arXiv:1009.3870v3 [math.SG] 2 Dec 2010

INDEX COMPUTATIONS IN RABINOWITZ FLOER HOMOLOGY

WILL J. MERRY AND GABRIEL P. PATERNAIN

ABSTRACT. In this note we study two index questions. In the first we establish the relationship between the Morse indices of the free time action functional and the fixed time action functional. The second is related to Rabinowitz Floer homology. Our index computations are based on a correction term which is defined as follows: around a non-degenerate Hamiltonian orbit lying in a fixed energy level a well-known theorem says that one can find a whole cylinder of orbits parametrized by the energy. The correction term is determined by whether the periods of the orbits are increasing or decreasing as one moves up the orbit cylinder. We also provide an example to show that, even above the Mañé critical value, the periods may be increasing thus producing a jump in the Morse index of the free time action functional in relation to the Morse index of the fixed time action functional.

Dedicated to Richard S. Palais on the occasion of his 80th birthday.

1. INTRODUCTION

Let (M, g) denote a closed connected orientable Riemannian manifold with cotangent bundle $\pi : T^*M \to M$. Let $\omega_0 = d\lambda_0$ denote the canonical symplectic form $dp \wedge dq$ on T^*M, where λ_0 is the Liouville 1-form. Let $\widetilde{M}$ denote the universal cover of M. Let $\sigma \in \Omega^2(M)$ denote a closed **weakly exact** 2-form, by this we mean that the pullback $\widetilde{\sigma} \in \Omega^2(\widetilde{M})$ is exact. We assume in addition that $\widetilde{\sigma}$ admits a **bounded** primitive. This means that there exists $\theta \in \Omega^1(\widetilde{M})$ with $d\theta = \widetilde{\sigma}$, and such that

$$\|\theta\|_\infty := \sup_{q \in \widetilde{M}} |\theta_q| < \infty,$$

where $|\cdot|$ denotes the lift of the metric g to $\widetilde{M}$. Let

$$\omega := \omega_0 + \pi^*\sigma$$

denote the **twisted symplectic form** determined by σ. We call the symplectic manifold (T^*M, ω) a **twisted cotangent bundle**. In fact, for the purposes of this paper the assumptions that σ is weakly exact and admits a bounded primitive are not strictly necessary; see Section 1.4 below.

Let us fix once and for all a **Tonelli Hamiltonian** $H \in C^\infty(T^*M, \mathbb{R})$. Here we recall that the classical Tonelli assumption means that H is **fibrewise strictly convex** and **superlinear**. In other words, the Hessian $\nabla^2 H$ of H restricted to each tangent space T_q^*M is positive definite, and

$$\lim_{|p| \to \infty} \frac{H(q,p)}{|p|} = \infty$$

uniformly for $q \in M$. Let X_H denote the symplectic gradient of H with respect to ω, and let $\phi_t : T^*M \to T^*M$ denote the flow of X_H. Denote by $L \in C^\infty(TM, \mathbb{R})$ the **Fenchel dual Lagrangian**. This is the unique Lagrangian L on TM defined by

$$L(q,v) := \max_{p \in T_q^*M} \{p(v) - H(q,p)\}.$$

The Lagrangian L is also fibrewise strictly convex and superlinear.

Assumption 1.1. Fix now once and for all a regular value k of H, and denote by $\Sigma := H^{-1}(k)$. Thus Σ is a closed hypersurface in T^*M. Since H is autonomous, ϕ_t preserves Σ. We shall consider orbits $y : \mathbb{R}/T\mathbb{Z} \to \Sigma$ of X_H that admit **non-degenerate orbit cylinders**.

The precise definition of a non-degenerate orbit cylinder is given below in Definition 1.5, but intutively we ask that y can be included in a whole family of orbits parametrized by energy, with the property that the periods are either increasing or decreasing (i.e. not constant) as one moves up the orbit cylinder. The key setting we have in mind where this condition is satisfied is when the orbit y is **strongly non-degenerate**, by which we mean that the **nullity** of y, $\nu(y)$ satisfies

$$\nu(y) := \dim \ker(d_{y(0)}\phi_T - \mathbb{1}) = 1. \tag{1.1}$$

In this case y admits a unique non-degenerate orbit cylinder (cf. the discussion after Definition 1.5 below). For an explanation as to the terminology "strongly non-degenerate" see Remark 1.6 below.

In this note we address two index questions from [12], which are both related to the **Rabinowitz Floer homology** $RFH_*(\Sigma, T^*M)$ of the hypersurface Σ (although both questions are valid in situations where the Rabinowitz Floer homology has not yet been defined, see Section 1.4).

1.1. The Morse index of the free time action functional.

The first circle of ideas we study here is only indirectly related to Rabinowitz Floer homology [3, 12]. We begin with some notation. Denote by ΛM the completion of $C^\infty(S^1, M)$ with respect to the Sobolev $W^{1,2}$-norm. The set ΛM carries the structure of a Hilbert manifold. Given a free homotopy class $\alpha \in [S^1, M]$, let $\Lambda_\alpha M$ denote the component of ΛM consisting of loops $q : S^1 \to M$ such that $[q] = \alpha$, and fix reference loops $q_\alpha \in \Lambda_\alpha M$. Given $q \in \Lambda_\alpha M$, let $\bar{q} \in C^0(S^1 \times [0,1], M) \cap W^{1,2}(S^1 \times [0,1], M)$ denote any map such that $\bar{q}(t,0) = q(t)$ and $\bar{q}(t,1) = q_\alpha(t)$. Then it is proved in [10, p194] that since we assume that σ is weakly exact and that $\widetilde{\sigma}$ admits a bounded primitive, the value of $\int_{S^1\times[0,1]} \bar{q}^*\sigma$ is independent of the choice of $\bar{q}$.

This allows us to define the **free time action functional** $\mathcal{S}_k$ on the product manifold $\Lambda M \times \mathbb{R}^+$:

$$\mathcal{S}_k : \Lambda M \times \mathbb{R}^+ \to \mathbb{R};$$

$$\mathcal{S}_k(q,T) := T\int_{S^1} \left(L\left(q, \frac{\dot{q}}{T}\right) + k\right) dt + \int_{S^1\times[0,1]} \bar{q}^*\sigma.$$

Denote by Crit$(\mathcal{S}_k)$ the set of critical points of $\mathcal{S}_k$. A pair (q,T) is a critical point of $\mathcal{S}_k$ if and only if the curve $\gamma : [0,T] \to M$ defined by $\gamma(t) := q(t/T)$ is the projection to M of a closed orbit of ϕ_t contained in Σ [10, Corollary 2.3].

We can also fix the period of the free time action functional, thus giving the **fixed period action functional**:

$$\mathcal{S}_k^T : \Lambda M \to \mathbb{R};$$

$$\mathcal{S}_k^T(q) := \mathcal{S}_k(q,T).$$

Let $\langle\langle \cdot, \cdot \rangle\rangle$ denote the $W^{1,2}$ metric on ΛM defined by

$$\left\langle\left\langle \zeta, \zeta' \right\rangle\right\rangle := \int_0^1 \left\langle \zeta, \zeta' \right\rangle + \left\langle \nabla_t \zeta, \nabla_t \zeta' \right\rangle dt,$$

where ∇ is the Levi-Civita connection of (M, g). We will use the same notation for the product metric on $\Lambda M \times \mathbb{R}^+$:

$$\left\langle\left\langle (\zeta, b), (\zeta', b') \right\rangle\right\rangle := \left\langle\left\langle \zeta, \zeta' \right\rangle\right\rangle + bb'.$$

We denote by $\nabla\mathcal{S}_k$ and $\nabla\mathcal{S}_k^T$ the gradient of $\mathcal{S}_k$ and $\mathcal{S}_k^T$ with respect to these metrics.

The **Morse index** $i(q,T)$ of a critical point $(q,T) \in \mathrm{Crit}(\mathcal{S}_k)$ is the maximal dimension of a subspace $W \subseteq W^{1,2}(S^1, q^*TM) \times \mathbb{R}$ on which $d^2_{(q,T)}\mathcal{S}_k(\cdot,\cdot)$ is negative definite. It is well known that for the Tonelli Lagrangians L we are working with the Morse index $i(q,T)$ is always finite. Similarly let $i_T(q)$ denote the Morse index of a critical point $q \in \mathrm{Crit}(\mathcal{S}_k^T)$, that is, the dimension of a maximal subspace of $W^{1,2}(S^1, q^*TM)$ on which $d_q^2\mathcal{S}_k^T(\cdot,\cdot)$ is negative definite.

Note that

$$d_q\mathcal{S}_k^T(\zeta) = d_{(q,T)}\mathcal{S}_k(\zeta, 0).$$

Thus if $(q,T) \in \mathrm{Crit}(\mathcal{S}_k)$ then $q \in \mathrm{Crit}(\mathcal{S}_k^T)$. It is therefore a natural question to ask how the two Morse indices $i(q,T)$ and $i_T(q)$ are related for $(q,T) \in \mathrm{Crit}(\mathcal{S}_k)$. It is clear that $0 \le i(q,T) - i_T(q) \le 1$, but the precise relationship is somewhat more complicated, as we now explain.

Definition 1.2. We say that a critical point $(q,T) \in \mathrm{Crit}(\mathcal{S}_k)$ admits an **orbit cylinder** if there exists $\varepsilon > 0$ and a smooth (in s) family $\{(q_{k+s}, T(k+s)\}_{s\in(-\varepsilon,\varepsilon)}$ of critical points of $\mathcal{S}_{k+s}$:

$$(q_{k+s}, T(k+s)) \in \mathrm{Crit}(\mathcal{S}_{k+s}),$$

where $(q_k, T(k)) = (q,T)$. We say that the orbit cylinder is **non-degenerate** if $T'(k) \neq 0$. In this case we define the **correction term**

$$\chi(q,T) := \mathrm{sign}(-T'(k)) \in \{-1, 1\}. \tag{1.2}$$

A priori, the correction term $\chi(q,T)$ depends on the choice of orbit cylinder. However one consequence of Theorem 1.3 below is that this is not the case.

This condition is the analogue of Assumption 1.1 in the Lagrangian setting. A sufficient condition for a critical point (q,T) to admit a non-degenerate orbit cylinder is that the corresponding periodic orbit $y(t) := x(t/T)$ of X_H (see Lemma 1.4 below) is strongly non-degenerate. In this case the orbit cylinder is actually **unique** (cf. the discussion surrounding (1.5) below).

Anyway, the precise relationship between the two Morse indices is given by the following result.

Theorem 1.3. *Let $(q,T) \in \mathrm{Crit}(\mathcal{S}_k)$ denote a critical point admitting a non-degenerate orbit cylinder. Then*

$$i(q,T) = i_T(q) + \frac{1}{2} - \frac{1}{2}\chi(q,T).$$

1.2. **Computing the virtual dimension of the moduli spaces of Rabinowitz Floer homology.**

The second functional we study is more directly related to Rabinowitz Floer homology. Denote by ΛT^*M the completion of $C^\infty(S^1, T^*M)$ with respect to the Sobolev $W^{1,2}$-norm. The **Rabinowitz action functional** $\mathcal{A}_k$ is defined on the product manifold $\Lambda T^*M \times \mathbb{R}$:

$$\mathcal{A}_k : \Lambda T^*M \times \mathbb{R} \to \mathbb{R};$$
$$\mathcal{A}_k(x,\eta) := \int_{S^1} x^*\lambda_0 + \int_{S^1\times[0,1]} \bar{q}^*\sigma - \eta\int_{S^1}(H(x)-k)dt,$$

where $q := \pi \circ x$, and $\bar{q} : S^1 \times [0,1] \to M$ is defined as before. The critical points of $\mathcal{A}_k$ are easily seen to satisfy:

$$\dot{x} = \eta X_H(x);$$
$$\int_{S^1}(H(x)-k)dt = 0.$$

Since H is invariant under its Hamiltonian flow, the second equation implies

$$x(S^1) \subseteq \Sigma.$$

Thus if Crit$(\mathcal{A}_k)$ denotes the set of critical points of $\mathcal{A}_k$, we can characterize Crit$(\mathcal{A}_k)$ by

$$\begin{aligned} \text{Crit}(\mathcal{A}_k) \quad = \quad & \big\{(x,\eta) \in \Lambda T^*M \times \mathbb{R} \,:\, x \in C^\infty(S^1, T^*M) \\ & \dot{x} = \eta X_H(x),\ x(S^1) \subseteq \Sigma\big\}\,. \end{aligned} \tag{1.3}$$

The critical points of $\mathcal{A}_k$ with $\eta > 0$ correspond bijectively to the orbits of ϕ_t contained in Σ: if $(x,\eta) \in$ Crit$(\mathcal{A}_k)$ with $\eta > 0$ and $y : \mathbb{R}/\eta\mathbb{Z} \to \Sigma$ is defined by $y(t) := x(t/\eta)$ then $y(t) = \phi_t(y(0))$. Thus the critical points of $\mathcal{A}_k$ are intimately related to those of $\mathcal{S}_k$. The following lemma [12, Lemma 4.1] makes this precise. Denote by $\mathfrak{L} : TM \to T^*M$ the **Legendre transform** of L, defined by

$$\mathfrak{L}(q,v) := \left(q, \frac{\partial L}{\partial v}(q,v)\right). \tag{1.4}$$

Lemma 1.4. *Given* $(q,T) \in$ Crit$(\mathcal{S}_k)$, *define*

$$x(t) := (q(t), \mathfrak{L}(q(t), \dot{q}(t))), \quad x^-(t) := x(-t).$$

Then

$$\text{Crit}(\mathcal{A}_k) = \big\{(x,T), (x^-, -T) \,:\, (q,T) \in \text{Crit}(\mathcal{S}_k)\big\} \cup \{(x,0) \in \Sigma \times \{0\}\}\,.$$

We now explain precisely our standing non-degeneracy assumption.

Definition 1.5. Given an orbit $y : \mathbb{R}/T\mathbb{Z} \to \Sigma$ of X_H, we say that y admits an **orbit cylinder** if there exists $\varepsilon > 0$ together with a smooth (in s) family $\mathcal{O} = (y_{k+s})_{s\in(-\varepsilon,\varepsilon)}$ of orbits of X_H

$$y_{k+s} : \mathbb{R}/T(k+s)\mathbb{Z} \to T^*M;$$
$$H(y_{k+s}) \equiv k+s,$$

with $y_k = y$. We say that the orbit cylinder is **non-degenerate** if $T'(k) \neq 0$. We define the **correction term** associated to an orbit admitting a non-degenerate orbit cylinder by

$$\chi(y) := \text{sign}(-T'(k)).$$

A priori, the correction term $\chi(y)$ depends on the choice of the orbit cylinder. However it follows from Theorem 1.3 and Lemma 1.4 that this is actually not the case.

A sufficient condition for an orbit y to admit an orbit cylinder is that y is **weakly non-degenerate**, by which we mean that y has exactly two **Floquet multipliers** equal to one (see for instance [8, Proposition 4.2]). In this case the orbit cylinder $\mathcal{O}$ is **unique**. If we assume in addition that y is **strongly non-degenerate** (i.e. (1.1) holds) then the orbit cylinder $\mathcal{O}$ is non-degenerate. Indeed, let N denote a hypersurface inside of Σ which is transverse to $y(\mathbb{R}/T\mathbb{Z})$ at the point $y(0)$ with $T_{y(0)}N$ equal to the symplectic orthogonal $(T_{y(0)}\mathcal{O})^\perp$ of the tangent space to the orbit cylinder at $y(0)$. Let $P_y : \mathcal{U} \to \mathcal{V}$ denote the associated **Poincaré map**, where $\mathcal{U}$ and $\mathcal{V}$ are neighborhoods of $y(0)$. P is a diffeomorphism that fixes $y(0)$. Then there exists a unique symplectic splitting of $T_{y(0)}T^*M$ such that $d_{y(0)}\phi_T$ is given by

$$d_{y(0)}\phi_T = \begin{pmatrix} 1 & -T'(k) & 0 \\ 0 & 1 & 0 \\ & & \\ 0 & 0 & d_{y(0)}P_y \\ & & \end{pmatrix}. \tag{1.5}$$

Here $\mathbb{1} - d_{y(0)}P_z$ is invertible. The assumption that $\nu(y) = 1$ therefore implies that $T'(k) \neq 0$.

Remark 1.6. A perhaps more standard name for "strongly non-degenerate" is **transversally non-degenerate**. Unfortunately it seems that the term "transversally non-degenerate" has been used in the literature to mean both "strongly non-degenerate" and "weakly non degenerate". We prefer to clearly differentiate the two conditions, as our results are only valid under the stronger one.

Now suppose $(x, \eta) \in \text{Crit}(\mathcal{A}_k)$ with $\eta \neq 0$. Let $y : \mathbb{R}/|\eta|\,\mathbb{Z} \to \Sigma$ be defined by $y(t) := x(t/|\eta|)$. By an abuse of language, we will say that (x, η) admits a non-degenerate orbit cylinder if same is true for the orbit y. In this case we may define the correction term

$$\chi(x, \eta) := \text{sign}(\eta)\chi(y).$$

Note that if $(q, T) \in \text{Crit}(\mathcal{S}_k)$ admits a non-degenerate orbit cylinder and $x, x^- \in \Lambda T^*M$ are defined as in Lemma 1.4 then both the critical points (x, T) and $(x^-, -T)$ of $\mathcal{A}_k$ admit non-degenerate orbit cylinders, and

$$\chi(q, T) = \chi(x, T) = -\chi(x^-, -T). \tag{1.6}$$

In the next definition we restrict to strongly non-degenerate critical points.

Definition 1.7. Given a critical point $(x, \eta) \in \text{Crit}(\mathcal{A}_k)$ admitting a non-degenerate orbit cylinder, we define an index $\mu_{\text{Rab}}(x, \eta) \in \mathbb{Z}$ by

$$\mu_{\text{Rab}}(x, \eta) := \mu_{\text{CZ}}(y) - \frac{1}{2}\chi(x, \eta)$$

Here $y : \mathbb{R}/|\eta|\,\mathbb{Z} \to \Sigma$ is defined as before by $y(t) := x(t/|\eta|)$, and $\mu_{\text{CZ}}(y) \in \left(\frac{1}{2}\mathbb{Z}\right) \setminus \mathbb{Z}$ denotes the **Conley-Zehnder index** of y. See [14] for the definition of the Conley-Zehnder index in the case where there exist Floquet multipliers of y that are equal to 1 (note however that our sign conventions match those of [1] not [14]).

In fact, the index $\mu_{\text{Rab}}(x, \eta)$ can be identified with a suitably defined **transverse Conley-Zehnder** index, and thus can be defined even when the orbit (x, η) does not admit a non-degenerate orbit cylinder. See Subsection 2.5. In fact, this identification proves the following result:

Proposition 1.8. *The index μ_{Rab} only depends on the hypersurface $\Sigma = H^{-1}(k)$ and not on the actual Hamiltonian defining it.*

Duistermaat's Morse index theorem [6] says that if $(q, T) \in \text{Crit}(\mathcal{S}_k)$ is strongly non-degenerate and $y(t) := x(t/T)$ where $x \in \Lambda T^*M$ is defined as in Lemma 1.4 then

$$i_T(q) = \mu_{\text{CZ}}(y) - \frac{1}{2}.$$

This is proved in the twisted case for mechanical Hamiltonians in [12, Appendix A]. The proof there goes through for an arbitrary Tonelli Hamiltonian; alternatively one could use the argument of [11, Lemma 3.12.5], which works directly for any Tonelli Hamiltonian. Combining Theorem 1.3 and (1.6) one sees that if $(q, T) \in \text{Crit}(\mathcal{S}_k)$ is a strongly non-degenerate critical point of $\mathcal{S}_k$ and $x, x^- \in \Lambda T^*M$ are defined as in Lemma 1.4 then

$$i(q, T) = \mu_{\text{Rab}}(x^+, T) = -\mu_{\text{Rab}}(x^-, -T) \tag{1.7}$$

(see [12, Corollary 4.5]). Moreover as a corollary of Proposition 1.8 we deduce:

Corollary 1.9. *The Morse index $i(q, T)$ depends only on the hypersurface Σ and not on the Lagrangian L.*

This is interesting, as the same result is **not** true for the fixed period Morse index $i_T(q)$.

Suppose now that J is an ω-compatible almost complex structure on T^*M. By this we mean that $\omega(J\cdot,\cdot)$ defines a Riemannian metric on T^*M. Using J we can build an L^2-metric $\langle\langle\cdot,\cdot\rangle\rangle_J$ on ΛT^*M via

$$\langle\langle\xi,\xi'\rangle\rangle_J := \int_{S^1}\omega(J\xi,\xi')dt.$$

We will use the same notation for the product metric on $\Lambda T^*M\times\mathbb{R}^+$:

$$\langle\langle(\xi,b),(\xi',b')\rangle\rangle_J := \langle\langle\xi,\xi'\rangle\rangle_J + bb'.$$

Denote by $\nabla\mathcal{A}_k$ the gradient of $\mathcal{A}_k$ with respect to $\langle\langle\cdot,\cdot\rangle\rangle_J$. Thus

$$\nabla\mathcal{A}_k(x,\eta) = \left(J(x)(\partial_t x - \eta X_H(x)), -\int_{S^1}(H(x)-k)dt\right).$$

Let us denote by $\bar{\partial}_{\mathcal{A}_k}$ the **Rabinowitz Floer operator** defined by

$$\bar{\partial}_{\mathcal{A}_k} : C^\infty(\mathbb{R}\times S^1, T^*M)\times C^\infty(\mathbb{R},\mathbb{R}) \to C^\infty(\mathbb{R}\times S^1, TT^*M)\times C^\infty(\mathbb{R},\mathbb{R})$$

$$\bar{\partial}_{\mathcal{A}_k}(u) := \partial_s u + \nabla\mathcal{A}_k(u(s,\cdot)), \quad u = (x,\eta).$$

The operator $\bar{\partial}_{\mathcal{A}_k}$ extends to define a section of a certain Banach bundle $\mathcal{E}$ over a certain Banach manifold $\mathcal{M}\subseteq C^0(\mathbb{R}\times S^1, T^*M\times\mathbb{R})\times C^0(\mathbb{R},\mathbb{R})$. Roughly speaking, $\mathcal{M}$ consists of those maps $u = (x,\eta)$ that are of class $W^{1,r}_{\text{loc}}$ for some $r > 2$, and satisfy a certain prescribed behavior at infinity. Any $u\in\mathcal{M}$ with $\bar{\partial}_{\mathcal{A}_k}(u) = 0$ is necessarily of class C^∞. Given two critical points $v_\pm := (x_\pm,\eta_\pm)$ of $\text{Crit}(\mathcal{A}_k)$ let us denote by $\mathcal{M}(v_-,v_+)\subseteq\mathcal{M}$ the set of zeros u of $\bar{\partial}_{\mathcal{A}_k}$ that submit to the asymptotic conditions

$$\lim_{s\to\pm\infty} u(s,\cdot) = v_\pm.$$

Note that in the definition of $\mathcal{M}(v_-,v_+)$ we are **not** dividing through by the $\mathbb{R}$-action given by translating along the gradient flow lines.
Recall that a space $\mathcal{N}$ is said to have **virtual dimension** $m\in\mathbb{Z}$ if $\mathcal{N}$ can be seen as the set of zeros of a smooth section of some Banach bundle, whose linearization is Fredholm of index m. If such a section is transverse to the zero-section then the implicit function theorem implies that $\mathcal{N}=\emptyset$ or $m\geq 0$ and $\mathcal{N}$ carries the structure of a smooth m-dimensional manifold. In our second result we extend Proposition 4.1 of [4] to the situation at hand and compute the virtual dimension of $\mathcal{M}(v_-,v_+)$ in the case where $v_\pm$ are strongly non-degenerate critical points.

Theorem 1.10. *Let $v_\pm = (x_\pm,\eta_\pm)\in\text{Crit}(\mathcal{A}_k)$ with $\eta_\pm\neq 0$ denote critical points of $\mathcal{A}_k$ that admit non-degenerate orbit cylinders. Then the space $\mathcal{M}(v_-,v_+)$ has virtual dimension*

$$\text{virdim}\,\mathcal{M}(v_-,v_+) = \mu_{\text{Rab}}(v_-) - \mu_{\text{Rab}}(v_+) - 1.$$

We emphasize that the “-1” in the formula above comes from the fact that we are in a Morse-Bott situation, where the critical manifolds have dimension 1. The method we use in order to compute the virtual dimension differs in some respect to that of [4, Proposition 4.1]. In both cases the key part of the calculation is the computation of a “correction term” between the spectral flow of the Rabinowitz action functional and the spectral flow of the classical Hamiltonian action functional. The difference is that in [4, Section 4] the authors first perturb the hypersurface in such a way that the “correction term” is easy to compute [4, Lemma C.6]. In contrast, we work directly with the Hamiltonian H. However as a consequence we need to assume that the given orbits are strongly non-degenerate, whereas in [4, Section 4] they require only that the Rabinowitz action functional is Morse-Bott on the component of the critical set containing the given orbits.

We remark that we are not assuming at any point that our hypersurfaces are of contact type or stable. We also note that most likely our orientability assumption is not necessary and the arguments of this paper can be extended using the trivializations from [18].

1.3. **Relationship with Rabinowitz Floer homology and Mañé's critical values.**

So far we have not placed any restriction on the value $k \in \mathbb{R}$, apart from asking for k to be a regular value of H. In fact, to be able to define and compute the Rabinowitz Floer homology of Σ, or to be able to do Morse homology with $\mathcal{S}_k$, we need to make an additional assumption on k. One option is to assume that k is sufficiently large. More precisely, there are two particular "critical values" c and c_0 of k, known as the **Mañé critical values**[1]. They are such that the dynamics of the hypersurface $H^{-1}(k)$ differs dramatically depending on the relation of k to these numbers. They satisfy $c < \infty$ if and only if $\widetilde{\sigma}$ admits a bounded primitive, and $c_0 < \infty$ if and only if σ is actually exact, and are defined as follows: let $\widetilde{H} : T^*\widetilde{M} \to \mathbb{R}$ denote the lift of H to $T^*\widetilde{M}$, and set

$$c := \inf_{\theta} \sup_{q \in \widetilde{M}} \widetilde{H}(q, \theta_q), \tag{1.8}$$

where the infimum is taken over all 1-forms θ on $\widetilde{M}$ with $d\theta = \widetilde{\sigma}$. If σ is not exact define $c_0 := \infty$. Otherwise define

$$c_0 := \inf_{\theta} \sup_{q \in M} H(q, \theta_q), \tag{1.9}$$

that is, the same definition only working directly on T^*M rather than lifting to $T^*\widetilde{M}$. See for instance [5, Section 5] for more information.

In [12] the first author computed[2] the Rabinowitz Floer homology of Σ for $k > c$, which builds on the earlier construction of Abbondandolo and Schwarz [3] which computed the Rabinowitz Floer homology of Σ for $k > c_0$.

To get started one first assumes that all periodic orbits of X_H contained in Σ are strongly non-degenerate. Then the key idea is to define chain maps between the Rabinowitz Floer chain complex, which is generated by the critical points of $\mathcal{A}_k$, and the Morse (co)chain complex generated by the critical points of $\mathcal{S}_k$. The motivation for such a construction is provided by Lemma 1.4 and the relationship (1.7).

A further remark to make is that in the setting of Rabinowitz Floer homology one is free to use any Hamiltonian $F : T^*M \to \mathbb{R}$ such that $F^{-1}(k) = \Sigma$ and $X_F|_\Sigma$ coincides with the Reeb vector field of the hypersurface. If Σ is of **restricted contact type** then one can choose such a Hamiltonian F such that F near Σ is radial and homogeneous of degree 2 in the radial direction provided by the symplectization of the hypersurface. In this case it is not hard to see that the correction term of a strongly non-degenerate orbit (x, η) is always given by the sign of η. If $k > c_0$ then one can always find such a Hamiltonian (see [3, Section 10]); this explains why in [3] the correction terms do not appear explicitly. If we just ask that $k > c$ then it is convenient for several reasons to keep the physical Hamiltonian H; thus in [12] the correction term does appear. One compelling reason to keep H unchanged is that for $k > c$, the hypersurface is now only virtually contact and not necessarily contact, see [5].

---

[1] c is sometimes also denoted as c_u.

[2] Technically speaking, in [12] only mechanical Hamiltonians, rather than Tonelli Hamiltonians are considered, although the proofs go through in this more general setting.

A natural question to ask therefore is whether if for $k > c$ the correction terms $\chi(q,T)$ and $\chi(x,\eta)$ (for $\eta > 0$) are always $+1$. In the last section of this note we answer this question in the negative. More precisely, we construct an example of a mechanical Hamiltonian H on (T^*S^2, ω_σ) (for a suitable choice of magnetic form σ) for which there exist a strongly non-degenerate orbit y of X_H with energy $k > c$ (in this case $c = c_0$) for which the correction term $\chi(y)$ is equal to -1. (It is easily seen that the example can be arranged so that it also carries orbits for which the correction term is $+1$.) In particular, for this example Theorem 1.3 implies that $i(q,T) = i_T(q) + 1$.

1.4. **Dropping the assumption that σ is weakly exact and admits a bounded primitive.**

We conclude this introduction with the following remark. None of the results in this paper actually use the fact that σ is weakly exact and admits a bounded primitive, at least if one is prepared to work with 1-forms rather than functionals. More precisely, suppose we consider the 1-form $\mathfrak{a}_k \in \Omega^1(\Lambda T^*M \times \mathbb{R})$ defined by

$$\mathfrak{a}_k(x,\eta)(\xi,b) := \int_{S^1} \omega(\xi, \dot{x} - \eta X_H(x))dt - b\int_{S^1} H(x)dt \quad \text{for} \quad (\xi,b) \in T_{(x,\eta)}(\Lambda T^*M \times \mathbb{R}).$$

Under the assumption that σ is closed, $\mathfrak{a}_k$ is a well defined closed 1-form on $\Lambda T^*M \times \mathbb{R}$. Of course, if σ is weakly exact and admits a bounded primitive then $\mathfrak{a}_k$ is exact, with

$$\mathfrak{a}_k(x,\eta) = d_{(x,\eta)}\mathcal{A}_k.$$

The critical points of $\mathfrak{a}_k$, that is, the points $(x,\eta) \in \Lambda T^*M \times \mathbb{R}$ with $a_k(x,\eta) = 0$, are still given by (1.3).

In a similar vein, if we only assume that σ is closed then we can define a closed 1-form $\mathfrak{s}_k$ on $\Lambda M \times \mathbb{R}^+$. As with $\mathfrak{a}_k$, if we make the additional assumptions that σ is weakly exact and admits a bounded primitive, then $\mathfrak{s}_k$ is exact and satisfies

$$\mathfrak{s}_k(q,T) = d_{(q,T)}\mathcal{S}_k.$$

As with $\mathcal{S}_k$, the critical points of $\mathfrak{s}_k$ are the pairs (q,T) such that if $\gamma : [0,T] \to M$ is defined by $\gamma(t) := q(t/T)$ then γ is the projection to M of a closed orbit of ϕ_t contained in Σ.

Both Theorem 1.3 and Theorem 1.10 continue to make sense if we work only with the 1-forms $\mathfrak{s}_k$ and $\mathfrak{a}_k$ respectively, and all the proofs in this note go through word for word in this more general setting. Hopefully this observation will be useful in defining a more general Rabinowitz Floer homology for twisted cotangent bundles where σ is only assumed closed.

Acknowledgement. We are deeply grateful to the anonymous referee for suggesting a considerable simplification to the proof of Theorem 1.3 (cf. Remark 2.3). We would also like to thank Alberto Abbondandolo for several helpful comments and suggestions on an earlier draft of this note and to Urs Frauenfelder for his remark that led to Subsection 2.5.

2. PROOFS

We denote by $\overline{\mathbb{R}}$ the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, with the differentiable structure induced by the bijection $[-\pi/2, \pi/2] \to \overline{\mathbb{R}}$ given by

$$s \mapsto \begin{cases} \tan s & s \in (-\pi/2, \pi/2) \\ \pm\infty & s = \pm\pi/2. \end{cases}$$

All the sign conventions used in this paper match those of [3, 12].

2.1. **The proof of Theorem 1.3.**

Denote by $\nabla^2\mathcal{S}_k(q,T)$ and $\nabla^2\mathcal{S}_k^T(q)$ the $W^{1,2}$-Hessians of $\mathcal{S}_k$ and $\mathcal{S}_k^T$, so that if $(q,T)\in\mathrm{Crit}(\mathcal{S}_k)$ then

$$d^2_{(q,T)}\mathcal{S}_k((\zeta,b),(\zeta',b')) = \left\langle\left\langle \nabla^2\mathcal{S}_k(q,T)(\zeta,b),(\zeta',b')\right\rangle\right\rangle;$$

$$d^2_{(q,T)}\mathcal{S}_k(\zeta,\zeta') = \left\langle\left\langle \nabla^2\mathcal{S}_k^T(q)(\zeta),\zeta'\right\rangle\right\rangle.$$

The assertion of the theorem is local, so without loss of generality we may suppose $M=\mathbb{R}^n$. Suppose $(q_{k+s},T(k+s))_{s\in(-\varepsilon,\varepsilon)}$ is an orbit cylinder about $(q_k,T(k))$. Let

$$\zeta_k(t) := \frac{\partial}{\partial s}q_{k+s}(t).$$

Since $(q_{k+s},T(k+s))$ is a critical point of $\mathcal{S}_{k+s}$, for any $(\zeta,b)\in W^{1,2}(S^1,\mathbb{R}^n)\times\mathbb{R}$ we have

$$d_{(q_{k+s},T(k+s))}\mathcal{S}_{k+s}(\zeta,b)=0.$$

We differentiate this equation with respect to s and evaluate at $s=0$ to obtain

$$\begin{aligned}
0 &= \frac{\partial}{\partial s}\Big|_{s=0} d_{(q_{k+s},T(k+s))}\mathcal{S}_{k+s}(\zeta,b)\\
&= \frac{\partial}{\partial s}\Big|_{s=0} d_{(q_k,T(k))}\mathcal{S}_{k+s}(\zeta,b) + d^2_{(q_k,T(k))}\mathcal{S}_k((\zeta_k,T'(k)),(\zeta,b)).
\end{aligned}$$

In order to compute $\frac{\partial}{\partial s}\Big|_{s=0} d_{(q_k,T(k))}\mathcal{S}_{k+s}(\zeta,b)$, choose a variation $(q_{k,r},T(k,r))$ for $r\in(-\varepsilon,\varepsilon)$ such that $(q_{k,0},T(k,0))=(q_k,T(k))$ and $\frac{\partial}{\partial r}\Big|_{r=0}(q_{k,r},T(k,r))=(\zeta,b)$. Then

$$\begin{aligned}
\frac{\partial}{\partial s}\Big|_{s=0} d_{(q_k,T(k))}\mathcal{S}_{k+s}(\zeta,b) &= \frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial}{\partial r}\Big|_{r=0}\mathcal{S}_{k+s}(q_{k,r},T(k,r))\\
&= \frac{\partial}{\partial r}\Big|_{r=0}\frac{\partial}{\partial s}\Big|_{s=0}\mathcal{S}_{k+s}(q_{k,r},T(k,r))\\
&= \frac{\partial}{\partial r}\Big|_{r=0}T(k,r)\\
&= b.
\end{aligned}\tag{2.1}$$

In other words, we have proved:

$$d^2_{(q_k,T(k))}\mathcal{S}_k((\zeta_k,T'(k)),\zeta,b)) = -b.$$

Taking $b=0$ we see that

$$d^2_{(q_k,T(k))}\mathcal{S}_k((\zeta_k,T'(k)),(\zeta,0)) = 0,\tag{2.2}$$

and taking $(\zeta,b)=(\zeta_k,T'(k))$ we see that

$$d^2_{(q_k,T(k))}\mathcal{S}_k((\zeta_k,T'(k)),(\zeta_k,T'(k))) = -T'(k).$$

Let us write $W^{1,2}(S^1,\mathbb{R}^n)\times\mathbb{R} = E^+\oplus\ker(\nabla^2\mathcal{S}_k(q_k,T(k))\oplus E^-$, where $E^{+(-)}$ is the positive (negative) eigenspace of $\nabla^2\mathcal{S}_k(q_k,T(k))$. Similarly write $W^{1,2}(S^1,\mathbb{R}^n) = E^+_{T(k)}\oplus\ker(\nabla^2\mathcal{S}_k^{T(k)}(q_k))\oplus E^-_{T(k)}$, where $E^{+(-)}_{T(k)}$ is the positive (negative) eigenspace of $\nabla^2\mathcal{S}_k^{T(k)}(q_k)$ (such spectral decompositions exist as $\nabla^2\mathcal{S}_k(q_k,T(k))$ and $\nabla^2\mathcal{S}_k^{T(k)}(q_k)$ are bounded self-adjoint operators). Clearly

$E^{\pm}_{T(k)} \times \{0\} \subseteq E^{\pm}$. Equation (2.2) tells us that the 1-dimensional vector space $W := \mathrm{span}_{\mathbb{R}} \langle (\zeta_k, T'(k)) \rangle$ is orthogonal to $W^{1,2}(S^1, \mathbb{R}^n) \times \{0\}$, and thus we must have

$$E^+ = E^+_{T(k)} \times \{0\}, \quad E^- = (E^-_{T(k)} \times \{0\}) \oplus W \quad \text{if } T'(k) > 0,$$

or

$$E^+ = (E^+_{T(k)} \times \{0\}) \oplus W, \quad E^- = E^-_{T(k)} \times \{0\} \quad \text{if } T'(k) < 0.$$

The theorem follows.

2.2. **The spectral flow.**

In this section we recall the definition of the **spectral flow**, following the exposition in [15], and then recall the statement of a theorem of Cieliebak and Frauenfelder [4, Theorem C.5], which we use to prove Theorem 1.10. Let W and H denote a pair of separable real Hilbert spaces with $W \subseteq H = H^* \subseteq W^*$ such that the inclusion $W \hookrightarrow H$ is compact with dense range. Let $\mathbf{L}(W, H)$ denote the set of bounded linear operators, and let $\mathbf{S}(W, H) \subseteq \mathbf{L}(W, H)$ denote the subspace of self-adjoint operators (considered as unbounded operators on H with dense domain W).

Denote by $\mathbf{A}(W, H)$ the set of continuous (with respect to the norm topologies) maps $A : \mathbb{R} \to \mathbf{S}(W, H)$ and such that the limits

$$A^{\pm} := \lim_{s \to \pm\infty} A(s)$$

exist and belong to $\mathbf{S}(W, H)$. Denote by $\mathbf{A}_0(W, H) \subseteq \mathbf{A}(W, H)$ the subset consisting of those elements $A \in \mathbf{A}(W, H)$ such that the limit operators $A^{\pm}$ are bijective onto W. The **spectral flow** is a map

$$\mu_{\mathrm{SF}} : \mathbf{A}_0(W, H) \to \mathbb{Z}$$

which is characterized by the following four properties.

- μ_{SF} is constant on the connected components of $\mathbf{A}_0(W, H)$.
- If A is a constant map, $\mu_{\mathrm{SF}}(A) = 0$.
- $\mu_{\mathrm{SF}}(A_0 \oplus A_1) = \mu_{\mathrm{SF}}(A_0) + \mu_{\mathrm{SF}}(A_1)$.
- If $W = H$ are finite dimensional then

  $$\mu_{\mathrm{SF}}(A) = \frac{1}{2}\mathrm{sign}(A^+) - \frac{1}{2}\mathrm{sign}(A^-),$$

  where for a matrix C, $\mathrm{sign}(C)$ denotes the number of positive eigenvalues of C minus the number of negative eigenvalues of C.

The spectral flow may be extended to a function $\mu_{\mathrm{SF}} : \mathbf{A}(W, H) \to \mathbb{Z}$ as follows. Suppose $A \in \mathbf{A}(W, H)$. Since the limit operators $A^{\pm}$ are self-adjoint, their spectrums $\sigma(A^{\pm})$ are discrete and contained in $\mathbb{R}$. Hence

$$\lambda := \inf \left\{ |\rho| \, : \, \rho \in (\sigma(A^+) \cup \sigma(A^-)) \backslash \{0\} \right\} > 0.$$

Thus if we choose $0 < \delta < \lambda$ then the operators

$$A^{\pm}_{\delta} := A^{\pm} \mp \delta \mathbb{1}$$

are bijective. Let $\beta \in C^{\infty}(\mathbb{R}, [-1, 1])$ denote a smooth cutoff function such that $\beta(s) = 1$ for $s \geq 1$ and $\beta(s) = -1$ for $s \leq 0$, and set

$$A_{\delta}(s) := A(s) - \delta\beta(s)\mathbb{1},$$

where $\delta > 0$ is as above. Then $A_{\delta} \in \mathbf{A}_0(W, H)$, and hence the spectral flow $\mu_{\mathrm{SF}}(A_{\delta})$ is well defined. We may therefore define

$$\mu_{\mathrm{SF}}(A) := \lim_{\delta \searrow 0} \mu_{\mathrm{SF}}(A_{\delta}).$$

Note that this limit stabilizes for $\delta > 0$ sufficiently small.

Suppose now we are given maps $A \in \mathbf{A}(W, H)$, $h \in \mathbf{A}(\mathbb{R}, H)$ and $\tau \in \mathbf{A}(\mathbb{R}, \mathbb{R})$. Define the map $A_{h,\tau} \in \mathbf{A}(W \oplus \mathbb{R}, H \oplus \mathbb{R})$ by:

$$A_{h,\tau}(s)\begin{pmatrix} v \\ b \end{pmatrix} = \begin{pmatrix} A(s) & h(s) \\ h^*(s) & \tau(s) \end{pmatrix}\begin{pmatrix} v \\ b \end{pmatrix} = \begin{pmatrix} A(s)(v) + bh(s) \\ \langle h(s), v\rangle_H + \tau(s)b \end{pmatrix}.$$

The spectral flow of the operator $A_{h,\tau}$ is then given by:

$$\mu_{\mathsf{SF}}(A_{h,\tau}) := \lim_{\delta \searrow 0} \mu_{\mathsf{SF}}((A_{h,\tau})_\delta).$$

If $\tau \equiv 0$ we write A_h instead of $A_{h,0}$.

Before stating the theorem of Cieliebak and Frauenfelder, we introduce the following terminology.

Definition 2.1. Say that the triple $(A, h, \tau) \in \mathbf{S}(W, H) \times H \times \mathbb{R}$ is **regular** if $h \in W \cap \mathrm{range}(A)$ and if $v \in W$ is any vector such that $Av = h$, then the real number $\lambda_{A,h} := \langle v, h\rangle_H$ is not equal to τ. Note h is orthogonal to $\ker(A)$ as A is self-adjoint. Thus $\lambda_{A,h}$ does not depend on the choice of v such that $Av = h$: if v' is another choice then $v - v' \in \ker(A)$ and hence $\langle v - v', h\rangle_H = 0$ as $h \perp \ker(A)$.

The next result is [4, Theorem C.5].

Theorem 2.2. (Cieliebak, Frauenfelder)
Let $A \in \mathbf{A}(W, H)$, $h \in \mathbf{A}(\mathbb{R}, H)$ and $\tau \in \mathbf{A}(\mathbb{R}, \mathbb{R})$. Assume that the limit operators $(A^\pm, h^\pm, \tau^\pm)$ are regular triples. Then

$$\mu_{\mathsf{SF}}(A_{h,\tau}) = \mu_{\mathsf{SF}}(A) + \frac{1}{2}\mathrm{sign}(\tau^+ - \lambda_{A^+,h^+}) - \frac{1}{2}\mathrm{sign}(\tau^- - \lambda_{A^-,h^-}).$$

Remark 2.3. In fact, we use Theorem 2.2 only in the case where $\tau \equiv 0$ (this is also how the theorem is stated in [4], although the proof goes through without change). In an earlier version of this paper we proved Theorem 1.3 by applying Theorem 2.2 to the Hessian of $\mathcal{S}_k$, and arguing as in [15, Corollary 6.41]. The referee pointed out that Theorem 1.3 follows from the simpler argument given in Subsection 2.1 above; we are very grateful to him or her for this observation.

2.3. **The proof of Theorem 1.10.**

In this section we prove Theorem 1.10. Before getting started, we introduce some more notation, Suppose $(x, \eta) \in \mathrm{Crit}(\mathcal{A}_k)$ with $\eta > 0$. Let us denote by

$$\mathcal{A}_k^\eta : \Lambda T^*M \to \mathbb{R}$$

the **classical action functional** of Hamiltonian mechanics:

$$\mathcal{A}_k^\eta(x) := \mathcal{A}_k(x, \eta).$$

Let $\nabla\mathcal{A}_k^\eta$ denote the L^2-gradient of $\mathcal{A}_k^\eta$ with respect to the metric $\langle\langle\cdot,\cdot\rangle\rangle_J$ on ΛT^*M, and as above denote by $\nabla^2\mathcal{A}_k(x, \eta)$ and $\nabla^2\mathcal{A}_k^\eta(x)$ the L^2-Hessians of $\mathcal{A}_k$ and $\mathcal{A}_k^\eta$ at a critical point (x, η) of $\mathcal{A}_k$. One checks that $\nabla^2\mathcal{A}_k(x, \eta)$ and $\nabla^2\mathcal{A}_k^\eta(x)$ are related as follows:

$$\nabla^2\mathcal{A}_k(x,\eta)(\xi, b) = \begin{pmatrix} \nabla^2\mathcal{A}_k^\eta(x) & -\nabla H(x) \\ -\nabla H(x)^* & 0 \end{pmatrix}\begin{pmatrix} \xi \\ b \end{pmatrix} = \left\langle\left\langle \nabla^2\mathcal{A}_k^\eta(x), \xi\right\rangle\right\rangle_J - b\left\langle\left\langle \nabla H(x), \xi\right\rangle\right\rangle_J. \tag{2.3}$$

Now fix two critical points $v_\pm = (x_\pm, \eta_\pm) \in \mathrm{Crit}(\mathcal{A}_k)$. In order to show that $\mathcal{M}(v_-, v_+)$ has virtual dimension $\mu_{\mathsf{Rab}}(v_-) - \mu_{\mathsf{Rab}}(v_+) - 1$ we need to show that the linearization D_u of $\bar{\partial}_{\mathcal{A}_k}$ at any

$u \in \mathcal{M}(v_-, v_+)$ is Fredholm, and compute its index. For simplicity we will prove the theorem only in the case where both η_- and η_+ are positive. The case when $\eta_\pm$ have arbitrary sign is analogous.

Suppose $u = (x, \eta) \in \mathcal{M}(v_-, v_+)$. It is well known that any such map u extends to a map $u : \overline{\mathbb{R}} \to \Lambda T^*M \times \mathbb{R}$, which we continue to denote by u. Fix $r \geq 2$. The linearization of D_u of $\bar{\partial}_{\mathcal{A}_k}$ at u is given by:

$$D_u : W^{1,r}(\mathbb{R} \times S^1, x^*TT^*M) \times W^{1,r}(\mathbb{R}, \mathbb{R}) \to L^r(\mathbb{R} \times S^1, x^*TT^*M) \times L^r(\mathbb{R}, \mathbb{R});$$

$$D_u \begin{pmatrix} \xi \\ b \end{pmatrix} = \begin{pmatrix} \nabla_s \xi + J(x)\nabla_t \xi + \nabla_\xi J(\dot{x}) - \eta \nabla_\xi \nabla H(x) - b\nabla H(x) \\ \partial_s b - \int_0^1 \langle \nabla H, \xi \rangle_J \, dt \end{pmatrix}.$$

In order to compute the index we express D_u in a local trivialization. Let $\varphi : \overline{\mathbb{R}} \times S^1 \times \mathbb{R}^{2n} \to x^*TT^*M$ denote a symplectic trivialization of the pullback bundle over $\overline{\mathbb{R}} \times S^1$, and denote by $\varphi_\pm := \varphi|_{\{\pm\infty\} \times S^1}$ the induced symplectic trivializations of the pullback bundles $x_\pm^* TT^*M$. Let $\widehat{\varphi} : \overline{\mathbb{R}} \times S^1 \times \mathbb{R}^{2n} \times \mathbb{R} \to x^*TT^*M \times \eta^*T\mathbb{R}$ denote the product trivialization of φ together with some trivialization of the bundle $\eta^*T\mathbb{R} \to \overline{\mathbb{R}}$. We now conjugate D_u via $\widehat{\varphi}$ to obtain an operator of the form

$$D_{A_h} := \widehat{\varphi}^{-1} \circ D_u \circ \widehat{\varphi};$$

$$D_{A_h} : W^{1,r}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \times W^{1,r}(\mathbb{R}, \mathbb{R}) \to L^r(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \times L^r(\mathbb{R}, \mathbb{R});$$

$$D_{A_h} \begin{pmatrix} v \\ b \end{pmatrix}(s) = (\partial_s + A_h) \begin{pmatrix} v \\ b \end{pmatrix}(s) = \begin{pmatrix} \partial_s v(s) + A(s)(v) + bh(s) \\ \partial_s b(s) + \int_0^1 \langle h(s), v \rangle \, dt \end{pmatrix}.$$

Here

$$A(s,t) := J_0 \partial_t + S(s,t), \tag{2.4}$$

where

$$S(s,t) \in C^\infty(\mathbb{R} \times S^1, \mathbf{L}(\mathbb{R}^{2n}, \mathbb{R}^{2n}))$$

is defined by

$$S(s,t) = \varphi^{-1} \circ (\nabla_s \varphi + J(x)\nabla_t \varphi - \nabla_\varphi J(\dot{x}) - \eta \nabla_\varphi \nabla H(x))\,(s,t), \tag{2.5}$$

and $J_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ is the standard almost complex structure on $\mathbb{R}^{2n}$.

Secondly

$$h \in C^\infty(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

is the vector valued function

$$h(s,t) := \varphi^{-1} \circ (-\nabla H(x))(s,t). \tag{2.6}$$

As it stands the operator D_{A_h} is **not** Fredholm. This can be rectified however by defining D_{A_h} instead on a suitably **weighted Sobolev space**. This is explained in [7, Lemma A.13]. Since the kernel and the cokernel of D_{A_h} are spanned by smooth elements, the index of D_{A_h} does not depend on r, and hence it suffices to consider the case $r = 2$. In this case a well known result of Robbin and Salamon [15, Theorem 4.21] tells us that the Fredholm index of D_{A_h} is given by spectral flow of A_h :

$$\mathrm{ind}(D_{A_h}) = \mu_{\mathtt{SF}}(A_h).$$

We therefore need to compute this spectral flow. Note that the limit operators are given by:

$$A^\pm = \varphi_\pm^{-1} \circ \nabla^2 \mathcal{A}_k^{\eta_\pm}(x_\pm) \circ \varphi_\pm;$$

$$h^\pm := \varphi_\pm^{-1} \circ (-\nabla H(x_\pm)).$$

The computation in the next subsection will show that the triple $(A^\pm, h^\pm, 0)$ is regular in the sense of Definition 2.1, that is, the correction terms $\lambda_{A^\pm,h^\pm}$ are non-zero. Thus by Theorem 2.2 the spectral flow of A_h is given by

$$\mu_{\mathrm{SF}}(A_h) = \mu_{\mathrm{SF}}(A) - \frac{1}{2}\mathrm{sign}(\lambda_{A^+,h^+}) + \frac{1}{2}\mathrm{sign}(\lambda_{A^-,h^-}). \tag{2.7}$$

A well known result of Salamon and Zehnder [17] tells us that

$$\mu_{\mathrm{SF}}(A) = \mu_{\mathrm{CZ}}(y_-) - \mu_{\mathrm{CZ}}(y_+) - 1,$$

where $y_\pm(t) := y_\pm(t/\eta_\pm)$. Here the "$-1$" in the formula comes from the fact that the $y_\pm$ have Floquet multipliers equal to 1 (see Theorem 4.2 and Lemma 4.4 in [4]). In order to complete the proof of Theorem 1.10 we therefore need to prove that

$$\mathrm{sign}(\lambda_{A^\pm,h^\pm}) = -\chi(x_\pm, \eta_\pm). \tag{2.8}$$

2.4. **Computing $\lambda_{A^\pm,h^\pm}$.**

Let us start now with a critical point $(x_k, \eta(k)) \in \mathrm{Crit}(\mathcal{A}_k)$ that admits a non-degenerate orbit cylinder. Thus there exists $\varepsilon > 0$ and a family $(x_{k+s}, \eta(k+s))$ of critical points of $\mathcal{A}_{k+s}$ for $s \in (-\varepsilon, \varepsilon)$, with $\eta(k+s) > 0$ for each $s \in (-\varepsilon,\varepsilon)$ and $\eta'(k) \neq 0$. Let

$$\xi_k(t) := \frac{\partial}{\partial s}\Big|_{s=0} x_{k+s}(t).$$

We will prove below that:

$$\nabla^2 \mathcal{A}_k^{\eta(k)}(x_k)(\xi_k) = \eta'(k)\nabla H(x_k). \tag{2.9}$$

Equation (2.8) readily follows from (2.9), as we compute:

$$\begin{aligned}
\lambda_{\nabla^2\mathcal{A}_k^{\eta(k)}(x_k), -\nabla H(x_k),} &:= \int_0^1 \left\langle -\frac{1}{\eta'(k)}\xi_k, -\nabla H(x_k)\right\rangle dt \\
&= \frac{1}{\eta'(k)}\int_0^1 d_{x_k}H\left(\frac{\partial}{\partial s}\Big|_{s=0} x_{k+s}(t)\right) dt \\
&= \frac{1}{\eta'(k)}\int_0^1 \frac{\partial}{\partial s}\Big|_{s=0} H(x_{k+s}(t))dt \\
&= \frac{1}{\eta'(k)}\int_0^1 \frac{\partial}{\partial s}\Big|_{s=0} \{k+s\}dt \\
&= \frac{1}{\eta'(k)},
\end{aligned}$$

and hence

$$\mathrm{sign}(\lambda_{\nabla^2\mathcal{A}_k^{\eta(k)}(x_k), -\nabla H(x_k)}) := \mathrm{sign}(\eta'(k)) = -\chi(x_k, \eta(k)).$$

It remains therefore to check (2.9). Set $\rho_k := \nabla^2\mathcal{A}_k^{\eta(k)}(x_k)(\xi_k) - \eta'(k)\nabla H(x_k)$; we show $\rho_k = 0$. We proceed as in Subsection 2.1. The assertion is local, and hence we may assume $T^*M = \mathbb{R}^{2n}$. For any $(\xi, b) \in W^{1,2}(S^1, \mathbb{R}^{2n}) \times \mathbb{R}$ we have

$$d_{(x_{k+s},\eta(k+s))}\mathcal{A}_{k+s}(\xi, b) = 0,$$

and differentiating this equation with respect to s and setting $s=0$ we discover

$$\begin{aligned}0 &= \frac{\partial}{\partial s}\Big|_{s=0} d_{(x_{k+s},\eta(k+s))}\mathcal{A}_{k+s}(\xi,b)\\ &= \frac{\partial}{\partial s}\Big|_{s=0} d_{(x_k,\eta(k))}\mathcal{A}_{k+s}(\xi,b) + d^2_{(x_k,\eta(k))}\mathcal{A}_k((\xi_k,\eta'(k)),(\xi,b))\\ &= b + d^2_{(x_k,\eta(k))}\mathcal{A}_k((\xi_k,\eta'(k)),(\xi,b)),\end{aligned}$$

where the last line used a similar argument to (2.1).

Taking $(\xi,b)=(\rho_k,0)$ we see that

$$\begin{aligned}0 &= \left\langle\left\langle \nabla^2\mathcal{A}_k(x_k,\eta'(k))(\xi_k,\eta'(k)),(\rho_k,0)\right\rangle\right\rangle_J\\ &\overset{(*)}{=} \left\langle\left\langle \begin{pmatrix} \nabla^2\mathcal{A}_k^{\eta(k)}(x_k) & -\nabla H(x_k)\\ -\nabla H(x_k)^* & 0\end{pmatrix}\begin{pmatrix}\xi_k\\ \eta'(k)\end{pmatrix},\begin{pmatrix}\rho_k\\ 0\end{pmatrix}\right\rangle\right\rangle_J\\ &= \left\langle\left\langle\rho_k,\rho_k\right\rangle\right\rangle_J,\end{aligned}$$

where $(*)$ used (2.3). Thus $\rho_k=0$ as desired. This conclues the proof of (2.8) and hence of Theorem 1.10.

2.5. Proving that the index μ_{Rab} does not depend on the choice of Hamiltonian.

In this section we prove that the grading μ_{Rab} from Definition 1.7 does not depend on the choice of Hamiltonian H representing Σ. We also explain how is μ_{Rab} related to a suitably defined **transverse** Conley-Zehnder index.

We begin in a more general situation. Let $\Sigma\subseteq T^*M$ be a closed oriented separating hypersurface of T^*M, where the latter is endowed with a twisted symplectic form ω. Let $\mathcal{L}$ denote the characteristic line bundle of Σ. Note that $\mathcal{L}$ carries a natural orientation. Denote by $\mathcal{Q}$ the quotient bundle $T\Sigma/\mathcal{L}$ with projection $p:T\Sigma\to\mathcal{Q}$. Thus $\mathcal{Q}$ is a rank $2n-2$ bundle over Σ, which carries the structure of a symplectic vector bundle. Let $T^vT^*M\subseteq TT^*M$ denote the vertical distribution defined by $T^v_{(q,p)}T^*M := T_{(q,p)}T^*_qM$, and denote by $\mathcal{V}\subseteq\mathcal{Q}$ the Lagrangian subbundle of $\mathcal{Q}$ defined by $\mathcal{V}_x := p\left(T^v_xT^*M\cap T_x\Sigma\right)$.

Suppose $y:\mathbb{R}/T\mathbb{Z}\to\Sigma$ is a smooth curve. The bundle $y^*\mathcal{Q}$ over $\mathbb{R}/T\mathbb{Z}$ is trivial. Let us say that a trivialization $\Psi:\mathbb{R}/T\mathbb{Z}\times\mathbb{R}^{2n-2}\to y^*\mathcal{Q}$ is **$\mathcal{V}$-preserving** if $\Psi(t)(V^{n-1})=\mathcal{V}_{y(t)}$ for each $t\in\mathbb{R}/T\mathbb{Z}$, where $V^{n-1}:=(0)\times\mathbb{R}^{n-1}$. Such trivializations always exist; this is proved by first choosing a trivialization Ψ' of the trivial bundle $y^*\mathcal{V}$ and extending it to a trivialization Ψ of $y^*\mathcal{Q}$ by making use of a compatible almost complex structure on $\mathcal{Q}$ to write $\mathcal{Q}=\mathcal{V}\oplus J\mathcal{V}$ (see for instance [2, Lemma 1.2]).

Definition 2.4. A **defining Hamiltonian** for Σ is an autonomous Hamiltonian $H\in C^\infty(T^*M,\mathbb{R})$ such that $\Sigma=H^{-1}(0)$ and $X_H|_\Sigma$ is a positively oriented non-vanishing section of $\mathcal{L}$. Denote by $\mathcal{D}(\Sigma)$ the set of all defining Hamiltonians. Note that $\mathcal{D}(\Sigma)$ is a convex set. If $H\in\mathcal{D}(\Sigma)$ and ϕ_t^H denotes the flow of X_H then $d\phi_t^H$ induces a linear map $Q_t^H:\mathcal{Q}_x\to\mathcal{Q}_{\phi_t^H(x)}$ for each $x\in\Sigma$.

Remark 2.5. Our definition of a "defining Hamiltonian" is weaker than the usual one in Rabinowitz Floer homology ([4]). There they ask that not only is $X_H|_\Sigma$ a positively oriented non-vanishing section of $\mathcal{L}$, but that it is actually equal to some given fixed section - in the contact case this is normally the Reeb vector field.

Note that if H and F are two elements of $\mathcal{D}(\Sigma)$ then there exists a smooth positive function $f\in C^\infty(\Sigma,\mathbb{R}^+)$ such that $X_H|_\Sigma = fX_F|_\Sigma$. Then the Hamiltonian flows $\phi_t^H|_\Sigma$ and $\phi_t^F|_\Sigma$ are related by

$$\phi_t^H(x) = \phi_{\alpha(t,x)}^F(x) \quad \text{for } x \in \Sigma.$$

where

$$\alpha(t,x) = \int_0^t f(\phi_s^H(x))ds.$$

Suppose $y : \mathbb{R}/T\mathbb{Z} \to \Sigma$ is a periodic orbit of X_H. Let $\beta : [0, \alpha(T, y(0))] \to [0, T]$ be the inverse of the function $t \mapsto \alpha(t, y(0))$. Then the curve $z : \mathbb{R}/(\alpha(T, y(0))\mathbb{Z} \to \Sigma$ defined by $z(t) = y(\beta(t))$ is a periodic orbit of X_F. We will say that z is the orbit of X_F corresponding to the orbit y of X_H.

Definition 2.6. Let $H \in \mathcal{D}(\Sigma)$. Suppose $y : \mathbb{R}/T\mathbb{Z} \to \Sigma$ is a periodic orbit of X_H. Let Ψ denote a $\mathcal{V}$-preserving trivialization of $y^*\mathcal{Q}$, and denote by $\lambda_{y,\Psi} : [0, T] \to \mathrm{Sp}(2n-2)$ the path

$$\lambda_{y,\Psi}(t) := \Psi(t)^{-1} \circ Q_t^H \circ \Psi(0).$$

We denote by $\mu_{\mathrm{CZ}}^\tau(y)$ the **transverse Conley-Zehnder index** of y, which by definition is given by

$$\mu_{\mathrm{CZ}}^\tau(y) = \mu_{\mathrm{Ma}}(\mathrm{graph}(\lambda_{y,\Psi}), \Delta_{\mathbb{R}^{2n-2}}) \in \frac{1}{2}\mathbb{Z},$$

where $\mu_{\mathrm{Ma}}(\cdot, \cdot)$ denotes the **relative Maslov index** as defined in [14] and $\Delta_{\mathbb{R}^{2n-2}}$ denotes the diagonal in $\mathbb{R}^{2n-2}$. Note however that our sign conventions for $\mu_{\mathrm{Ma}}(\cdot, \cdot)$ match those of [1] not [14]. As the notation suggests, the index $\mu_{\mathrm{CZ}}^\tau(y)$ is independent of the choice of $\mathcal{V}$-preserving trivialization Ψ; this can be proved in exactly the same way as [2, Lemma 1.3].

The following lemma is immediate from the homotopy invariance of $\mu_{\mathrm{Ma}}(\cdot, \cdot)$ and the fact that $\mathcal{D}(\Sigma)$ is convex.

Lemma 2.7. *Let $H, F \in \mathcal{D}(\Sigma)$. Let y denote a periodic orbit of X_H contained in Σ, and let z denote the periodic orbit of X_F corresponding to y. Then*

$$\mu_{\mathrm{CZ}}^\tau(y) = \mu_{\mathrm{CZ}}^\tau(z).$$

Assume now that $H \in \mathcal{D}(\Sigma)$ and y is a periodic orbit of X_H contained in Σ admitting an orbit cylinder $\mathcal{O} = (y_s)_{s \in (-\varepsilon, \varepsilon)}$. Associated to such an orbit cylinder $\mathcal{O}$ is the map

$$\xi(t) := \frac{\partial}{\partial s}\Big|_{s=0} y_s(t).$$

By differentiating the equation $H(y_s) \equiv s$ we see that

$$d_{y(t)}H(\xi(t)) \equiv 1.$$

The tangent space $T_{y(t)}\mathcal{O}$ is spanned by $X_H(y(t))$ and $\xi(t)$, and forms a symplectic vector subspace of $T_{y(t)}T^*M$. Moreover by differentiating the equation

$$\phi_{T(s)}^H(y_s(0)) = y_s(T(s)) = y_s(0)$$

with respect to s and setting $s = 0$ we obtain

$$\xi(T) + T'(0)X_H(y(T)) = \xi(0). \tag{2.10}$$

Also, by differentiating the equation

$$\phi_t^H(y_s(0)) = y_s(t)$$

with respect to s and setting $s = 0$ we obtain

$$d_{y(0)}\phi_t^H(\xi(0)) = \xi(t). \tag{2.11}$$

Let us denote by $E_{y(t)} := (T_{y(t)}\mathcal{O})^{\perp}$ the ω-orthogonal complement of $T_{y(t)}\mathcal{O}$ in $T_{y(t)}T^*M$. We denote by $y^*T\mathcal{O}$ and y^*E the induced bundles over $\mathbb{R}/T\mathbb{Z}$. The following lemma is clear.

Lemma 2.8. *There is a natural isomorphism of symplectic vector bundles over* $\mathbb{R}/T\mathbb{Z}$*:*

$$y^*\mathcal{Q} \cong y^*E.$$

Define a trivialization $\Phi : \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^2 \to y^*T\mathcal{O}$ by

$$\Phi(t)(0,1) := X_H(y(t)), \quad \Phi(t)(1,0) := \xi(t) + \frac{tT'(0)}{T} X_H(y(t))$$

(equation (2.10) shows that this is well defined, i.e. $\Phi(T) = \Phi(0)$). Using (2.11) and the fact that $d_{y(0)}\phi_t^H(X_H(y(0))) = X_H(y(t))$, we deduce that the curve $\lambda_{y,\Phi} : [0,T] \to \mathrm{Sp}(2)$ defined by

$$\lambda_{y,\Phi}(t) := \Phi(t)^{-1} \circ d_{y(0)}\phi_t^H|_{T\mathcal{O}} \circ \Phi(0)$$

is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ -\frac{tT'(0)}{T} & 1 \end{pmatrix}.$$

By a well-known computation (see for instance [4, Lemma 4.3]):

$$\mu_{\mathrm{Ma}}(\mathrm{graph}(\lambda_{y,\Phi}), \Delta_{\mathbb{R}^2}) = \begin{cases} \frac{1}{2}\mathrm{sign}(-T'(0)) & T'(0) \neq 0 \\ 0 & T'(0) = 0. \end{cases} \tag{2.12}$$

We can identify $\mathcal{V}$-preserving trivializations of $y^*\mathcal{Q}$ with trivializations $\Psi : \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^{2n-2} \to y^*E$ with the property that

$$\Psi(t)(V^{n-1}) = E_{y(t)} \cap ((T_{y(t)}\Sigma \cap T^v_{y(t)}T^*M) \oplus \mathcal{L}_{y(t)}).$$

Thus if Ψ is a such a trivialization and $\lambda_{y,\Psi} : [0,T] \to \mathrm{Sp}(2n-2)$ denotes the curve defined by

$$\lambda_{y,\Psi}(t) := \Psi(t)^{-1} \circ d_{y(0)}\phi_t^H|_{E_{y(t)}} \circ \Psi(0).$$

then

$$\mu^{\tau}_{\mathrm{CZ}}(y) = \mu_{\mathrm{Ma}}(\mathrm{graph}(\lambda_{y,\Psi}), \Delta_{\mathbb{R}^{2n-2}}).$$

We can glue the two trivializations Φ and Ψ together to get a trivialization $\Phi \oplus \Psi$ of the full pullback bundle y^*TT^*M. This trivialization has the property that

$$(\Phi \oplus \Psi)(t)(V^n) = (T_{y(t)}\Sigma \cap T^v_{y(t)}T^*M) \oplus \mathcal{L}_{y(t)}. \tag{2.13}$$

As before let $\lambda_{y,\Phi\oplus\Psi} : [0,T] \to \mathrm{Sp}(2n)$ denote the curve defined by

$$\lambda_{y,\Phi\oplus\Psi}(t) := (\Phi \oplus \Psi)(t)^{-1} \circ d_{y(0)}\phi_t^H \circ (\Phi \oplus \Psi)(0).$$

Let us denote by

$$\widetilde{\mu}_{\mathrm{CZ}}(y) := \mu_{\mathrm{Ma}}(\mathrm{graph}(\lambda_{\Phi\oplus\Psi}), \Delta_{\mathbb{R}^{2n}}) \in \frac{1}{2}\mathbb{Z}.$$

This half-integer is independent of the trivialization $\Phi \oplus \Psi$ in the sense that if Θ is any other trivialization of y^*TT^*M satisfying (2.13) then

$$\mu_{\mathrm{Ma}}(\mathrm{graph}(\lambda_{\Theta}), \Delta_{\mathbb{R}^{2n}}) = \mu_{\mathrm{Ma}}(\mathrm{graph}(\lambda_{\Phi\oplus\Psi}), \Delta_{\mathbb{R}^{2n}}).$$

Moreover we have by (2.12) and the product axiom of $\mu_{\mathrm{Ma}}(\cdot,\cdot)$ ([14, Theorem 4.1]) that the following holds.

Lemma 2.9. *Let y be a periodic orbit of X_H contained in Σ admitting an orbit cylinder. Then*

$$\widetilde{\mu}_{\mathrm{CZ}}(y) = \mu^{\tau}_{\mathrm{CZ}}(y) + \frac{1}{2}\mathrm{sign}(-T'(0)).$$

In the statement of the lemma, if $T'(0) = 0$ we understand that $\mathrm{sign}(-T'(0)) = 0$.

We now relate $\widetilde{\mu}_{\mathrm{CZ}}(y)$ with the **Conley-Zehnder index** $\mu_{\mathrm{CZ}}(y)$. By definition

$$\mu_{\mathrm{CZ}}(y) = \mu_{\mathrm{Ma}}(\mathrm{graph}(\lambda_\Omega), \Delta_{\mathbb{R}^{2n}})$$

where $\Omega : \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^{2n} \to y^*TT^*M$ is any **vertical preserving trivialization**, that is for all $t \in \mathbb{R}/T\mathbb{Z}$,

$$\Omega(t)(V^n) = T^v_{y(t)}T^*M, \tag{2.14}$$

and $\lambda_\Omega : [0,T] \to \mathrm{Sp}(2n)$ is defined analogously to before. As before $\mu_{\mathrm{CZ}}(y)$ is independent of the vertical preserving trivialization Ω.

Let us denote by $m : \pi_1(\mathrm{Sp}(2n)) \to \mathbb{Z}$ is the classical Maslov index. See for instance [9, Theorem 2.29] for the definition of m. The loop axion of the Conley-Zehnder index (see [16, Section 2.4]) together with the product axiom of m implies that the difference of the indices is given by

$$\widetilde{\mu}_{\mathrm{CZ}}(y) - \mu_{\mathrm{CZ}}(y) = 2m(\Theta^{-1} \circ \Omega),$$

where Θ (resp. Ω) is any trivialization of y^*TT^*M satisfying (2.13) (resp. (2.14)). This integer visibly depends only on Σ and the free homotopy class of y; in fact the function $[y] \mapsto 2m(\Theta^{-1} \circ \Omega)$ defines a class $c(\Sigma) \in H^1(\Sigma, \mathbb{Z})$.

Corollary 2.10. *Suppose $H \in \mathcal{D}(\Sigma)$ and y is a periodic orbit of X_H contained in Σ which admits an orbit cylinder. Then*

$$\mu_{\mathrm{CZ}}(y) - \frac{1}{2}\mathrm{sign}(-T'(0)) = \mu^{\tau}_{\mathrm{CZ}}(y) - c(\Sigma)([y]).$$

The right-hand side is independent of the choice of $H \in \mathcal{D}(\Sigma)$ by Lemma 2.7 (and the right-hand side is defined even when y does not admit an orbit cylinder), and hence so is the left-hand side.

In the case we will be most interested in $c(\Sigma)$ is always zero.

Lemma 2.11. *Assume that Σ is transverse to each fibre T^*_qM along a closed curve $y : S^1 \to \Sigma$. Then $c(\Sigma)([y]) = 0$.*

Proof. By [14, Remark 5.3] one has

$$2m(\Theta^{-1} \circ \Omega) = \mu_{\mathrm{Ma}}(\Theta^{-1} \circ \Omega(V^n), V^n).$$

If Σ is transverse to T^vT^*M then the path $\Theta(t)^{-1} \circ \Omega(t)$ takes values in $\mathrm{Sp}_{n-1}(2n) \subseteq \mathrm{Sp}(2n)$, that is,

$$\dim \Theta(t)^{-1} \circ \Omega(t)(V^n) \cap V^n = n - 1 \quad \text{for all } t \in [0,1].$$

Thus by the zero axiom of $\mu_{\mathrm{Ma}}(\cdot,\cdot)$ (see [14, Theorem 4.1]) $\mu_{\mathrm{Ma}}(\Theta^{-1} \circ \Omega(V^n), V^n) = 0$. ☐

Let us now revert back to the case studied in this note, where $\Sigma = H^{-1}(k)$ for H a Tonelli Hamiltonian. The previous corollary proves that the grading μ_{Rab} from Definition 1.7 depends only on Σ and not on H. In fact in this case $c(\Sigma) = 0$ always. The points $(q,p) \in \Sigma$ where Σ is not transverse to T^*_qM have the form $(q, \mathfrak{L}(q,0))$ (where $\mathfrak{L}$ is the Legendre transform, see (1.4)) and they form a submanifold Σ_0 of Σ of codimension n if not empty. For $n \geq 2$, any closed curve $y : S^1 \to \Sigma$ can be slightly deformed in Σ to a curve that misses Σ_0 and hence from Lemma

2.11 we conclude that $c(\Sigma)([y]) = 0$. Thus for the class of systems considered in this paper, if $(x,\eta) \in \mathrm{Crit}(\mathcal{A}_k)$ with $\eta > 0$, and $y(t) := x(t/\eta)$ then:

$$\mu_{\mathrm{Rab}}(x,\eta) = \mu_{\mathrm{CZ}}(y) - \frac{1}{2}\mathrm{sign}(-T'(0)) = \mu^{\tau}_{\mathrm{CZ}}(y).$$

3. An example

In this section we construct an example of a Hamiltonian H on T^*S^2 with a non-degenerate periodic orbit y of X_H with energy $k > c$ for which the correction term $\chi(y) = -1$. Let (r,θ) denote polar coordinates on $\mathbb{R}^2$, and let $D := \{(r,\theta) : r \in (0,4)\}$. Let $f : (0,\infty) \to \mathbb{R}$ denote a smooth function such that $f \equiv 0$ on $(0,1]$ and $[3,\infty)$, which will be specified precisely later. Define $L : TD \to \mathbb{R}$ by

$$L(r,\theta,\dot{r},\dot{\theta}) = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - f(r)\dot{\theta}.$$

Let $E : TD \to \mathbb{R}$ be defined by $E(r,\theta,\dot{r},\dot{\theta}) = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$.

L is the Lagrangian associated to the flat metric and the exact magnetic form $\sigma = d(-f d\theta) = -f'(r)dr \wedge d\theta$. Let σ_{area} denote the area form, so $\sigma_{\mathrm{area}} = r dr \wedge d\theta$. Then $\sigma = -\frac{f'(r)}{r}\sigma_{\mathrm{area}}$. Let $F : D \to \mathbb{R}$ be defined by $F(r,\theta) := -f'(r)/r$. Let $\mathtt{i} : TD \to TD$ denote rotation by $+\pi/2$.

Suppose $\gamma(t) = (r(t),\theta(t))$. Then γ satisfies the Euler-Lagrange equations for L if and only if

$$\ddot{r} = \dot{\theta}(r\dot{\theta} - f'(r)), \quad r^2\dot{\theta} - f(r) = \text{const}.$$

Suppose $\gamma : \mathbb{R}/T\mathbb{Z} \to D$ is a curve such that $(\gamma,\dot{\gamma})$ is a closed orbit of the Euler-Lagrange flow of L contained in $E^{-1}(k)$. Let $(\gamma_s : \mathbb{R}/T\mathbb{Z} \to D)_{s\in(-\varepsilon,\varepsilon)}$ denote a variation along γ such that $(\gamma_s,\dot{\gamma}_s) \subseteq E^{-1}(k)$ for each $s \in (-\varepsilon,\varepsilon)$, and denote by $\zeta = \frac{\partial}{\partial s}\big|_{s=0}\gamma_s$ the associated Jacobi field. Since $\{\dot{\gamma}(t), \mathtt{i}\dot{\gamma}(t)\}$ constitutes an orthogonal frame along γ, there exist unique functions $x,y : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}$ such that

$$\zeta(t) = x(t)\dot{\gamma}(t) + y(t)\mathtt{i}\dot{\gamma}.$$

Set

$$K(\sigma) := \langle \nabla F, \mathtt{i}\dot{\gamma}\rangle + F^2.$$

Then the functions x and y satisfy the **Jacobi equations**

$$\dot{x} + Fy = 0.$$

$$\ddot{y} + K(\sigma)y = 0$$

(see for instance [13, p495]). Provided $K(\sigma) > 0$, we claim that if

$$\frac{2\pi}{T\sqrt{K(\sigma)}} \notin \mathbb{Q}$$

then the orbit $(\gamma,\dot{\gamma})$ is weakly non-degenerate, that is, the space of periodic Jacobi fields (with initial conditions tangent to the energy level) along γ is 1-dimensional, spanned by $\dot{\gamma}$. Indeed, if a Jacobi field $\zeta = x\dot{\gamma} + y\mathtt{i}\dot{\gamma}$ existed with y not identically zero then the period of y must be commensurable with T, which is equivalent to asking that $\frac{2\pi}{T\sqrt{K(\sigma)}} \in \mathbb{Q}$. (This latter condition will imply in fact that all iterates of γ are non-degenerate.)

Now let us suppose that

$$\gamma_k(t) = (r_k(t),\theta_k(t)) : \mathbb{R}/T(k)\mathbb{Z} \to D$$

is a loop in D such that $(\gamma_k,\dot{\gamma}_k)$ is an orbit of the Euler-Lagrange flow of L contained in $E^{-1}(k)$. Let us suppose moreover that

$$r_k(t) \equiv \rho(k), \quad \theta_k(t) = a(k)t$$

for some constants $a(k) > 0$ and $\rho(k) \in (0,4)$.

For such a curve γ_k to be an orbit we need $\rho(k)a(k) - f'(\rho(k)) = 0$, and in order to have energy k we need $\rho(k)^2 a(k)^2 = 2k$. Thus

$$\sqrt{2k} = \rho(k)a(k) = f'(\rho(k)). \tag{3.1}$$

Thus

$$T(k) = \frac{2\pi}{a(k)} = \frac{2\pi\rho(k)}{\sqrt{2k}} = \frac{2\pi\rho(k)}{f'(\rho(k))}.$$

With this choice of γ_k we have $\dot{\gamma}_k = (0, a(k))$ and $|\dot{\gamma}_k| = \rho(k)a(k)$. Thus $\mathtt{i}\dot{\gamma}_k = (-\rho(k)a(k), 0)$, and thus

$$K(\sigma) = -\rho(k)a(k)F'(\rho(k)) + F(\rho(k))^2.$$

Substituting $F = -f'(r)/r$ and simplifying we obtain

$$K(\sigma) = a(k)f''(\rho(k)).$$

Let us now focus on $k = 1/2$ and choose as our initial condition to define γ that

$$a(1/2) = 1/2.$$

Then (3.1) implies that

$$\rho(1/2) = 2, \quad f'(\rho(1/2)) = f'(2) = 1.$$

Then $\sqrt{K(\sigma)} = \sqrt{\frac{f''(2)}{2}}$, and thus provided $f''(2) > 0$, we see that in order for $(\gamma, \dot{\gamma})$ to be weakly non-degenerate suffices to have

$$\frac{2\pi}{T(k)\sqrt{K(\sigma)}} = \frac{2\pi}{4\pi\sqrt{\frac{f''(2)}{2}}} = \frac{1}{\sqrt{2f''(2)}} \notin \mathbb{Q}.$$

If in addition $T'(1/2) \neq 0$, then the orbit will also be strongly non-degenerate. Since $T(k) = \frac{2\pi\rho(k)}{f'(\rho(k))}$, we compute that

$$T'(k) = \frac{f'(\rho(k))2\pi\rho'(k) - 2\pi\rho(k)f''(\rho(k))\rho'(k)}{f'(\rho(k))^2},$$

and hence

$$T'(1/2) = 2\pi\rho'(1/2)(1 - 2f''(2)).$$

Note also from (3.1) we have

$$\frac{1}{\sqrt{k}} = f''(\rho(k))\rho'(k),$$

and hence if $0 < f''(2) < 1/2$ then we have $\rho'(1/2) > 0$, and hence also $T'(1/2) > 0$.

Since we have chosen f such that f vanishes outside $\{1 < r < 3\}$, we can embed D as a subset of S^2 and find a 1-form ψ on S^2 such that ψ coincides with $-f d\theta$ on D and vanishes on $S^2 \backslash D$. Let g denote a metric on S^2 that restricts to D to define the standard flat metric, and let $H : T^*S^2 \to \mathbb{R}$ be defined by $H(q,p) = \frac{1}{2}|p|_g^2$. Set $\Sigma_k := H^{-1}(k)$. The Mañé critical value $c = c_0$ of H can be estimated by

$$c \leq \sup_{q \in S^2} \frac{1}{2}|\psi_q|^2 = \sup_{r \in (1,3)} \frac{1}{2}\frac{|f(r)|^2}{r^2} \leq \frac{1}{2}\sup_{r \in (1,3)} |f(r)|^2$$

(see equations (1.8) and (1.9)). Putting this together, suppose we choose our function $f : (0,\infty) \to \mathbb{R}$ such that:

(1) $f \equiv 0$ on $(0,\infty)\backslash(1,3)$ and $0 \leq f(r) < 1$ for all $r \in (0,\infty)$;
(2) $f'(2) = 1$;

(3) $0 < f''(2) < 1/2$ and $1/\sqrt{2f''(2)} \notin \mathbb{Q}$.

Such functions f clearly exist. Let $\omega := dp \wedge dq + \pi^*(d\psi)$, and write as usual X_H for the symplectic gradient of H with respect to ω. With this choice of f, we have shown that:

(1) $c < 1/2$.

(2) There exists a closed non-degenerate orbit $y : \mathbb{R}/T\mathbb{Z} \to \Sigma_{1/2}$ of X_H with $\chi(y) \overset{\text{def}}{=} \text{sign}(-T'(1/2)) < 0$.

Finally if we consider a new orbit α with initial condition $a(1/2) = 2/5$ then a similar computation shows that if in addition we ask that $f'(5/2) = 1$, $f''(5/2) > 2/5$ and $\sqrt{2/5f''(5/2)} \notin \mathbb{Q}$ then the corresponding orbit z of X_H has $\chi(z) > 0$.

REFERENCES

1. A. Abbondandolo, A. Portaluri, and M. Schwarz, *The homology of path spaces and Floer homology with conormal boundary conditions*, J. Fixed Point Theory Appl. **4** (2008), no. 2, 263–293.
2. A. Abbondandolo and M. Schwarz, *On the Floer homology of cotangent bundles*, Comm. Pure Appl. Math. **59** (2006), 254–316.
3. ______, *Estimates and computations in Rabinowitz-Floer homology*, Jour. Topology Analysis **1** (2009), no. 4, 307–405.
4. K. Cieliebak and U. Frauenfelder, *A Floer homology for exact contact embeddings*, Pacific J. Math. **239** (2009), no. 2, 216–251.
5. K. Cieliebak, U. Frauenfelder, and G. P. Paternain, *Symplectic topology of Mañé's critical value*, Geometry and Topology **14** (2010), 1765–1870.
6. J. J. Duistermaat, *On the Morse Index in Variational Calculus*, Adv. Math. **21** (1976), 173–195.
7. U. Frauenfelder, *The Arnold-Givental conjecture and moment Floer homology*, Int. Math. Res. Not. **42** (2004), 2179–2269.
8. H. Hofer and E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser Verlag, 1994.
9. D. McDuff and D. Salamon, *Introduction to symplectic topology*, 2 ed., Oxford Mathematical Monographs, Oxford University Press, 1998.
10. W. J. Merry, *Closed orbits of a charge in a weakly exact magnetic field*, Pacific J. Math. **247** (2010), 189–212.
11. ______, *Lagrangian Rabinowitz Floer homology of twisted cotangent bundles*, arXiv:1010.4190 (2010).
12. ______, *On the Rabinowitz Floer homology of twisted cotangent bundles*, arXiv:1002.0162 (2010).
13. G. P. Paternain and M. Paternain, *Critical values of autonomous Lagrangians*, Comment. Math. Helv. **72** (1997), 481–499.
14. J. Robbin and D. Salamon, *The Maslov index for paths*, Topology **32** (1993), 827–844.
15. ______, *The spectral flow and the Maslov index*, Bull. London Math. Soc. **27** (1995), 1–33.
16. D. Salamon, *Lectures on Floer Homology*, Symplectic Geometry and Topology (Y. Eliashberg and L. Traynor, eds.), IAS/Park City Math. Series, vol. 7, Amer. Math. Soc., 1999, pp. 143–225.
17. D. Salamon and E. Zehnder, *Morse Theory for Periodic Solutions of Hamiltonian Systems and the Maslov Index*, Comm. Pure Appl. Math. **45** (1992), 1303–1360.
18. J. Weber, *Perturbed closed geodesics are periodic orbits: Index and transversality*, Math. Z. **241** (2002), 45–81.

DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS, UNIVERSITY OF CAMBRIDGE, CAMBRIDGE CB3 0WB, UK

E-mail address: w.merry@dpmms.cam.ac.uk, g.p.paternain@dpmms.cam.ac.uk