsf{J}'$ is totally ramified over T. Thus, B is DSR for T/R. □

The next proposition shows that all graded division algebras N which are DSR for T/R are obtainable from those in Ex. 4.2 by iterated application of Prop. 4.3. This justifies the term "decomposably semiramified" for such N.

Proposition 4.4. *Let* N *be a graded division algebra which is DSR for* T/R*. Take any decomposition* $\mathsf{N}_0 = L_1 \otimes_{\mathsf{T}_0} \ldots \otimes_{\mathsf{T}_0} L_k$ *with each* L_i *cyclic Galois over* T_0*, and choose correspondingly* $\sigma_1 \ldots, \sigma_k \in \mathsf{Gal}(\mathsf{N}_0\mathsf{T}/\mathsf{T}) \cong \mathrm{Gal}(\mathsf{N}_0/\mathsf{T}_0)$ *such that* $\sigma_i|_{L_j} = \mathrm{id}$ *whenever* $j \neq i$ *and* $\mathrm{Gal}(L_i/\mathsf{T}_0) = \langle \sigma_i|_{L_i}\rangle$ *for each* i*. (So* $\mathsf{Gal}(\mathsf{N}_0\mathsf{T}/\mathsf{T}) = \langle\sigma_1\rangle \times \ldots \times \langle\sigma_k\rangle$*.) Let* r_i *be the order of* σ_i*. For each* i *choose* $\gamma_i \in \Gamma_{\mathsf{N}}$ *with* $\Theta_{\mathsf{N}}(\gamma_i) = \sigma_i$*. Then, there exist* $b_1, \ldots, b_k \in \mathsf{R}^*$ *such that* $\deg(b_i) = r_i\gamma_i$ *and*

$$\mathsf{N} \;\cong_g\; (L_1\mathsf{T}/\mathsf{T}, \sigma_1, b_1) \otimes_{\mathsf{T}} \ldots \otimes_{\mathsf{T}} (L_k\mathsf{T}/\mathsf{T}, \sigma_k, b_k) \;\cong_g\; \mathsf{A}(\mathsf{N}_0\mathsf{T}/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{1}, \mathbf{b}).$$

Proof. Since N is DSR for T/R, there is a graded T/R-involution τ of N and a maximal graded subfield J of N with J totally ramified over T and $\tau(\mathsf{J}) = \mathsf{J}$. As noted earlier, we have $\Gamma_{\mathsf{J}} = \Gamma_{\mathsf{N}}$. Since τ is a graded automorphism of J of order 2, the fixed set $\mathsf{S} = \mathsf{J}^\tau = \{a \in \mathsf{J} \mid \tau(a) = a\}$ is a graded subfield of J with $2 = [\mathsf{J} : \mathsf{S}] = [\mathsf{J}_0 : \mathsf{S}_0]\,|\Gamma_{\mathsf{J}} : \Gamma_{\mathsf{S}}|$. Since $\mathsf{S}_0 \cap \mathsf{T}_0 = \mathsf{R}_0 \subsetneqq \mathsf{T}_0 = \mathsf{J}_0 \cap \mathsf{T}_0$ we have $\mathsf{S}_0 \subsetneqq \mathsf{J}_0$, so $[\mathsf{J}_0 : \mathsf{S}_0] = 2$, and hence $\Gamma_{\mathsf{S}} = \Gamma_{\mathsf{J}}$ $(= \Gamma_{\mathsf{N}})$. Thus, for each i there is a nonzero $x_i \in \mathsf{S}_{\gamma_i}$, and for any such x_i, $\mathrm{int}(x_i)|_{\mathsf{N}_0\mathsf{T}} = \sigma_i$ as $\Theta_{\mathsf{N}}(\gamma_i) = \sigma_i$. Let $b_i = x_i^{r_i} \in \mathsf{S}^*$. Then, $\Theta_{\mathsf{N}}(\deg(b_i)) = \sigma_i^{r_i} = \mathrm{id}$, so $\deg(b_i) \in \ker(\Theta_{\mathsf{N}}) = \Gamma_{\mathsf{T}}$; hence, $b_i \in \mathsf{J}_{\deg(b_i)} = \mathsf{T}_{\deg(b_i)}$ as J is totally ramified over T. Therefore, $b \in \mathsf{S}^* \cap \mathsf{T} = \mathsf{R}^*$. Let C_i be the graded T-subalgebra of N generated by L_i and x_i. Since $\mathrm{int}(x_i)|_{L_i\mathsf{T}} = \sigma_i|_{L_i\mathsf{T}}$, there is a graded T-algebra epimorphism $(L_i\mathsf{T}/\mathsf{T}, \sigma_i, b_i) \to \mathsf{C}_i$, which is a graded isomorphism as the domain is graded simple. Since the x_i all lie in the graded field S and $\sigma_i|_{L_j\mathsf{T}} = \mathrm{id}$ for $j \neq i$, the distinct C_i centralize each other. Hence, there is a graded T-algebra homomorphism $(L_1\mathsf{T}/\mathsf{T}, \sigma_1, b_1) \otimes_{\mathsf{T}} \ldots \otimes_{\mathsf{T}} (L_k\mathsf{T}/\mathsf{T}, \sigma_k, b_k) \to \mathsf{N}$ which is injective as the domain is graded simple, and surjective by dimension count. Clearly also, $(L_1\mathsf{T}/\mathsf{T}, \sigma_1, b_1) \otimes_{\mathsf{T}} \ldots \otimes_{\mathsf{T}} (L_k\mathsf{T}/\mathsf{T}, \sigma_k, b_k) \cong_g \mathsf{A}(\mathsf{N}_0\mathsf{T}/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{1}, \mathbf{b})$. □

Proposition 4.5. *Let* E *be a semiramified central graded division algebra over* T*, and suppose* E *has a graded* T/R*-involution, where* T *is inertial over* R*. Then,* E_0 *is* $\mathsf{T}_0/\mathsf{R}_0$*-generalized dihedral and*

(i) $\mathsf{E} \sim_g \mathsf{I} \otimes_{\mathsf{T}} \mathsf{N}$ *in* $\mathsf{Br}(\mathsf{T})$ *for some* T*-central graded division algebras* I *and* N *with* I *inertial and* N *DSR for* T/R*.*

(ii) *Take any decomposition* $\mathsf{T} \sim_g \mathsf{I}' \otimes_{\mathsf{T}} \mathsf{N}'$ *in* $\mathsf{Br}(\mathsf{T})$ *with graded* T*-central division algebras* I' *and* N' *with* I' *inertial and* N' *DSR for* T/R*. Then,* $\mathsf{N}'_0 \cong \mathsf{E}_0$*,* $\Gamma_{\mathsf{N}'} = \Gamma_{\mathsf{E}}$*,* $\Theta_{\mathsf{N}'} = \Theta_{\mathsf{E}}$*, and* $[\mathsf{I}'_0] \in \mathrm{Br}(\mathsf{E}_0/\mathsf{T}_0; \mathsf{R}_0)$*. Furthermore,* I'_0 *is uniquely determined modulo* $\mathrm{Dec}(\mathsf{E}_0/\mathsf{T}_0; \mathsf{R}_0)$*.*

Proof. (i) Since E is semiramified, $\mathsf{E}_0\mathsf{T}$ is an inertial maximal graded subfield of E. Moreover, as E has a graded T/R-involution, E_0 is $\mathsf{T}_0/\mathsf{R}_0$-generalized dihedral, by [HW$_2$, Lemma 4.6(ii)]. Because E has an inertial graded maximal subfield, it is a graded abelian crossed product: Say $\mathsf{E}_0 = L_1 \otimes_{\mathsf{T}_0} \ldots \otimes_{\mathsf{T}_0} L_k$, where each field L_i is cyclic Galois over T_0 (so dihedral over R_0). Then $G = \mathsf{Gal}(\mathsf{E}_0\mathsf{T}/\mathsf{T}) \cong \mathsf{Gal}(\mathsf{E}_0/\mathsf{T}_0)$ has a corresponding cyclic decomposition $G = \langle\sigma_1\rangle \times \ldots \times \langle\sigma_k\rangle$, where each $\sigma_i|_{L_j\mathsf{T}} = \mathrm{id}$ for $j \neq i$, and $\sigma_i|_{L_i\mathsf{T}}$ generates $\mathsf{Gal}(L_i\mathsf{T}/\mathsf{T})$. Let $r_i = |\langle\sigma_i\rangle| = [L_i : \mathsf{T}_0]$. By Lemma 3.4, $\mathsf{E} = \mathsf{A}(\mathsf{E}_0\mathsf{T}/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$ where each $u_{ij} \in \mathsf{E}_0^*$, $b_i \in \mathsf{E}_0\mathsf{T}^*$, $\frac{1}{r_i}\deg(b_i) + \Gamma_{\mathsf{T}}$ has order r_i in $\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$, and

$$\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}} \;=\; \langle \tfrac{1}{r_1}\deg(b_1) + \Gamma_{\mathsf{T}}\rangle \times \ldots \times \langle \tfrac{1}{r_k}\deg(b_k) + \Gamma_{\mathsf{T}}\rangle. \tag{4.4}$$

So, $\deg(b_i) \in \Gamma_{\mathsf{E}_0\mathsf{T}} = \Gamma_{\mathsf{T}} = \Gamma_{\mathsf{R}}$ and the image of $\deg(b_i)$ has order r_i in $\Gamma_{\mathsf{T}}/r_i\Gamma_{\mathsf{T}}$. For each i, choose $c_i \in \mathsf{R}^*$ with $\deg(c_i) = \deg(b_i)$. Let

$$\mathsf{N} \;=\; \mathsf{C}_1 \otimes_{\mathsf{T}} \ldots \otimes_{\mathsf{T}} \mathsf{C}_k, \quad \text{where each} \;\; \mathsf{C}_i = (L_i\mathsf{T}/\mathsf{T}, \sigma_i, c_i).$$

By Ex. 4.2 each C_i is DSR for T/R with $(\mathsf{C}_i)_0 \cong L_i$ and $\Gamma_{\mathsf{C}_i} = \langle \frac{1}{r_i}\deg(c_i)\rangle + \Gamma_{\mathsf{T}} = \langle \frac{1}{r_i}\deg(b_i)\rangle + \Gamma_{\mathsf{T}}$. It follows by induction on k using Lemma 4.3 and (4.4) that N is a graded division algebra which is DSR for T/R. Choose $z_i \in \mathsf{C}_i^*$ with $\mathrm{int}(z_i)|_{L_i\mathsf{T}} = \sigma_i$ and $z_i^{r_i} = c_i$. Then, when we view $z_i \in \mathsf{N}^*$, we have $\mathrm{int}(z_i) = \sigma_i$ on all of $\mathsf{N}_0\mathsf{T}$. Since further $z_iz_j = z_jz_i$ for all i, j, our N is the graded abelian crossed product $\mathsf{N} = \mathsf{A}(\mathsf{E}_0\mathsf{T}/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{1}, \mathbf{c})$. For its opposite algebra N^{op} we then have $\mathsf{N}^{\mathrm{op}} \cong_g \mathsf{A}(\mathsf{E}_0\mathsf{T}/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{1}, \mathbf{d})$ where each $d_i = c_i^{-1}$. Let $\widehat{\mathsf{I}} = \mathsf{A}(\mathsf{E}_0\mathsf{T}/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{e})$ where each $e_i = b_id_i = b_ic_i^{-1} \in \mathsf{E}_0^*$. The u_{ij} and b_i satisfy conditions (3.1) and (3.2), as do the c_i with the corresponding $u_{ij} = 1$; hence the u_{ij} here and e_i satisfy (3.1) and (3.2); also, $\deg(u_{ij}) = 0$ for all i, j. So, $\widehat{\mathsf{I}}$ is a well-defined graded abelian crossed product. By Remark 3.3, we have $\widehat{\mathsf{I}} \sim_g \mathsf{E} \otimes_{\mathsf{T}} \mathsf{N}^{\mathrm{op}}$. There are homogeneous $x_1, \ldots, x_k \in \widehat{\mathsf{I}}^*$ such that $\mathrm{int}(x_i)|_{\mathsf{E}_0\mathsf{T}} = \sigma_i$, $x_i^{r_i} = e_i$, and $x_i x_j x_i^{-1} x_j^{-1} = u_{ij}$ for all i, j. Then, $\deg(x_i) = \frac{1}{r_i}\deg(e_i) = 0$; hence, $\deg(x^{\mathbf{i}}) = 0$ for each $\mathbf{i} \in \mathcal{I} = \prod_{i=1}^k \{0, 1, 2, \ldots, r_i - 1\}$. Thus, in $\widehat{\mathsf{I}} = \bigoplus_{\mathbf{i} \in \mathcal{I}} \mathsf{E}_0\mathsf{T}x^{\mathbf{i}}$ we have $\widehat{\mathsf{I}}_0 = \bigoplus_{\mathbf{i}\in\mathcal{I}} \mathsf{E}_0 x^{\mathbf{i}} \cong A(\mathsf{E}_0/\mathsf{T}_0, \boldsymbol{\sigma}, \mathbf{u}, \mathbf{e})$, which is a central simple T_0-algebra with $\dim_{\mathsf{T}_0}(\widehat{\mathsf{I}}_0) = [\mathsf{E}_0 : \mathsf{T}_0]^2 = \dim_{\mathsf{T}}(\widehat{\mathsf{I}})$. Hence, $\widehat{\mathsf{I}}$ is inertial over T. Since $\widehat{\mathsf{I}}$ is simple, by Lemma 2.2 $\widehat{\mathsf{I}} \cong_g M_\ell(\mathsf{I})$ for a graded division algebra I with $\widehat{\mathsf{I}}_0 \cong M_\ell(\mathsf{I}_0)$. Then, $[\mathsf{I}_0 : \mathsf{T}_0] = \frac{1}{\ell^2}\dim_{\mathsf{T}_0}(\widehat{\mathsf{I}}_0) = \frac{1}{\ell^2}\dim_{\mathsf{T}}(\widehat{\mathsf{I}}) = [\mathsf{I} : \mathsf{T}]$, showing that I is inertial over T. Since $\mathsf{I} \sim_g \widehat{\mathsf{I}}$, we have in $\mathsf{Br}(\mathsf{T})$,

$$[\mathsf{E}] = [\mathsf{E}]\,[\mathsf{N}]^{-1}[\mathsf{N}] = [\mathsf{E} \otimes_{\mathsf{T}} \mathsf{N}^{\mathrm{op}}]\,[\mathsf{N}] = [\widehat{\mathsf{I}}]\,[\mathsf{N}] = [\,\mathsf{I}\,]\,[\mathsf{N}] = [\,\mathsf{I} \otimes_{\mathsf{T}} \mathsf{N}\,],$$

i.e., $\mathsf{E} \sim_g \mathsf{I} \otimes_{\mathsf{T}} \mathsf{N}$, proving (i). Also, N has a graded T/R-involution τ_{N}, which is also a graded involution for N^{op}, and E has a graded T/R-involution τ_{E}. So, $\tau = \tau_{\mathsf{E}} \otimes \tau_{\mathsf{N}}$ is a graded T/R-involution on $\widehat{\mathsf{I}}$, and $\tau_0 = \tau|_{\widehat{\mathsf{I}}_0}$ is a $\mathsf{T}_0/\mathsf{R}_0$-involution on $\widehat{\mathsf{I}}_0$. So, in $\mathrm{Br}(\mathsf{T}_0)$ we have $[\,\mathsf{I}_0] = [\widehat{\mathsf{I}}_0] \in \mathrm{Br}(\mathsf{E}_0/\mathsf{T}_0; \mathsf{R}_0)$.

(ii) Take any decomposition $\mathsf{E} \sim_g \mathsf{I}' \otimes \mathsf{N}'$ as in (ii). Since I' is inertial and E is the graded division algebra with $\mathsf{E} \sim_g \mathsf{I}' \otimes_{\mathsf{T}} \mathsf{N}'$, Cor. 2.3 yields $\mathsf{E}_0 \sim \mathsf{I}'_0 \otimes_{\mathsf{T}_0} \mathsf{N}'_0$ and $\mathsf{E}_0 = Z(\mathsf{E}_0) \cong Z(\mathsf{N}'_0) = \mathsf{N}'_0$, so E_0 splits I'_0; furthermore, $\Gamma_{\mathsf{E}} = \Gamma_{\mathsf{N}'}$ and $\Theta_{\mathsf{E}} = \Theta_{\mathsf{N}'}$. We now use the b_i, c_i, N, and I of part (i). Because N' is DSR with $\mathsf{N}'_0 \cong \mathsf{E}_0$ and $\Theta_{\mathsf{N}'}(\frac{1}{r_i}\deg(c_i)) = \Theta_{\mathsf{E}}(\frac{1}{r_i}\deg(b_i)) = \sigma_i$, by Prop. 4.4 there exist $c'_1, \ldots, c'_k \in \mathsf{R}^*$ with $\deg(c'_i) = \deg(c_i)$ such that $\mathsf{N}' \cong_g \mathsf{A}(\mathsf{E}_0\mathsf{T}/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{1}, \mathbf{c}')$. Let $\mathsf{B} = \mathsf{A}(\mathsf{E}_0\mathsf{T}/\mathsf{T}, \boldsymbol{\sigma}, \mathbf{1}, \mathbf{f})$ where each $f_i = c_i {c'_i}^{-1} \in \mathsf{R}_0^*$. So, in $\mathsf{Br}(\mathsf{T})$, $\mathsf{B} \sim_g \mathsf{N} \otimes_{\mathsf{T}} \mathsf{N}'^{\mathrm{op}} \sim_g \mathsf{I}' \otimes_{\mathsf{T}} \mathsf{I}^{\mathrm{op}}$. Because $\deg(f_i) = 0$ for each i, the argument for $\widehat{\mathsf{I}}$ in (i) shows that B is inertial over T with

$$\mathsf{B}_0 \;\cong\; A(\mathsf{E}_0/\mathsf{T}_0, \boldsymbol{\sigma}, \mathbf{1}, \mathbf{f}) \;\cong\; (L_1/\mathsf{T}_0, \sigma_1, f_1) \otimes_{\mathsf{T}_0} \ldots \otimes_{\mathsf{T}_0} (L_k/\mathsf{T}_0, \sigma_k, f_k).$$

Thus, $[\mathsf{B}_0] \in \mathrm{Dec}(\mathsf{E}_0/\mathsf{T}_0; \mathsf{R}_0)$, as each $f_i \in \mathsf{R}_0^*$ (see Ex. 4.2). Let C be the graded division algebra with $\mathsf{C} \sim_g \mathsf{B} \sim_g \mathsf{I}' \otimes_{\mathsf{T}} \mathsf{I}^{\mathrm{op}}$. Since B_0 is simple and I' is inertial, Lemma 2.2 and Cor. 2.3 yield $\mathsf{C}_0 \sim \mathsf{B}_0$ and $\mathsf{C}_0 \sim (\mathsf{I}' \otimes_{\mathsf{T}} \mathsf{I}^{\mathrm{op}})_0 \cong \mathsf{I}'_0 \otimes_{\mathsf{T}_0} \mathsf{I}_0^{\mathrm{op}}$; so, in $\mathrm{Br}(\mathsf{T}_0)$,

$$[\mathsf{I}'_0] \;=\; [\mathsf{C}_0]\,[\,\mathsf{I}_0] \;=\; [\mathsf{B}_0]\,[\,\mathsf{I}_0] \;=\; [\mathsf{B}_0]\,[\widehat{\mathsf{I}}_0] \in \mathrm{Br}(\mathsf{E}_0/\mathsf{T}_0; \mathsf{R}_0).$$

Since $[\mathsf{B}_0] \in \mathrm{Dec}(\mathsf{E}_0/\mathsf{T}_0; \mathsf{R}_0)$, we have $\mathsf{I}'_0 \equiv \mathsf{I}_0 \pmod{\mathrm{Dec}(\mathsf{E}_0/\mathsf{T}_0; \mathsf{R}_0)}$. This yields the uniqueness of I'_0 modulo $\mathrm{Dec}(\mathsf{E}_0/\mathsf{T}_0; \mathsf{R}_0)$ independent of the choice of decomposition of E as $\mathsf{I}' \otimes_{\mathsf{T}} \mathsf{N}'$. □

Remark 4.6. The $\mathsf{I} \otimes \mathsf{N}$ decomposition described in Prop. 4.5 for E semiramified actually holds more generally for E inertially split (with graded T/R-involution), i.e., when E has a maximal graded subfield inertial over T. One then has $\mathsf{N}_0 \cong Z(\mathsf{E}_0)$ and $\mathsf{I}_0 \otimes_{\mathsf{T}_0} Z(\mathsf{E}_0) \sim \mathsf{E}_0$. See [JW, Lemma 5.14, Th. 5.15] for the nonunitary nongraded Henselian valued analogue of this.

5. Galois cohomology with twisted coefficients

Where $\widehat{H}^{-1}(H, M^*)$ occurs in formulas for SK_1 as in §3, analogous formulas for the unitary SK_1 involve $\widehat{H}^{-1}(G, \widetilde{M^*})$ for a twisted action of G on the multiplicative group M^*. In this section, we recall the relevant twisted action, and give some calculations concerning $\widehat{H}^{-1}$ which will be used later. The cohomology with twisted action also allows us to give a new interpretation of Albert's corestriction condition for an algebra to have a unitary involution, see Prop. 5.1 below.

Let G be a profinite group with a closed subgroup H with $|G : H| = 2$. From the mappping $G/H \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\sim} \mathrm{Aut}(\mathbb{Z})$ we obtain a nontrivial discrete G-module structure on $\mathbb{Z}$ for which for $g \in G$, $j \in \mathbb{Z}$,

$$g * j \;=\; \begin{cases} j, & \text{if } g \in H, \\ -j, & \text{if } g \notin H. \end{cases}$$

Let $\widetilde{\mathbb{Z}}$ denote $\mathbb{Z}$ with this new G-action. Then, for any discrete G-module A we have an associated discrete G-module $\widetilde{A} = A \otimes_{\mathbb{Z}} \widetilde{\mathbb{Z}}$. That is, $\widetilde{A} = A$ as an abelian group, but the G-action on $\widetilde{A}$ (denoted by $*$, while $\cdot$ denotes the G-action on A) is given by

$$g * a \;=\; \begin{cases} g\cdot a, & \text{if } g \in H, \\ -g\cdot a, & \text{if } g \notin H, \end{cases} \quad \text{for all } g \in G,\ a \in A. \tag{5.1}$$

So, the actions of H on $\widetilde{A}$ and on A coincide, and $\widetilde{\widetilde{A}} = A$ as G-modules. The cohomology of such modules is discussed in [AE, Appendix], [KMRT, §30.B], [HKRT, §5]. Notably, there is a canonical short exact sequence of G-modules

$$0 \longrightarrow \widetilde{A} \longrightarrow \mathrm{Ind}_{H\to G}(A) \longrightarrow A \longrightarrow 0$$

Since Shapiro's Lemma says that $\widehat{H}^i(G, \mathrm{Ind}_{H\to G}(A)) \cong \widehat{H}^i(H,A)$ for all $i \in \mathbb{Z}$, this yields a long exact sequence of Tate cohomology groups:

$$\ldots \longrightarrow \widehat{H}^{i-1}(G,A) \longrightarrow \widehat{H}^i(G,\widetilde{A}) \longrightarrow \widehat{H}^i(H,A) \longrightarrow \widehat{H}^i(G,A) \longrightarrow \widehat{H}^{i+1}(G,\widetilde{A}) \longrightarrow \ldots \tag{5.2}$$

(This is stated in [KMRT, (30.10)] and [AE] for nonnegative indices, but it is valid for $i < 0$ as well.) For the trivial G-module $\mathbb{Z}$ we have $|H^1(G,\widetilde{\mathbb{Z}})| = 2$, as (5.2) shows, and each connecting homomorphism $\delta\colon \widehat{H}^{i-1}(G,A) \to \widehat{H}^i(G,\widetilde{A})$ is given by the cup product with the nontrivial element of $H^1(G,\widetilde{\mathbb{Z}})$.

We will invoke the twisted cohomology typically in the following setting: Let $F \subseteq K \subseteq M$ be fields with $[K:F] = 2$, and M Galois over F. Let $G = \mathrm{Gal}(M/F)$ and $H = \mathrm{Gal}(M/K)$, which is a closed subgroup of G of index 2. Then, M^* is a discrete G-module, and $\widetilde{M^*}$ denotes M^* with the twisted G-action relative to H described above. Recall that $\mathrm{Br}(M/K;F)$ denotes the subgroup of $\mathrm{Br}(M/K)$ consisting of classes of central simple K-algebras split by M and having a unitary K/F-involution.

Proposition 5.1. *$H^2(G,\widetilde{M^*}) \cong \mathrm{Br}(M/K;F)$.*

Proof. Part of the long exact sequence (5.2) is

$$H^1(G,M^*) \longrightarrow H^2(G,\widetilde{M^*}) \longrightarrow H^2(H,M^*) \xrightarrow{\ \mathrm{cor}\ } H^2(G,M^*) \tag{5.3}$$

By Albert's theorem [KMRT, Th. 3.1(2)], for $[A] \in \mathrm{Br}(M/K)$, the algebra A has a K/F-involution iff $\mathrm{cor}_{K\to F}(A)$ is split. Thus, in the isomorphism $\mathrm{Br}(M/K) \cong H^2(H,M^*)$, $\mathrm{Br}(M/K;F)$ maps isomorphically to $\ker\big(H^2(H,M^*) \xrightarrow{\mathrm{cor}} H^2(G,M^*)\big)$. Because $H^1(G,M^*) = 0$ by the homological Hilbert 90, the exact sequence (5.3) above yields the desired isomorphism. □

Remark 5.2. Here are formulas for $\widehat{H}^i(G,\widetilde{M^*})$ for small i, which are easily derived from standard group cohomology formulas and (5.2) above. We assume $[M:K] < \infty$, and let θ be any element of $G \setminus H$. So, $\mathrm{Gal}(K/F) = \{\mathrm{id}_K, \theta|_K\}$. We write $b^{1-\theta}$ for $b/\theta(b)$.

(i) $H^1(G,\widetilde{M^*}) \;\cong\; F^*/N_{K/F}(K^*) \;\cong\; \widehat{H}^0(\mathrm{Gal}(K/F),K^*)$.

(ii) $H^0(G,\widetilde{M^*}) \;\cong\; \{c \in K^* \mid N_{K/F}(c) = 1\}$.

(iii) $\widehat{H}^0(G,\widetilde{M^*}) \;\cong\; \{c \in K^* \mid N_{K/F}(c) = 1\} \big/ \{N_{M/K}(m)^{1-\theta} \mid m \in M^*\}$
$\qquad = \{b^{1-\theta} \mid b \in K^*\} \big/ \{N_{M/K}(m)^{1-\theta} \mid m \in M^*\}$.

We will be working particularly with $\widehat{H}^{-1}(G,\widetilde{M^*})$. For this, let $\widetilde{N}\colon \widetilde{M^*} \to K^*$ be given by

$$\widetilde{N}(m) \;=\; \prod_{g\in G} g * m \;=\; \prod_{h\in H} h(m)\cdot(\theta h)(m)^{-1} \;=\; N_{M/K}(m)/\theta(N_{M/K}(m)).$$

So, $\widetilde{N}$ is the norm map for $\widetilde{M^*}$ as a G-module. Note that

$$\ker(\widetilde{N}) \;=\; \{m \in M^* \mid N_{M/K}(m) \in F^*\}. \tag{5.4}$$

Also, let

$$I_G(\widetilde{M^*}) \;=\; \big\langle\, (g*m)m^{-1} \mid m\in M^*, g\in G\big\rangle \;=\; \big\langle\, h(m)/m,\; h\theta(m)\,m \mid m\in M^*, h\in H\big\rangle. \tag{5.5}$$

Then, by definition,

$$\widehat{H}^{-1}(G,\widetilde{M^*}) \;\cong\; \ker(\widetilde{N})\big/ I_G(\widetilde{M^*}). \tag{5.6}$$

In the following useful lemma, part (ii) is an abstraction of an argument of Yanchevskiĭ [Y3, proof of Cor. 4.13].

Lemma 5.3. *Let D be a finite dihedral group, i.e., $D=\langle h,\theta\rangle$ where $\theta^2=1$, $\theta\neq 1$, and $\theta h\theta^{-1}=h^{-1}$, and h has finite order. Let $H=\langle h\rangle$. Let A be a D-module such that $H^1(H,A)=0$ and $H^1(\langle\theta\rangle,A^H)=0$. Let $A^\theta=\{a\in A\mid \theta\cdot a=a\}$ and $N_H(a)=\sum_{h\in H}h\cdot a$. Then,*

(i) $A^H+A^\theta \;=\; \{a\in A\mid a-\theta\cdot a\in A^H\}$.
(ii) $A^\theta+A^{h\theta} \;=\; \{a\in A\mid N_H(a)\in A^\theta\} \;=\; A^\theta+A^{\theta h}$.
(iii) *The map* $\mathrm{cor}_{\langle\theta\rangle\to D}\times \mathrm{cor}_{\langle h\theta\rangle\to D}\colon \widehat{H}^{-1}(\langle\theta\rangle,\widetilde{A})\times \widehat{H}^{-1}(\langle h\theta\rangle,\widetilde{A})\to \widehat{H}^{-1}(D,\widetilde{A})$ *is surjective.*

Proof. (i) We have the short exact sequence of $\langle\theta\rangle$-modules $0\to A^H\to A\to A/A^H\to 0$. Since $H^1(\langle\theta\rangle,A^H)=1$, the long exact cohomology sequence shows that A^θ maps onto $(A/A^H)^\theta$, which yields (i).

(ii) Note that for $a\in A$, $N_H(\theta{\cdot}a)=\sum_{k\in H}(k\theta)\cdot a=\sum_{k\in H}(\theta k^{-1})\cdot a=\theta\cdot N_H(a)$. The left inclusion $\subseteq$ in (ii) follows immediately. For the inverse inclusion, take $a\in A$ with $N_H(a)\in A^\theta$. Then, $N_H(a-\theta{\cdot}a)=N_H(a)-\theta{\cdot}N_H(a)=0$. Since $H^1(H,A)=0$, with $H=\langle h\rangle$, there is $c\in A$ with $a-\theta{\cdot}a=c-h{\cdot}c$. So,

$$\begin{aligned} 0 \;&=\; a-\theta{\cdot}a+\theta{\cdot}(a-\theta{\cdot}a) \;=\; c-h{\cdot}c+\theta{\cdot}c-(\theta h){\cdot}c\\ &=\; c-h{\cdot}c+(h\theta h){\cdot}c-(\theta h){\cdot}c \;=\; [c-(\theta h){\cdot}c]-h{\cdot}[c-(\theta h){\cdot}c], \end{aligned}$$

i.e., $c-(\theta h){\cdot}c\in A^H$. Since the group action of $\langle\theta h\rangle$ on A^H coincides with the action of $\langle\theta\rangle$ on A^H, we have $H^1(\langle\theta h\rangle,A^H)\cong H^1(\langle\theta\rangle,A^H)=0$. Therefore, part (i) applies, with θh replacing θ. Thus, we can write $c=d+e$ with $d\in A^H$ and $e\in A^{\theta h}$, hence $\theta{\cdot}e=h{\cdot}e=(h\theta){\cdot}(\theta{\cdot}e)$. Now, as $d=h{\cdot}d$,

$$a-\theta{\cdot}a \;=\; c-h{\cdot}c \;=\; e-h{\cdot}e \;=\; e-\theta{\cdot}e,$$

showing that $a+\theta{\cdot}e\in A^\theta$. Thus, $a=[a+\theta{\cdot}e]-\theta{\cdot}e\in A^\theta+A^{h\theta}$, completing the proof of the first equality in (ii). Since $\theta h=h^{-1}\theta$, the second equality in (ii) follows from the first by replacing h by h^{-1}.

(iii) We have $\widehat{H}^{-1}(\langle\theta\rangle,\widetilde{A})\cong A^\theta\big/\{a+\theta{\cdot}a\mid a\in A\}$, $\widehat{H}^{-1}(\langle h\theta\rangle,\widetilde{A})\cong A^{h\theta}\big/\{a+(h\theta){\cdot}a\mid a\in A\}$, and

$$\widehat{H}^{-1}(D,\widetilde{A}) \;\cong\; \{a\in A\mid N_H(a)\in A^\theta\}\,\big/\,\langle a-k{\cdot}a,\; a+(k\theta){\cdot}a\mid a\in A,\; k\in H\rangle.$$

The map $\mathrm{cor}_{\langle\theta\rangle\to D}\colon \widehat{H}^{-1}(\langle\theta\rangle,\widetilde{A})\to\widehat{H}^{-1}(D,\widetilde{A})$ arises from the inclusion $A^\theta\hookrightarrow\{a\in A\mid N_H(a)\in A^\theta\}$; likewise for $\mathrm{cor}_{\langle h\theta\rangle\to D}\colon \widehat{H}^{-1}(\langle h\theta\rangle,\widetilde{A})\to\widehat{H}^{-1}(D,\widetilde{A})$. Thus, the surjectivity asserted in part (iii) is immediate from part (ii). □

Proposition 5.4. *Let $F\subseteq K\subseteq M$ be fields with $[M:F]<\infty$ and M a K/F-generalized dihedral exten-sion. Let $G=\mathrm{Gal}(M/F)$ and $H=\mathrm{Gal}(M/K)$. Take any $\theta\in G\setminus H$. Then there is an exact sequence:*

$$\prod_{h\in H}\widehat{H}^{-1}(\langle h\theta\rangle,\widetilde{M^*}) \;\longrightarrow\; \widehat{H}^{-1}(G,\widetilde{M^*}) \;\longrightarrow\; \ker(\widetilde{N})\,\big/\,\Pi \;\longrightarrow\; 1 \tag{5.7}$$

where $\ker(\widetilde{N})=\{m\in M^*\mid N_{M/K}(m)\in F^*\}$ and $\Pi=\prod_{h\in H}M^{*h\theta}$. In particular, if M/K is cyclic Galois, then $\ker(\widetilde{N})/\Pi=1$.

Proof. Here, $M^{*h\theta} = \{m \in M^* \mid h\theta(m) = m\}$. We have $\widehat{H}^{-1}(G, \widetilde{M^*}) \cong \ker(\widetilde{N})/I_G(\widetilde{M^*})$ as in (5.4)–(5.6). For any $h \in H$ and $m \in M^*$,

$$m/h(m) \,=\, [m\cdot\theta(m)]/[\theta(m)\cdot h(m)] \,=\, [m\cdot\theta(m)]/[\theta(m)\cdot h\theta(\theta(m))] \,\in\, M^{*\theta}M^{*h\theta}$$

and $m\cdot h\theta(m) \in M^{*h\theta}$. Hence, by (5.5),

$$I_G(\widetilde{M^*}) \,\subseteq\, \prod_{h\in H} M^{*h\theta} \,=\, \Pi. \tag{5.8}$$

Thus, there is a well-defined epimorphism $\zeta\colon \widehat{H}^{-1}(G, \widetilde{M^*}) \to \ker(\widetilde{N})\,/\,\Pi$, with $\ker(\zeta) = \Pi/I_G(\widetilde{M^*})$. Now, for $h \in H$, we have $\widehat{H}^{-1}(\langle h\theta\rangle, \widetilde{M^*}) \cong M^{*h\theta}/N_{M/M^{\langle h\theta\rangle}}(M^*)$. So, $\prod_{h\in H} \widehat{H}^{-1}(\langle h\theta\rangle, \widetilde{M^*})$ clearly maps onto $\ker(\zeta)$, proving the exactness of (5.7). If H is cyclic, then G is dihedral, and $\ker(\widetilde{N}) = \Pi$ by Lemma 5.3(ii). □

Remark 5.5. In the context of Prop. 5.4, suppose $H = \langle h_1, \ldots, h_m\rangle$. Then, the following lemma shows that

$$\prod_{h\in H} M^{*h\theta} \,=\, \prod_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{0,1\}^m} M^{*\,h_1^{\varepsilon_1}\ldots h_m^{\varepsilon_m}\theta}, \tag{5.9}$$

so the left term in (5.7) could be replaced by $\prod_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{0,1\}^m} \widehat{H}^{-1}(\langle h_1^{\varepsilon_1}\ldots h_m^{\varepsilon_m}\theta\rangle, \widetilde{M^*})$. One can see by looking at examples that the product on the right in (5.9) is minimal in that if we delete any of the terms in that product, then the equality no longer holds in general.

Lemma 5.6. *Let $G = \langle H, \theta\rangle$ be a generalized dihedral group, where H is an abelian subgroup of G with $|G{:}H| = 2$, θ has order 2, and $\theta h\theta = h^{-1}$ for all $h \in H$. Let A be any G-module. Suppose $H = \langle h_1, \ldots, h_m\rangle$. Then,*

$$\sum_{h\in H} A^{h\theta} \,=\, \sum_{(\varepsilon_1,\ldots\varepsilon_m)\in\{0,1\}^m} A^{h_1^{\varepsilon_1}\ldots h_m^{\varepsilon_m}\theta}.$$

Proof. This follows from [HW$_2$, Lemma 4.9] (with A for U, H for the abelian group A and $W_h = A^{h\theta}$ for all $h \in H$), once we establish that $A^{h\theta} \subseteq A^{k\theta} + A^{k^2h^{-1}\theta}$ for all $h, k \in H$. For this, take any $a \in A^{h\theta}$. Then $\theta(a) = h^{-1}(a)$. Hence, $k^2h^{-1}\theta(k\theta(a)) = k^2h^{-1}k^{-1}(a) = k\theta(a)$, showing that $k\theta(a) \in A^{k^2h^{-1}\theta}$. Thus $a = [a + k\theta(a)] - k\theta(a) \in A^{k\theta} + A^{k^2h^{-1}\theta}$, proving the required inclusion. □

6. Unitary relative Brauer Groups, bicylic case

In this section we prove a unitary version of the formula $\mathrm{Br}(M/K)/\,\mathrm{Dec}(M/K) \cong \widehat{H}^{-1}(\mathrm{Gal}(M/K), M^*)$, for M a bicyclic Galois extension of K, see (3.9) above. The unitary version was inspired by the result of Yanchevskiĭ [Y$_3$, Prop. 5.5], which was a key part of his proof in [Y$_4$, Th. A] that any finite abelian group can be realized as the unitary SK_1 of some division algebra with involution of the second kind.

Let $F \subseteq K \subseteq M$ be fields with $[K:F] = 2$ and K Galois over F, and $M = L_1 \otimes_K L_2$ with each L_i cyclic Galois over F. Assume M is K/F-generalized dihedral, as described at the beginning of §4. Let $G = \mathrm{Gal}(M/F)$ and $H = \mathrm{Gal}(M/K)$, and choose and fix an element $\theta \in G\backslash H$. So, $\mathrm{Gal}(K/F) = \{\theta|_K, \mathrm{id}_K\}$. To simplify notation, let σ (not σ_1) be a fixed generator of $\mathrm{Gal}(M/L_2)$, and ρ (not σ_2) a fixed generator of $\mathrm{Gal}(M/L_1)$; so, $H = \langle\sigma\rangle \times \langle\rho\rangle$. Let $n = [L_1:K]$, which is the order of σ in H, and let $\ell = [L_2:K]$, which is the order of ρ. As in Prop. 5.4, let

$$\ker(\widetilde{N}) \,=\, \{a \in M^* \mid N_{M/K}(a) \in F^*\}$$

and

$$\Pi \,=\, \textstyle\prod_{h\in H} M^{*h\theta} \,=\, M^{*\theta}M^{*\rho\theta}M^{*\sigma\theta}M^{*\rho\sigma\theta}. \tag{6.1}$$

(See (5.9) for the second equality.)

Proposition 6.1. *We have*

$$\mathrm{Br}(M/K;F)\big/\,\mathrm{Dec}(M/K;F)\ \cong\ \ker(\widetilde{N})/\Pi.$$

Proof. This follows by combining the formulas for unitary SK_1 given in [Y3, Prop. 5.5] with the Henselian version of the formula in [HW2, Cor. 4.11]. However, we give a direct proof avoiding the use of Yanchevskiĭ's special unitary conorms, since we will later need an explicit description of the isomorphism.

Define a map

$$\Psi\colon\ \mathrm{Br}(M/K;F)\longrightarrow \ker(\widetilde{N})/\Pi$$

as follows: By Lemma 4.1, a Brauer class in $\mathrm{Br}(M/K;F)$ is represented by an algebra $A = A(u,b_1,b_2)$, where u, b_1, b_2 satisfy the conditions in (3.6) and $b_1 \in L_2^{*\theta}, b_2 \in L_1^{*\theta}$, and $u\,\rho\sigma\theta(u) = 1$. By Hilbert 90 (for the group $\langle\rho\sigma\theta\rangle$), there is $q \in M^*$ with $u = q/\rho\sigma\theta(q)$. Define

$$\Psi\big(A(u,b_1,b_2)\big)\ =\ q\,\Pi\ \in\ \ker(\widetilde{N})/\Pi.$$

We will show that Ψ is a well-defined, surjective homomorphism with kernel $\mathrm{Br}(L_1/K;F)\,\mathrm{Br}(L_2/K;F)$, which equals $\mathrm{Dec}(M/K;F)$ (see (4.3)).

For the well-definition of Ψ, first note that

$$1\ =\ N_{M/K}(u)\ =\ N_{M/K}(q/\rho\sigma\theta(q))\ =\ N_{M/K}(q)\big/N_{M/K}(\theta(q))\ =\ N_{M/K}(q)\big/\theta(N_{M/K}(q)),$$

so, $q \in \ker(\widetilde{N})$. Also, given u, the choice of q with $q/\rho\sigma\theta(q) = u$ is unique up to a multiple in $M^{*\rho\sigma\theta}$. Since $M^{*\rho\sigma\theta} \subseteq \Pi$, $\Psi\big(A(u,b_1,b_2)\big)$ is independent of the choice of q from u. Now, suppose $A(u,b_1,b_2) \cong A(u',b_1',b_2')$, with u, b_1, b_2 and u', b_1', b_2' each satisfying the conditions of Lemma 4.1(iii). We have the presentation $A(u,b_1,b_2) = \bigoplus_{i=0}^{n-1}\bigoplus_{j=0}^{\ell-1} Mx^iy^j$, where $\mathrm{int}(x)|_M = \sigma$, $x^n = b_1$, $\mathrm{int}(y)|_M = \rho$, $y^\ell = b_2$, and $xyx^{-1}y^{-1} = u$, so, (see (3.6))

$$b_1 \in M^{\langle\sigma\rangle}\ =\ L_2,\quad b_2 \in M^{\langle\rho\rangle}\ =\ L_1,\quad N_{M/L_2}(u)\ =\ b_1/\rho(b_1),\quad N_{M/L_1}(u)\ =\ \sigma(b_2)/b_2. \tag{6.2}$$

The conditions of Lemma 4.1(iii) we are also assuming are that

$$b_1\ \in\ L_2^\theta,\qquad b_2 \in L_1^\theta,\qquad \text{and}\qquad u\,\rho\sigma\theta(u)\ =\ 1. \tag{6.3}$$

The corresponding conditions in (6.2) and (6.3) hold for b_1', b_2' and u'. By Lemma 4.1, there is a K/F-involution τ of $A = A(u,b_1,b_2)$ with $\tau|_M = \theta, \tau(x) = x, \tau(y) = y$. We have an isomorphism $A(u,b_1,b_2) \cong A(u',b_1',b_2')$, and by Skolem-Noether there is such an isomorphism which restricts to the identity on M. Therefore, there exist x' and y' in A^* such that $\mathrm{int}(x')|_M = \sigma$, $x'^n = b_1'$, $\mathrm{int}(y')|_M = \rho$, $y'^\ell = b_2'$, and $x'y'x'^{-1}y'^{-1} = u'$. Since $\mathrm{int}(x')|_M = \mathrm{int}(x)|_M$ there is $c_1 \in C_A(M)^* = M^*$ with $x' = c_1x$, and likewise $c_2 \in M^*$ with $y' = c_2y$. By simplifying the expressions $b_1' = (c_1x)^n$, $b_2' = (c_2y)^\ell$, and $u' = (c_1x)(c_2y)(c_1x)^{-1}(c_2y)^{-1}$, we find that

$$b_1'\ =\ N_{M/L_2}(c_1)\,b_1,\qquad b_2'\ =\ N_{M/L_1}(c_2)\,b_2,\qquad u'\ =\ \big(c_1/\rho(c_1)\big)\big(\sigma(c_2)/c_2\big)\,u. \tag{6.4}$$

By Lemma 4.1, there is a K/F-involution τ' on A with $\tau'(x') = x', \tau'(y') = y'$, and $\tau'|_M = \theta$. Since $\tau'\tau^{-1}$ is a K-automorphism of A, there exists $e \in A^*$ with $\tau' = \mathrm{int}(e)\tau$. Because $\tau'|_M = \tau|_M$, $e \in C_A(M) = M$. The condition that $\tau'^2 = \mathrm{id}_A$ implies that $e/\theta(e) \in K^*$. Since $e/\theta(e)\big(\theta(e/\theta(e))\big) = 1$, Hilbert 90 for K/F shows that there is $d \in K^*$ with $d/\theta(d) = e/\theta(e)$. By replacing e by e/d, we may assume that $\theta(e) = e$. The conditions that $c_1x = \tau'(c_1x) = \mathrm{int}(e)\tau(c_1x)$ and $c_2y = \tau'(c_2y) = \mathrm{int}(e)\tau(c_2y)$ yield

$$c_1\ =\ \sigma\theta(c_1)\,e/\sigma(e)\qquad\text{and}\qquad c_2\ =\ \rho\theta(c_2)\,e/\rho(e),$$

hence,

$$\rho(c_1)\ =\ \rho\sigma\theta(c_1)\,\rho(e)/\rho\sigma(e)\qquad\text{and}\qquad \sigma(c_2)\ =\ \rho\sigma\theta(c_2)\,\sigma(e)/\rho\sigma(e). \tag{6.5}$$

The equations (6.5) yield

$$c_1/\rho(c_1)\ =\ \big(c_1/\rho\sigma\theta(c_1)\big)\big(\rho\sigma(e)/\rho(e)\big)\qquad\text{and}\qquad \sigma(c_2)/c_2\ =\ \big(\rho\sigma\theta(c_2)/c_2\big)\big(\sigma(e)/\rho\sigma(e)\big). \tag{6.6}$$

Let $\widetilde{q} = (c_1/c_2)\sigma(e)$. Then, using (6.6), (6.4) and $\theta(e) = e$,

$$\begin{aligned}\widetilde{q}/\rho\sigma\theta(\widetilde{q}) &= \big(c_1/\rho\sigma\theta(c_1)\big)\big(\rho\sigma\theta(c_2)/c_2\big)\big(\sigma(e)/\rho\sigma\theta\sigma(e)\big)\\ &= \big(c_1/\rho(c_1)\big)\,\big(\rho(e)/\rho\sigma(e)\big)\,\big(\sigma(c_2)/c_2\big)\,\big(\rho\sigma(e)/\sigma(e)\big)\,\big(\sigma(e)/\rho\theta(e)\big)\\ &= \big(c_1/\rho(c_1)\big)\,\big(\sigma(c_2)/c_2\big) \;=\; u'/u.\end{aligned} \tag{6.7}$$

When $q \in M^*$ is chosen so that $q/\rho\sigma\theta(q) = u$, set $q' = \widetilde{q}q$; then (6.7) shows that $q'/\rho\sigma\theta(q') = u'$. We check that $\widetilde{q} \in \Pi$: We have (see (6.4) and (6.3)) $N_{M/L_2}(c_1) = b_1'/b_1 \in L_2^{*\theta}$. Therefore, by Lemma 5.3(ii) applied to the dihedral group $\langle\sigma,\theta\rangle = \mathrm{Gal}(M/L_2^\theta)$, $c_1 \in M^{*\theta}M^{*\sigma\theta} \subseteq \Pi$. Likewise, $c_2 \in M^{*\theta}M^{*\rho\theta} \subseteq \Pi$ as $N_{M/L_1}(c_2) = b_2'/b_2 \in L_1^{*\theta}$. Finally, since $\theta(e) = e$, we have $\sigma(e) = \sigma\theta(e) = \sigma\theta\sigma^{-1}(\sigma(e)) = \sigma^2\theta(\sigma(e))$. So, $\sigma(e) \in M^{*\sigma^2\theta} \subseteq \Pi$. Thus, $q' \equiv q \pmod{\Pi}$, which shows that Ψ is well-defined independent of the choice of presentation of A as $A(u,b_1,b_2)$ with u,b_1,b_2 as in Lemma 4.1(iii).

For the surjectivity of Ψ, take any $q \in \ker(\widetilde{N})$ and set $u = q/\rho\sigma\theta(q)$. So, $u\,\rho\sigma\theta(u) = 1$. Furthermore, as $N_{M/K}(q) \in F^*$,

$$N_{M/K}(u) \;=\; N_{M/K}(q)/N_{M/K}(\rho\sigma\theta(q)) \;=\; N_{M/K}(q)/\theta(N_{M/K}(q)) \;=\; 1.$$

Since $N_{L_2/K}(N_{M/L_2}(u)) = N_{M/K}(u) = 1$, by Hilbert 90 for L_2/K there is $b_1 \in L_2^*$ with $b_1/\rho(b_1) = N_{M/L_2}(u)$. Then,

$$b_1/\rho(b_1) \;=\; N_{M/L_2}(q)\big/N_{M/L_2}(\rho\sigma\theta(q)) \;=\; N_{M/L_2}(q)/\rho\theta(N_{M/L_2}(q)).$$

Hence,

$$1 \;=\; (b_1/\rho(b_1))\,\rho\theta(b_1/\rho(b_1)) \;=\; (b_1/\theta(b_1))/\rho(b_1/\theta(b_1)),$$

which shows that $b_1/\theta(b_1) \in L_2^\rho = K$. By Lemma 5.3(i) applied to the dihedral group $\mathrm{Gal}(L_2/F) = \langle\rho|_{L_2},\theta|_{L_2}\rangle$, it follows that $b_1 = k\widehat{b_1}$ with $k \in K^*$ and $\widehat{b_1} \in L_2^{*\theta}$. By replacing b_1 with $\widehat{b_1}$, we may assume that $b_1 \in L_2^{*\theta}$. Likewise, there is $b_2 \in L_1^{*\theta}$ with $N_{M/L_1}(u^{-1}) = b_2/\sigma(b_2)$. Then, as u,b_1,b_2 satisfy the conditions of (3.6) (where $\sigma_1 = \sigma$ and $\sigma_2 = \rho$) the algebra $A(u,b_1,b_2)$ exists, and by Lemma 4.1 $[A(u,b_1,b_2)] \in \mathrm{Br}(M/K;F)$. Clearly, $\Psi[A(u,b_1,b_2)] = q\,\Pi$.

Finally, we determine $\ker(\Psi)$: If $[B] \in \mathrm{Br}(L_1/K;F)$ then we can assume that B has L_1 as a maximal subfield. Then, by Lemma 4.1, $B \cong (L_1/K,\sigma,b_1)$, where $b_1 \in K^{*\theta} = F^*$. Likewise, for any $[C] \in \mathrm{Br}(L_2/K;F)$, we have $C \sim (L_2/K,\rho,b_2)$ for some $b_2 \in F^*$. Then,

$$\big[B \otimes_K C\big] = \big[(L_1/K,\sigma,b_1) \otimes_K (L_2/K,\rho,b_2)\big] = \big[A(1,b_1,b_2)\big] \in \ker(\Psi),$$

since when $u = 1$ we can take $q = 1$. So $\mathrm{Br}(L_1/K;F)\,\mathrm{Br}(L_2/K;F) \subseteq \ker(\Psi)$. For the reverse inclusion, take any $A = A(u,b_1,b_2)$ with $[A] \in \ker(\Psi)$. Since $[A] \in \mathrm{Br}(M/K;F)$, by Lemma 4.1 we may assume that $b_1 \in L_2^{*\theta}, b_2 \in L_1^{*\theta}$ and $u\,\rho\sigma\theta(u) = 1$. Since, $[A] \in \ker(\Psi)$, we have $u = q/\rho\sigma\theta(q)$ with $q \in \Pi$, so $q = q_\theta q_{\rho\theta} q_{\sigma\theta} q_{\rho\sigma\theta}$, where $q_\theta \in M^{*\theta}, q_{\rho\theta} \in M^{*\rho\theta}$, $q_{\sigma\theta} \in M^{*\sigma\theta}$, and $q_{\rho\sigma\theta} \in M^{*\rho\sigma\theta}$. Thus,

$$\begin{aligned}u &= q/\rho\sigma\theta(q) \;=\; \big(q_\theta/\rho\sigma(q_\theta)\big)\big(q_{\rho\theta}/\sigma(q_{\rho\theta})\big)\big(q_{\sigma\theta}/\rho(q_{\sigma\theta})\big)\\ &= \big(q_\theta q_{\rho\theta}/\sigma(q_\theta q_{\rho\theta})\big)\big(q_{\sigma\theta}\sigma(q_\theta)/\rho(q_{\sigma\theta}\sigma(q_\theta))\big) \;=\; \big(c_2/\sigma(c_2)\big)\big(\rho(c_1)/c_1\big),\end{aligned}$$

where $c_2 = q_\theta q_{\rho\theta}$ and $c_1 = (q_{\sigma\theta}\sigma(q_\theta))^{-1}$. Then by (3.7), $A = A(u,b_1,b_2) \cong A(u',b_1',b_2')$ where $u' = (c_1/\rho(c_1))(\sigma(c_2)/c_2)u = 1$, and $b_1' = N_{M/L_2}(c_1)b_1$ and $b_2' = N_{M/L_1}(c_2)b_2$. Since $c_2 \in M^{*\theta}M^{*\rho\theta}$, an easy calculation or an application of Lemma 5.3(ii) for the dihedral group $\mathrm{Gal}(M/L_1^\theta) = \langle\rho,\theta\rangle$ shows that $N_{M/L_1}(c_2) \in L_1^{*\theta}$. Therefore, $b_2' = N_{M/L_1}(c_2)b_2 \in L_1^{*\theta}$, as $b_2 \in L_1^{*\theta}$. But also, as in (6.2), $\sigma(b_2')/b_2' = N_{M/L_1}(u') = N_{M/L_1}(1) = 1$. Hence, $b_2' \in L_1^{*\theta} \cap L_1^{*\sigma} = K^{*\theta} = F^*$. Likewise, as $q_{\sigma\theta} \in M^{*\theta}$ and $\sigma(q_\theta) \in M^{*\sigma^2\theta} \subseteq M^{*\theta}M^{*\sigma\theta}$ (see (5.9)), we have $c_1 \in M^{*\theta}M^{*\sigma\theta}$. Therefore, an easy calculation or Lemma 5.3(ii) for the dihedral group $\mathrm{Gal}(M/L_2^\theta) = \langle\sigma,\theta\rangle$ shows that $N_{M/L_2}(c_1) \in L_2^{*\theta}$. So, arguing just as for b_2', we find that $b_1' \in F^*$. Thus,

$$A \;\cong\; A(1,b_1',b_2') \;\cong\; (L_1/K,\sigma,b_1') \otimes_K (L_2/K,\rho,b_2'),$$

and since the $b_i' \in F^*$, $\big[(L_1/K,\sigma,b_1')\big] \in \mathrm{Br}(L_1/K;F)$ and $\big[(L_2/K,\rho,b_2')\big] \in \mathrm{Br}(L_2/K;F)$, by Lemma 4.1. Thus, $\ker(\Psi) = \mathrm{Br}(L_1/K;F)\,\mathrm{Br}(L_2/K;F) = \mathrm{Dec}(M/K;F)$. □

This yields our unitary analogue to (3.9) above.

Proposition 6.2. *For M bicyclic Galois over K with M K/F-generalized dihedral, setting $G = \mathrm{Gal}(M/F)$, $H = \mathrm{Gal}(M/K)$, and θ any element of $G \setminus H$ as above, there is an exact sequence*

$$\prod_{h\in H} \widehat{H}^{-1}(\langle h\theta\rangle, \widetilde{M^*}) \;\longrightarrow\; \widehat{H}^{-1}(G,\widetilde{M^*}) \;\longrightarrow\; \mathrm{Br}(M/K;F)/\,\mathrm{Dec}(M/K;F) \;\longrightarrow\; 0 \tag{6.8}$$

Proof. This follows from Prop. 6.1 and Prop. 5.4. □

7. Semiramfied algebras

We now apply the results of the preceding sections to the calculation of unitary SK_1 for semiramified graded division algebras with graded T/R-involution Throughout this section, fix a graded field T and a graded subfield R of T with $[\mathsf{T}\!:\!\mathsf{R}] = 2$ and T Galois over R, say with $\mathsf{Gal}(\mathsf{T}/\mathsf{R}) = \{\mathrm{id},\psi\}$. Assume further that T is inertial over R. Thus, $\Gamma_{\mathsf{T}} = \Gamma_{\mathsf{R}}$, $[\mathsf{T}_0\!:\!\mathsf{R}_0] = 2$, T_0 is Galois over with $\mathrm{Gal}(\mathsf{T}_0/\mathsf{R}_0) = \{\mathrm{id},\psi_0\}$, where $\psi_0 = \psi|_{\mathsf{T}_0}$, and $\psi = \psi_0 \otimes \mathrm{id}_{\mathsf{R}}$ when we identify T with $\mathsf{T}_0 \otimes_{\mathsf{R}_0} \mathsf{R}$. By definition, for a central simple graded division algebra B over T with a graded unitary T/R-involution τ, the unitary SK_1 is given by

$$\mathrm{SK}_1(\mathsf{B},\tau) \;=\; \Sigma'_\tau(\mathsf{B})\,/\,\Sigma_\tau(\mathsf{B}),$$

where

$$\Sigma'_\tau(\mathsf{B}) \;=\; \{b\in\mathsf{B}^* \mid \mathrm{Nrd}_{\mathsf{B}}(b)\in\mathsf{R}\} \qquad\text{and}\qquad \Sigma_\tau(\mathsf{B}) \;=\; \big\langle\{b\in\mathsf{B}^* \mid \tau(b)=b\}\big\rangle$$

We are assuming that T/R is inertial because otherwise T/R is totally ramified and $\mathrm{SK}_1(\mathsf{B},\tau) = 1$, by [$\mathrm{HW}_2$, Th. 4.5]. It is known by [HW_2, Lemma 2.3(iii)] that $[\mathsf{B}^*,\mathsf{B}^*] \subseteq \Sigma_\tau(\mathsf{B})$, so $\mathrm{SK}_1(\mathsf{B},\tau)$ is an abelian group. Also, if τ' is another graded T/R-involution on B, then $\Sigma'_{\tau'}(\mathsf{B}) = \Sigma'_\tau(\mathsf{B})$ and $\Sigma_{\tau'}(\mathsf{B}) = \Sigma_\tau(\mathsf{B})$, so $\mathrm{SK}_1(\mathsf{B},\tau') = \mathrm{SK}_1(\mathsf{B},\tau)$. The easy proof is analogous to the ungraded proof given in [Y_1, Lemma 1].

Let E be a semiramified T-central graded division algebra. So, as we have seen, E_0 is a field abelian Galois over T_0, and $\overline{\Theta}_{\mathsf{E}}\colon \Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}} \to \mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0)$ is a canonical isomorphism. Suppose E has a graded T/R-involution τ; so $\tau|_{\mathsf{T}_0} = \psi_0$. We have seen in Prop. 4.5 that E_0 is then a $\mathsf{T}_0/\mathsf{R}_0$-generalized dihedral Galois extension. Let $H = \mathrm{Gal}(\mathsf{E}_0/\mathsf{T}_0)$ and $G = \mathrm{Gal}(\mathsf{E}_0/\mathsf{R}_0)$, and let $\overline{\tau} = \tau|_{\mathsf{E}_0} \in G\setminus H$. For each $\gamma\in\Gamma_{\mathsf{E}}$ choose and fix $x_\gamma \in \mathsf{E}_\gamma$ with $x_\gamma \neq 0$ and $\tau(x_\gamma) = x_\gamma$. (Such x_γ exist, by [HW_2, Lemma 4.6(i)].) Our starting point is the formula proved in [HW_2, Th. 4.7]

$$\mathrm{SK}_1(\mathsf{E},\tau) \;\cong\; \big(\Sigma_\tau(\mathsf{E})'\cap\mathsf{E}_0^*\big)\,/\big(\Sigma_\tau(\mathsf{E})\cap\mathsf{E}_0^*\big) \;=\; \ker(\widetilde{N})/\big(\Pi\cdot X\,\big), \tag{7.1}$$

where

$$\begin{aligned}\ker(\widetilde{N}) \;&=\; \{a\in\mathsf{E}_0^* \mid N_{\mathsf{E}_0/\mathsf{T}_0}(a)\in\mathsf{R}_0\};\\ \Pi \;&=\; \prod_{h\in H}\mathsf{E}_0^{*h\overline{\tau}}, \quad\text{where}\quad \mathsf{E}_0^{*h\overline{\tau}} \;=\; \{a\in\mathsf{E}_0^* \mid h\overline{\tau}(a)=a\};\\ X \;&=\; \big\langle x_\gamma x_\delta x_{\gamma+\delta}^{-1} \mid \gamma,\delta\in\Gamma_{\mathsf{E}}\big\rangle \;\subseteq\; \mathsf{E}_0^*.\end{aligned}$$

Note that H maps $\ker(\widetilde{N})$ (resp. Π) to itself, so H acts on $\ker(\widetilde{N})/\Pi$. But this action is trivial since $I_H(\ker(\widetilde{N})) \subseteq I_G(\widetilde{\mathsf{E}_0}^*) \subseteq \Pi$ (see (5.8) above).

Theorem 7.1. *Suppose E is DSR for T/R, i.e., in addition to the hypotheses above, E has a maximal graded subfield J with $\tau(\mathsf{J}) = \mathsf{J}$. Then,*

(i) $\mathrm{SK}_1(\mathsf{E},\tau) \cong \ker(\widetilde{N})/\Pi$, *and there is an exact sequence*

$$\prod_{h\in H} \widehat{H}^{-1}(\langle h\overline{\tau}\rangle, \widetilde{\mathsf{E}_0^{\,*}}) \longrightarrow \widehat{H}^{-1}(G, \widetilde{\mathsf{E}_0^{\,*}}) \longrightarrow \mathrm{SK}_1(\mathsf{E},\tau) \longrightarrow 1.$$

(ii) *If* $\mathsf{E}_0 = L_1 \otimes_{\mathsf{T}_0} L_2$ *with each* L_i *cyclic Galois over* T_0, *then*

$$\mathrm{SK}_1(\mathsf{E},\tau) \;\cong\; \mathrm{Br}(\mathsf{E}_0/\mathsf{T}_0;\mathsf{R}_0)\big/\operatorname{Dec}(\mathsf{E}_0/\mathsf{T}_0;\mathsf{R}_0).$$

Proof. (i) The first formula for $\mathrm{SK}_1(\mathsf{E},\tau)$ was given in [HW$_2$, Cor. 4.11]. The point is that the x_γ can all be chosen in J; then $X \subseteq \mathsf{J}_0^{*\tau} = \mathsf{R}_0{}^* \subseteq \Pi$, so the X term in (7.1) drops out. The exact sequence in (i) then follows by Prop. 5.4. Part (ii) is immediate from (i) and Prop. 6.1. □

Note that Th. 7.1 is the unitary analogue to Prop. 3.2 for nonunitary SK_1 in the DSR case.

To improve the formula (7.1) in the manner of Th. 7.1 for E semiramified but not DSR we need more information on the contribution of the X term. This contribution is measured by $(\Pi\cdot X)/\Pi$. For $\gamma\in\Gamma_{\mathsf{E}}$ we write $\overline{\gamma}$ for $\gamma+\Gamma_{\mathsf{T}}\in\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$.

Proposition 7.2. *There is a well-defined* 2*-cocycle* $g\in Z^2(\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}, \ker(\widetilde{N})/\Pi)$ *given by*

$$g(\overline{\gamma},\overline{\delta}) \;=\; x_\gamma x_\delta x_{\gamma+\delta}^{-1}\,\Pi. \tag{7.2}$$

This g *is independent of the choice of nonzero symmetric elements* $x_\gamma, x_\delta, x_{\gamma+\delta}$ *in* $\mathsf{E}_\gamma, \mathsf{E}_\delta, \mathsf{E}_{\gamma+\delta}$. *Furthermore, for all* $\overline{\gamma},\overline{\delta}\in\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$ *and* $i,j,k,\ell\in\mathbb{Z}$, *we have*

$$g(i\overline{\gamma}+j\overline{\delta}, k\overline{\gamma}+\ell\overline{\delta}) \;=\; g(\overline{\gamma},\overline{\delta})^{\Delta} \quad \textit{where}\quad \Delta = \det\left(\begin{smallmatrix} i & j\\ k & \ell\end{smallmatrix}\right). \tag{7.3}$$

(*In particular,* $g(\overline{\gamma},\overline{\gamma}) = 1\,\Pi$ *and* $g(\overline{\delta},\overline{\gamma}) = g(\overline{\gamma},\overline{\delta})^{-1}$.) *Moreover,* $\langle \operatorname{im}(g)\rangle = \big(\Pi\cdot X\big)/\Pi$, *which is a finite group.*

Proof. For $\gamma,\delta\in\Gamma_{\mathsf{E}}$, set

$$c_{\gamma,\delta} \;=\; x_\gamma x_\delta x_{\gamma+\delta}^{-1} \in \mathsf{E}_0^*.$$

Note that $c_{\gamma,\delta}\in\ker(\widetilde{N})$, since it is a product of τ-symmetric elements of E^*. For notational convenience we work with the function

$$f\colon \Gamma_{\mathsf{E}}\times\Gamma_{\mathsf{E}} \longrightarrow \ker(\widetilde{N})/\Pi \quad\text{given by}\quad f(\gamma,\delta) \;=\; c_{\gamma,\delta}\,\Pi.$$

Thus, $g(\overline{\gamma},\overline{\delta}) = f(\gamma,\delta)$ We first show that the definition of f is independent of the choices made of $x_\gamma, x_\delta, x_{\gamma+\delta}$. Fix γ and δ in Γ_{E} for the moment. Take any $a\in\mathsf{E}_0^*$ with $\tau(ax_\gamma) = ax_\gamma$. Then, $ax_\gamma = \tau(ax_\gamma) = x_\gamma\overline{\tau}(a) = \Theta_{\mathsf{E}}(\gamma)(\overline{\tau}(a))x_\gamma$; so $a = \Theta_{\mathsf{E}}(\gamma)(\overline{\tau}(a))$, i.e. $a\in\mathsf{E}_0^{*\Theta_{\mathsf{E}}(\gamma)\overline{\tau}}\subseteq\Pi$. Hence, if we let $x'_\gamma = ax_\gamma$, then $x'_\gamma x_\delta x_{\gamma+\delta}^{-1} \equiv x_\gamma x_\delta x_{\gamma+\delta}^{-1} \pmod{\Pi}$. Likewise, if we take any $b\in\mathsf{E}_0^*$ with $\tau(bx_\delta) = bx_\delta$, then $\Theta_{\mathsf{E}}(\gamma)(b)\in\mathsf{E}_0^{*\Theta_E(2\gamma+\delta)\overline{\tau}}\subseteq\Pi$ so $x_\gamma x'_\delta x_{\gamma+\delta}^{-1} = \Theta_{\mathsf{E}}(\gamma)(b)x_\gamma x_\delta x_{\gamma+\delta}^{-1} \equiv x_\gamma x_\delta x_{\gamma+\delta}^{-1} \pmod{\Pi}$. Again, for $d\in\mathsf{E}_0^*$ with $\tau(dx_{\gamma+\delta}) = dx_{\gamma+\delta}$, we have $d\in\mathsf{E}_0^{*\Theta_{\mathsf{E}}(\gamma+\delta)\overline{\tau}}\subseteq\Pi$, so for $x'_{\gamma+\delta} = dx_{\gamma+\delta}$, we have $x_\gamma x_\delta x'^{-1}_{\gamma+\delta} \equiv x_\gamma x_\delta x_{\gamma+\delta}^{-1} \pmod{\Pi}$. Thus, each such change does not affect the value of $f(\gamma,\delta)$, and we are free to make such changes when convenient.

We prove further identities for the function f which hold for all $\gamma,\delta,\varepsilon\in\Gamma_{\mathsf{E}}$ and $i,j,k,\ell\in\mathbb{Z}$:

(i) $\quad f(\gamma+\beta,\delta) \;=\; f(\gamma,\delta) \;=\; f(\gamma,\delta+\beta)\quad$ for any $\beta\in\Gamma_{\mathsf{T}}$.

For, as $\Gamma_{\mathsf{R}} = \Gamma_{\mathsf{T}}$, there is a nonzero $a\in\mathsf{R}_\beta$. Since $a\in Z(\mathsf{E})$ and $\tau(a) = a$, we could have chosen $x_{\gamma+\beta} = ax_\gamma$, $x_{\delta+\beta} = ax_\delta$, and $x_{\gamma+\delta+\beta} = ax_{\gamma+\delta}$. Then,

$$f(\gamma+\beta,\delta) \;=\; (ax_\gamma)x_\delta(ax_{\gamma+\delta})^{-1}\,\Pi \;=\; x_\gamma x_\delta x_{\gamma+\delta}^{-1}\,\Pi \;=\; f(\gamma,\delta),$$

and likewise $f(\gamma,\delta+\beta) = f(\gamma,\delta)$. This proves (i), which shows that the g of the Prop. is well-defined.

(ii) $\quad f(i\gamma, j\gamma) \;=\; 1\,\Pi.$

For, we can choose $x_{i\gamma} = x_\gamma^i$, $x_{j\gamma} = x_\gamma^j$, and $x_{i\gamma+j\gamma} = x_\gamma^{i+j}$. Then, $c_{i\gamma,j\gamma} = 1$.

(iii) $\qquad f(\delta,\gamma) \;=\; f(\gamma,\delta)^{-1}.$

For, by applying τ to the equation $x_\gamma x_\delta = c_{\gamma,\delta} x_{\gamma+\delta}$, we obtain

$$x_\delta x_\gamma \;=\; x_{\gamma+\delta}\overline{\tau}(c_{\gamma,\delta}) \;=\; \Theta_{\mathsf{E}}(\gamma+\delta)(\overline{\tau}(c_{\gamma,\delta}))x_{\delta+\gamma},$$

yielding $c_{\delta,\gamma} = \Theta_{\mathsf{E}}(\gamma+\delta)(\overline{\tau}(c_{\gamma,\delta}))$, so $c_{\delta,\gamma}c_{\gamma,\delta} \in \mathsf{E}_0^{*\Theta_{\mathsf{E}}(\gamma+\delta)\overline{\tau}} \subseteq \Pi$. Formula (iii) then follows.

(iv) $\qquad f(\gamma,\delta)f(\gamma+\delta,\varepsilon) \;=\; f(\gamma,\delta+\varepsilon)f(\delta,\varepsilon),$

i.e., $f \in Z^2(\Gamma_{\mathsf{E}}, \ker(\widetilde{N})/\Pi)$, since Γ_{E} (acting via $\Theta_{\mathsf{E}}(\Gamma_{\mathsf{E}}) = H$) acts trivially on $\ker(\widetilde{N})/\Pi$. This identity follows from $(x_\gamma x_\delta)x_\varepsilon = x_\gamma(x_\delta x_\varepsilon)$, which yields $c_{\gamma,\delta}\, c_{\gamma+\delta,\varepsilon} = \Theta_{\mathsf{E}}(\gamma)(c_{\delta,\varepsilon})\, c_{\gamma,\delta+\varepsilon}$. Then (iv) follows, given the trivial action of H on $\ker(\widetilde{N})/\Pi$.

(v) $\qquad f(\gamma+\delta,\delta) \;=\; f(\gamma,\delta) \quad \text{and} \quad f(\gamma,\gamma+\delta) \;=\; f(\gamma,\delta).$

For, as $\tau(x_\delta x_\gamma x_\delta) = x_\delta x_\gamma x_\delta$, we can take $x_{\gamma+2\delta} = x_\delta x_\gamma x_\delta$. Then,

$$x_\delta x_\gamma x_\delta \;=\; c_{\delta,\gamma}\, c_{\delta+\gamma,\delta}\, x_{\gamma+2\delta} \;=\; c_{\delta,\gamma}\, c_{\delta+\gamma,\delta}\, x_\delta x_\gamma x_\delta.$$

Hence, $1\,\Pi = f(\delta,\gamma)f(\delta+\gamma,\delta)$, so $f(\delta+\gamma,\delta) = f(\delta,\gamma)^{-1} = f(\gamma,\delta)$, using (iii). This proves the first formula in (v), and the second formula follows analogously, or from the first by using (iii).

(vi) $\qquad f(\gamma+j\delta,\delta) \;=\; f(\gamma,\delta) \;=\; f(\gamma,j\gamma+\delta) \quad \text{for all } j \in \mathbb{Z}.$

This follows from (v) by induction on j.

(vii) $\qquad f(i\gamma,j\delta) \;=\; f(\gamma,\delta)^{ij}.$

For, by (iv) with $j\delta$ for δ and δ for ε,

$$f(\gamma,j\delta)f(\gamma+j\delta,\delta) \;=\; f(\gamma,(j+1)\delta)f(j\delta,\delta),$$

which by (vi) and (ii) reduces to $f(\gamma,j\delta)f(\gamma,\delta) = f(\gamma,(j+1)\delta)$. Then (vii) for $i = 1$ follows by induction on j with the initial case $j = 0$ given by (ii). From the $i = 1$ case the result for arbitrary i follows by using (iii).

(viii) $\qquad f(i\gamma+j\delta,k\gamma+\ell\delta) \;=\; f(\gamma,\delta)^{\Delta} \quad \text{where } \Delta = \det\left(\begin{smallmatrix} i & j \\ k & \ell \end{smallmatrix}\right).$

For this note first that this is true if $i = 0$, as

$$f(j\delta,k\gamma+\ell\delta) \;=\; f(\delta,k\gamma+\ell\delta)^j \;=\; f(\delta,k\gamma)^j \;=\; f(\gamma,\delta)^{-jk},$$

by (vii), (vi), (vii), and (iii). Analogously, (viii) is true if $k = 0$. To verify (viii) in general, we argue by induction on $|i| + |k|$. By invoking (iii) and interchanging $i\gamma + j\delta$ with $k\gamma + \ell\delta$ if necessary, we can assume $|i| \le |k|$. We can assume $|i| \ge 1$, since the case $|i| = 0$ is already done. Let $\eta = \pm 1$, with the sign chosen so that $|k - \eta i| = |k| - |i|$. Since $|i| + |k - \eta i| = |k| < |i| + |k|$, we have by (vi) and induction,

$$\begin{aligned} f(i\gamma+j\delta,k\gamma+\ell\delta) \;&=\; f\big(i\gamma+j\delta,(k\gamma+\ell\delta) - \eta(i\gamma+j\delta)\big) \;=\; f\big(i\gamma+j\delta,(k-\eta i)\gamma + (\ell-\eta j)\delta\big) \\ &=\; f(\gamma,\delta)^{\Delta'} \quad \text{where } \; \Delta' = \det\begin{pmatrix} i & j \\ k-\eta i & \ell-\eta j \end{pmatrix} = \det\left(\begin{smallmatrix} i & j \\ k & \ell \end{smallmatrix}\right). \end{aligned}$$

Thus, (viii) is proved, and when (viii) is restated in terms of g, it is formula (7.3). It is clear from the definition and well-definition of g that $\langle \mathrm{im}(g)\rangle = (\Pi \cdot X)/\Pi$. This abelian group is finite since the domain of g is finite, and each $g(\overline{\gamma},\overline{\delta})$ has finite order by formula (7.3). Identity (iv) above shows that f is a 2-cocycle, so g is also a 2-cocycle. □

Remark. If the finite abelian group $\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$ has exponent e, then formula (7.3) shows that $\langle \mathrm{im}(g)\rangle$ has exponent dividing e. So, we have the crude upper bound $|\langle \mathrm{im}(g)\rangle| \le e^{|\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}|^2}$.

We can now prove a formula for unitary SK_1 of semiramified graded algebras. This is a unitary analogue to Th. 3.7 above.

Theorem 7.3. *Let E be a semiramified T-central graded division algebra with a unitary graded T/R-involution τ, where T is unramified over R. Take any decomposition $\mathsf{E} \sim_g \mathsf{I} \otimes_{\mathsf{T}} \mathsf{N}$ where I is inertial with $[\mathsf{I}_0] \in \mathrm{Br}(\mathsf{E}_0/\mathsf{T}_0;\mathsf{R}_0)$ and N is DSR for T/R, as in Prop. 4.5 above. Then,*

(i) $\mathrm{SK}_1(\mathsf{E},\tau) \cong \big(\ker(\widetilde{N})/\Pi\big)/\langle \mathrm{im}(g)\rangle$, *where g is the function of Prop. 7.2. If $\mathsf{I}_0 \sim A(\mathsf{E}_0/\mathsf{T}_0,\boldsymbol{\sigma},\mathbf{u},\mathbf{b})$ as in Lemma 4.1(iii) with $\theta = \tau|_{\mathsf{E}_0}$, then $\mathrm{im}(g)$ is computable from the u_{ij}.*

(ii) *If $\mathsf{E}_0 \cong L_1 \otimes_{\mathsf{T}_0} L_2$ with each L_i cyclic Galois over T_0, then*

$$\mathrm{SK}_1(\mathsf{E},\mathsf{T}) \;\cong\; \mathrm{Br}(\mathsf{E}_0/\mathsf{T}_0;\mathsf{R}_0)\big/[\mathrm{Dec}(\mathsf{E}_0/\mathsf{T}_0;\mathsf{R}_0)\cdot\langle[\mathsf{I}_0]\rangle].$$

Proof. (i) From (7.1) and Prop. 7.2, we have

$$SK(\mathsf{E},\tau) \;\cong\; \big(\ker(\widetilde{N})/\Pi\big)\big/\big[(\Pi\cdot X)/\Pi\big] \;\cong\; \big(\ker(\widetilde{N})/\Pi\big)/\langle \mathrm{im}(g)\rangle.$$

It remains to relate $\mathrm{im}(g)$ to the u_{ij} describing I_0.

We have $\mathsf{I}_0 \sim A(\mathsf{E}_0/\mathsf{T}_0,\boldsymbol{\sigma},\mathbf{u},\mathbf{b})$, as in Lemma 4.1(iii), with $\theta = \overline{\tau} = \tau|_{\mathsf{T}_0}$. Since N is DSR for T/R with $\mathsf{N}_0 \cong \mathsf{E}_0$ and $\Theta_{\mathsf{N}} = \Theta_{\mathsf{E}}$ by Prop. 4.5, Prop. 4.4 yields $\mathsf{N} \cong_g \mathsf{A}(\mathsf{E}_0\mathsf{T}/\mathsf{T},\boldsymbol{\sigma},\mathbf{1},\mathbf{c})$, with each $c_i \in \mathsf{R}^*$ with $\deg(c_i) = r_i\gamma_i$ for some $\gamma_i \in \Gamma_{\mathsf{N}} = \Gamma_{\mathsf{E}}$ with $\Theta_{\mathsf{E}}(\gamma_i) = \sigma_i$. Therefore, by Remark 3.3, $\mathsf{E} \sim_g \mathsf{E}'$, where $\mathsf{E}' = \mathsf{A}(\mathsf{E}_0\mathsf{T}/\mathsf{T},\boldsymbol{\sigma},\mathbf{u},\mathbf{d})$, with the same $\mathbf{u}$ as for I_0 and each $d_i = b_ic_i \in \mathsf{E}_0^*\mathsf{R}^*$. So, $\tau(d_i) = \tau(c_i)\tau(b_i) = c_ib_i = b_ic_i = d_i$. Since N is a semiramified graded division algebra and $\deg(d_i) = \deg(c_i)$ for each i, Lemma 3.4 applied to N and to E' shows that $\Gamma_{\mathsf{E}'} = \Gamma_{\mathsf{N}}$ and E' is a semiramified graded division algebra. Therefore, as E and E' are each graded division algebras with $\mathsf{E} \sim_g \mathsf{E}'$, we have $\mathsf{E} \cong_g \mathsf{E}'$ by the graded Wedderburn Theorem. So, we may assume $\mathsf{E} = \mathsf{E}' = \mathsf{A}(\mathsf{E}_0\mathsf{T}/\mathsf{T},\boldsymbol{\sigma},\mathbf{u},\mathbf{d})$. Take $y_1,\ldots,y_k \in \mathsf{E}^*$ with $\mathrm{int}(y_i)|_{\mathsf{E}_0\mathsf{T}} = \sigma_i$, $y_i^{r_i} = d_i$, and $y_iy_jy_i^{-1}y_j^{-1} = u_{ij}$. Now, the graded field $\mathsf{E}_0\mathsf{T}$ is T/R generalized dihedral, and $\theta = \tau|_{\mathsf{E}_0\mathsf{T}}$ lies in $\mathsf{Gal}(\mathsf{E}_0\mathsf{T}/\mathsf{R}) \setminus \mathsf{Gal}(\mathsf{E}_0\mathsf{T}/\mathsf{T})$. Therefore, the proof of Lemma 4.1 (iii) $\Rightarrow$ (i) shows that there is a graded T/R-involution τ' of E with each $y_i = \tau'(y_i)$ and $\tau'|_{\mathsf{E}_0\mathsf{T}} = \theta$. Since $\mathrm{SK}_1(\mathsf{E},\tau) = \mathrm{SK}_1(\mathsf{E},\tau')$ we may replace τ by τ', so each $y_i = \tau(y_i)$, while $\overline{\tau}$ is unchanged.

Fix any $\eta \in \Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$, and let $\sigma_\eta = \overline{\Theta}_E(\eta) \in H$. Take the unique $\mathbf{i} \in \mathcal{I}$ with $\sigma^{\mathbf{i}} = \sigma_\eta$ (notation as in §3), let $\gamma = \deg(y^{\mathbf{i}}) \in \Gamma_{\mathsf{E}}$, and set $y_\gamma = y^{\mathbf{i}}$. Since $\Theta_{\mathsf{E}}(\gamma) = \mathrm{int}(y_\gamma)|_{\mathsf{E}_0} = \overline{\Theta}_{\mathsf{E}}(\eta)$ and $\overline{\Theta}_{\mathsf{E}}\colon \Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}} \to H$ is an isomorphism for E semiramified (see §2), $\eta = \overline{\gamma}$ in $\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$. Since $\tau(y_i) = y_i$ for each i, $\tau(y_\gamma)$ is the product of the y_i appearing in y_γ but with the order reversed. Hence, the commutator identities show that $\tau(y_\gamma) = a_\gamma y_\gamma$ where a_γ in E_0 is a computable product of the u_{ij} and their conjugates under the y_i. Since each $y_\ell u_{ij} y_\ell^{-1} = \sigma_\ell(u_{ij})$, a_γ is a computable product of terms $\sigma_\ell(u_{ij})$. (For example, $\tau(y_1y_2y_3) = y_3y_2y_1 = [u_{32}\sigma_2(u_{31})u_{21}]y_1y_2y_3$.) By applying τ to the equation $\tau(y_\gamma) = a_\gamma y_\gamma$, we find

$$a_\gamma\,\sigma_\eta\tau(a_\gamma) \;=\; 1.$$

Therefore, from Hilbert 90 for the quadratic extension $\mathsf{E}_0/\mathsf{E}_0^{\sigma_\eta\tau}$, there is $t_\gamma \in \mathsf{E}_0^*$ with

$$t_\gamma\,[\sigma_\eta\tau(t_\gamma)]^{-1} \;=\; a_\gamma.$$

Then, $\tau(t_\gamma y_\gamma) = t_\gamma y_\gamma$, so for the x_γ in X we can set $x_\gamma = t_\gamma y_\gamma$. Now take any $\zeta \in \Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$ and carry out the same process for ζ as we have just done for η, obtaining $\delta \in \Gamma$ with $\overline{\delta} = \zeta$, and y_δ with $\deg(y_\delta) = \delta$ and $\mathrm{int}(y_\delta)|_{\mathsf{E}_0} = \sigma_\zeta$, then determining a_δ, t_δ, x_δ. Then set $y_{\gamma+\delta} = y_\gamma y_\delta$, so $\mathrm{int}(y_{\gamma+\delta})|_{\mathsf{E}_0} = \sigma_\eta\sigma_\zeta$. Let $a_{\gamma+\delta} = \tau(y_{\gamma+\delta})y_{\gamma+\delta}^{-1} \in \mathsf{E}_0^*$. Since $a_{\gamma+\delta}\sigma_\eta\sigma_\zeta\tau(a_{\gamma+\delta}) = 1$, by Hilbert 90 there is $t_{\gamma+\delta} \in \mathsf{E}_0^*$ with $t_{\gamma+\delta}[\sigma_\eta\sigma_\zeta\tau(b_{\gamma+\delta})]^{-1} = a_{\gamma+\delta}$. Then set $x_{\gamma+\delta} = t_{\gamma+\delta}y_{\gamma+\delta}$, so that $\tau(x_{\gamma+\delta}) = x_{\gamma+\delta}$. By the definition of the function g of Prop. 7.2, we have in $\ker(\widetilde{N})/\Pi$,

$$g(\eta,\zeta) \;=\; x_\gamma x_\delta x_{\gamma+\delta}^{-1}\,\Pi = \;(t_\gamma y_\gamma)(t_\delta y_\delta)(t_{\gamma+\delta}y_\gamma y_\delta)^{-1}\,\Pi \;=\; t_\gamma \sigma_\eta(t_\delta)t_{\gamma+\delta}^{-1}\,\Pi.$$

Since the t's are determined by the a's, which are determined by the u_{ij}, this shows that $\mathrm{im}(g)$ is determined by the u_{ij}.

(ii) Suppose now that $\mathsf{E}_0 = L_1 \otimes_{\mathsf{T}_0} L_2$ with each L_i cyclic Galois over T_0, and let $\sigma = \sigma_1$ and $\rho = \sigma_2$, as in §6. The isomorphism

$$\mathrm{Br}(M/K;F)\big/\,\mathrm{Dec}(M/K;F) \;\cong\; \ker(\widetilde{N})/\Pi \tag{7.4}$$

of Prop. 6.1 maps $[\mathsf{I}_0] = [A(u,b_1,b_2)]$ to $q\,\Pi$, where $q \in \mathsf{E}_0^*$ with $u = q[\rho\sigma\overline{\tau}(q)]^{-1}$. Take standard generators y_1, y_2 of $A(u,b_1,b_2)$. As noted for (i), we can assume after modifying τ (without changing $\overline{\tau}$) that $\tau(y_1) = y_1$ and $\tau(y_2) = y_2$. Let $\gamma = \deg(y_1)$ and $\delta = \deg(y_2)$ in Γ_{E}, so $\Theta_{\mathsf{E}}(\gamma) = \mathrm{int}(y_1)|_{\mathsf{E}_0} = \sigma$ and $\Theta_{\mathsf{E}}(\delta) = \mathrm{int}(y_2)|_{\mathsf{E}_0} = \rho$. Since $\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}} \cong H = \langle\sigma,\rho\rangle$, we have $\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}} = \langle\overline{\gamma},\overline{\delta}\rangle$. As $\tau(y_1) = y_1$, we can take $x_\gamma = y_1$, and likewise $x_\delta = y_2$. Because $\tau(y_2y_1) = uy_2y_1 = q[\rho\sigma\overline{\tau}(q)]^{-1}y_2y_1$, we have $\tau(qy_2y_1) = qy_2y_1$; thus, we can take $x_{\delta+\gamma} = qy_2y_1$. Then,

$$g(\overline{\delta},\overline{\gamma}) \;=\; x_\delta x_\gamma x_{\delta+\gamma}^{-1}\,\Pi \;=\; y_2y_1(qy_2y_1)^{-1}\,\Pi \;=\; q^{-1}\,\Pi.$$

Since $\overline{\delta}$ and $\overline{\gamma}$ generate $\Gamma_{\mathsf{E}}/\Gamma_{\mathsf{T}}$ formula (7.3) shows that $\mathrm{im}(g) = \langle g(\overline{\delta},\overline{\gamma})\rangle = \langle q^{-1}\Pi\rangle = \langle q\,\Pi\rangle$. Therefore, the isomorphism of (7.4) maps $\langle[\mathsf{I}_0]\rangle$ to $\langle q\,\Pi\rangle = \langle\mathrm{im}(g)\rangle$. Thus, the isomorphism asserted for (ii) follows from (i). □

Example 7.4. Here is a unitary version of Ex. 3.9. Take any integer $n \geq 2$, and let $F \subseteq K$ be fields with $[K\!:\!F] = 2$, K Galois over F, and $K = F(\omega)$ where ω is a primitive n^2-root of unity. Suppose further that for the nonidentity element ψ_0 of $\mathrm{Gal}(K/F)$ we have $\psi_0(\omega) = \omega^{-1}$. (For example, we could take $K = \mathbb{Q}(\omega)$, the n^2-cyclotomic extension of $\mathbb{Q}$, and $F = K\cap\mathbb{R}$.) Let $\mathsf{T} = K[x,x^{-1},y,y^{-1}]$, the Laurent polynomial ring, with its usual grading by $\mathbb{Z}\times\mathbb{Z}$; so, T is a graded field. Let $\mathsf{R} = F[x,x^{-1},y,y^{-1}]$, which is a graded subfield of T with $[\mathsf{T}\!:\!\mathsf{R}] = 2$, T Galois over R, and T inertial over R. Also, $\mathsf{Gal}(\mathsf{T}/\mathsf{R}) = \{\psi, \mathrm{id}_{\mathsf{T}}\}$, where $\psi = \psi_0\otimes\mathrm{id}_{\mathsf{R}}$ on $\mathsf{T} = \mathsf{T}_0\otimes_{\mathsf{R}_0}\mathsf{R}$. Take any $a,b \in F^*$ such that $[K(\sqrt[n]{a},\,\sqrt[n]{b}\,)\!:\!K] = n^2$, and let $M = K(\sqrt[n]{a},\,\sqrt[n]{b}\,)$. Then, it is easy to check that M is K/F-generalized dihedral. (One can think of such field extensions M/F as the generalized dihedral analogue to Kummer extensions.) Indeed, ψ_0 on K extends to $\theta \in \mathrm{Gal}(M/F)$ given by $\theta(\sqrt[n]{a}) = \sqrt[n]{a}$, $\theta(\sqrt[n]{b}) = \sqrt[n]{b}$, and $\theta|_K = \psi_0$; so, $\theta^2 = \mathrm{id}_M$, and for $h \in \mathrm{Gal}(M/K)$, we have $\theta h\theta = h^{-1}$. As in Ex. 3.9, take the graded symbol algebra $\mathsf{E} = (ax^n, by^n, \mathsf{T})_\omega$ of degree n^2, with its generators i,j satisfying $i^{n^2} = ax^n$, $j^{n^2} = by^n$, $ij = \omega ji$. For σ_1,σ_2 as in Ex. 3.9, it was noted there that $\mathsf{E} = \mathsf{A}(M\mathsf{T}/\mathsf{T},\boldsymbol{\sigma},\mathbf{u},\mathbf{d})$ where $u_{12} = \omega$, and $d_1 = 1/(y\sqrt[n]{b})$ and $d_2 = x\sqrt[n]{a}$. We extend θ to an element of $\mathsf{Gal}(M\mathsf{T}/\mathsf{R})$ by setting $\theta|_{\mathsf{R}} = \mathrm{id}$. Since $\theta(d_1) = d_1$, $\theta(d_2) = d_2$, and $u_{12}\,\sigma_1\sigma_2\theta(u_{12}) = \omega\omega^{-1} = 1$, the graded version of Lemma 4.1 shows that there is a graded T/R-involution τ on E given by $\tau(j^{-1}) = j^{-1}$, $\tau(i) = i$, and $\tau|_{M\mathsf{E}} = \theta$. That is, τ is the R-linear map $\mathsf{E}\to\mathsf{E}$ such that $\tau(c\,i^\ell j^m) = \psi(c)j^m i^\ell$ for all $c \in \mathsf{T}$, $\ell, m \in \mathbb{Z}$. We have the decomposition of E noted in Ex. 3.9,

$$\mathsf{E} \sim_g \mathsf{I}\otimes_{\mathsf{T}}\mathsf{N} \qquad \text{where} \qquad \mathsf{I} = (a,b,\mathsf{T})_\omega \quad \text{and} \quad \mathsf{N} = (x,b,\mathsf{T})_{\omega^n}\otimes_{\mathsf{T}}(a,y,\mathsf{T})_{\omega^n}.$$

These I and N are T-central graded division algebras with I inertial and N DSR. Furthermore, as $a,b,x,y \in \mathsf{R}^*$, there are unitary graded T/R-involutions τ_{I} on I and τ_{N} on N defined analogously to τ on E. So, by Th. 7.1(ii)

$$\mathrm{SK}_1(\mathsf{N},\tau_{\mathsf{N}}) \;\cong\; \mathrm{Br}\left(M/K;F\right)\big/\,\mathrm{Dec}\left(M/K;F\right), \quad \text{where} \quad M \;=\; K(\sqrt[n]{a},\,\sqrt[n]{b}\,),$$

with $\mathrm{Dec}\left(M/K;F\right) = \mathrm{Br}\left(K(\sqrt[n]{a}\,)/K;F\right)\cdot\mathrm{Br}\left(K(\sqrt[n]{b}\,)/K;F\right)$ by (4.3). Since $\mathsf{I}_0 \cong (a,b,K)_\omega$, Th. 7.3(ii) yields

$$\mathrm{SK}_1(\mathsf{E},\tau) \;\cong\; \mathrm{Br}(M/K;F)\big/\big[\,\mathrm{Dec}(M/K;F)\cdot\langle(a,b,K)_\omega\rangle\big].$$

Note that E is semiramified, but it may or may not be DSR. Indeed, by Prop. 4.5(ii) E is DSR if and only if $\mathsf{I}_0 \in \mathrm{Dec}\big(M/K;F\big)$; the formulas above show that this holds if and only if the obvious surjection $\mathrm{SK}_1(\mathsf{N},\tau_N) \to \mathrm{SK}_1(\mathsf{E},\tau)$ is an isomorphism. Note also that $\mathrm{Dec}(M/K;F)$ may be strictly smaller than $\mathrm{Dec}(M/K)\cap \mathrm{Br}(M/K;F)$, i.e., there may be an algebra in $\mathrm{Br}(M/K)$ which decomposes according to M and has a K/F-involution, but in any decomposition the factors do not have K/F-involutions. Examples of this are given in Remark 8.2 below.

For an ungraded version of this example, let K, F, a, and b be as above; then let $K' = K((x))((y))$ and $F' = F((x))((y))$, and $D = (ax^n, by^n, K')_\omega$. Then, with respect to the usual rank 2 Henselian valuations $v_{K'}$ on K' and $v_{F'}$ on F', K' is inertial of degree 2 over F'. Furthermore, with respect to the valuation v_D on D extending $v_{K'}$ on K', D is a semiramified K'-central division algebra with a unitary K'/F'-involution τ_D defined just as for τ on E. For the associated graded ring $\mathsf{gr}(D)$ of D determined by v_D, we have $\mathsf{gr}(D) \cong_g \mathsf{E}$, so by [HW$_2$, Th. 3.5] $\mathrm{SK}_1(D,\tau_D) \cong \mathrm{SK}_1(\mathsf{E},\tau)$.

8. Noninjectivity

For any T-central graded division algebra B with unitary T/R-involution τ, there are well-defined canonical homomorphisms

$$\alpha\colon\ \mathrm{SK}_1(\mathsf{B},\tau) \to \mathrm{SK}_1(\mathsf{B}) \quad \text{given by} \ \ a\,\Sigma_\tau(\mathsf{B}) \mapsto \tau(a)a^{-1}\,[\mathsf{B}^*,\mathsf{B}^*] \ \text{ for } a \in \Sigma'_\tau(\mathsf{B}), \tag{8.1}$$

and

$$\beta\colon\ \mathrm{SK}_1(\mathsf{B}) \to \mathrm{SK}_1(\mathsf{B},\tau) \quad \text{given by} \ \ b\,[\mathsf{B}^*,\mathsf{B}^*] \mapsto b\,\Sigma_\tau(\mathsf{B}) \ \text{ for } b \in \mathsf{B}^* \text{ with } \mathrm{Nrd}_{\mathsf{B}}(b) = 1.$$

It is easy to check that $\beta\circ\alpha$ and $\alpha\circ\beta$ are each the squaring map. As pointed out in [Y$_3$, Lemma, p. 185], since the exponent of the abelian group $\mathrm{SK}_1(\mathsf{B},\tau)$ divides $\deg(\mathsf{B})$, if $\deg(\mathsf{B})$ is odd, then α must be injective. It seems to have been an open question up to now whether α is always injective, even when $\deg(\mathsf{B})$ is even. We now settle this question by using some of the results above to give examples of B of degree 4 with α not injective. We thank J.-P. Tignol for pointing out the relevance of indecomposable division algebras of degree 8 and exponent 2, and for calling his paper [T$_1$] to our attention.

Let F be a field with $\mathrm{char}(F) \neq 2$. Let $M = F(\sqrt{a},\sqrt{b},\sqrt{c}\,)$ with $a,b,c \in F^*$ and $[M:F] = 8$. Let $K = F(\sqrt{a}\,)$. We write $\mathrm{Br}_2(F)$ for the 2-torsion subgroup of $\mathrm{Br}(F)$, and set $\mathrm{Br}_2(M/F) = \mathrm{Br}(M/F)\cap\mathrm{Br}_2(F)$, $\mathrm{Br}_2(M/K;F) = \mathrm{Br}(M/K;F)\cap\mathrm{Br}_2(K)$, etc. Note that as $\mathrm{Gal}(M/F)$ is an elementary abelian 2-group, M is a K/F-generalized dihedral extension. Also, $\mathrm{res}_{F\to K}$ maps $\mathrm{Br}_2(M/F)$ to $\mathrm{Br}(M/K;F)$, since for $[A]\in\mathrm{Br}_2(M/F)$, $\mathrm{cor}_{K\to F}[A\otimes_F K] = [A]^{[K:F]} = 1$ in $\mathrm{Br}(F)$, so by Albert's Theorem $A\otimes_F K$ has a unitary K/F-involution.

Proposition 8.1. *There is an exact sequence:*

$$0 \longrightarrow \mathrm{Br}_2(M/F)\big/\,\mathrm{Dec}(M/F) \longrightarrow \mathrm{Br}(M/K;F)\big/\,\mathrm{Dec}(M/K;F) \longrightarrow \mathrm{Br}(M/K)\big/\,\mathrm{Dec}(M/K) \tag{8.2}$$

Proof. The kernel of the right map in (8.2) is $\big[\,\mathrm{Br}(M/K;F)\cap\mathrm{Dec}(M/K)\big]\big/\,\mathrm{Dec}(M/K;F)$. So, the exactness of (8.2) is equivalent to two assertions:

(a) $\quad \mathrm{Br}(M/K;F)\cap\mathrm{Dec}(M/K) \ = \ \mathrm{Br}_2(M/K;F).$

and

(b) $\quad \mathrm{Br}_2(M/F)\big/\,\mathrm{Dec}(M/F) \ \cong\ \mathrm{Br}_2(M/K;F)\big/\,\mathrm{Dec}(M/K;F)$

The equality (a) is immediate from the fact that $\mathrm{Dec}(M/K) = \mathrm{Br}_2(M/K)$, as M is a biquadratic extension of K. (This is well-known, and is deducible, e.g., by refining the argument in [KMRT, Prop. 16.2]. It also appears in [T$_1$, Cor. 2.8] as the assertion that property $\mathrm{P}_2(2)$ holds for K.) The isomorphism (b) appears in [T$_1$, Prop.2.2] as the isomorphism $N_2(M/F) \cong M_2(M/K/F)$, see the conmments on p. 14 of [T$_1$]. Since the isomorphism (b) is somewhat buried in the general arguments of [T$_1$], we give a short and direct proof of it: If $[A]\in\mathrm{Dec}(M/F)$, then $A \sim Q_1\otimes_F Q_2\otimes_F Q_3$, where Q_1 is the quaternion algebra

$\big(\frac{a,r}{F}\big)$, $Q_2 = \big(\frac{b,s}{F}\big)$, and $Q_3 = \big(\frac{c,t}{F}\big)$, for some $r,s,t \in F^*$. So, $A \otimes_F K \sim (Q_2 \otimes_F K) \otimes_K (Q_3 \otimes_F K)$. Here, $Q_2 \otimes_F K$ has the unitary K/F-involution $\eta \otimes \psi$, where η is any involution of the first kind on Q_2 and ψ is the nonidentity F-automorphism of K. So $[Q_2 \otimes_F K] \in \mathrm{Br}(K(\sqrt{b})/K;F) \subseteq \mathrm{Dec}(M/K;F)$; likewise $[Q_3 \otimes_F K] \in \mathrm{Br}(K(\sqrt{c})/K;F) \subseteq \mathrm{Dec}(M/K;F)$, and hence $[A \otimes_F K] \in \mathrm{Dec}(M/K;F)$. Thus, $\mathrm{res}_{F\to K}$ induces a well-defined map $f\colon \mathrm{Br}_2(M/F)/\mathrm{Dec}(M/F) \to \mathrm{Br}_2(M/K;F)/\mathrm{Dec}(M/K;F)$. From Arason's long exact sequence (see, e.g., [KMRT, Cor. 30.12(1)] or (5.2) above)

$$\ldots \to H^2(F,\mu_2) \to H^2(K,\mu_2) \to H^2(F,\mu_2) \to \ldots,$$

f is surjective. For injectivity of f, take any $[A] \in \mathrm{Br}_2(M/F)$ with $\mathrm{res}_{F\to K}[A] \in \mathrm{Dec}(M/K;F)$. We need to show $[A] \in \mathrm{Dec}(M/F)$. We have $A \otimes_F K \sim Q_2' \otimes_K Q_3'$ where the Q_i' are quaternion algebras over K with $Q_2' \in \mathrm{Br}(K(\sqrt{b})/K;F)$ and $Q_3' \in \mathrm{Br}(K(\sqrt{c})/K;F)$. By a result of Albert [KMRT, Prop. 2.22], the quaternion algebra Q_2' with K/F-involution has the form $Q_2' \cong Q_2'' \otimes_F K$, where Q_2'' is a quaternion algebra over F. Then, $[Q_2''] \in \mathrm{Br}_2(K(\sqrt{b})/F) = \mathrm{Dec}(K(\sqrt{b})/F)$, as noted for (a) above. Likewise, $Q_3' \cong Q_3'' \otimes_F K$, where $[Q_3''] \in \mathrm{Dec}(K(\sqrt{c})/F)$. Since $[A \otimes_F Q_2'' \otimes_F Q_3''] \in \mathrm{Br}(K/F) = \mathrm{Dec}(K/F)$, we have

$$[A] = [A \otimes_F Q_2'' \otimes_F Q_3''][Q_2''][Q_3''] \in \mathrm{Dec}(K/F)\cdot \mathrm{Dec}(K(\sqrt{b})/F)\cdot \mathrm{Dec}(K(\sqrt{c})/F) \subseteq \mathrm{Dec}(M/F).$$

Thus, f is an isomorphism, proving (b). □

Remark 8.2. The term $\mathrm{Br}_2(M/F)/\mathrm{Dec}(M/F)$ for M/F triquadratic has arisen in the study of indecomposable algebras A of degree 8 and exponent 2. Note first that for any A of degree 8 and exponent 2, by Rowen's theorem [R, Th. 6.2] there is a triquadratic field extension M of the center F of A, such that M is a maximal subfield of A. If A is indecomposable, then $[A]$ yields a nontrivial element of $\mathrm{Br}_2(M/F)/\mathrm{Dec}(M/F)$. Examples of indecomposables if degree 8 and exponent 2 were first given in [ART, Th. 5.1]. Subsequently, Karpenko showed in [Kar, Cor. 5.4] that if B is a division algebra with center F of degree 8 and exponent 8, and F' is a field generically reducing the exponent of B to 2, then $B \otimes_F F'$ is an indecomposable division algebra of degree 8 and exponent 2. Also, K. McKinnie in her thesis (unpublished), using lattice methods, gave another example of indecomposables of degree 8 and exponent 2. There is a kind of converse to this as well: Given a division algebra A with $[A] \in \mathrm{Br}_2(M/F) \setminus \mathrm{Dec}(M/F)$, Amitsur, Rowen, and Tignol showed in [ART, Th. 3.3] that the associated generic abelian crossed product algebra A' of A is indecomposable of degree 8 and exponent 2. (It is not stated this way in [ART], but made explicit in [T_2, § 2].) This A' is the ring of quotients of a semiramified graded division algebra E of the type considered in previous sections: E is graded Brauer equivalent to $\mathsf{I} \otimes_{\mathsf{T}} \mathsf{N}$, where T is a graded field with $\mathsf{T}_0 \cong F$, I is an inertial graded division algebra over T with $\mathsf{I}_0 \cong A$, and N is DSR over T with $\mathsf{N}_0 \cong M$.

Using Prop. 8.1 we now construct biquaterion graded algebras where the map α of (8.1) above is not injective.

Example 8.3. Let M be a triquadratic extension of a field F ($\mathrm{char}(F) \neq 2$) with $\mathrm{Br}_2(M/F)/\mathrm{Dec}(M/F) \neq 0$. (Such F and M exist, as noted in Remark 8.2.) Say $M = F(\sqrt{a},\sqrt{b},\sqrt{c})$ for $a,b,c \in F^*$. Let $K = F(\sqrt{a})$, and let $H = \mathrm{Gal}(M/K)$. Let $\mathsf{R} = F[x,x^{-1},y,y^{-1}]$, the Laurent polynomnial ring in indeterminates x and y, with its usual grading in which $\mathsf{R}_{(k,\ell)} = Fx^ky^\ell$ for all $(k,\ell) \in \mathbb{Z}\times\mathbb{Z}$. So, R is a graded field with $R_0 = F$ and $\Gamma_{\mathsf{R}} = \mathbb{Z}\times\mathbb{Z}$. Let $\mathsf{T} = K[x,x^{-1},y,y^{-1}]$, a graded field with $[\mathsf{T}:\mathsf{R}] = 2$, and let $\mathsf{E} = \mathsf{Q} \otimes_{\mathsf{T}} \mathsf{Q}'$, where Q and Q' are the following semiramified graded quaternion division algebras over T: $\mathsf{Q} = \big(\frac{b,x}{\mathsf{T}}\big)$, which is generated over T by homogeneous elements i and$j$ with relations $i^2 = b$, $j^2 = x$, and $ij = -ji$, with $\deg(i) = 0$ and $\deg(j) = (\frac12, 0)$. So, $\mathsf{Q}_0 \cong K(\sqrt{b})$ and $\Gamma_{\mathsf{Q}} = \frac12\mathbb{Z}\times\mathbb{Z}$. Likewise, set $\mathsf{Q}' = \big(\frac{c,y}{\mathsf{T}}\big)$ with standard generators i' and j', with $\deg(i') = 0$ and $\deg(j') = (0,\frac12)$, so $\mathsf{Q}_0 \cong K(\sqrt{c})$ and $\Gamma_{\mathsf{Q}'} = \mathbb{Z}\times\frac12\mathbb{Z}$. Since $\mathsf{Q} \cong \big(\frac{b,x}{\mathsf{R}}\big) \otimes_{\mathsf{R}} \mathsf{T}$, Q has the graded T/R-involution $\tau_{\mathsf{Q}} = \eta\otimes\psi$, where η is the canonical symplectic graded involution on $\big(\frac{b,x}{\mathsf{R}}\big)$, for which $\eta(i) = -i$ and $\eta(j) = -j$, and ψ is the nonidentity graded R-automorphism of T. Likewise Q' has a graded T/R-involution $\tau_{\mathsf{Q}'}$ with $\tau_{\mathsf{Q}'}(i') = -i'$ and $\tau_{\mathsf{Q}'}(j') = -j'$. By Lemma 4.3, E is a graded division algebra which is DSR for T/R with $\mathsf{E}_0 \cong \mathsf{Q}_0 \otimes_{\mathsf{T}_0} \mathsf{Q}_0' \cong K(\sqrt{b}\,) \otimes_K K(\sqrt{c}\,) \cong M$ and $\Gamma_{\mathsf{E}} = \Gamma_{\mathsf{Q}} + \Gamma_{\mathsf{Q}'} = \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$; our graded T/R-involution on E is $\tau = \tau_{\mathsf{Q}} \otimes \tau_{\mathsf{Q}'}$. (Explicitly, $\mathsf{S} = \mathsf{T}[i, i'] \cong_g M[x, x^{-1}, y, y^{-1}]$ is a maximal graded subfield of E with S inertial over T, and $\mathsf{J} = \mathsf{T}[j, j'] \cong_g \mathsf{T}[\sqrt{x}, \sqrt{x}^{-1}, \sqrt{y}, \sqrt{y}^{-1}]$ is a maximal graded subfield of E which is totally ramified over T with $\tau(\mathsf{J}) = \mathsf{J}$.) We claim that the following diagram is commutative with all horizontal maps isomorphisms and vertical maps described below:

$$\begin{array}{ccccc} \operatorname{Br}(M/K;F)\big/\operatorname{Dec}(M/K;F) & \longrightarrow & \ker(\widetilde{N})/\Pi & \longrightarrow & \operatorname{SK}_1(\mathsf{E},\tau) \\ \big\downarrow & & \big\downarrow & & \alpha\big\downarrow \\ \operatorname{Br}(M/K)\big/\operatorname{Dec}(M/K) & \longrightarrow & \widehat{H}^{-1}(H, M^*) & \longrightarrow & \operatorname{SK}_1(\mathsf{E}) \end{array} \tag{8.3}$$

The left vertical map is the map in Prop. 8.1, whose kernel is there shown to be isomorphic to $\operatorname{Br}_2(M/F)/\operatorname{Dec}(M/F)$. Since we have assumed this kernel is nontrivial, once the claim is established the right vertical map α, which is the map of (8.1) must also have nontrivial kernel, as desired.

We now verify the claim. In the top line of (8.3), $\ker(\widetilde{N}) = \{a \in M^* \mid N_{M/K}(a) \in F\}$ and $\Pi = \prod_{h \in H} M^{*h\overline{\tau}}$, where $H = \operatorname{Gal}(M/K)$ and $\overline{\tau} = \tau|_{\mathsf{E}_0}$. The middle vertical map sends $a\,\Pi \mapsto a/\overline{\tau}(a)\, I_H(M^*)$. It is well defined since if $a \in \ker(\widetilde{N})$, we have $N_{K/F}(a/\tau(a)) = N_{K/F}(a)/\tau(N_{K/F}(a)) = 1$, and if $b \in M^{*h\overline{\tau}}$, then $b/\overline{\tau}(b) = h\overline{\tau}(b)/\overline{\tau}(b) \in I_H(M^*)$. In the right rectangle of (8.3), the top map sends $a\,\Pi \mapsto a\Sigma_\tau(\mathsf{E})$, and the bottom map sends $b\, I_H(M^*) \mapsto b\,[\mathsf{E}^*, \mathsf{E}^*]$, so the right rectangle is clearly commutative. The horizontal maps in this rectangle are the isomorphisms given in Th. 7.1(i) and Prop. 3.2(i). For the left vertical map take an arbitrary element of $\operatorname{Br}(M/K;F)$, which has the form $[A]$, where $A = A(u, b_1, b_2)$ in the notation of §6, with $u,\ b_1,\ b_2$ satisfying the relations in (3.1) and (3.2) and the added relations in Lemma 4.1(iii), notably $u\,\sigma\rho\overline{\tau}(u) = 1$. The horizontal map in the left rectangle is the isomorphism of Th. 6.1 which sends $[A]$ mod $\operatorname{Dec}(M/K;F)$ to $q\,\Pi$ for any $q \in M^*$ with $q/\sigma\rho\overline{\tau}(q) = u$. This is mapped downward to $u\, I_H(M^*)$, since $q/\overline{\tau}(q) = u\,\sigma\rho\overline{\tau}(q)/\overline{\tau}(q) \equiv u \pmod{I_H(M^*)}$. On the other hand, $[A]$ mod $\operatorname{Dec}(M/K;F)$ is mapped downward to $[A]$ mod $\operatorname{Dec}(M/K)$, which is mapped to the right to $u\, I_H(M^*)$ by the isomorphism of (3.9). Thus, the left rectangle of (8.3) is commutative, and its horizontal maps are isomorphisms, completing the proof of the claim.

Remark 8.4. For the preceding example with the α of (8.1) noninjective, we have worked with graded division algebras. There are corresponding examples of division algebras over a Henselian valued field with the corresponding α not injective, obtainable as follows: With fields $F \subseteq K \subseteq M$ as in Ex. 8.3, let $F' = F((x))((y))$, $K' = K((x))((y))$, and $M' = M((x))((y))$, which are twice iterated Laurent power series fields each with it standard Henselian valuation with value group $\mathbb{Z} \times \mathbb{Z}$ (with right-to-left lexicographic ordering) and residue fields $\overline{F'} \cong F$, $\overline{K'} \cong K$, and $\overline{M'} \cong M$. Let $D = \left(\frac{b,x}{K'}\right) \otimes_{K'} \left(\frac{c,y}{K'}\right)$, which is a division algebra over K', and the Henselian valuation $v_{K'}$ on K' extends uniquely to a valuation v_D on D, for which $\overline{D} \cong M$ and $\Gamma_D = \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$. For the associated graded ring of D determined by v_D, we have $\mathsf{gr}(D) \cong_g \mathsf{E}$ and, as D is tame over K', $Z(\mathsf{gr}(D)) = \mathsf{gr}(K') \cong_g \mathsf{T}$, for the E and T of Ex. 8.3. Also, $\mathsf{gr}(F') \cong_g \mathsf{R}$ for the R of Ex. 8.3. This D has a unitary K'/F'-involution τ_D, since each constituent quaternion algebra has such an involution. Because the Henselian valuation $v_{F'}$ on F' has a unique extension to K', namely $v_{K'}$, and v_D is the unique extension of $v_{K'}$ to D, we must have $v_D \circ \tau_D = v_D$. Therefore, τ_D induces a graded involution $\widetilde{\tau}$ on E, which is a unitary T/R-involution. By [HW$_2$, Th. 3.5] and [HW$_1$, Th. 4.8], $\operatorname{SK}_1(D, \tau_D) \cong \operatorname{SK}_1(\mathsf{E}, \widetilde{\tau})$ and $\operatorname{SK}_1(D) \cong \operatorname{SK}_1(\mathsf{E})$. These isomorphisms are compatible with the map $\alpha_{\widetilde{\tau}}\colon \operatorname{SK}_1(\mathsf{E}, \widetilde{\tau}) \to \operatorname{SK}_1(\mathsf{E})$ and the corresponding map $\alpha_D\colon \operatorname{SK}_1(D, \tau_D) \to \operatorname{SK}_1(D)$. Also, because $\widetilde{\tau}$ and the τ of Ex. 8.3 are each graded T/R-involutions on E, we have $\operatorname{SK}_1(\mathsf{E}, \widetilde{\tau}) \cong \operatorname{SK}_1(\mathsf{E}, \tau)$, and it is easy to check that under this isomorphism $\alpha_{\widetilde{\tau}}$ corresponds to the α of Ex. 8.3. Since this α is not injective, α_D is also noninjective.

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