A MAXIMUM RANK PROBLEM FOR DEGENERATE ELLIPTIC FULLY NONLINEAR EQUATIONS

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Abstract. The solutions to the Dirichlet problem for two degenerate elliptic fully nonlinear equations in $n+1$ dimensions, namely the real Monge-Ampère equation and the Donaldson equation, are shown to have maximum rank in the space variables when $n \leqslant 2$. A constant rank property is also established for the Donaldson equation when $n=3$.

1. Introduction

The solutions of elliptic partial differential equations are known to have many remarkable convexity properties, under suitable structure conditions. Some early works are those of Brascamp-Lieb [5], Caffarelli and Friedman [6], Yau [26], with many important subsequent developments (see [1, 18, 7, 16, 20, 2, 3] and also references therein). The constant rank theorem has been established for a general class of fully elliptic nonlinear equations. But the situation for degenerate elliptic fully nonlinear equations has remained largely unexplored, despite its considerable interest for example in geometry. One exception is the beautiful work of Lempert [19] on the solution to the homogeneous complex Monge-Ampère equation on convex domains in $\mathbf{C}^n$ with prescribed log singularity at an interior point (the pluri-Green's function). Using a complex foliation, he showed that the solution is smooth and the complex Hessian has maximum rank $n-1$. Even for that result, there is no known PDE proof.

In this paper, we study a maximum rank problem for the Dirichlet problem for two basic models of such equations, on the space $X^n \times T$, where $X^n = (\mathbf{R}/\mathbf{Z})^n$ is the n-dimensional torus and $T=(0,1)$ is the unit interval. The first model is the Monge-Ampère equation

$$\det(D^2_{x,t} u + I_{n+1}) = \varepsilon \tag{1.1}$$

and the second is the equation introduced by Donaldson [13]

$$u_{tt}(n+\Delta u) - \sum_{j=1}^{n} u_{jt}^2 = \varepsilon. \tag{1.2}$$

Here the variables in $X \times T$ have been denoted by (x,t), and I_{n+1} is the $(n+1)\times(n+1)$ matrix with the $n \times n$ identity matrix I_n as its upper left block, and all zeroes on its $(n+1)$ row and

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its $(n+1)$ column. The right hand side ε is a strictly positive constant, but may be arbitrarily small. One imposes the Dirichlet condition

$$u(x,0)=u^0(x), \qquad u(x,1)=u^1(x), \tag{1.3}$$

where the boundary data u^0 and u^1 are assumed to be C^∞. For the Monge-Ampère equation, the solution u is required to satisfy $D^2_{xt}u+I_{n+1}\geqslant 0$, while for the Donaldson equation, it is required to satisfy $n+\Delta u\geqslant 0$.

Both cases of right hand side $0<\varepsilon<<1$ and $\varepsilon=0$ are of importance in geometry and physics: the geodesic and approximate geodesic equations for the space of Kähler potentials coincide with a complex version of (1.1) [12, 21, 25], and for toric varieties, they reduce to a real version of (1.1) on polytopes with Guillemin boundary conditions. The equation (1.2) has a similar interpretation as the geodesic and approximate geodesic equations for the space of volume forms on a Riemannian manifold. It coincides with (1.1) when $n=1$, but is more closely related to Nahm's equation in theoretical physics as well as to some free boundary problems in applied mathematics [13].

For fixed $\varepsilon>0$, the equations (1.1) and (1.2) are elliptic. The existence of a unique smooth solution for $\varepsilon>0$ is a consequence of the general theory [11, 14] for the Monge-Amp'ere equation (1.1), and it has been established in [9, 17] for the Donaldson equation (1.1). For $\varepsilon=0$, it has been shown by D. Guan [15] that the partial Legendre transform of the solution of (1.1) is a linear function of t, and thus the equation admits a smooth solution which is strictly convex in x. The existence of $C^{1,\alpha}$ solutions for (1.2) is in [9, 17]. We refer to [8, 22, 23, 24] for various regularity results on the complex Monge-Ampère equation on Kähler manifolds.

The main question of interest is whether the equations (1.1) and (1.2) have maximum rank in the space directions, in the sense that the Hessian satisfies $D^2_x u+I_n>0$ for all t. Of course, the boundary data have to satisfy the maximum rank property first. That is, for some strictly positive constant λ,

$$D^2_x u^0+I_n\geqslant\lambda, \qquad D^2_x u^1+I_n\geqslant\lambda. \tag{1.4}$$

We shall be interested in when there is an estimate

$$D^2_x u+I_n\geqslant\widetilde{\lambda}, \tag{1.5}$$

with some constant $\widetilde{\lambda}>0$ *uniformly in* ε? It does not appear that the Monge-Ampère and Donaldson equations (1.1, 1.2) fall under the scope of the broad structure conditions which have been introduced for partial convexity properties in [3]. However, by building on the strong maximum principle methods of [6, 26, 7, 2, 3], exploiting the specific form of the Monge-Ampère and the Donaldson equations, and pushing the desired estimates to the boundary, we can establish the following:

Theorem 1. *Let u be the solution of the Monge-Ampère equation (1.1) on $X^n\times T$ with $D^2_{xt}u+I_{n+1}\geqslant 0$ and Dirichlet data (1.3) satisfying the strict convexity condition (1.4). Assume that*

$n \leqslant 2$. *Then for all* $t \in T$ *and all* $\varepsilon > 0$, $u(\cdot, t)$ *satisfies the same strict convexity condition in the interior,*

$$D_x^2 u(x,t) + I_n \geqslant \lambda, \tag{1.6}$$

with the same $\lambda > 0$ *as in (1.4).*

When $\varepsilon = 0$, the solutions of (1.1) are explicit and manifestly satisfy the inequality (1.6) [15], so the interest in Theorem 1 lies in the solutions for $\varepsilon > 0$ themselves. They can be easier to use than the solutions for $\varepsilon = 0$, see for example the complex case treated in [8, 10, 4]. For the equation (1.2), lower bounds for $D_x^2 u + I_n$ in both cases $\varepsilon > 0$ and $\varepsilon = 0$ were not known. We have

Theorem 2. *Let* u *be the solution of the Donaldson equation (1.2) on* $X^n \times T$ *with* $n + \Delta u \geqslant 0$ *and Dirichlet data (1.3) satisfying the strict convexity condition (1.4). Assume that* $n \leqslant 2$. *Then the strict convexity condition (1.4) with the same lower bound* λ *is preserved in the interior, that is, for all* $t \in T$ *and all* $\varepsilon > 0$, $u(\cdot, t)$ *satisfies*

$$D_x^2 u(x,t) + I_n \geqslant \lambda. \tag{1.7}$$

It would be interesting to determine whether one can lift the restriction of $n \leqslant 2$ in Theorem 1 and Theorem 2. For the Donaldson equation (1.2), we can prove the following partial constant rank theorem.

Theorem 3. *Suppose* Ω *is a domain in* $\mathbf{R}^n$ *and* $\delta > 0$. *Let* u *be a solution of the Donaldson equation (1.2) on* $\Omega \times (0, \delta)$ *satisfying* $D_x^2 u + I_n \geqslant 0$ *for each* $t \in (0, \delta)$. *Assume that* $n \leqslant 3$. *Then the rank of* $(D_x^2 u + I_n)$ *is constant for all* $(x,t) \in \Omega \times (0, \delta)$.

For the Monge-Ampère equation, $\varepsilon > 0$ and the convexity condition $D_{xt}^2 u + I_{n+1} \geqslant 0$ imply trivially the strict convexity condition $D_{xt}^2 u + I_{n+1} > 0$ and hence the strict space convexity condition $D_x^2 u + I_n > 0$. Thus the main interest in Theorem 1 lies in the fact that the lower bound for $D_x^2 u + I_n$ depends only on the boundary data. For the Donaldson equation, even the mere space convexity of the solution does not seem so easy. We observe that it follows from Theorem 3 when $n \leqslant 3$:

Theorem 4. *Let* u *be the solution of the Donaldson equation (1.2) on* $X^n \times T$ *with* $n + \Delta u \geqslant 0$ *and Dirichlet data (1.3) satisfying the strict convexity condition (1.4). Assume that* $n \leqslant 3$. *Then the strict convexity of* u *is preserved in the interior, that is, for all* $t \in T$ *and all* $\varepsilon > 0$,

$$D_x^2 u(x,t) + I_n > 0. \tag{1.8}$$

There are many important questions related to the maximum rank problem which should be investigated. Perhaps of greatest interest is the question of whether maximum rank theorems such as Theorem 1 hold for the complex Monge-Ampère equation, i.e., the geodesic and approximate geodesic equations for the space of Kähler metrics. It is not clear whether the techniques developed in [20, 16] for complex nonlinear equations can be adapted to treat the maximum rank problem for the complex Monge-Ampère equation. One would also like to generalize Theorem 1 to general Riemannian manifolds of arbitrary dimension. The results of this paper should be thought of as experimental. It is our hope that the paper can generate some interest for the study of the maximum rank problem, as we believe that it is an important topic in PDE and differential geometry.

The proof of Theorems 1, 2 and 3 is given in §5. The essential part is contained in Propositions 2 and 3, which are proved in §3 and §4 respectively.

2. The general set-up

Both the Monge-Ampère and the Donaldson equations (1.1)(1.2) are equations of the form

$$F(D_{xt}^2 u + I_{n+1}) = \varepsilon, \tag{2.1}$$

where $F(M)$ is a function of the symmetric $n \times n$ matrix $M = (M_{\alpha\beta})$. Recall that u is a function on $(x,t) \in X^n \times T$. It is convenient to denote by Latin letters $i, j, \cdots$ the n indices for the "space" variables $x = (x_j)$, and by Greek letters $\alpha, \beta, \cdots$ the $n+1$ indices for the "space-time" variables (x,t). As usual, we denote by $F^{\alpha\beta}$ and $F^{\alpha\beta,\gamma\delta}$ the derivatives of F with respect to $M_{\alpha\beta}$,

$$F^{\alpha\beta} = \frac{\partial F}{\partial M_{\alpha\beta}}, \qquad F^{\alpha\beta,\gamma\delta} = \frac{\partial^2 F}{\partial M_{\alpha\beta} \partial M_{\gamma\delta}}. \tag{2.2}$$

Let μ_0 be the minimum over $X^n \times T$ of the lowest eigenvalue of $D_x^2 u + I_n$,

$$\mu_0 = \min_{(x,t)\in X^n \times T} \min_{|\xi|^2=1} \langle (D_x^2 u(x,t) + I_n)\xi, \xi \rangle. \tag{2.3}$$

We would like to show that μ_0 is attained at the boundary. For this, it suffices to show that the set where the matrix $D_x^2 u + I_n - \mu_0 I_n$ has a zero eigenvalue is open. In practice, it suffices to show that for each K, the set where the matrix $D_x^2 u + I_n - \mu_0 I_n$ has a zero eigenvalue of multiplicity K is open. Let x_0 be an interior point of $X^n \times T$ where $D_x^2 + I_n - \mu_0 I_n$ has a zero eigenvalue of multiplicity K. Set

$$\varphi = \sum_{i_1 < \cdots < i_{n-K+1}} \lambda_{i_1} \cdots \lambda_{i_{n-K+1}} \equiv \sigma_{n-K+1}(\lambda_1, \cdots, \lambda_n) \tag{2.4}$$

where λ_i are the eigenvalues of $D_x^2 u + I_n - \mu_0 I_n$ *. The strong maximum principle reduces the desired statement to a key local, elliptic inequality near x_0. The precise formulation we need is the following:

Proposition 1. (a) *Let x_0 be an interior point where $D_x^2 u + I_n - \mu_0 I_n$ has a zero eigenvalue of some multiplicity K, and let φ be defined as in (2.4). If there is a constant C so that*

$$F^{\alpha\beta} \varphi_{\alpha\beta} \leqslant C(\varphi + |\nabla \varphi|) \tag{2.5}$$

*The function φ depends obviously on the choice of order K. To lighten the notation for φ, we have not indicated this explicitly.

for all points in a neighborhood of x_0, then φ vanishes in a neighborhood of x_0.

(b) *If (2.5) holds for an arbitrary interior point x_0 where $D_x^2 u + I_n - \mu_0 I_n$ has a zero of multiplicity K, and if φ vanishes at some point, then φ vanishes identically on $X \times T$. In particular, λ is the largest lower bound for the boundary data $D_x^2 u^0 + I_n$ and $D_x^2 u^1 + I_n$, and we have $\mu_0 = \lambda$ and, for all $(x,t) \in X^n \times T$,*

$$D_x^2 u + I_n \geqslant \lambda I_n. \tag{2.6}$$

Thus we need to investigate estimates of the form (2.5). Let x_0 be an interior point as in Proposition 1, (a), and let x be an arbitrary point in a neighborhood of x_0. Choose a parametrization $\lambda_1, \cdots \lambda_n$ of the eigenvalues of the matrix $D_x^2 u + I_n - \mu_0 I_n$ which is continuous in a neighborhood of x_0. For each fixed x in this neighborhood, we can choose a coordinate system with $D_x^2 u$ diagonal at x. Thus at x, we have

$$u_{ij} + (1-\mu_0)\delta_{ij} = \lambda_i \delta_{ij}. \tag{2.7}$$

Define the matrix v_{ij} by

$$v_{ij} = u_{ij} + (1-\mu_0)\delta_{ij}. \tag{2.8}$$

We divide the indices i, $1 \leqslant i \leqslant n$, into two sets of indices,

$$\{1, \cdots, n\} = G \cup B \tag{2.9}$$

with the "good" set G consisting of those indices i for which $\lambda_i(x_0) \neq 0$, and the "bad" set B consisting of those indices i for which $\lambda_i(x_0) = 0$. Note that $\#G = n - K$ and $\#B = K$, where $\#G, \#B$ denote the cardinalities of G and B. The starting point of our considerations is the following

Lemma 1. *Let $\#G$ be the number of good directions, and set $\varphi = \sigma_{\#G+1}(\lambda_1, \cdots, \lambda_n)$. Then we have*

(a) *The function φ is of size*

$$c_1 \sum_{m\in B} v_{mm} \leqslant \varphi \leqslant c_2 \sum_{m \in B} v_{mm} \tag{2.10}$$

for some strictly positive constants c_1, c_2.

(b) *The first derivatives of φ are given by*

$$\varphi_\alpha = (\prod_{g\in G} v_{gg}) \sum_{m\in B} u_{mm\alpha} + O(\varphi) \tag{2.11}$$

(c) *The second derivatives of φ are given by*

$$\varphi_{\alpha\beta} = (\prod_{g\in G} v_{gg})(\sum_{m\in B} u_{\alpha\beta mm} - 2\sum_{m\in B}\sum_{g\in G} \frac{u_{mg\alpha}u_{mg\beta}}{v_{gg}}) + O(\varphi + |\nabla\varphi|). \tag{2.12}$$

(d) *The linearized operator* $F^{\alpha\beta}\varphi_{\alpha\beta}$ *is given by*

$$\begin{aligned} F^{\alpha\beta}\varphi_{\alpha\beta} &= -(\prod_{g\in G} v_{gg})(\sum_{m\in B} F^{\alpha\beta,\gamma\delta}u_{\alpha\beta m}u_{\gamma\delta m} + 2\sum_{m\in B}\sum_{g\in G}\frac{F^{\alpha\beta}}{v_{gg}}u_{mg\alpha}u_{mg\beta}) \\ &\quad +O(\varphi+|\nabla\varphi|) \end{aligned} \tag{2.13}$$

Proof: The function φ is a linear superposition of terms, each of which is a product of $\#G+1$ eigenvalues of $D_x^2 u + I_n - \mu_0 I_n$. Thus it is of the size of the sum of the terms with exactly one eigenvalue in B. This establishes (a). Formally, (b) and (c) can be established in the same way, by differentiation of the eigenvalues if they are smooth. More generally, the same proof can be adapted by expanding φ in terms of minors. As for (d), successive differentiations of the equation $F(D_{xt}^2 u + I_{n+1}) = \varepsilon$ gives

$$\begin{aligned} F^{\alpha\beta}u_{\alpha\beta\mu} &= 0 \\ F^{\alpha\beta}u_{\alpha\beta\mu\mu} &= -F^{\alpha\beta,\gamma\delta}u_{\alpha\beta\mu}u_{\gamma\delta\mu}. \end{aligned} \tag{2.14}$$

Multiplying the expression for $\varphi_{\alpha\beta}$ in (c) by $F^{\alpha\beta}$, and making use of this last identity gives (d). Q.E.D.

3. The Monge-Ampère equation

We consider now more specifically the Monge-Ampère equation, where

$$F(M) = \det M_{\alpha\beta} \tag{3.1}$$

and ε is a strictly positive constant. Our main results in this situation can be stated as follows:

Proposition 2. *Let u be a solution of the equation $F(D_{xt}^2 u + I_n) = 0$ on $X^n \times T$, which is convex in the sense that $D_{xt}^2 u + I_{n+1} \geqslant 0$. Define μ_0 as in (2.3), and let K be either n or $n-1$. Then the set of interior points x_0 where the matrix $D_x^2 u + I_n - \mu_0 I_n$ has a zero eigenvalue of multiplicity K is open.*

Proof: Let x_0 be an interior point where $D^2 u + I_n - \mu_0 I_n$ has a zero eigenvalue of multiplicity K. We treat first the easier case when $K = n$. In this case, the function φ is, explicitly,

$$\varphi = \sum_{m=1}^{n} v_{mm}. \tag{3.2}$$

Consider next the expression (2.13) for the function $F^{\alpha\beta}\varphi_{\alpha\beta}$. Since $|\nabla u_{ij}| \leqslant C\,\varphi^{\frac{1}{2}}$ for $1 \leqslant i,j \leqslant n$, we obtain modulo $O(\varphi + |\nabla\varphi|)$,

$$F^{\alpha\beta}\varphi_{\alpha\beta} = -\sum_{m\in B} F^{\alpha\beta,\gamma\delta}u_{\alpha\beta m}u_{\gamma\delta m} = -2\sum_{m\in B} F^{tt,\gamma\delta}u_{ttm}u_{\gamma\delta m} = -2\sum_{m\in B} u_{ttm}\partial_m F^{tt}. \tag{3.3}$$

For the Monge-Ampère equation, F^{tt} is simply the determinant

$$F^{tt} = \prod_{i=1}^{n}\mu_i. \tag{3.4}$$

where μ_i denotes the eigenvalues of D^2u+I. Since we have then $\lambda_i=\mu_i-\mu_0$, we can write

$$F^{tt}=\sum_{p=0}^{n}\sigma_{n-p}(\lambda_1,\cdots,\lambda_n)\mu_0^p. \tag{3.5}$$

Since $0\leqslant\lambda_i\leqslant\varphi$ for all i, we have

$$\sum_{p=0}^{n-2}|\nabla\sigma_{n-p}(\lambda_1,\cdots,\lambda_n)|\leqslant C\,\varphi,\qquad |\nabla\sigma_1(\lambda_1,\cdots,\lambda_n)|=|\nabla\varphi|. \tag{3.6}$$

Thus $|\partial_m F^{tt}|\leqslant C(\varphi+|\nabla\varphi|)$. Altogether, we obtain the inequality (2.5) and the desired statement follows from Proposition 1.

We consider now the case when the matrix $D_x^2u+I_n-\mu_0 I_n$ admits at an interior point x_0 a zero eigenvalue of multiplicity $n-1$. Thus there is only one good direction, which we label g, $G=\{g\}$, and all other space directions, $B=\{1\leqslant m\leqslant n;\ m\neq g\}$ are bad. The expression (c) for $F^{\alpha\beta}\varphi_{\alpha\beta}$ in Lemma 1 becomes in this case

$$F^{\alpha\beta}\varphi_{\alpha\beta}=-\sum_{m\in B}\big(v_{gg}\sum_{\alpha\beta,\gamma\delta}F^{\alpha\beta,\gamma\delta}v_{\alpha\beta m}v_{\gamma\delta m}+2\sum_{\alpha\beta}F^{\alpha\beta}u_{mg\alpha}u_{mg\beta})+O(|\nabla\varphi|+|\varphi|). \tag{3.7}$$

The next step is to derive an identity for the term $F^{\alpha\beta}u_{mg\alpha}u_{mg\beta}$:

Lemma 2. *We have*

$$F^{\alpha\beta}u_{mg\alpha}u_{mg\beta}=(u_{gg}+1)\sum_{\alpha,\beta\neq g}F^{\alpha\beta,gg}(u_{mg\alpha}u_{mg\beta}-u_{mgg}u_{\alpha\beta m})+O(|\nabla\varphi|+|\varphi|) \tag{3.8}$$

Proof: Recall that differentiating the equation gives

$$F^{\alpha\beta}u_{\alpha\beta m}=0 \tag{3.9}$$

Extracting the terms involving the good direction g gives,

$$F^{gg}u_{ggm}+2\sum_{\alpha\neq g}F^{\alpha g}u_{\alpha gm}=-\sum_{\alpha,\beta\neq g}F^{\alpha\beta}u_{\alpha\beta m}. \tag{3.10}$$

Returning to the expression $F^{\alpha\beta}u_{mg\alpha}u_{mg\beta}$, we can write

$$\begin{aligned}F^{\alpha\beta}u_{mg\alpha}u_{mg\beta} &= F^{gg}u_{mgg}u_{mgg}+2\sum_{a\neq g}F^{\alpha g}u_{mg\alpha}u_{mgg}+\sum_{\alpha,\beta\neq g}F^{\alpha\beta}u_{mg\alpha}u_{mg\beta}\\ &= u_{mgg}(F^{gg}u_{mgg}+2\sum_{\alpha\neq g}F^{\alpha g}u_{mg\alpha})+\sum_{\alpha,\beta\neq g}F^{\alpha\beta}u_{mg\alpha}u_{mg\beta}\\ &= \sum_{\alpha,\beta\neq g}F^{\alpha\beta}(u_{mg\alpha}u_{mg\beta}-u_{mgg}u_{\alpha\beta m}). \end{aligned} \tag{3.11}$$

We exploit the fact that F is an affine function of any of the entries to write

$$F^{\alpha\beta}=F^{\alpha\beta,gg}(u_{gg}+1)+F^{\alpha\beta}_{|u_{gg}+1=0}. \tag{3.12}$$

The identity in the lemma follows then from the following claim

$$\begin{aligned}\sum_{m\in B}\sum_{\alpha,\beta\neq g} F^{\alpha\beta}_{|_{u_{gg}+1=0}} u_{mg\alpha}u_{mg\beta} &= O(|\nabla\varphi|+\varphi)\\ \sum_{m\in B}\sum_{\alpha,\beta\neq g} F^{\alpha\beta}_{|_{u_{gg}+1=0}} u_{mgg}u_{\alpha\beta m} &= O(|\nabla\varphi|+\varphi).\end{aligned}\tag{3.13}$$

To see the first identity above, we note that the terms with $\alpha\in B$ and $\beta\in B$ are $O(\varphi)$. Thus we need only consider the terms with at least α or β equal to t. But then the cofactor $F^{\alpha\beta}_{|_{u_{gg}+1=0}}$ has either a full column or a full row of zeroes, and must be 0.

Next, we consider the second identity above. For the same reason as above, $F^{\alpha\beta}_{|_{u_{gg}+1=0}}=0$ if either α or β is equal to t. Thus we can restrict to $\alpha,\beta\in B$. Write now

$$\begin{aligned}\sum_{\alpha,\beta\neq g} F^{\alpha\beta}_{|_{u_{gg}+1=0}} u_{\alpha\beta m} &= \sum_{\alpha,\beta\in B} F^{\alpha\beta}_{|_{u_{gg}+1=0}} u_{\alpha\beta m} + O(|\nabla\varphi|+\varphi)\\ &= \sum_{i\in B} F^{ii}_{|_{u_{gg}+1=0}} u_{iim} + 2\sum_{i,j\in B, i<j} F^{ij}_{|_{u_{gg}+1=0}} u_{ijm} + O(|\nabla\varphi|+\varphi).\end{aligned}\tag{3.14}$$

By inspection, we observe that

- If $i,j\in B$, then $F^{ij}_{|_{u_{gg}+1=0}}=0$ unless $i=j$.
- If $i\in B$, then $F^{ii}_{|_{u_{gg}+1=0}}=u_{gt}^2\prod_{j\in B,j\neq i}(u_{jj}+1)$.

The last identity implies

$$\begin{aligned}\sum_{m\in B} F^{ii}_{|_{u_{gg}+1=0}} u_{iim} &= u_{gt}^2\mu_0^{n-2}\sum_{m\in B} u_{iim} + O(\varphi)\\ &= O(|\nabla\varphi|+\varphi).\end{aligned}\tag{3.15}$$

The lemma is proved. Q.E.D.

For our purposes, it is convenient to rewrite the identity in the preceding lemma in the following form: note that $u_{\alpha\beta m}u_{\gamma\delta m}=O(|\nabla\varphi|+\varphi)$ if both α and β are in B. Since neither of them is g, we can assume that at least one of them is t. Thus

Lemma 3. *We have*

$$\begin{aligned}F^{\alpha\beta}u_{mg\alpha}u_{mg\beta} &= (u_{gg}+1)\big(F^{tt,gg}u_{tgm}^2 + 2\sum_{i\in B} F^{it,gg}u_{mgi}u_{mgt}\big)\\ &\quad -(u_{gg}+1)\sum_{\alpha,\beta\neq g} F^{\alpha\beta,gg}u_{mgg}u_{\alpha\beta m} + O(|\nabla\varphi|+|\varphi|).\end{aligned}\tag{3.16}$$

Our next task is to simplify the expression

$$\sum_{\alpha\beta,\gamma\delta} F^{\alpha\beta,\gamma\delta}u_{\alpha\beta m}u_{\gamma\delta m}.\tag{3.17}$$

First, we isolate the contribution of the index (gg), which will cancel out with the corresponding term from the first identity,

$$\sum_{\alpha\beta,\gamma\delta} F^{\alpha\beta,\gamma\delta} u_{\alpha\beta m} u_{\gamma\delta m} = 2\sum_{\alpha\beta} F^{\alpha\beta,gg} u_{\alpha\beta m} u_{ggm} + \sum_{(\alpha\beta),(\gamma\delta)\neq(gg)} F^{\alpha\beta,\gamma\delta} u_{\alpha\beta m} u_{\gamma\delta m}. \tag{3.18}$$

Next, we work out the remaining contributions. For this, it is convenient to introduce the following sets of indices, $\mathcal{A} = \{(gt),(tg),(tt)\}$, and $\mathcal{B} = \mathcal{A}^c$, so that $(\alpha\beta)$ is in $\mathcal{B}$ if and only if at least one of the indices α or β is in B.

- If both $(\alpha\beta) \in \mathcal{B}$ and $(\gamma\delta) \in \mathcal{B}$, then $|u_{\alpha\beta m}| + |u_{\gamma\delta m}| = O(\varphi^{\frac{1}{2}})$, and thus these contributions are $O(\varphi)$ and can be neglected.

- The contributions when both $(\alpha\beta)$ and $(\gamma\delta)$ are in $\mathcal{A}$ can be worked out explicitly,

$$\sum_{(\alpha\beta)\in\mathcal{A},(\gamma\delta)\in\mathcal{A}} F^{\alpha\beta,\gamma\delta} u_{\alpha\beta m} u_{\gamma\delta m} = F^{tg,gt} u_{tgm}^2 + F^{gt,tg} u_{tgm}^2 = -2u_{tgm}^2 \prod_{j\in B} u_{jj} = -2F^{tt,gg} u_{tmg}^2. \tag{3.19}$$

- The remaining contributions are

$$2\sum_{(\alpha\beta)\in\mathcal{B},(\gamma\delta)\in\mathcal{A}} F^{\alpha\beta,\gamma\delta} u_{\alpha\beta m} u_{\gamma\delta m}. \tag{3.20}$$

To identify these terms, we divide the set $\mathcal{B}$ of indices $(\alpha\beta)$ with at least one index in B into three mutually disjoint sets:

$\mathcal{B}_0 = \{(\alpha\beta); \alpha \in B, \quad \beta \in B\}$
$\mathcal{B}_1 = \{(\alpha\beta); \alpha \in \{g,t\}, \quad \beta \in B\}$
$\mathcal{B}_2 = \{(\alpha\beta); \alpha \in B, \quad \beta \in \{g,t\}\}$

The sum breaks up correspondingly

$$\begin{aligned} 2\sum_{(\alpha\beta)\in\mathcal{B},(\gamma\delta)\in\mathcal{A}} F^{\alpha\beta,\gamma\delta} u_{\alpha\beta m} u_{\gamma\delta m} &= 2\sum_{a=0,1,2}\sum_{(\alpha\beta)\in\mathcal{B}_a} \big(F^{\alpha\beta,tt} u_{\alpha\beta m} u_{ttm} \\ &\quad + (F^{\alpha\beta,gt} + F^{\alpha\beta,tg}) u_{\alpha\beta m} u_{gtm}\big). \end{aligned} \tag{3.21}$$

Each of these terms can now be worked out explicitly. First, we have

$$\sum_{(\alpha\beta)\in\mathcal{B}_1\cup\mathcal{B}_2} F^{\alpha\beta,tt} u_{\alpha\beta m} u_{ttm} = 0 \tag{3.22}$$

because we can see by inspection that $F^{\alpha\beta,tt}$ is given then by a matrix with a column or a row of 0 and hence must be 0.

Next, we have

$$\sum_{(ij)\in\mathcal{B}_0} F^{ij,tt} u_{ijm} u_{ttm} = \sum_{i\in B} u_{iim} u_{ttm} \prod_{j\neq i, j\in B} (u_{jj}+1). \tag{3.23}$$

This is because $F^{ij,tt} = 0$ unless $i = j$, and the entries $F^{ii,tt}$ can be easily computed, giving the formula above. Since we can replace $u_{jj} + 1$ by μ_0 modulo $O(\varphi)$, we obtain

$$\sum_{(ij)\in\mathcal{B}_0} F^{ij,tt} u_{ijm} u_{ttm} = \mu_0^{n-2} \sum_{i\in B} u_{iim} u_{ttm} + O(\varphi) = O(|\nabla\varphi| + \varphi). \tag{3.24}$$

It remains only to determine the sum

$$\sum_{a=0,1,2} \sum_{(\alpha\beta)\in\mathcal{B}_a} (F^{\alpha\beta,tg} + F^{\alpha\beta,gt}) u_{\alpha\beta m} u_{gtm}. \tag{3.25}$$

Consider first the contributions from $(\alpha\beta) \in \mathcal{B}_2$, i.e. $(\alpha\beta) = (\alpha g)$ or $(\alpha\beta) = (\alpha t)$. Then it is clear that we obtain

$$\begin{aligned} (\alpha\beta) = (\alpha g): &\qquad F^{\alpha g,tg} + F^{\alpha g,gt} = F^{\alpha g,gt} = -F^{\alpha t,gg} \\ (\alpha\beta) = (\alpha t): &\qquad F^{\alpha t,tg} + F^{\alpha t,gt} = F^{\alpha t,tg} = -F^{\alpha g,tt} = 0. \end{aligned} \tag{3.26}$$

Thus we find

$$\sum_{(\alpha\beta)\in\mathcal{B}_2} (F^{\alpha\beta,tg} + F^{\alpha\beta,gt}) u_{\alpha\beta m} u_{gtm} = -\sum_{\alpha\in B} F^{\alpha t,gg} u_{\alpha gm} u_{gtm}. \tag{3.27}$$

Similarly, the contributions from $(\alpha\beta) \in \mathcal{B}_1$ correspond to $(\alpha\beta) = (g\beta)$ or $(\alpha\beta) = (t\beta)$, and work out to be

$$\begin{aligned} (\alpha\beta) = (g\beta): &\qquad F^{g\beta,tg} + F^{g\beta,gt} = F^{g\beta,tg} = -F^{gg,t\beta} \\ (\alpha\beta) = (t\beta): &\qquad F^{t\beta,tg} + F^{t\beta,gt} = -F^{tt,g\beta} = 0, \end{aligned} \tag{3.28}$$

and

$$\sum_{(\alpha\beta)\in\mathcal{B}_1} (F^{\alpha\beta,tg} + F^{\alpha\beta,gt}) u_{\alpha\beta m} u_{gtm} = -\sum_{\beta\in B} F^{gg,t\beta} u_{g\beta m} u_{tgm}. \tag{3.29}$$

Finally, we come to the contributions from $(\alpha\beta) \in \mathcal{B}_0$. Here it is seen by inspection that only $(\alpha\beta) = (\alpha\beta)$ will contribute, and thus

$$\begin{aligned} \sum_{(\alpha\beta)\in\mathcal{B}_0} (F^{\alpha\beta,tg} + F^{\alpha\beta,gt}) u_{\alpha\beta m} u_{gtm} &= \sum_{\alpha\in B} (F^{\alpha\alpha,tg} + F^{\alpha\alpha,gt}) u_{\alpha\alpha m} u_{gtm} \\ &= 2\sum_{\alpha\in B} F^{\alpha\alpha,tg} u_{\alpha\alpha m} u_{gtm}. \end{aligned} \tag{3.30}$$

But an inspection shows that for $\alpha = i \in B$,

$$F^{ii,tg} = u_{gt} \prod_{j\neq i, j\in B} (u_{jj} + 1) \tag{3.31}$$

so that

$$\begin{aligned}\sum_{(\alpha\beta)\in\mathcal{B}_0}(F^{\alpha\beta,tg}+F^{\alpha\beta,gt})u_{\alpha\beta m}u_{gtm} &= 2u_{gt}\sum_{i\in B}u_{iim}u_{gtm}\prod_{j\neq i,j\in B}(u_{jj}+1)\\ &= 2u_{gt}\mu_0^{n-2}\sum_i u_{iim}u_{gtm}+O(\varphi)\\ &= O(|\nabla\varphi|+\varphi).\end{aligned}$$

(3.32)

In summary, we have proved

Lemma 4. *We have the following identity*

$$\begin{aligned}\sum_{\alpha\beta,\gamma\delta}F^{\alpha\beta,\gamma\delta}u_{\alpha\beta m}u_{\gamma\delta m} &= 2\sum_{\alpha\beta}F^{\alpha\beta,gg}u_{\alpha\beta m}u_{ggm}\\ &\quad -2(F^{tt,gg}u_{tmg}^2+2\sum_{i\in B}F^{it,gg}u_{igm}u_{gtm})+O(|\nabla\varphi|+\varphi).\end{aligned} \tag{3.33}$$

Comparing the identities in Lemmas 3 and 4, we obtain the main lemma in this case,

Lemma 5. *We have*

$$F^{\alpha\beta}u_{mg\alpha}u_{mg\beta}=-(u_{gg}+1)\,F^{\alpha\beta,\gamma\delta}u_{\alpha\beta m}u_{\gamma\delta m}+O(|\nabla\varphi|+\varphi). \tag{3.34}$$

and, finally, since $v_{gg}=u_{gg}+1-\mu_0$,

$$F^{\alpha\beta}\varphi_{\alpha\beta}=2\mu_0\,F^{\alpha\beta,\gamma\delta}u_{\alpha\beta m}u_{\gamma\delta m}+O(|\nabla\varphi|+\varphi). \tag{3.35}$$

The first term on the right hand side is negative, modulo $O(\varphi+|\nabla\varphi|)$: indeed, Lemma 5 shows that it is given by $-(u_{gg}+1)^{-1}F^{\alpha\beta}u_{mg\alpha}u_{mg\beta}$. But $u_{gg}+1>0$ and, in the case of the Monge-Ampère equation, the matrix $F^{\alpha\beta}$ is just the matrix of minors of $D_{xt}^2u+I_{n+1}$, which is positive. Thus we obtain again the key estimate (2.5). Q.E.D.

4. The Donaldson equation

In our notation, the Donaldson equation (1.2) is an equation of the form (2.1), with $F(M)$ given by

$$F(M)=M_{tt}(1+\sum_{j=1}^n M_{jj})-\sum_{j=1}^n M_{jt}^2. \tag{4.1}$$

We again consider the Dirichlet problem on the space $X^n\times T$, with the usual boundary condition (1.3). Our main result is the following:

Proposition 3. *Let u be a solution of the Donaldson equation $F(D^2u+I')=0$ on $X^n\times T$ satisfying $D_x^2u+I_n\geqslant 0$, with F as in (4.1). Define $\mu_0\geqslant 0$ as in (2.3), and let K be either n or $n-1$. Then the set of interior points x_0 where the matrix $D_x^2u+I_n-\mu_0 I_n$ has a zero eigenvalue* *of multiplicity K is open. If $\mu_0 = 0$ in (2.3), then the set of interior points x_0 where the matrix $D_x^2 u + I_n$ has a zero eigenvalue of multiplicity $K \geqslant n-2$ is open.*

Proof: As in the previous sections, we work at an arbitrary point x in a neighborhood of a given point x_0 where the matrix $D^2u + I_n - \mu_0 I_n$ admits a zero eigenvalue of multiplicity K. The three values $K = n, n-1, n-2$ correspond respectively to $\#G = 0, 1, 2$, where $\#G$ is the number of good directions. The most difficult case is $\#G = 2$ (corresponding to the case $K = n-2$). Thus we write down the calculations for general $\#G$, and then specialize to the cases of interest.

We use the same notations as in Section §2. If we use $\varphi = \sigma_{n-K+1}$ as before, and apply Lemma 1 with the function $F(M)$ corresponding to Donaldson's equation, we would find

$$F^{\alpha\beta}\varphi_{\alpha\beta} = -2\sum_{m\in B}\tilde{Q}_m\,(\prod_{g\in G} v_{gg}) + O(\varphi + |\nabla\varphi|),$$

with

$$\tilde{Q}_m = u_{ttm}\Delta u_m - \sum_{j\in G} u_{tjm}^2 + u_{tt}\sum_{j,k\in G}\frac{u_{jkm}^2}{v_{jj}} + (n+\Delta u)\sum_{j\in G}\frac{u_{tjm}^2}{v_{jj}} - 2\sum_{j\in G}\sum_{k=1}^{n}\frac{u_{tk}u_{tjm}u_{jkm}}{v_{jj}}.$$

We notice that there are linear terms of the form $\nabla u_{km}, k, m \in B$. When $n \geqslant 3$ and $K = n-2$, these linear terms cannot be bounded by $\varphi + |\nabla\varphi|$. To overcome this obstacle, we use instead

$$\varphi = \sigma_{n-K+1} + q, \quad \text{where } q = \tfrac{\sigma_{n-K+2}}{\sigma_{n-K+1}}. \tag{4.2}$$

The regularity and strong concavity of q was proved in [2]. Following the arguments in [2, 3] (e.g, see eq. (60) in [3], in our case, F is independent of $\nabla u, u, x$) for φ defined in (4.2), we obtain

$$\begin{aligned}\sum_{\alpha,\beta=1}^{N} F^{\alpha\beta}\varphi_{\alpha\beta} &= -2\sum_{m\in B}[\sigma_l(G) + \frac{\sigma_1^2(B|m) - \sigma_2(B|m)}{\sigma_1^2(B)}]\tilde{Q}_m + O(\varphi + \sum_{i,j\in B}|\nabla u_{ij}|)\\ &\quad - \sum_{\alpha,\beta=1}^{N} F^{\alpha\beta}[\frac{\sum_{i\in B} V_{i\alpha}V_{i\beta}}{\sigma_1^3(B)} + \frac{\sum_{i,j\in B, i\neq j} u_{ij\alpha}u_{ji\beta}}{\sigma_1(B)}],\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}\tilde{Q}_m &= u_{ttm}\Delta u_m - \sum_{j\in G} u_{tjm}^2 + u_{tt}\sum_{j,k\in G}\frac{u_{jkm}^2}{v_{jj}} + (n+\Delta u)\sum_{j\in G}\frac{u_{tjm}^2}{v_{jj}} - 2\sum_{j\in G}\sum_{k=1}^{n}\frac{u_{tk}u_{tjm}u_{jkm}}{v_{jj}}\\ &= Q_m + Q_m^*,\end{aligned} \tag{4.4}$$

$$\begin{aligned}Q_m &= u_{ttm}\Delta u_m - \sum_{k\in G} u_{ktm}^2 + u_{tt}\sum_{j,k\in G}\frac{u_{mjk}^2}{1+u_{jj}}\\ &\quad + (n+\Delta u)\sum_{j\in G}\frac{u_{mjt}^2}{1+u_{jj}} - 2\sum_{j,k\in G}\frac{u_{tk}u_{tjm}u_{mjk}}{1+u_{jj}},\end{aligned} \tag{4.5}$$

$$Q_m^* = \sum_{m\in B}\sum_{j\in G} F^{\alpha\beta}\left(\frac{1}{v_{jj}} - \frac{1}{1+u_{jj}}\right)u_{mj\alpha}u_{mj\beta}, \tag{4.6}$$

and $\sigma_l(B) = \sum_{i\in B} v_{ii}$, $\sigma_l(B|m) = \sum_{i\neq m, i\in B} v_{ii}$,

$$V_{i\alpha} = u_{ii\alpha}\sigma_1(B) - u_{ii}\Big(\sum_{j\in B} u_{jj\alpha}\Big). \tag{4.7}$$

The term $\sum_{i,j\in B}|\nabla u_{ij}|$ in (4.3) can be controlled by $\varphi, |\nabla\varphi|$ and the last term in (4.3) in the same way as in [2, 3]. We obtain, for some $C > 0$,

$$\sum_{\alpha,\beta=1}^{N} F^{\alpha\beta}\phi_{\alpha\beta} \leqslant -2C\sum_{m\in B}(Q_m + Q_m^*) + O(\varphi + |\nabla\varphi|). \tag{4.8}$$

Since $\mu_0 \geqslant 0$, we have $\frac{1}{v_{jj}} - \frac{1}{1+u_{jj}} \geqslant 0$. It is easy to see that Q_m^* in (4.6) is nonnegative by the positivity of $(F^{\alpha\beta})$. Thus we would be done if we can show that $Q_m \geqslant 0$, modulo $O(\varphi + |\nabla\varphi|)$. Since the contributions of each index $m \in B$ are entirely similar, we can consider them individually. To simplify the notation, we set $m = 1$, and drop the subindex m from Q_m.

4.1. **Using the equation.** Differentiating the equation gives

$$u_{tt1} = -u_{tt}\frac{\Delta u_1}{n+\Delta u} + \frac{2}{n+\Delta u}\sum u_{tj}u_{tj1}. \tag{4.9}$$

Thus Q can be written as

$$Q = A + B + C \tag{4.10}$$

with A, B, C defined by

$$\begin{aligned} A &\equiv -u_{tt}\frac{(\Delta u_1)^2}{n+\Delta u} + u_{tt}\sum_{j,k\in G}\frac{u_{1jk}^2}{1+u_{jj}} \\ B &\equiv \frac{2}{n+\Delta u}\Delta u_1\sum u_{tj}u_{tj1} - 2\sum_{j,k\in G}\frac{u_{tk}u_{tj1}u_{jk1}}{1+u_{jj}} \\ C &\equiv \sum_{j\in G} u_{1jt}^2\left(\frac{n+\Delta u}{1+u_{jj}} - 1\right). \end{aligned} \tag{4.11}$$

4.2. **The A terms.** The A terms can be re-written as follows, modulo $O(\varphi + |\nabla\varphi|)$,

$$A = \frac{u_{tt}}{2(n+\Delta u)}\sum_{j,k\in G, j\neq k}\frac{((1+u_{kk})u_{jj1} - (1+u_{jj})u_{kk1})^2}{(1+u_{jj})(1+u_{kk})} + u_{tt}\sum_{j,k\in G, j\neq k}\frac{u_{1jk}^2}{1+u_{jj}}. \tag{4.12}$$

To see this, just write

$$
\begin{aligned}
-\frac{(\Delta u_1)^2}{n+\Delta u} + \sum_{j,k\in G} \frac{u_{1jk}^2}{1+u_{jj}} &= \sum u_{1jj}^2 \left(\frac{1}{1+u_{jj}} - \frac{1}{n+\Delta u}\right) - \sum_{j\neq k} \frac{u_{jj1}u_{kk1}}{n+\Delta u} + \sum_{j\neq k} \frac{u_{1jk}^2}{1+u_{jj}} \\
&= \sum \frac{u_{1jj}^2}{(1+u_{jj})(n+\Delta u)} \sum_{k\neq j}(1+u_{kk}) - \sum_{j\neq k} \frac{u_{jj1}u_{kk1}}{n+\Delta u} + \sum_{j\neq k} \frac{u_{1jk}^2}{1+u_{jj}} \\
&= \frac{1}{2}\sum_{j\neq k} \frac{1}{n+\Delta u}\left(\frac{1+u_{kk}}{1+u_{jj}} u_{jj1}^2 + \frac{1+u_{jj}}{1+u_{kk}} u_{kk1}^2\right) \\
&\quad - \frac{1}{n+\Delta u}\sum_{j\neq k} u_{jj1}u_{kk1} + \sum_{j\neq k} \frac{u_{1jk}^2}{1+u_{jj}} \\
&= \frac{1}{2(n+\Delta u)} \sum_{j\neq k} \frac{((1+u_{kk})u_{jj1} - (1+u_{jj})u_{kk1})^2}{(1+u_{jj})(1+u_{kk})} + \sum_{j\neq k} \frac{u_{1jk}^2}{1+u_{jj}}.
\end{aligned}
$$

4.3. **The B terms.** The B terms can be re-written as

$$
B = -\sum_{j\neq k} \frac{u_{tj}u_{tj1}}{(1+u_{jj})(n+\Delta u)} (v_{jj1}(u_{kk}+1) - v_{kk1}(u_{jj}+1)) - 2\sum_{j\neq k} u_{tk}u_{tj1}u_{1jk}\frac{1}{1+u_{jj}}. \tag{4.13}
$$

To see this, we decompose the B terms as follows

$$
\frac{2}{n+\Delta u}\Delta u_1 \sum_{k\in G} u_{tk}u_{tk1} = \frac{2}{n+\Delta u}\sum u_{jj1}\sum_{k\in G} u_{tk}u_{tk1} = B_1 + B_2 \tag{4.14}
$$

and

$$
-2\sum_{j,k\in G} \frac{u_{tk}u_{tj1}u_{jk1}}{1+u_{jj}} = B_3 - 2\sum_{j,k\in G, j\neq k} \frac{u_{tk}u_{tj1}u_{jk1}}{u_{jj}} \tag{4.15}
$$

with the terms B_1, B_2, B_3 defined by

$$
\begin{aligned}
B_1 &= 2\sum_k \frac{u_{tk}u_{tk1}u_{kk1}}{n+\Delta u} \\
B_2 &= 2\sum_{j\neq k} \frac{u_{tk}u_{tk1}u_{jj1}}{n+\Delta u} \\
B_3 &= -2\sum \frac{u_{tk}u_{tk1}u_{kk1}}{1+u_{kk}}.
\end{aligned} \tag{4.16}
$$

The terms B_1 and B_3 can be regrouped as

$$
\begin{aligned}
B_1 + B_3 &= -2\sum u_{tk}u_{tk1}u_{kk1}\left(\frac{1}{1+u_{kk}} - \frac{1}{n+\Delta u}\right) \\
&= -2\sum u_{tk}u_{tk1}u_{kk1}\frac{1}{(1+u_{kk})(n+\Delta u)}\sum_{j\neq k}(1+u_{jj}),
\end{aligned} \tag{4.17}
$$

and, combined with B_2, as

$$\begin{aligned} B_1+B_2+B_3 &= -2\sum u_{tk}u_{tk1}u_{kk1}\frac{1}{(1+u_{kk})(n+\Delta u)}\sum_{j\neq k}(1+u_{jj})+2\sum_{j\neq k}\frac{u_{tk}u_{tk1}u_{jj1}}{n+\Delta u} \\ &= 2\sum_{j\neq k}\frac{u_{tk}u_{tk1}}{(1+u_{kk})(n+\Delta u)}(u_{jj1}(1+u_{kk})-u_{kk1}(1+u_{jj})). \end{aligned}$$

4.4. **A second formula for Q.** For convenience, we write here the formula for Q obtained in this manner

$$\begin{aligned} Q &= u_{tt}\{\frac{1}{2(n+\Delta u)}\sum_{j\neq k}\frac{((1+u_{kk})u_{jj1}-(1+u_{jj})u_{kk1})^2}{(1+u_{jj})(1+u_{kk})}+\sum_{j\neq k}\frac{u_{1jk}^2}{1+u_{jj}}\} \\ &\quad -\frac{2\,u_{tj}u_{tj1}}{(1+u_{jj})(n+\Delta u)}(u_{jj1}(1+u_{kk})-(1+u_{jj})u_{kk1})-2\sum_{j\neq k}\frac{u_{tk}u_{tj1}u_{1jk}}{1+u_{jj}} \\ &\quad +\sum_{j\in G}u_{1jt}^2(\frac{n+\Delta u}{1+u_{jj}}-1). \end{aligned} \tag{4.18}$$

We can complete the square in $u_{kk}u_{jj1}-u_{jj}u_{kk1}$, and obtain

$$\begin{aligned} Q &= \frac{1}{2}\sum_{j\neq k}\{(\frac{u_{tt}}{(1+u_{jj})(1+u_{kk})(n+\Delta u)})^{\frac{1}{2}}((1+u_{kk})u_{jj1}-(1+u_{jj})u_{kk1}) \\ &\quad +(\frac{u_{tk}u_{tk1}}{1+u_{kk}}-\frac{u_{tj}u_{tj1}}{1+u_{jj}})(\frac{(1+u_{jj})(1+u_{kk})}{u_{tt}(n+\Delta u)})^{\frac{1}{2}}\}^2 \\ &\quad +D+u_{tt}\sum_{j\neq k}\frac{u_{1jk}^2}{1+u_{jj}}-2\sum_{j\neq k}\frac{u_{tk}u_{tj1}u_{jk1}}{1+u_{jj}} \end{aligned} \tag{4.19}$$

where we have introduced the D terms

$$\begin{aligned} D &\equiv -\frac{1}{2}\sum_{j\neq k}\frac{(1+u_{jj})(1+u_{kk})}{u_{tt}(n+\Delta u)}(\frac{u_{tk}u_{tk1}}{1+u_{kk}}-\frac{u_{tj}u_{tj1}}{1+u_{jj}})^2+C \\ &= -\frac{1}{2}\sum_{j\neq k}\frac{(1+u_{jj})(1+u_{kk})}{u_{tt}(n+\Delta u)}(\frac{u_{tk}u_{tk1}}{1+u_{kk}}-\frac{u_{tj}u_{tj1}}{1+u_{jj}})^2+\sum_{j\in G}u_{1jt}^2(\frac{n+\Delta u}{1+u_{jj}}-1). \end{aligned} \tag{4.20}$$

When $\#G\leqslant 1$, the term Q reduces to the term D, and D reduces in turn to a manifestly positive expression. Thus the cases $\#G\leqslant 1$ are now proved.

4.5. **The D terms.** In the rest of the proof, we will assume $\mu_0=0$ and $\#G=2$.

Expanding the squares in the D terms gives

$$\begin{aligned} D &= -\frac{1}{2}\sum_{j\neq k}\frac{1}{u_{tt}(n+\Delta u)}(\frac{1+u_{jj}}{1+u_{kk}}u_{tk}^2u_{tk1}^2+\frac{1+u_{kk}}{1+u_{jj}}u_{tj}^2u_{tj1}^2) \\ &\quad +\sum_{j\neq k}\frac{1}{u_{tt}(n+\Delta u)}u_{tk}u_{tk1}u_{tj}u_{tj1}+\sum u_{1jt}^2\frac{1}{1+u_{jj}}\sum_{k\neq j}(1+u_{kk}) \\ &= \sum u_{1jt}^2(\frac{1}{(1+u_{jj})}\sum_{k\neq j}(1+u_{kk})-\sum_{k\neq j}\frac{u_{tj}^2(1+u_{kk})}{u_{tt}(n+\Delta u)})+\sum_{j\neq k}\frac{u_{tk}u_{tk1}u_{tj}u_{tj1}}{u_{tt}(n+\Delta u)} \\ &= \sum_{j\in G}u_{1jt}^2\sum_{k\neq j}\frac{1+u_{kk}}{1+u_{jj}}\frac{u_{tt}(1+\Delta u)-u_{tj}^2}{u_{tt}(n+\Delta u)}+\sum_{j\neq k}\frac{u_{tk}u_{tk1}u_{tj}u_{tj1}}{u_{tt}(n+\Delta u)}. \end{aligned}$$

We can now make use of the equation

$$u_{tt}(n+\Delta u)-u_{tj}^2=\sum_{\ell\neq j}u_{t\ell}^2+\varepsilon \tag{4.21}$$

and obtain

$$\begin{aligned} D &= \sum u_{1jt}^2\sum_{k\neq j}\frac{1+u_{kk}}{1+u_{jj}}\frac{u_{tk}^2}{u_{tt}(n+\Delta u)}+\sum_{j\neq k}\frac{1}{u_{tt}(n+\Delta u)}u_{tk}u_{tk1}u_{tj}u_{tj1} \\ &\quad +\sum u_{1jt}^2\sum_{k\neq j}\frac{1+u_{kk}}{1+u_{jj}}\frac{1}{u_{tt}(n+\Delta u)}(\sum_{\ell\neq j,k}u_{t\ell}^2+\varepsilon). \end{aligned} \tag{4.22}$$

Thus we arrive at

$$\begin{aligned} D &= \frac{1}{2}\sum_{j\neq k}\frac{1}{(1+u_{jj})(1+u_{kk})u_{tt}(n+\Delta u)}(u_{tj1}u_{tk}(1+u_{kk})+u_{tk1}u_{tj}(1+u_{jj}))^2 \\ &\quad +\sum u_{1jt}^2\sum_{k\neq j}\frac{1+u_{kk}}{1+u_{jj}}\frac{1}{u_{tt}(n+\Delta u)}(\sum_{\ell\neq j,k}u_{t\ell}^2+\varepsilon). \end{aligned} \tag{4.23}$$

4.6. **A third formula for Q.** We summarize the expression for Q obtained in this manner

$$\begin{aligned} Q &= \frac{1}{2}\sum_{j\neq k}\{(\frac{u_{tt}}{(1+u_{jj})(1+u_{kk})(n+\Delta u)})^{\frac{1}{2}}((1+u_{kk})u_{jj1}-(1+u_{jj})u_{kk1}) \\ &\quad +(\frac{u_{tk}u_{tk1}}{1+u_{kk}}-\frac{u_{tj}u_{tj1}}{1+u_{jj}})(\frac{(1+u_{jj})(1+u_{kk})}{u_{tt}(n+\Delta u)})^{\frac{1}{2}}\}^2 \\ &\quad +\frac{1}{2}\sum_{j\neq k}\frac{1}{(1+u_{jj})(1+u_{kk})u_{tt}(n+\Delta u)}(u_{tj1}u_{tk}(1+u_{kk})+u_{tk1}u_{tj}(1+u_{jj}))^2 \\ &\quad +\sum\frac{1}{u_{tt}(n+\Delta u)}u_{1jt}^2\sum_{k\neq j}\frac{1+u_{kk}}{1+u_{jj}}(\sum_{\ell\neq j,k}u_{t\ell}^2+\varepsilon)+E \end{aligned} \tag{4.24}$$

where the term E is defined by

$$E \equiv u_{tt} \sum_{j \neq k} \frac{u_{1jk}^2}{1+u_{jj}} - 2 \sum_{j \neq k} u_{tk} u_{tj1} u_{1jk} \frac{1}{1+u_{jj}}. \tag{4.25}$$

4.7. **The E terms.** We rewrite the E term as follows,

$$\begin{aligned} E &= \frac{1}{2} \sum_{j \neq k} u_{tt} \Big(\frac{u_{1jk}^2}{1+u_{jj}} + \frac{u_{1kj}^2}{1+u_{kk}} \Big) - \sum_{j \neq k} u_{1jk} \Big(\frac{u_{tk} u_{tj1}}{1+u_{jj}} + \frac{u_{tj} u_{tk1}}{1+u_{kk}} \Big) \\ &= \frac{1}{2} \sum_{j \neq k} \frac{1}{(1+u_{jj})(1+u_{kk})} \big\{ u_{1jk}^2 ((1+u_{jj}) + (1+u_{kk})) u_{tt} \\ &\quad -2u_{1jk} (u_{tk} u_{tj1} (1+u_{kk}) + u_{tj} u_{tk1} (1+u_{jj})) \big\}. \end{aligned} \tag{4.26}$$

4.8. **Differences when $\#G \geqslant 3$.** It is here that there seems to be a significant difference between the cases of $\#G = 2$ and $\#G > 2$. When $\#G = 2$ and $\mu_0 = 0$, we have, modulo $O(\varphi)$,

$$(1+u_{jj}) + (1+u_{kk}) = n + \Delta u, \forall j \neq k \in G, \tag{4.27}$$

but not otherwise.

4.9. **Case $\#G = 2$.** When $\#G = 2$, the argument can be completed as follows. Using the fact that $(1+u_{jj}) + (1+u_{kk}) = n + \Delta u$, we can write

$$E = \frac{1}{2} \sum_{j \neq k} \frac{u_{1jk}^2 (n+\Delta u) u_{tt} - 2u_{1jk} (u_{tk} u_{tj1} (1+u_{kk}) + u_{tj} u_{tk1} (1+u_{jj}))}{(1+u_{jj})(1+u_{kk})} \tag{4.28}$$

and hence

$$\begin{aligned} E &= \frac{1}{2} \sum_{j \neq k} \frac{1}{(1+u_{jj})(1+u_{kk})} \Big\{ u_{1jk} \sqrt{(n+\Delta u) u_{tt}} - \frac{u_{tk} u_{tj1} (1+u_{kk}) + u_{tj} u_{tk1} (1+u_{jj})}{\sqrt{u_{tt}(n+\Delta u)}} \Big\}^2 \\ &\quad - \frac{1}{2} \sum_{j \neq k} \frac{1}{(1+u_{jj})(1+u_{kk})} \frac{1}{u_{tt}(n+\Delta u)} (u_{tk} u_{tj1} (1+u_{kk}) + u_{tj} u_{tk1} (1+u_{jj}))^2 \end{aligned}$$

Note now that the next to last term in E cancels a term in the third formula for Q.

4.10. **A fourth formula for Q.** Thus we have obtained, when $\#G = 2$,

$$\begin{aligned} Q &= \frac{1}{2} \sum_{j \neq k} \Big\{ \Big(\frac{u_{tt}}{(1+u_{jj})(1+u_{kk})(n+\Delta u)} \Big)^{\frac{1}{2}} ((1+u_{kk}) u_{jj1} - (1+u_{jj}) u_{kk1}) \\ &\quad + \Big(\frac{u_{tk} u_{tk1}}{(1+u_{kk})} - \frac{u_{tj} u_{tj1}}{(1+u_{jj})} \Big) \Big(\frac{(1+u_{jj})(1+u_{kk})}{u_{tt}(n+\Delta u)} \Big)^{\frac{1}{2}} \Big\}^2 \\ &\quad + \frac{1}{2} \sum_{j \neq k} \frac{\Big\{ u_{1jk} \sqrt{(n+\Delta u) u_{tt}} - \frac{u_{tk} u_{tj1} (1+u_{kk}) + u_{tj} u_{tk1} (1+u_{jj})}{\sqrt{u_{tt}(n+\Delta u)}} \Big\}^2}{(1+u_{jj})(1+u_{kk})} \\ &\quad + \sum \frac{1}{u_{tt}(n+\Delta u)} u_{1jt}^2 \sum_{k \neq j} \frac{1+u_{kk}}{(1+u_{jj})} \Big(\sum_{\ell \neq j,k} u_{t\ell}^2 + \varepsilon \Big). \end{aligned} \tag{4.29}$$

This expression for Q shows that it is non-negative. The proof of Proposition 3 is complete.

5. Proof of the main theorems

Theorems 1 is a consequence of Proposition 2. In this case, since $\varepsilon > 0$, we have $D^2_{tx}u + I_{n+1} > 0$. If the matrix $D^2_x u + I_n - \mu_0 I_n$ has a zero eigenvalue on the boundary of $X \times I$, there is nothing to prove. Otherwise, if x_0 is an interior point with a zero eigenvalue of multiplicity $K \geqslant 1$, then all possible values of K are covered by Proposition 2 when $n \leqslant 2$. Thus $D^2_x u + I_n - \mu_o I_n$ vanishes everywhere, and in particular again on the boundary.

The argument for Theorem 2 is similar using Proposition 3, except that we need to show first that the solution u satisfies the space convexity condition $D^2_x u + I_n > 0$ for each t. In view of Proposition 3, we need to create a homotopic deformation path. Note that $u = 1 + t^2$ is the solution to the equation

$$u_{tt}(n + \Delta u) - \sum_{k=1}^{n} u_{tk}^2 = 2n,$$

with boundary data

$$u(x, 0) = 1, \quad u(x, 1) = 2, \quad \forall x \in X.$$

For $\varepsilon > 0$ and given boundary data u^0, u^1 satisfying (1.4), for each $0 \leqslant s \leqslant 1$, we consider the following family of equations

$$u_{tt}(n + \Delta u) - \sum_{k=1}^{n} u_{tk}^2 = s\varepsilon + 2n(1 - s), \tag{5.1}$$

with boundary data

$$u(x, 0) = su^0(x) + 1 - s, \qquad u(x, 1) = su^1(x) + 2(1 - s), \quad \forall x \in X. \tag{5.2}$$

It is easy to see that boundary data (5.2) satisfies the condition (1.4) (possibly with different $\lambda > 0$, but independent of s). By [17], the equation (5.1) has a unique smooth solution for each $s \in [0, 1]$. By continuity, the solution u satisfies $D^2_x u + I_n > 0$ for each t when $s > 0$ is small. Let $s_0 > 0$ be the first value of s where $D^2_x u + I_n$ has a zero eigenvalue for some $(x, t) \in X \times T$, if such a point exists. Then for all $s < s_0$, $D^2_x u + I_n > 0$ and we can apply Proposition 3 to the equation with Dirichlet data corresponding to this value of s. It follows that $D^2_x u + I_n \geqslant \lambda$ for $s < s_0$. By continuity, this inequality still holds at $s = s_0$. This is a contradiction, and thus no point with $D^2_x u + I_n$ with a zero eigenvalue exists. This establishes the strict space convexity of the solution for all $0 \leqslant s \leqslant 1$. By applying again Proposition 3, we obtain the precise lower bound $D^2_x u + I_n \geqslant \lambda$ for all $t, s \in [0, 1]$. Theorem 2 is proved.

Finally, Theorem 3 follows directly from the last statement of Proposition 3, and Theorem 4 follows directly from Theorem 3.

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