Covariant Dirac Operators on Quantum Groups

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Abstract

We give a construction of a Dirac operator on a quantum group based on any simple Lie algebra of classical type. The Dirac operator is an element in the vector space $\mathrm{cl}_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, where the first tensor factor is a deformation of the classical Clifford algebra. The tensor space $\mathrm{cl}_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is given a structure of the adjoint module of the quantum group and the Dirac operator is invariant under this action. This work generalizes the operator introduced by Bibikov and Kulish in [1].

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Introduction

The Dirac operator on a semisimple Lie group G is an element in the noncommutative Weyl algebra $\mathrm{cl}(\mathfrak{g}) \otimes U(\mathfrak{g})$, where $U(\mathfrak{g})$ is the enveloping algebra for the Lie algebra of G. The vector space $\mathfrak{g}$ generates a Clifford algebra $\mathrm{cl}(\mathfrak{g})$ whose structure is determined by the Killing form of $\mathfrak{g}$. Since $\mathfrak{g}$ acts on itself by the adjoint action, $\mathrm{cl}(\mathfrak{g}) \otimes U(\mathfrak{g})$ is a $\mathfrak{g}$-module. The Dirac operator on G spans a one dimensional invariant submodule of $\mathrm{cl}(\mathfrak{g}) \otimes U(\mathfrak{g})$ which is of the first order in $\mathrm{cl}(\mathfrak{g})$. Kostant's Dirac operator [13] has an additional cubical term in $\mathrm{cl}(\mathfrak{g})$ which is constant in $U(\mathfrak{g})$.

The noncommutative Weyl algebra acts on a Hilbert space which is also a $\mathfrak{g}$-module. Since the Dirac operator is the invariant subspace it follows that it commutes with the action of $\mathfrak{g}$ and hence the Dirac operator acts as a constant on each irreducible component in the representation of $\mathfrak{g}$ on the Hilbert space. The spectrum of the Dirac operator captures the metric properties of the the Riemannian manifold G [3].

In [1] Bibikov and Kulish considered the quantum group deformation of $\mathfrak{su}(2)$ and constructed a Dirac operator which is invariant under the adjoint action of the quantum group. In this approach, however, the spectrum of the operator grows exponentially as a function of the highest weight of the representation of the quantum group. This Dirac operator cannot be applied in the noncommutative differential calculus because the metric properties do not deform in a reasonable way. This led to a new approach to build a Dirac operator on a quantum group as an operator on a Hilbert space with a classical spectrum and in this case the axioms of the noncommutative geometry were fulfilled. This was first applied for $U_q(\mathfrak{su}(2))$ in [2, 4]. More generally a geometric Dirac operator was constructed using the Drindfeld's twist in [15] where it was defined for any compact quantum group.

The purpose of this paper is to study further the approach of the reference [1]. We are looking for an operator in the vector space $\mathrm{cl}_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, where $\mathrm{cl}_q(\mathfrak{g})$ is a trivial q-deformation of the Clifford algebra which transforms covariantly under the action of the quantum group. Furthermore, we postulate the following defining principles for the covariant Dirac operator D:

1. D transforms covariantly in the one dimensional trivial module under the adjoint action of $U_q(\mathfrak{g})$.

2. D commutes with the representation of $U_q(\mathfrak{g})$.

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For the Lie algebras the property 2. is a consequence of 1. However, if we let a quantum group act on the tensor product $\mathrm{cl}_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ with its coproduct, we need to choose the module structures in $\mathrm{cl}_q(\mathfrak{g})$ and $U_q(\mathfrak{g})$ carefully to make an operator with property 1. verify 2. This is not a general fact and will be discussed in Section 2.

Even though this Dirac operator is not suitable for a spectral triple, it can be used to study topological properties of quantum groups. The covariant Dirac operator can be used in a Fredholm or Kasparov module and it carries K-theoretical and homological (in the sense of cyclic cohomology) information of a quantum group.

We consider the adjoint representation of the quantum group and define a bilinear form in this module which is invariant under the action. The braiding operator $\check{R}$ commutes with the coproduct of $U_q(\mathfrak{g})$ and it can be considered as a q-analogue of the permutation of a tensor product. We let $\check{R}$ act on a tensor product of adjoint representations and use the spectral decomposition of this action to define the q-Clifford algebra. The eigenvectors of $\check{R}$ split into two parts which can be considered as q-deformations of symmetric and antisymmetric tensor products. We identify the 'q-symmetric' tensors with their image in the bilinear form. The practical difficulty in this approach is that there is no general formulas for the spectral decompositions. However, the explicit form of the $\check{R}$-matrix is well known and so one can solve the eigenvalue problem with some mathematical software for any chosen quantum group and apply our results.

We give a constructive proof for the existence of the covariant Dirac operator with the properties 1. and 2. on any quantum group based on any complex simple Lie algebra of classical type. The deformation parameter q is supposed to be strictly positive real number. We write an explicit formula for the operator on $U_q(\mathfrak{sl}(n))$.

Conventions. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra with a set of simple roots $\Delta = \{\alpha_i : 1 \leq i \leq n\}$ and Cartan matrix a_{ij}. Let $q \neq 1$ be a complex number. The quantum group $U_q(\mathfrak{g})$ is the unital associative algebra with generators t_i, t_i^{-1}, e_i, f_i $(1 \leq i \leq n)$ subject to [6, 10]

$$[t_i, t_j] = 0, \quad t_i t_i^{-1} = 1 \quad t_i e_j t_i^{-1} = q_i^{a_{ij}} e_j, \quad t_i f_j t_i^{-1} = q_i^{-a_{ij}} f_j,$$
$$[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} e_i^s e_j e_i^{1-a_{ij}-s} = 0 = \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} f_i^s f_j f_i^{1-a_{ij}-s} \qquad (i \neq j),$$

where $q_i = q^{d_i}$, d_i's being the coprime integers such that $d_i a_{ij}$ is a symmetric matrix and the q-binomial coefficients are defined by

$$[m]_{q_i} = (q_i - q_i^{-1})(q_i^2 - q_i^{-2}) \cdots (q_i^m - q_i^{-m}), \qquad [0]_{q_i} = 1$$
$$\begin{bmatrix} m \\ n \end{bmatrix}_{q_i} = \frac{[m]_{q_i}}{[n]_{q_i}[m-n]_{q_i}}.$$

In the limit $q \to 1$ the algebra $U_q(\mathfrak{g})$ reduces to $U(\mathfrak{g})$. $U_q(\mathfrak{g})$ is a Hopf algebra with a coproduct $\triangle : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, an antipode $S : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ and a counit $\epsilon : U_q(\mathfrak{g}) \to \mathbb{C}$. The vectors t_i and t_i^{-1} are grouplike so that $\epsilon(t_i) = \epsilon(t_i^{-1}) = 1$ whereas $\epsilon(e_i) = \epsilon(f_i) = 0$. The coproduct $\triangle$ is noncocommutative and there exists a universal R-matrix such that

$$R\triangle(x)R^{-1} = \sigma\triangle(x),$$

where σ permutes the tensor product. The R-matrix is an infinite sum defined in some completion of the tensor product $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ but only a finite number of terms are nonzero in any finite dimensional representation. Here we always assume that the parameter q is a strictly positive real number. In this case the braiding operator $\check{R} = \sigma R$ is a selfadjoint operator in any finite dimensional representation. In the limit $q \to 1$, $\check{R}$ becomes the permutation operator.

If V is a module for the algebra $U_q(\mathfrak{g})$ then a weight of $v \in V$ is a linear functional $\mu \in U_q^*(t^{\pm 1}, \dots, t^{\pm n})$ defined in a dual space of the quantized Cartan subalgebra so that $x.v = q^{\mu(x)}v$ for all $x \in U_q(t^{\pm 1}, \dots, t^{\pm n})$.

As was shown in [14, 16] the theory of finite dimensional representations of semisimple Lie algebra $\mathfrak{g}$ and quantum group $U_q(\mathfrak{g})$ are identical in the case q is not a root of unity. A highest weight module L_λ^q of $U_q(\mathfrak{g})$ is finite dimensional if and only if $\lambda \in P^+$. If $\lambda \in P^+$ then the dimension of each weight space is equal to the dimension of the corresponding weight space in the highest weight module module L_λ of $U(\mathfrak{g})$. The category of representations is semisimple with simple objects L_λ^q, $\lambda \in P^+$.

The adjoint action of the quantum group on itself is an algebra homomorphism $U_q(\mathfrak{g}) \times U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined by

$$x \overset{\mathrm{ad}}{\blacktriangleright} y = x' y S(x''). \tag{1}$$

One can find a finite dimensional submodule in $U_q(\mathfrak{g})$ which is isomorphic to the adjoint representation of the quantum group [5]. Let $q = e^h$, $Z = h^{-1}(R^t R - 1)$ where $R^t = \sigma R \sigma$ and

$$Z_{lk} = (\pi_{lk} \otimes \mathrm{id}) Z \in \mathbb{C} \otimes U_q(\mathfrak{g}) \simeq U_q(\mathfrak{g}), \tag{2}$$

where π_{lk} are the matrix elements of the defining representation of $U_q(\mathfrak{g})$. Denote by π^* the dual representation

$$\pi^*_{il}(x) = \pi_{li}(S(x)).$$

The vectors Z_{lk} transform covariantly under the adjoint action

$$x \overset{\mathrm{ad}}{\blacktriangleright} Z_{lk} = Z_{ij} \pi^*_{il}(x') \pi_{jk}(x''), \qquad \text{for all} \qquad x \in U_q(\mathfrak{g}).$$

According to the termionology of [5], a (weak) quantum Lie algebra is an invariant submodule in $U_q(\mathfrak{g})$ which is a deformation of $\mathfrak{g}$ and transforms covariantly under the adjoint action. Denote by $\{u_i\}$ and $\{u_i^*\}$ the basis vectors of the defining representation and its dual and by $\{v_i\}$ the basis of the adjoint representation. Using the matrix coefficients of the module isomorphism $v_a \mapsto K_a^{ij}(u_i^* \otimes u_j)$ we define

$$Z_a = K_a^{ij} (\pi_{ij} \otimes \mathrm{id}) Z.$$

Z_a's span a quantum Lie algebra $\mathfrak{L}_q(\mathfrak{g})$ inside $U_q(\mathfrak{g})$ which is isomorphic to the adjoint representation of $U_q(\mathfrak{g})$.

1 Covariant Clifford algebra

Let $\mathfrak{g}$ be a simple Lie algebra of classical type. Denote by (U, π) and (U^*, π^*) the defining representation of $U_q(\mathfrak{g})$ and its dual. The adjoint representation is an invariant submodule $V \subset U^* \otimes U$. The action is given by

$$x \overset{\mathrm{ad}}{\triangleright} (u_l^* \otimes u_k) = \pi_{li}(S(x')) u_i^* \otimes \pi_{jk}(x'') u_j$$

for all $x \in U_q(\mathfrak{g})$ and $v = u_l^* \otimes u_k \in V$.

Proposition. There exists a nondegenerate bilinear form $B_q : V \otimes V \to \mathbb{C}$ which is invariant under the adjoint action of $U_q(\mathfrak{g})$, i.e.

$$B_q(x \overset{\mathrm{ad}}{\triangleright} (v \otimes w)) := B_q(x' \overset{\mathrm{ad}}{\triangleright} v \otimes x'' \overset{\mathrm{ad}}{\triangleright} w) = \epsilon(x) B_q(v \otimes w).$$

B_q is unique up to a multiplicative constant.

Proof. There exists a module isomorphism $f : V \to V^*$ defined as follows. The action of $\check{R}$ gives a module isomorphism $\check{R} : U^* \otimes U \to U \otimes U^*$. Composing this with the identification $U \otimes U^* \simeq (U^* \otimes U)^*$ and restricting the composition to the invariant submodule V defines the module isomorphism f.

Choose a basis $\{v_i\}$ of V and let $\{v_i^*\}$ be the dual basis. The canonical pairing defined on the generators by $\mathrm{eval}(v_j^* \otimes v_k) = v_j^*(v_k) = \delta_{jk}$ is nondegenerate. Thus, the composition

$$\begin{aligned} B_q : V \otimes V &\overset{\sim}{\to} V^* \otimes V \overset{\mathrm{eval}}{\to} \mathbb{C}, \\ v \otimes w &\mapsto \mathrm{eval}(f(v) \otimes (w)) \end{aligned}$$

is nondegenerate. B_q is invariant because

$$B_q(x' \overset{\mathrm{ad}}{\triangleright} v \otimes x'' \overset{\mathrm{ad}}{\triangleright} w) = \mathrm{eval}(x' \overset{\mathrm{ad}}{\triangleright} f(v) \otimes x'' \overset{\mathrm{ad}}{\triangleright} w) = f(v)(S(x')x'' \overset{\mathrm{ad}}{\triangleright} (w)) = \epsilon(x) B_q(v \otimes w),$$

for all $x \in U_q(\mathfrak{g})$ and $v, w \in V$.

Let ϕ be the map $V \to V^*$ which sends $v \in V$ to the functional $\phi(v)(w) = B_q(v \otimes w) \in V^*$. Using the Hopf algebra axioms we see that ϕ is a module homomorphism:

$$\begin{aligned} \phi(x \overset{\mathrm{ad}}{\triangleright} v)(w) &= \phi(x' \overset{\mathrm{ad}}{\triangleright} v)(\epsilon(x'')w) = \phi(x' \overset{\mathrm{ad}}{\triangleright} v)((x'' S(x''')) \overset{\mathrm{ad}}{\triangleright} w) \\ &= B_q(x' \overset{\mathrm{ad}}{\triangleright} v \otimes x'' \overset{\mathrm{ad}}{\triangleright} (S(x''') \overset{\mathrm{ad}}{\triangleright} w)) = \epsilon(x') B_q(v \otimes S(x'') \overset{\mathrm{ad}}{\triangleright} w) \\ &= \phi(v)(S(x) \overset{\mathrm{ad}}{\triangleright} w) \end{aligned}$$

for all $x \in U_q(\mathfrak{g})$ and $v, w \in V$. Furthermore, ϕ is a module isomorphism because B_q is nondegenerate. If θ is another module isomorphism $\theta : V \to V^*$ we can define a module isomorphism $\theta^{-1} \circ \phi : V \to V$ which commutes with the action of the quantum group. $\mathfrak{g}$ is simple and so the adjoint module V is irreducible and the uniqueness follows from the Schur's lemma. □

In practical calculations the form B_q is easiest to find by fixing the costants directly from the invariance condition.

Let $\check{R}_i = \sigma_i R_i$ ($1 \le i \le N-1$) be the linear operator where R_i is the R matrix acting on the i'th and $(i+1)$'th component in the tensor product space $V^{\otimes N}$ and σ_i permutes the tensor components. The braiding operator $\check{R}_i$ commutes with the action of $U_q(\mathfrak{g})$ on the tensor product and thus the eigenspaces of $\check{R}_i$ are invariant subspaces of $U_q(\mathfrak{g})$. $\check{R}_i$ is a selfadjoint operator and its eigenvalues are real. Furthermore, the eigenvalue of a nonzero eigenspace is not equal to zero for any $q > 0$ because $\check{R}_i$ is an isomorphism. Thus, the tensor product splits into parts consisting of the vectors with strictly positive eigenvalues and strictly negative eigenvalues for any allowed value of q. In the classical limit $q \to 1$ these eigenspaces become the symmetric and antisymmetric tensor products in the i'th and $(i+1)$'th component.

Given a spectral resolution of $\check{R}_i$ denote by $\{a_{i,k} : k \in I\}$ the negative eigenvalues and by $\{b_{i,k} : k \in J\}$ the positive eigenvalues of $\check{R}_i$. The brading operators form a generalized Hecke-algebra with relations

$$\begin{aligned} &\check{R}_i \check{R}_{i+1} \check{R}_i = \check{R}_{i+1} \check{R}_i \check{R}_{i+1} \\ &\check{R}_i \check{R}_j = \check{R}_j \check{R}_i, \\ &\prod_{k \in I} (\check{R}_i - a_{i,k}) \prod_{l \in J} (\check{R}_i - b_{i,l}) = 0 \end{aligned}$$

for all $1 \le i, j \le n-1$ and $|i-j| > 1$.

Let $T(V)$ be the tensor algebra of V. We define the covariant Clifford algebra as a projection of $T(V)$ on the q-antisymmetric tensor products by

$$\mathrm{cl}_q(\mathfrak{g}) = T(V)/\mathfrak{I}$$

where the ideal $\mathfrak{I}$ is defined by

$$\mathfrak{I} = \{(\mathrm{id} - B_q^i)v : v \in \mathrm{Ker}(\check{R}_i - b_{i,k}) \quad \text{for some} \quad i \in \mathbb{N}, k \in J\}.$$

B_q^i is the invariant pairing of i'th and $(i+1)$'th tensor component. $\mathrm{cl}_q(\mathfrak{g})$ transforms covariantly under the adjoint action of $U_q(\mathfrak{g})$ because B_q^i is invariant and the operators $\check{R}_i - b_{i,k}$ commutes with the action of $U_q(\mathfrak{g})$.

A finite dimensional classical Clifford algebra considered as an assosiative algebra is isomorphic to a finite dimensional matrix algebra with entries in $\mathbb{C}$. The obstruction to infinitesimal deformation of associative algebras is given by the condition $H^2(A, A) = 0$ in the Hochschild cohomology group [7, 8]. For any associative separable semisimple algebra A and two sided module P the cohomology groups $H^n(A, P)$ $(n > 0)$ are trivial. Especially the classical Clifford algebras are rigid algebras i.e., do not deform. The deformations $\mathrm{cl}_q(\mathfrak{g})$ are isomorphic to matrix algebras.

It follows that the irreducible representations of $\mathrm{cl}_q(\mathfrak{g})$ can be constructed as in the classical case (see, e.g. [9]). If V is $2n$-dimensional, then there exists linearly independent vectors $\hat{\psi}_1, \ldots, \hat{\psi}_{2n} \in \mathrm{cl}_q(\mathfrak{g})$ which generate $\mathrm{cl}_q(\mathfrak{g})$ as an algebra and satisfy $\{\hat{\psi}_i, \hat{\psi}_j\} = 0 = \{\hat{\psi}_{i+n}, \hat{\psi}_{j+n}\}$ and $\{\hat{\psi}_i, \hat{\psi}_{n+j}\} = \delta_{ij}$ for all $1 \le i, j \le n$. The vectors $\hat{\psi}_1, \ldots, \hat{\psi}_n$ generate a subalgebra which we denote by $\mathrm{cl}_q(\hat{\psi}_1, \ldots, \hat{\psi}_n)$. If $\mathfrak{J}$ is the smallest left ideal in $\mathrm{cl}_q(\mathfrak{g})$ containing the elements $\hat{\psi}_{n+1}, \ldots, \hat{\psi}_{2n}$ then the vector space

$$S = \mathrm{cl}_q(\hat{\psi}_1, \ldots, \hat{\psi}_n)/\mathfrak{J}$$

is an irreducible $\mathrm{cl}_q(\mathfrak{g})$-module. The vectors $\{\psi_i : 1 \le i \le n\}$ can be considered as creation operators and $\{\psi_i : n+1 \le i \le 2n\}$ as annihilation operators in a finite dimensional Fock space which satisfy the CAR algebra relations.

If V is $(2n+1)$-dimensional then one can choose $\hat{\psi}_1, \ldots, \hat{\psi}_{2n} \in \mathrm{cl}_q(\mathfrak{g})$ as above and form the Clifford module S. Now $\mathrm{cl}_q(\mathfrak{g})$ contains an additional element $\hat{\psi}_{2n+1}$ anticommuting with each $\hat{\psi}_i$ and $\hat{\psi}_{2n+1}^2 = -1$. This polarizes S into two eigenspaces with eigenvalues $\pm i$. Thus, S is an irreducible $\mathrm{cl}_q(\mathfrak{g})$-module.

Proposition. There exists a representation of $U_q(\mathfrak{g})$ on S which is compatible with the action of the Clifford algebra $\mathrm{cl}_q(\mathfrak{g})$ in the sense that

$$x'.\psi.S(x'').\Psi = (x \overset{\mathrm{ad}}{\triangleright} \psi).\Psi \tag{3}$$

for any $x \in U_q(\mathfrak{g})$, $\psi \in \mathrm{cl}_q(\mathfrak{g})$ and $\Psi \in S$.

Proof. The Clifford algebra $\mathrm{cl}(\mathfrak{g})$ is a Hopf module algebra for the envoloping algebra $U(\mathfrak{g})$ and the generators of $U(\mathfrak{g})$ act as derivations on $\mathrm{cl}(\mathfrak{g})$. The isomorphism of associative algebras $\phi : \mathrm{cl}(\mathfrak{g}) \to \mathrm{Mat}(n \times n)$ induces a Hopf module algebra structure on $\mathrm{Mat}(n \times n)$. All the derivations of $\mathrm{Mat}(n \times n)$ are matrix commutators. Thus there exists a representation χ of $U(\mathfrak{g})$ on $\mathbb{C}^n$ so that the action on $\mathrm{Mat}(n \times n) = \mathbb{C}^n \otimes (\mathbb{C}^n)^*$ is defined by

$$x \triangleright (w \otimes f) = \chi(x')w \otimes \chi^*(x'')f = \chi(x')w \otimes f \circ \chi(S(x'')), \tag{4}$$

for all $w \otimes f \in \mathrm{Mat}(n \times n)$.

The algebra $\mathrm{cl}(\mathfrak{g})$ is generated by the vectors ψ_i' which span the adjoint representation as a vector space. Let $\mathrm{Mat}^0(n \times n)$ denote the subspace of $\mathrm{Mat}(n \times n)$ spanned by $\phi(\psi_i')$'s. This is also a submodule of $U_q(\mathfrak{g})$ isomorphic to the adjoint representation. The fact that $\mathrm{cl}(\mathfrak{g})$ and $\mathrm{Mat}(n \times n)$ are equivalent Hopf module algebras implies the following: The projection of $\mathrm{Mat}^0(n \times n) \otimes \mathrm{Mat}^0(n \times n)$ on the symmetric tensor products is a module homomorpshim

$$\phi(\psi_i') \otimes \phi(\psi_j') \mapsto \phi(\psi_i')\phi(\psi_j') + \phi(\psi_j')\phi(\psi_i') = B(\psi_i', \psi_j')\mathbf{1},$$

which gets values in the trivial module $\mathbb{C}\mathbf{1}$, i.e. B is an invariant bilinear form.

Since the categories of the representations of $U(\mathfrak{g})$ and $U_q(\mathfrak{g})$ have the same morphisms, we have a $U_q(\mathfrak{g})$-module homomorphism $\phi_q : \mathrm{cl}_q(\mathfrak{g}) \to \mathrm{Mat}(n \times n)$ so that the action on the matrix algebra is the action (4) applied to the coproduct, antipode and a representation χ_q of $U_q(\mathfrak{g})$. Furhermore, if ψ_i's are the generators of $\mathrm{cl}_q(\mathfrak{g})$ the matrices $\phi_q(\psi_i)$ span a submodule $\mathrm{Mat}_q^0(n \times n)$, which is isomorphic to the adjoint representation of $U_q(\mathfrak{g})$. The projection on the q-symmetric subspace (the invariant eigenspace of $\check{R}$ with positive eigenvalues) is a module homomorphism and gets values in the trivial module $\mathbb{C}\mathbf{1}$. It follows from the homomorpshim property that the trivial module is determined by an invariant bilinear form which is unique. Thus, the map ϕ_q is a Hopf module algebra isomorphism and satisfies

$$\phi_q(x \overset{\mathrm{ad}}{\triangleright} \psi) = \chi_q(x')\phi_q(\psi)\chi_q(S(x'')).$$

The representation χ_q is given by $n \times n$ matrices and thus can be embedded into the Clifford algebra $\mathrm{cl}_q(\mathfrak{g})$ to define (3). $\square$

2 Covariant Dirac operator

Let (V, ρ) and (V^*, ρ^*) denote an adjoint representation of $U_q(\mathfrak{g})$ and its dual. We first define an invariant one dimensional subspace in the vector space $V \otimes V^*$ and then define D in the image of this subspace in a module isomorphism from $V \otimes V^*$ to $\mathrm{cl}_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$. The module structure of $\mathrm{cl}_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is chosen so that D commutes with the representation.

Proposition. Let $\{v_i\}$ and $\{v_i^*\}$ be bases of V and V^*. The vector $\widetilde{D} \in V \otimes V^*$ defined by

$$\widetilde{D} = \sum_i v_i \otimes v_i^*$$

is invariant under the action of $U_q(\mathfrak{g})$.

Proof. For all $x \in U_q(\mathfrak{g})$

$$\begin{aligned} x \overset{\mathrm{ad}}{\triangleright} \widetilde{D} &= \sum_{i,k,l} \rho_{ki}(x')v_k \otimes \rho^*_{li}(x'')v_l^* \\ &= \sum_{i,k,l} \rho_{ki}(x')\rho_{il}(S(x''))v_k \otimes v_l^* \\ &= \sum_{k,l} \rho_{lk}(x'S(x''))v_k \otimes v_l^* \\ &= \sum_{k,l} \epsilon(x)\delta_{kl}v_k \otimes v_l^* = \epsilon(x)\widetilde{D}. \qquad \square \end{aligned}$$

The representation is dependent on the choice of a Hopf structure for the quantum group. If we fix a Hopf structure, then by definition, the adjoint action on a tensor product $\psi \otimes Z \in \mathrm{cl}_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is

$$x \overset{\mathrm{ad}}{\triangleright} (\psi \otimes Z) := x' \overset{\mathrm{ad}}{\triangleright} \psi \otimes x'' \overset{\mathrm{ad}}{\blacktriangleright} Z. \tag{5}$$

Let $\mathrm{cl}_q^V(\mathfrak{g})$ be a submodule of $\mathrm{cl}_q(\mathfrak{g})$ which is isomorphic to the adjoint representation. If $\phi_1 : V \to \mathrm{cl}_q^V(\mathfrak{g})$ and $\phi_2 : V^* \to \mathfrak{L}_q(\mathfrak{g}) \subset U_q(\mathfrak{g})$ are module isomorphisms we could try to define the Dirac operator by $A = (\phi_1 \otimes \phi_2)(\widetilde{D})$. This is certainly invariant under the action (5) of the quantum group. Let us write $A = \sum_{i,j} \alpha_{ij}\psi_i \otimes Z_j$ for some complex numbers α_{ij}. Assume that the elements of $\mathrm{cl}_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ act on a vector space $H = S \otimes W$, where W is a finite dimensional $U_q(\mathfrak{g})$-module and S is an irreducible

Clifford module with compatible quantum group action. Even though A is invariant it fails to commute with the representation of $U_q(\mathfrak{g})$ on H. This can be seen using $x = x'\epsilon(x'')$ and $\epsilon(x) = S(x')x''$ twice

$$
\begin{aligned}
x.A.h = (x' \otimes x'').(\sum_{i,j} \alpha_{ij}\psi_i \otimes Z_j).h = \sum_{i,j} \alpha_{ij}(x' \overset{\mathrm{ad}}{\triangleright} \psi_i \otimes x''' \overset{\mathrm{ad}}{\blacktriangleright} Z_j).(x'' \otimes x'''').h \\
\neq \epsilon(x')(\sum_{i,j} \alpha_{ij}\psi_i \otimes Z_j).(x'' \otimes x''').h = A.x.h
\end{aligned}
$$

for some $x \in U_q(\mathfrak{g})$ and $h \in H$ because of the noncocommutativity of the coproduct. This problem can be cured by forcing the quantum group act on the modules $\mathrm{cl}_q(\mathfrak{g})$ and $U_q(\mathfrak{g})$ with opposite Hopf algebra conventions.

Let us choose the primary Hopf algebra structure for $U_q(\mathfrak{g})$ by

$$
\begin{aligned}
&\triangle(e_i) = e_i \otimes t_i + 1 \otimes e_i, \quad \triangle(f_i) = f_i \otimes 1 + t_i^{-1} \otimes f_i, \quad \triangle(t_i) = t_i \otimes t_i \\
&S(t_i) = t_i^{-1}, \quad S(e_i) = -e_i t_i^{-1}, \quad S(f_i) = -t_i f_i
\end{aligned}
$$

The Sweedler's notation $\triangle(x) = x' \otimes x''$ is always applied to this one. Denote by $\{Z_i\}$ a basis of the quantum Lie algebra $\mathfrak{L}_q(\mathfrak{g})$. The adjoint action of the quantum group on $\mathfrak{L}_q(\mathfrak{g})$ is given by (1).

The following opposite Hopf algebra can also be applied to construct a representations on tensor products and duals

$$
\begin{aligned}
&\underset{\leftarrow}{\triangle}(x) = x'' \otimes x', \\
&\underset{\leftarrow}{S}(t_i) = t_i^{-1}, \quad \underset{\leftarrow}{S}(e_i) = -t_i^{-1} e_i, \quad \underset{\leftarrow}{S}(f_i) = -f_i t_i
\end{aligned}
$$

Denote by V^{op} the adjoint representation applied to the opposite Hopf algebra conventions. The action is defined by

$$
x \overset{\mathrm{o-ad}}{\triangleright} (u_l^* \otimes u_k) = \pi_{li}(\underset{\leftarrow}{S}(x''))u_i^* \otimes \pi_{jk}(x')u_j \tag{6}
$$

for all $x \in U_q(\mathfrak{g})$ and $u_l^* \otimes u_k \in V^{\mathrm{op}} \subset U^* \otimes U$. $\underset{\leftarrow}{\check{R}} = \sigma R^t$ is a brading operator which commutes with the opposite coproducts. Denote by B_q the bilinear form invariant under the opposite adjoint representation and by $\mathrm{cl}_q(\mathfrak{g})$ the corresponding Clifford algebra which, by construction, transforms covarianly under the following action of $U_q(\mathfrak{g})$

$$
x \overset{\mathrm{o-ad}}{\triangleright} \psi_{i_1}\psi_{i_2}\cdots\psi_{i_n} = (x^{(n)} \overset{\mathrm{o-ad}}{\triangleright} \psi_{i_1})\cdots(x^{(1)} \overset{\mathrm{o-ad}}{\triangleright} \psi_{i_n}).
$$

Theorem. Let $\mathrm{cl}_q^1(\mathfrak{g}) \simeq V^{\mathrm{op}}$ denote the embedding of V^{op} in $\mathrm{cl}_q(\mathfrak{g})$. Define

$$
D = (\phi_1 \otimes \phi_2)(\widetilde{D}) \in \mathrm{cl}_q(\mathfrak{g}) \otimes \mathfrak{L}_q(\mathfrak{g})
$$

where $\phi_1 : V \to \mathrm{cl}_q^1(\mathfrak{g})$ and $\phi_2 : V^* \to \mathfrak{L}_q(\mathfrak{g})$ are module isomorphisms. D spans a one dimensional invariant subspace in $\mathrm{cl}_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ and the representation of $U_q(\mathfrak{g})$ and the action of D on the vecor space $H = S \otimes W$ commute if the actions of $U_q(\mathfrak{g})$ and $\mathrm{cl}_q(\mathfrak{g})$ on S are compatible.

Proof. D clearly spans a one dimensional invariant subspace because $\widetilde{D}$ does. Let us write $D = \sum_{i,j} \alpha_{ij}\psi_i \otimes Z_j$. Let $\Psi \otimes w \in S \otimes W$ and $x \in U_q(\mathfrak{g})$. Using the Hopf algebra properties

$$
\epsilon(x) = S(x')x'' = \underset{\leftarrow}{S}(x'')x'
$$

we find that

$$\begin{aligned}
xD.(\Psi\otimes w) &= x(\sum_{i,j}\alpha_{ij}\psi_i\otimes Z_j).(\Psi\otimes w)\\
&=(\sum_{i,j}\alpha_{ij}\epsilon(x')x''\psi_i\otimes x'''\epsilon(x'''')Z_j).(\Psi\otimes w)\\
&=(\sum_{i,j}\alpha_{ij}x^{(3)}\psi_i\underleftarrow{S}(x^{(2)})x^{(1)}\otimes x^{(4)}Z_jS(x^{(5)})x^{(6)}).(\Psi\otimes w)\\
&=(\sum_{i,j}\alpha_{ij}(x''\overset{\text{o-ad}}{\triangleright}\psi_i)\otimes(x'''\overset{\text{ad}}{\blacktriangleright}Z_j))(x'\otimes x'''').(\Psi\otimes w)\\
&=\epsilon(x'')D(x'\otimes x''').(\Psi\otimes w)=Dx.(\Psi\otimes w).\qquad\square
\end{aligned}$$

We can also add a cubical term in $\mathrm{cl}_q(\mathfrak{g})$ to the Dirac operator. The tensor product $V^{\mathrm{op}}\otimes V^{\mathrm{op}}$ contains a submodule isomorphic to the adjoint representation which lies in the q-antisymmetric part and thus is mapped nontrivially into the algebra $\mathrm{cl}_q(\mathfrak{g})$. Let $\mathfrak{A}\subset\mathrm{cl}_q(\mathfrak{g})$ denote this submodule and $\phi_3: V^*\to\mathfrak{A}$ a module isomorphism. The cubical part of the Dirac operator is defined by

$$D_c = m(\phi_1\otimes\phi_3)(\widetilde{D})\otimes 1,$$

where m is the product of $\mathrm{cl}_q(\mathfrak{g})$. D_c is invariant and commutes with the representation.

D spans an invariant subspace in $\mathrm{cl}_q(\mathfrak{g})\otimes U_q(\mathfrak{g})$ which is a q-deformation of the noncommutative Weyl algebra $\mathrm{cl}(\mathfrak{g})\otimes U(\mathfrak{g})$. In the limit $q\to 1$ we get the classical Dirac operator on a Lie group with a suitable choice of the coefficients.

Example: $U_q(\mathfrak{sl(n)})$**.** The defining representation of $U_q(\mathfrak{sl(n)})$ on $U=\mathbb{C}^n=\bigoplus_{i=1}^n\mathbb{C}u_i$ can be written by

$$e_i.u_k=\delta_{k,i+1}u_{k-1},\qquad f_i.u_k=\delta_{k,i}u_{k+1},\qquad t_i.u_k=q^{\delta_{k,i}-\delta_{k,i+1}}u_k.$$

Let $\Phi=\{\xi_i-\xi_j : i\neq j; 1\leq i,j\leq n\}$ be the set of roots of the Lie algebra $\mathfrak{sl(n)}$. The adjoint representation of $U_q(\mathfrak{sl(n)})$ is the invariant subspace of $U^*\otimes U$ with the highest weight $\xi_1-\xi_n\in\Phi$. The highest weight vector is $u_n^*\otimes u_1$.

Let us choose two sets of basis vectors for $U^*\otimes U$:

$$\begin{aligned}
&u^+_{\xi_k-\xi_l}=u^-_{\xi_k-\xi_l}=u_l^*\otimes u_k,\qquad 1\leq k\neq l\leq n\\
&u^+_{0,k}=q^{-1}(u_k\otimes u_k^*-u_{k+1}\otimes u^*_{k+1})\\
&u^-_{0,k}=u_k\otimes u_k^*-u_{k+1}\otimes u^*_{k+1},\qquad 1\leq k\leq n-1.
\end{aligned}$$

We consider the (n^2-1)-dimensional vector space spanned by $\{u^+_{\xi_k-\xi_l},u^+_{0,k}\}$ as a module where the quantum group acts with the primary Hopf algebra conventions. Let us denote this module by V. Similarly, $\{u^-_{\xi_k-\xi_l},u^-_{0,k}\}$ is a module (V^{op}) where the action is given by the opposite Hopf algebra. The quantum group acts on these vectors as follows

$$\begin{aligned}
&e_i\triangleright u^\pm_{\xi_k-\xi_l}=\delta_{k,i+1}u^\pm_{\xi_{k-1}-\xi_l}-q^{\pm1}\delta_{i,l}u^\pm_{\xi_k-\xi_{l+1}},\qquad l>k\quad\text{or}\quad k>l+1\\
&e_i\triangleright u^\pm_{\xi_{k+1}-\xi_k}=q^{\pm1}\delta_{i,k}u^\pm_{0,k}\\
&e_i\triangleright u^\pm_{0,k}=q^{\mp1}\delta_{k,i+1}u^\pm_{\xi_{k-1}-\xi_k}-(q+q^{-1})\delta_{i,k}u^\pm_{\xi_k-\xi_{k+1}}+q^{\pm1}\delta_{k,i-1}u^\pm_{\xi_{k+1}-\xi_{k+2}}\\
&f_i\triangleright u^\pm_{\xi_k-\xi_l}=\delta_{i,k}u^\pm_{\xi_{k+1}-\xi_l}-q^{\mp}\delta_{l,i+1}u^\pm_{\xi_k-\xi_{l-1}},\qquad k>l\quad\text{or}\quad l>k+1\\
&f_i\triangleright u^\pm_{\xi_k-\xi_{k+1}}=-\delta_{i,k}u^\pm_{0,k}\\
&f_i\triangleright u^\pm_{0,k}=-q^{\mp2}\delta_{k,i+1}u^\pm_{\xi_k-\xi_{k-1}}+(1+q^{\mp2})\delta_{i,k}u^\pm_{\xi_{k+1}-\xi_k}-\delta_{k,i-1}u^\pm_{\xi_{k+2}-\xi_{k+1}}\\
&t_i\triangleright u^\pm_{\xi_k-\xi_l}=q^{\langle\xi_k-\xi_l,\alpha_i\rangle}u^\pm_{\xi_k-\xi_l},\qquad t_i.u^\pm_{0,k}=u^\pm_{0,k},
\end{aligned}\tag{7}$$

where $\triangleright$ means $\overset{\mathrm{ad}}{\triangleright}$ or $\overset{\mathrm{o-ad}}{\triangleright}$. The dual module V^* of the module V is given by

$$
\begin{aligned}
&e_i.u^*_{\xi_k-\xi_l} = -q\delta_{i,k}u^*_{\xi_{k+1}-\xi_l} + q^2\delta_{i,l-1}u^*_{\xi_k-\xi_{l-1}}, \qquad k>l \quad \text{or} \quad l>k+1\\
&e^*_i.u^*_{\xi_k-\xi_{k+1}} = \delta_{i,k}(-qu^*_{0,k-1} + (q+q^{-1})u^*_{0,k} - q^{-1}u^*_{0,k+1})\\
&e^*_i.u^*_{0,k} = -q^3\delta_{i,k}u^*_{\xi_{k+1}-\xi_k}\\
&f^*_i.u^*_{\xi_k-\xi_l} = -q^{-1}\delta_{i,k-1}u^*_{\xi_{k-1}-\xi_l} + q^{-2}\delta_{i,l}u^*_{\xi_k-\xi_{l+1}}, \qquad l>k \quad \text{or} \quad k>l+1\\
&f^*_i.u^*_{\xi_{k+1}-\xi_k} = q^{-2}\delta_{i,k}(u^*_{0,k-1} - (1+q^{-2})u^*_{0,k} + q^{-2}u^*_{0,k+1})\\
&f^*_i.u^*_{0,k} = \delta_{i,k}u^*_{\xi_k-\xi_{k+1}}\\
&t^*_i.u^*_{\xi_k-\xi_l} = q^{-\langle\xi_k-\xi_l,\alpha_i\rangle}u^*_{\xi_k-\xi_l}, \qquad t^*_i.u^*_{0,k} = u^*_{0,k}.
\end{aligned}
$$

Let $\mathrm{cl}_q(\mathfrak{sl}(\mathfrak{n}))$ denote the Clifford algebra transforming covariantly under the opposite adjoint representation. The module $\mathrm{cl}^1_q(\mathfrak{sl}(\mathfrak{n}))$ is spanned by the vectors $\{\psi_{\xi_k-\xi_l}, \psi_{0,k}\}$ in $\mathrm{cl}_q(\mathfrak{sl}(\mathfrak{n}))$, where the vectors $\psi_{\xi_k-\xi_l}$ and $\psi_{0,k}$ are the images of $u^-_{\xi_k-\xi_l}$ and $u^-_{0,k}$ in the embedding of $V^{\mathrm{op}} \to \mathrm{cl}_q(\mathfrak{sl}(\mathfrak{n}))$.

Let us choose a basis for $\mathfrak{L}_q(\mathfrak{sl}(\mathfrak{n}))$ by

$$
\begin{aligned}
&T_{\xi_k-\xi_l} = Z_{lk}, \qquad 1 \le k \ne l \le n\\
&T_{0,k} = q^{-1}(Z_{kk} - Z_{k+1,k+1}),
\end{aligned}
$$

where $Z_{ij} \in U_q(\mathfrak{sl}(\mathfrak{n}))$ are given by (2). These vectors transforms in the adjoint action (1) precisely as the vectors $\{u^+_{\xi_k-\xi_l}, u^+_{0,k}\}$ in (7). To write down explicite formulas for these vectors one can use the R-matrix from [12].

Next we need module isomorphisms $\phi_1 : V \to \mathrm{cl}^1_q(\mathfrak{sl}(\mathfrak{n}))$ and $\phi_2 : V^* \to \mathfrak{L}_q(\mathfrak{sl}(\mathfrak{n}))$. The first isomorpshism is the diagonal map

$$\phi_1(u^+_{\xi_k-\xi_l}) = \alpha_{k,l}\psi_{\xi_k-\xi_l}, \qquad \phi_1(u^+_{0,k}) = \alpha^k_0\psi_{0,k}$$

where the coefficients are defined, up to a multiplicative constant, by

$$\alpha_{k,l} = \alpha_{k+1,l}, \qquad \alpha_{k,l} = q^2\alpha_{k,l+1}, \qquad \alpha_{k,k+1} = q^{-2}\alpha_{k+1,k}, \qquad \alpha^k_0 = \alpha_{k,k+1}.$$

The second isomorphism is given by

$$\phi_2(u^*_{\xi_k-\xi_l}) = \beta_{k,l}T_{\xi_l-\xi_k}, \qquad \phi_2(u^*_{0,k}) = \sum_{l=1}^{n-1}\beta^{k,l}_0 T_{0,l}$$

and the coefficients can be fixed from the relations

$$
\begin{aligned}
&\beta_{k,l} = \beta_{k+1,l}, \qquad \beta_{k,l} = q^{-2}\beta_{k,l+1}, \qquad \beta_{k,k+1} = q^2\beta_{k+1,k}\\
&\beta^{k,l}_0 = q^{-2(k-1)}\frac{(-2(n-1-k))_q(2(l-1))_q}{(-2(n-1))_q}\beta_{k,k+1} \qquad (l \le k)\\
&\beta^{k,l}_0 = \frac{(-2(n-1-l))_q(-2(k-1))_q}{(-2(n-1))_q}\beta_{k,k+1} \qquad (k \le l)\\
&(\pm m)_q = 1 + q^{\pm 2} + \ldots + q^{\pm 2m}.
\end{aligned}
$$

The Dirac operator is defined by

$$D = \sum_{k\ne l}\alpha_{k,l}\beta_{k,l}\psi_{\xi_k-\xi_l}\otimes T_{\xi_l-\xi_k} + \sum_{k=1}^{n-1}\alpha^k_0\psi_{0,k}\otimes(\sum_{l=1}^{n-1}\beta^{k,l}_0 T_{0,l}).$$

The spectrum. To understand the behaviour of the spectrum of D it is sufficient to study $U_q(\mathfrak{sl}(2))$. The quantum Lie algebra $\mathfrak{L}_q(\mathfrak{sl}(2))$ is spanned by

$$\begin{aligned} hT_\alpha &= -q(q-q^{-1})e \\ hT_0 &= q^{-1}(t^{-1}-t+q(q-q^{-1})^2 fe) \\ hT_{-\alpha} &= -(q-q^{-1})ft, \end{aligned}$$

where $\xi_1 - \xi_2 = \alpha$ is the simple root. The $\underleftarrow{R}$-matrix in the opposite adjoint representation (7) is

$$\begin{aligned} \underleftarrow{R} = {} & e_{11}\otimes(q^2e_{11}+e_{22}+q^{-2}e_{33})+e_{22}\otimes 1+e_{33}\otimes(q^{-2}e_{11}+e_{22}+q^2e_{33}) \\ & +(q^2-q^{-2})e_{21}\otimes e_{12}-(1-q^{-2})e_{12}\otimes e_{23}-(1+q^{-2})(q^2-q^{-2})e_{32}\otimes e_{12} \\ & +(q^2-q^{-2})e_{32}\otimes e_{23}+(q-q^{-1})^2(1+q^{-2})e_{31}\otimes e_{13}. \end{aligned}$$

There are three invariant eigenspaces in $V^{\rm op}\otimes V^{\rm op}$ for the operator $\check{\underleftarrow{R}}$. The q-antisymmetric part is three dimensional and its eigenvalue is $-q^{-2}$. There are also one and five dimensional q-symmetric eigenspaces with eigenvalues q^{-4} and q^2. The invariant q-antisymmetric subspace has a basis

$$\begin{aligned} A_\alpha &= \psi_\alpha\otimes\psi_0 - q^2\psi_0\otimes\psi_\alpha \\ A_0 &= -(1+q^2)\psi_\alpha\otimes\psi_{-\alpha}-(q^2-1)\psi_0\otimes\psi_0+(1+q^2)\psi_{-\alpha}\otimes\psi_\alpha \\ A_{-\alpha} &= \psi_0\otimes\psi_{-\alpha}-q^2\psi_{-\alpha}\otimes\psi_0. \end{aligned}$$

The coefficients are chosen so that these transform as the opposite adjoint representation in (7). We shall use the same symbols for these vectors after they have been embedded into the algebra $\mathrm{cl}_q(\mathfrak{sl}(2))$.

The invariant bilinear form $B_q : V^{\rm op}\otimes V^{\rm op}\to\mathbb{C}$ is given by

$$B_q(\psi_\alpha\otimes\psi_{-\alpha}) = b, \qquad B_q(\psi_0\otimes\psi_0)=(1+q^2)b, \qquad B_q(\psi_{-\alpha}\otimes\psi_\alpha)=q^2b,$$

and the rest of the generators are in the kernel of B_q. b is an arbitrary constant. The following Clifford algebra relations can be written down immediately from the spectral decomposition by identifying the q-symmetric vectors with their image in B_q:

$$\begin{aligned} &\psi_\alpha\psi_\alpha = \psi_{-\alpha}\psi_{-\alpha} = 0 \\ &q^4\psi_\alpha\psi_{-\alpha} - q^2\psi_0\psi_0 + \psi_{-\alpha}\psi_\alpha = 0 \\ &q^2\psi_\alpha\psi_0 + \psi_0\psi_\alpha = 0 \\ &q^2\psi_0\psi_{-\alpha} + \psi_{-\alpha}\psi_0 = 0 \\ &\psi_\alpha\psi_{-\alpha} + \psi_{-\alpha}\psi_\alpha = (1+q^2)b. \end{aligned}$$

Let us normalize the form B_q so that $b=(1+q^2)^{-1}$. The irreducible representation of $\mathrm{cl}_q(\mathfrak{sl}(2))$ on $S=\bigoplus_{i=0,1}\mathbb{C}v_i$ which is compatible with the defining representation of the quantum group (the $n=1$ representation in (8)) is given by

$$\begin{aligned} &\psi_\alpha.v_0 = 0, \qquad \psi_\alpha.v_1 = v_0 \\ &\psi_0.v_0 = qv_0, \qquad \psi_0.v_1 = -q^{-1}v_1 \\ &\psi_{-\alpha}.v_0 = v_1, \qquad \psi_{-\alpha}.v_1 = 0. \end{aligned}$$

Let W_n be the $(n+1)$-dimensional $U_q(\mathfrak{sl}(2))$-module of the highest weight n defined by [11]

$$\begin{aligned} &e.v_p = [n-p+1]v_{p-1}, \qquad e.v_0 = 0, \\ &f.v_p = [p+1]v_{p+1}, \qquad f.v_p = 0, \qquad t.v_p = q^{n-2p}v_p \end{aligned} \tag{8}$$

where $0\le p\le n$ and $[k]=(q^k-q^{-k})/(q-q^{-1})$.

For each $n \in \mathbb{N}$ one can define a vector space $H_n = S \otimes W_n$. As a $U_q(\mathfrak{sl}(2))$-module this decomposes as

$$\mathbf{1} \otimes \mathbf{n} = \mathbf{n{+}1} \oplus \mathbf{n{-}1}.$$

The covariant Dirac operator with a cubical term is defined by

$$\begin{aligned} D = &(1+q^{-2})(\psi_\alpha \otimes 1)(A_{-\alpha} \otimes 1 + 1 \otimes hT_{-\alpha}) + (1+q^{-2})(\psi_{-\alpha} \otimes 1)(q^2 A_\alpha \otimes 1 + 1 \otimes hT_\alpha) \\ &+ (\psi_0 \otimes 1)(A_0 \otimes 1 + 1 \otimes hT_0), \end{aligned}$$

where we have fixed the coefficients of the Dirac operator and its cubical part, of course, any other choice can be made. The Dirac operator has $(n+2)$ and n-dimensional eigenspaces H_n^+ and H_n^- and its action on H_n is given by

$$\begin{aligned} D.h_+ &= (q^{-n} - q^n - (q^{-3} + q^{-1} + 2q + q^3 + q^5))h_+, \qquad h_+ \in H_n^+ \\ D.h_- &= (-(q^{-n-2} - q^{n+2}) - (q^{-3} + q^{-1} + 2q + q^3 + q^5))h_-, \qquad h_- \in H_n^-. \end{aligned}$$

The spectrum grows exponentially as a function of n. The operator in [1] has the same spectrum as D except for the term independent of n which comes from the cubical term.

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