Vanishing Viscosity Limit for Isentropic Navier-Stokes Equations with Density-dependent Viscosity

Feimin Huang[a], Ronghua Pan[b], Tianyi Wang[a], Yong Wang[a,*], Xiaoyun Zhai[a]

[a]*Institute of Applied Mathematics, Academy of Mathematics and Systems Science*
Chinese Academy of Sciences, Beijing, 100190, China
[b]*School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332*

Abstract

In this paper, we study the vanishing viscosity limit of one-dimensional isentropic compressible Navier-Stokes equations with density-dependent viscosity, to the isentropic compressible Euler equations. Based on several new uniform estimates to the viscous systems, in addition to the framework recently established by G. Chen and M. Perepelitsa [10], we justify that the finite energy solution of the isentropic compressible Euler equations for a large class of initial data can be obtained as the inviscid limit of the compressible Navier-Stokes equations even when the viscosity depends on the density.

Keywords: compressible Navier-Stokes, compressible Euler equations.

AMS: 35L50, 35L60, 35L65, 76R50

1 Introduction

When the fluid density experiences large scale dropping, especially when vacuum is concerned, the motion of isentropic compressible viscous fluids is modeled by the following compressible Navier-Stokes equations with the density-dependent viscosity, in the Eulerian coordinates,

$$\begin{cases} \rho^\varepsilon_t + (\rho^\varepsilon u^\varepsilon)_x = 0, \\ (\rho^\varepsilon u^\varepsilon)_t + (\rho^\varepsilon (u^\varepsilon)^2 + p(\rho^\varepsilon))_x = \varepsilon((\rho^\varepsilon)^\alpha u^\varepsilon_x)_x, \end{cases} \tag{1.1}$$

*Corresponding author.
Email addresses: fhuang@amt.ac.cn(F. Huang), panrh@math.gatech.edu(R.Pan), wangtianyi@amss.ac.cn(T. Wang), yongwang@amss.ac.cn(Y. Wang), zhaixy@amt.ac.cn(X. Zhai)

where ρ^ε and u^ε denote the density and the velocity of the fluid, respectively. $m^\varepsilon = \rho^\varepsilon u^\varepsilon$ represents the momentum. $p = p(\rho)$ is pressure function of the density. In this paper, we consider the polytropic perfect gas, i.e.

$$p(\rho) = \kappa \rho^\gamma,$$

where $\gamma > 1$ is the adiabatic exponent, and the constant κ is chosen as $\kappa = \frac{(\gamma-1)^2}{4\gamma}$ up to a scaling. While $\varepsilon > 0$ is adpated to the system as the controlling parameter on the amplitude of viscosity, for which we assume $\varepsilon \in (0, \varepsilon_0]$ for some fixed $\varepsilon_0 > 0$ without loss of generality; $\alpha \geq 0$ is a constant which models the dependence of viscosity on density.

When $\alpha = 0$, (1.1) reduces to the classical compressible Navier-Stokes equation, called **CNS**. The case of $\alpha > 0$ occurs for non-uniform gases [5], and (1.1) can be formally derived by Chapman-Enskog expansion from the Boltzmann equation for (at least) hard sphere model and cut-off inverse power force model. A formal derivation can be found in [39]. It is also interesting to note that when $\alpha = 1$ and $\gamma = 2$, (1.1) recovers the "viscous Saint-Venant" system for shallow water without bottom friction [19], see also [11]. In this paper, we will focus on (1.1) with positive α, for which we call it **α-CNS**, distinguishing from the case of $\alpha = 0$, which is called **CNS**. On the other hand, the studies in [24] and [37] indicate the failure of CNS at vacuum and the validity of α-CNS at least at the level of local well-posedness theory. We therefore devote our efforts to this model in current paper.

We now consider the Cauchy problem of (1.1) when the far fields of the fluid are away from vacuum. Namely, we shall study the α-CNS (1.1) with the following initial data

$$\rho^\varepsilon(0, x) = \rho_0^\varepsilon(x) > 0, \quad u^\varepsilon(0, x) = u_0^\varepsilon(x), \tag{1.2}$$

such that

$$\lim_{x \to \pm\infty} (\rho_0^\varepsilon(x), u_0^\varepsilon(x)) = (\rho^\pm, u^\pm), \ \textit{with}\ \rho^\pm > 0.$$

In the past decades, the study of the mathematical theory on (1.1)–(1.2) has attracted a lot attention. Many interesting results were established for the local and global existence of both classical and weak solutions, we refer the readers to some of them such as, [2], [3], [17], [18], [22], [27], [28], [29], [30], [33], [37], [39], [40], [41] and [51]. It is equally interesting to study the inviscid limit for (1.1)–(1.2) as $\varepsilon \to 0$ toward the following one-dimensional isentropic Euler equations

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0. \end{cases} \tag{1.3}$$

It is a general belief that the physcial weak solution of (1.3) can be obtained in such a process, see [4], where a vanishing artificial viscosity limit for general hyperbolic system with small BV data is proved. This problem is closely related to the existence of weak solutions to (1.3) through a limitting process of physical approximation. In this paper, we will address this problem and study the vanishing viscosity limit for (1.1)–(1.2).

In BV framework, when the initial data is away from vacuum, the existence of global BV solution to (1.3) was established by [44] for $\gamma > 1$ and by [45] for $\gamma = 1$ using Glimm's method. (1.3) shows singular behavior when vacuum occurs which causes difficulties to mathematical analysis. It is still a major open problem on how to perform BV estimate when the solution may contain vacuum states. Instead, the L^∞ framework is successfully achieved using the theory of compensated compactness [43], [52]. The existence of L^∞ weak entropy solution of (1.3) was established by [12] for $\gamma = 1 + \frac{2}{2n+1}, n \geq 2$; by [13] for $\gamma \in (1, \frac{5}{3}]$; by [35] and [36] for $\gamma > \frac{5}{3}$; and finally by [26] for $\gamma = 1$. Recently, [32] further constructed the finite-energy solutions to the isentropic Euler equations with finite-energy initial data. We remark that these results are achieved through the vanishing artificial viscosity.

The problem of vanishing physical viscosity limit is more subtle and the progress has been less satisfactory, and the problem of vanishing viscosity limit of Navier-Stokes equations to Euler equations has been open for long time, though some interesting results are proved when restrictive initial data is assigned, see [23] and [53]. Recently, G. Chen and M. Perepelitsa [10] proved that the solutions of Navier-Stokes ($\alpha = 0$), whose viscosity is independent of density, converge to the finite energy solution of Euler equations as viscosity vanishes. This is a major breakthrough in this aspect.

Inspired by [10], we study the problem of vanishing viscosity for the α-CNS (1.1)–(1.2) in this paper with positive α. It is clear that for any fixed positive ε, the visocity coefficient with positive α experiences degeneracy near vacuum states. An obvious obstacle is the dissipation term in the energy identity contains only the weighted norm of velocity gradient which degenerates at vacuum. Such a singular behavior causes the major difficulty in the analysis and introduced the different behavior of solutions compared with CNS where $\alpha = 0$. The analysis exibits quite different flavor and requires very different ingredients. Fortunately, by a deep observation, we obtained several key uniform estimates. Based on these uniform estimates and the framework of [10], we are able to show that, when viscosity parameter ε tends to zero, the solutions of α-CNS (1.1)-(1.2) converge to the finite-energy solution of Euler equations for general initial data.

We now prepare to state our main result.

A pair of functions $(\eta(\rho, u), q(\rho, u))$, or $(\eta(\rho, m), q(\rho, m))$ for $m = \rho u$, is called an entropy-entropy flux pair of system (1.3), if the following holds

$$[\eta(\rho, u)]_t + [q(\rho, u)]_x = 0,$$

for any smooth solutions of (1.3). Furthermore, $\eta(\rho, m)$ is called a weak entropy if

$$\eta(0, u) = 0, \text{ for any fixed } u.$$

An entropy $\eta(\rho, m)$ is convex if the Hessian $\nabla^2 \eta(\rho, m)$ is nonnegative definite in the region under consideration.

From [36], it is well known that any week entropy (η, q) can be represented by

$$\begin{cases} \eta^{\psi}(\rho,\rho u)=\eta^{\psi}(\rho,m)=\displaystyle\int_{\mathbb{R}}\chi\left(\rho;s-u\right)\psi\left(s\right)ds, \\ q^{\psi}(\rho,\rho u)=q^{\psi}(\rho,m)=\displaystyle\int_{\mathbb{R}}\left(\theta s+\left(1-\theta\right)u\right)\chi\left(\rho;s-u\right)\psi\left(s\right)ds. \end{cases} \tag{1.4}$$

where the kernel is $\chi(\rho; s-u) = [\rho^{2\theta} - (s-u)^2]_+^{\lambda}$, $\lambda = \frac{3-\gamma}{2(\gamma-1)} > -\frac{1}{2}$, and $\theta = \frac{\gamma-1}{2}$. For instance, when $\psi(s) = \frac{1}{2}s^2$, the entropy pair is the mechanical energy and the associated flux

$$\eta^*(\rho,m)=\frac{m^2}{2\rho}+e(\rho), \quad q^*(\rho,m)=\frac{m^3}{2\rho^2}+me'(\rho), \tag{1.5}$$

where $e(\rho) = \frac{\kappa}{\gamma-1}\rho^{\gamma}$ represents the gas internal energy in physics.

Let $(\bar{\rho}(x), \bar{u}(x))$ be a pair of smooth monotone functions satisfying $(\bar{\rho}(x), \bar{u}(x)) = (\rho^{\pm}, u^{\pm})$, when $\pm x \geq L_0$ for some large $L_0 > 0$. The total mechanical energy for (1.1) in $\mathbb{R}$ with respect to the pair of reference function $(\bar{\rho}(x), \bar{u}(x))$ is

$$E\left[\rho,u\right](t)=\int_{\mathbb{R}}\left(\eta^*\left(\rho,m\right)-\eta^*\left(\bar{\rho},\bar{m}\right)-\nabla\eta^*(\bar{\rho},\bar{m})\cdot(\rho-\bar{\rho},m-\bar{m})\right)dx, \tag{1.6}$$

where $\bar{m} = \bar{\rho}\bar{u}$. After some calculations, we obtain that

$$E\left[\rho,u\right](t)=\int_{\mathbb{R}}\left(\frac{1}{2}\rho\left(t,x\right)\left|u(t,x)-\bar{u}(x)\right|^2+e^*\left(\rho(t,x),\bar{\rho}(x)\right)\right)dx \tag{1.7}$$

where $e^*(\rho, \bar{\rho}) = e(\rho) - e(\bar{\rho}) - e'(\bar{\rho})(\rho - \bar{\rho}) \geq 0$.

Definition 1.1 *Let (ρ_0, u_0) be given initial data with finite-energy with respect to the end states $(\rho^{\pm}, u^{\pm})$ at infinity, and $E[\rho_0, u_0] \leq E_0 < \infty$. A pair of measurable functions $(\rho, u) : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ is called a finite-energy entropy solution of the Cauchy problem (1.3) if the following holds:*

(i) The total energy in bounded in time: There is a bounded function $C(E,t)$, defined on $\mathbb{R}^+ \times \mathbb{R}^+$ and continuous in t for each $E \in \mathbb{R}^+$, such that, for a.e. $t > 0$,

$$E\left[\rho,u\right](t)\leq C\left(E_0,t\right);$$

(ii) The entropy inequality:

$$\eta^{\psi}\left(\rho,u\right)_t+q^{\psi}\left(\rho,u\right)_x\leq 0,$$

is satisfied in the sense of distributions for all test functions $\psi(s) \in \{\pm 1, \pm s, s^2\}$;

(iii) The initial data (ρ_0, u_0) are attained in the sense of distributions.

We now state our main conditions on the initial data (1.2), which is motivated from [10].

Condition 1.1 *Let* $(\bar{\rho}(x), \bar{u}(x))$ *be some pair of smooth monotone functions satisfying* $(\bar{\rho}(x), \bar{u}(x)) = (\rho^{\pm}(x), u^{\pm}(x))$ *when* $\pm x \geq L_0$ *for some large* $L_0 > 0$*. For positive constants* E_0*,* E_1 *and* M_0 *independent of* ε*, and* $c_0^{\varepsilon} > 0$ *the initial functions* $(\rho_0^{\varepsilon}, u_0^{\varepsilon})$ *are smooth satisfying the following properties*

(i) $\rho_0^{\varepsilon} \geq c_0^{\varepsilon} > 0, \quad \int_{\mathbb{R}} \rho_0^{\varepsilon}(x) |u_0^{\varepsilon}(x) - \bar{u}(x)| dx \leq M_0 < \infty$ *;*

(ii) The total mechanical energy with respect to $(\bar{\rho}, \bar{u})$ *is finite:*

$$\int_{\mathbb{R}} \left(\frac{1}{2} \rho_0^{\varepsilon} |u_0^{\varepsilon}(x) - \bar{u}(x)|^2 + e^*(\rho_0^{\varepsilon}(x), \bar{\rho}(x)) \right) dx \leq E_0 < \infty;$$

(iii) $\varepsilon^2 \int_{\mathbb{R}} \frac{|\rho_{0,x}^{\varepsilon}(x)|^2}{\rho_0^{\varepsilon}(x)^{3-2\alpha}} dx \leq E_1 < \infty$*;*

(iv)$(\rho_0^{\varepsilon}(x), \rho_0^{\varepsilon}(x) u_0^{\varepsilon}(x)) \to (\rho_0(x), \rho_0(x) u_0(x))$ *in the sense of distributions as* $\varepsilon \to 0$*, with* $\rho_0(x) \geq 0 \quad$ *a.e..*

Our main results are stated in the following Theorem.

Theorem 1.1 *Assume* $\frac{2}{3} \leq \alpha \leq \gamma$*,* $\gamma > 1$*. Let* $(\rho^{\varepsilon}, u^{\varepsilon})$*,* $m^{\varepsilon} = \rho^{\varepsilon} u^{\varepsilon}$ *be the solution of the Cauchy problem (1.1)-(1.2) with initial data* $(\rho_0^{\varepsilon}, u_0^{\varepsilon})$ *which satisfies Condition 1.1 for each fixed* $\varepsilon > 0$*. Then, when* $\varepsilon \to 0$*, there exists a subsequence of* $(\rho^{\varepsilon}, m^{\varepsilon})$ *that converges almost everywhere to a finite-energy entropy solution* (ρ, m) *to the Cauchy problem (1.3) with initial data* $(\rho_0, \rho_0 u_0)$ *for the isentropic Euler equations.*

Remark 1.1 *Due to some technical difficulty, we can only prove the result for* $\frac{2}{3} \leq \alpha \leq \gamma$*. In fact, for many physical gases the Chapman-Enskog viscosity predicts that* $\alpha \geq \frac{\gamma-1}{2}$*, see [5], and [39]. Our condition* $\frac{2}{3} \leq \alpha \leq \gamma$ *is valid for many physical cases including the shallow water model, but it did not cover the case of monoatomic gas where* $\gamma = \frac{5}{3}$ *and* $\alpha = \frac{1}{2}$*. It is very interesting to prove the result for* $\alpha \in [0, \frac{2}{3}]$*, which will be addressed later.*

One important basis of our proof for Theorem 1.1 is the following compactness theorem, established in [10].

Theorem 1.2 (Chen-Perepelitsa [10]) *Let* $\psi \in C_0^2(\mathbb{R})$*,* (η^{ψ}, q^{ψ}) *be a weak entropy pair generated by* ψ*. Assume that the sequences* $(\rho^{\varepsilon}(x,t), u^{\varepsilon}(x,t))$ *defined on* $\mathbb{R} \times \mathbb{R}_+$ *with* $m^{\varepsilon} = \rho^{\varepsilon} u^{\varepsilon}$*, satisfies the following conditions:*

(i). For any $-\infty < a < b < \infty$ *and all* $t > 0$*, it holds that*

$$\int_0^t \int_a^b (\rho^{\varepsilon})^{\gamma+1} dx d\tau \leq C(t, a, b), \tag{1.8}$$

where $C(t) > 0$ *is independent of* ε*.*

(ii).For any compact set $K \subset \mathbb{R}$*, it holds that*

$$\int_0^t \int_K (\rho^{\varepsilon})^{\gamma+\theta} + \rho^{\varepsilon} |u^{\varepsilon}|^3 dx d\tau \leq C(t, K), \tag{1.9}$$

where $C = C(t, K) > 0$ is independent of ε.

(iii). The sequence of entropy dissipation measures

$$\eta^\psi(\rho^\varepsilon, m^\varepsilon)_t + q^\psi(\rho^\varepsilon, m^\varepsilon)_x \text{ are compact in } H^{-1}_{loc}(\mathbb{R}^2_+). \tag{1.10}$$

Then there is a subsequence of $(\rho^\varepsilon, m^\varepsilon)$(still denoted as $(\rho^\varepsilon, m^\varepsilon)$) and a pair of measurable functions (ρ, m) such that

$$(\rho^\varepsilon, m^\varepsilon) \to (\rho, m), \quad a.e. \text{ as } \varepsilon \to 0. \tag{1.11}$$

In section 2 and section 3 below, we will verify conditions (1.8), (1.9) and (1.10) to prove our main theorem 1.1.

The rest of this paper is arranged as follows. In section 2, we make some new uniform estimates for the solutions of Navier-Stokes equations (1.1) which are independent of ε. These estimates are essential to show the convergence of the vanishing viscosity limit to the Euler equations. In section 3, using the estimate we obtained in section 2, we prove the $H^{-1}_{loc}(\mathbb{R}^2_+)-$ compactness for the solutions of (1.1). In section 4, based on the framework in [10], we prove our main Theorem1.1.

2 Uniform Estimates for the Solutions of α-CNS

First, we assume that $(\rho^\varepsilon, u^\varepsilon)$ is the global smooth solutions of Navier-Stokes equations (1.1)–(1.2), satifying

$$\rho^\varepsilon(x, t) \geq c^\varepsilon(t), \text{ for some } c^\varepsilon(t) > 0 \tag{2.1}$$

and

$$\lim_{x \to \pm\infty} (\rho^\varepsilon, u^\varepsilon)(x, t) = (\rho^\pm, u^\pm). \tag{2.2}$$

For the existence of global smooth solutions, the reders are referred to [28], [39] and [41]. Based on the above preparation, we now make some new the uniform estimates with respect to ε for the solutions $(\rho^\varepsilon, u^\varepsilon)$ of the α-CNS (1.1)–(1.2).

For simplicity, throughout this section, we denote $(\rho, u) = (\rho^\varepsilon, u^\varepsilon)$ without causing confusion and $C > 0$ denote the constant independent of ε.

Lemma 2.1 *(Energy Estimates) Suppose that $0 \leq \alpha \leq \gamma$, and $E[\rho_0, u_0] \leq E_0 < \infty$ for some $E_0 > 0$ independent of ε. It holds that*

$$\sup_{0 \leq \tau \leq t} E[\rho, u](\tau) + \varepsilon \int_0^t \int_{\mathbb{R}} \rho^\alpha u_x^2 dx d\tau \leq C(t), \tag{2.3}$$

where $C(t)$ depends on E_0, t, $\bar{\rho}$, and $\bar{u}$, but not on ε.

Proof. From the definition, we have

$$\frac{dE(t)}{dt} = \frac{d}{dt}\int_{\mathbb{R}} \eta^*(\rho, m) - \eta^*(\bar\rho, \bar m)\ dx - \int_{\mathbb{R}} \nabla\eta^*(\bar\rho, \bar m)(\rho_t, m_t)\ dx. \tag{2.4}$$

Since (η^*, q^*) is an entropy pair, we have

$$\eta^*(\rho, m)_t + q^*(\rho, m)_x - \varepsilon\eta^*_m(\rho, m)(\rho^\alpha u_x)_x = 0. \tag{2.5}$$

Integrate (2.5) with respect to x over $\mathbb{R}$, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} \eta^*(\rho, m) - \eta^*(\bar\rho, \bar m)\ dx + \varepsilon\int_{\mathbb{R}} \rho^\alpha u_x^2\ dx = q^*(\rho^-, m^-) - q^*(\rho^+, m^+). \tag{2.6}$$

Since we have

$$e^*(\rho, \bar\rho) \geq \rho(\rho^\theta - \bar\rho^\theta)^2,\ \theta = \frac{\gamma - 1}{2}. \tag{2.7}$$

Utilizing (2.7), we obtain

$$\begin{aligned}
&|\int_{\mathbb{R}} \nabla\eta^*(\bar\rho, \bar m)(\rho_t, m_t)dx| \\
&= |\int_{\mathbb{R}} \nabla\eta^*(\bar\rho, \bar m)(m_x, (p(\rho) + \rho u^2 - \varepsilon\rho^\alpha u_x)_x)dx| \\
&= |\int_{\mathbb{R}} \nabla\eta^*(\bar\rho, \bar m)_x(m, p(\rho) + \rho u^2 - \varepsilon\rho^\alpha u_x)\ dx| \\
&\leq \frac{\varepsilon}{4}\int_{\mathbb{R}} \rho^\alpha u_x^2\ dx + C\int_{\mathbb{R}} \rho|u - \bar u|^2\ dx + C\int_{-L_0}^{L_0} (\rho + p(\rho) + \rho^\alpha)dx + C \\
&\leq C + CE + \frac{\varepsilon}{4}\int_{\mathbb{R}} \rho^\alpha u_x^2 dx,
\end{aligned}$$

where we have used

$$\int_{-L_0}^{L_0} (\rho + p(\rho) + \rho^\alpha)dx \leq CE, \quad \text{for} \ \ 0 \leq \alpha \leq \gamma. \tag{2.8}$$

Substituting (2.6) and (2.8) into (2.4), we obtain

$$\frac{dE(t)}{dt} + \frac{3\varepsilon}{4}\int_{\mathbb{R}} \rho^\alpha u_x^2\ dx \leq C + CE, \tag{2.9}$$

Then Gronwall's inequality implies Lemma2.1.

Remark 2.1 *Since vacuum could occur in our solution, the inequality*

$$\int_0^t\int_{\mathbb{R}} \rho^\alpha u_x^2\ dxd\tau \leq C(t)$$

in (2.3) *is much weaker than the corresponding one*

$$\int_0^t \int_{\mathbb{R}} u_x^2 dx d\tau \le C(t).$$

in [10]. This will cause a great difficulty to prove Lemma 2.3 below, which is an essential step to verify the condition i) of Theorem 1.2, i.e. (1.8).

We now derive some higher order estimates.

Lemma 2.2 *If* $0 < \alpha \le \gamma$, *and* $(\rho_0(x), u_0(x))$ *satisfies*

$$\varepsilon^2 \int_{\mathbb{R}} \frac{|\rho_{0x}(x)|^2}{\rho_0(x)^{3-2\alpha}} \; dx \le E_1 < \infty,$$

for some E_1 *independent of* ε. *Then, for any* $t > 0$, *it holds that*

$$\varepsilon^2 \int_{\mathbb{R}} \rho^{2\alpha-3} \rho_x^2 \; dx + \varepsilon \int_0^t \int_{\mathbb{R}} \rho^{\alpha+\gamma-3} \rho_x^2 \; dx d\tau \le C(t), \tag{2.10}$$

where $C(t)$ *depends on* E_0, E_1, t, $\bar{\rho}$, $\bar{u}$, *but not on* ε.

Proof. Through (1.1), we have

$$\begin{cases} \rho_{xt} + \rho_{xx} u + 2\rho_x u_x + \rho u_{xx} = 0, \\ \rho u_t + \rho u u_x + p(\rho)_x = \varepsilon \rho^\alpha u_{xx} + \varepsilon \alpha \rho^{\alpha-1} u_x \rho_x. \end{cases} \tag{2.11}$$

Multiplying $(2.11)_1$ with $\rho^{2\alpha-3}\rho_x$, after some calculation, we obtain

$$\left(\frac{\rho^{2\alpha-3}\rho_x^2}{2}\right)_t + \left(\frac{\rho^{2\alpha-3}u\rho_x^2}{2}\right)_x + \alpha \rho^{2\alpha-3}\rho_x^2 u_x + \rho^{2\alpha-2}\rho_x u_{xx} = 0. \tag{2.12}$$

From $(2.11)_2 \times \rho^{\alpha-2}\rho_x$, after some calculation, we reach

$$\rho^{\alpha-1}\rho_x u_t + \rho^{\alpha-1} u \rho_x u_x + \kappa \rho^{\alpha+\gamma-3}\rho_x^2 = \varepsilon\alpha\rho^{2\alpha-3}\rho_x^2 u_x + \varepsilon\rho^{2\alpha-2}\rho_x u_{xx}. \tag{2.13}$$

The combination $\varepsilon^2(2.12) + \varepsilon(2.13)$ gives that

$$\begin{aligned} &\left(\frac{\varepsilon^2\rho^{2\alpha-3}\rho_x^2}{2}\right)_t + \left(\frac{\varepsilon^2\rho^{2\alpha-3}u\rho_x^2}{2}\right)_x + \varepsilon\kappa\rho^{\alpha+\gamma-3}\rho_x^2 \\ &\quad + \left(\varepsilon\rho^{\alpha-1}\rho_x u\right)_t - \left(\varepsilon\rho^{\alpha-1}\rho_t u\right)_x = \varepsilon\rho^\alpha u_x^2. \end{aligned} \tag{2.14}$$

Integrating (2.14) over $[0,t] \times \mathbb{R}$, we obtain

$$\begin{aligned} &\varepsilon^2 \int_{\mathbb{R}} \frac{\rho^{2\alpha-3}\rho_x^2}{2} \; dx + \varepsilon\kappa \int_0^t \int_{\mathbb{R}} \rho^{\alpha+\gamma-3}\rho_x^2 \; dx d\tau \\ &= \varepsilon^2 \int_{\mathbb{R}} \frac{\rho_0^{2\alpha-3}\rho_{0x}^2}{2} dx + \varepsilon \int_0^t \int_{\mathbb{R}} \rho^\alpha u_x^2 dx d\tau - \varepsilon \int_0^t \int_{\mathbb{R}} \left(\rho^{\alpha-1}\rho_x u\right)_\tau dx d\tau \\ &\le C(t) - \varepsilon \int_0^t \int_{\mathbb{R}} \left(\rho^{\alpha-1}\rho_x u\right)_\tau dx d\tau. \end{aligned} \tag{2.15}$$

Noticing that

$$\left(\rho^{\alpha-1}\rho_x u\right)_t = \left(\rho^{\alpha-1}\rho_x(u-\bar{u})\right)_t + \frac{1}{\alpha}\left((\rho^\alpha)_x\bar{u}\right)_t, \tag{2.16}$$

Integrating (2.16) with respect to x over $\mathbb{R}$, we have

$$\begin{aligned}
&\varepsilon\int_0^t\int_{\mathbb{R}}\left(\rho^{\alpha-1}\rho_x u\right)_\tau\, dxd\tau\\
&=\varepsilon\int_{\mathbb{R}}\rho^{\alpha-1}\rho_x(u-\bar{u})\ dx-\varepsilon\int_{\mathbb{R}}\rho_0^{\alpha-1}\rho_{0x}(u_0-\bar{u})\ dx\\
&\qquad+\frac{\varepsilon}{\alpha}\int_{\mathbb{R}}(\rho^\alpha)_x\bar{u}dx+\frac{\varepsilon}{\alpha}\int_{\mathbb{R}}(\rho_0^\alpha)_x\bar{u}\ dx\\
&\le C(t),
\end{aligned} \tag{2.17}$$

where we have used the following estimates (2.18)-(2.20)

$$\begin{aligned}
&\varepsilon\int_{\mathbb{R}}\rho^{\alpha-1}\rho_x(u-\bar{u})\ dx\\
&\le\frac{\varepsilon^2}{8}\int_{\mathbb{R}}\rho^{2\alpha-3}\rho_x^2dx+C\int_{\mathbb{R}}\rho(u-\bar{u})^2\ dx\\
&\le\frac{\varepsilon^2}{8}\int_{\mathbb{R}}\rho^{2\alpha-3}\rho_x^2dx+C(t),
\end{aligned} \tag{2.18}$$

$$\varepsilon\int_{\mathbb{R}}\rho_0^{\alpha-1}\rho_{0x}(u_0-\bar{u})\ dx\le\frac{\varepsilon^2}{8}\int_{\mathbb{R}}\rho_0^{2\alpha-3}\rho_{0x}^2dx+C(t), \tag{2.19}$$

$$\begin{aligned}
\frac{\varepsilon}{\alpha}\int_{\mathbb{R}}(\rho^\alpha)_x\bar{u}\ dx&=-\frac{\varepsilon}{\alpha}\int_{\mathbb{R}}\rho^\alpha\bar{u}_xdx+\frac{\varepsilon}{\alpha}\Big((\rho^+)^\alpha\bar{u}^+-(\rho^-)^\alpha\bar{u}^-\Big)\\
&\le\frac{C\varepsilon}{\alpha}\int_{-L_0}^{L_0}\rho^\alpha dx+C(t)\\
&\le C(t),
\end{aligned} \tag{2.20}$$

Substituting (2.17)–(2.20) into (2.15), we obtain Lemma2.2.

The following higher order integrability estimate is crucial in compactness argument.

Lemma 2.3 *If the conditions of Lemma 2.1 hold and $0<\alpha\le\gamma$, then for any $-\infty<a<b<\infty$ and all $t>0$, it holds that*

$$\int_0^t\int_a^b\rho^{\gamma+1}\ dxd\tau\le C(t,a,b), \tag{2.21}$$

where $C(t)>0$ depends on E_0, a, b, γ, t, $\bar{\rho}$, $\bar{u}$, but not on ε.

Proof. Choose

$$w(x) \in C_0^\infty(\mathbb{R}),\ 0 \le w(x) \le 1,\ w(x) = 1 \text{ for } x \in [a,b],\ \text{and } supp\{w\} = (a-1, b+1)$$

By $(1.1)_2 \times w$, we have

$$(p(\rho)w)_x = -(\rho u^2 w)_x + (p(\rho) + \rho u^2)w_x - (\rho u)_t w + \varepsilon(\rho^\alpha u_x w)_x - \varepsilon \rho^\alpha u_x w_x \tag{2.22}$$

Integrating (2.22) with respect to spatial variable over $(-\infty, x)$, we obtain

$$\begin{aligned} p(\rho)w = -\rho u^2 w + \varepsilon \rho^\alpha u_x w - \left(\int_{-\infty}^x \rho u w \; dy \right)_t \\ + \int_{-\infty}^x [(\rho u^2 + p(\rho))w_x - \varepsilon \rho^\alpha u_x w_x] \; dy. \end{aligned} \tag{2.23}$$

Multiplying (2.23) by ρw, we have

$$\begin{aligned} \rho p(\rho) w^2 &= -\rho^2 u^2 w^2 + \varepsilon \rho^{\alpha+1} u_x w^2 - \left(\rho w \int_{-\infty}^x \rho u w \; dy \right)_t \\ &\quad - (\rho u)_x w \int_{-\infty}^x \rho u w \; dy + \rho w \int_{-\infty}^x [(\rho u^2 + p(\rho))w_x - \varepsilon \rho^\alpha u_x w_x] \; dx \\ &= \varepsilon \rho^{\alpha+1} u_x w^2 - \left(\rho w \int_{-\infty}^x \rho u w \; dy \right)_t - \left(\rho u w \int_{-\infty}^x \rho u w \; dy \right)_x \\ &\quad + \rho u w_x \int_{-\infty}^x \rho u w \; dy + \rho w \int_{-\infty}^x [(\rho u^2 + p(\rho))w_x - \varepsilon \rho^\alpha u_x w_x] \; dy \end{aligned} \tag{2.24}$$

Integrating (2.24) over $(0,t) \times \mathbb{R}$, we have

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} \kappa \rho^{\gamma+1} w^2 \; dx d\tau \\ &= \varepsilon \int_0^t \int_{\mathbb{R}} \rho^{\alpha+1} u_x w^2 \; dx d\tau - \int_{\mathbb{R}} \left(\rho w \int_{-\infty}^x \rho u w \; dy \right) dx \\ &\quad + \int_{\mathbb{R}} \left(\rho_0 w \int_{-\infty}^x \rho_0 u_0 w dy \right) dx + \int_0^t \int_{\mathbb{R}} \left(\rho u w_x \int_{-\infty}^x \rho u w \; dy \right) dx d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}} \left(\rho w \int_{-\infty}^x [(\rho u^2 + p(\rho))w_x - \varepsilon \rho^\alpha u_x w_x] dy \right) dx d\tau. \end{aligned} \tag{2.25}$$

Let

$$A = \{x : \rho(x,t) \ge \hat{\rho}\},\ \text{where } \hat{\rho} = 2\max\{\rho+, \rho-\}, \tag{2.26}$$

then we have the following estimates by (2.3)

$$|A| \le \frac{C(t)}{e^*(2\hat{\rho}, \bar{\rho})} =: d(t). \tag{2.27}$$

By (2.26), we know that for any (x,t) there exists a point $x_0 = x_0(x,t)$ such that $|x - x_0| \le d(t)$ and $\rho(x_0,t) = \hat{\rho}$. Here we choose $\beta = \alpha + \frac{\gamma-1}{2} > 0$,

$$\begin{aligned}
&\sup_{x\in supp\{w\}} \varepsilon\rho^{\beta}(x,t) \le \varepsilon\hat{\rho}^{\beta} + \sup_{x\in supp\{w\}\cap A} \varepsilon\rho^{\beta}(x,t) \\
&\le 2\varepsilon\hat{\rho}^{\beta} + \sup_{x\in supp\{w\}\cap A} |\varepsilon\rho^{\beta}(x,t) - \varepsilon\rho^{\beta}(x_0,t)| \\
&\le 2\varepsilon\hat{\rho}^{\beta} + \sup_{x\in supp\{w\}\cap A} \int_{x_0-d(t)}^{x_0+d(t)} |\beta||\varepsilon\rho^{\beta-1}\rho_x|\ dx \\
&\le 2\varepsilon\hat{\rho}^{\beta} + \int_{a-1-2d(t)}^{b+1+2d(t)} |\beta||\varepsilon\rho^{\beta-1}\rho_x|\ dx \qquad (2.28) \\
&\le 2\varepsilon\hat{\rho}^{\beta} + \int_{a-1-2d(t)}^{b+1+2d(t)} |\beta|\rho^{2\beta-2\alpha+1}\ dx + \int_{\mathbb{R}} \varepsilon^2\rho^{2\alpha-3}\rho_x^2\ dx \\
&\le C(t) + \int_{a-1-2d(t)}^{b+1+2d(t)} \rho^{\gamma}\ dx \\
&\le C(t)
\end{aligned}$$

Using (2.28), the first term on the right hand side of (2.25) can be estimated as following

$$\begin{aligned}
&\varepsilon\int_0^t\int_{\mathbb{R}} \rho^{\alpha+1}u_x w^2 dxd\tau \\
&\le \varepsilon\int_0^t\int_{\mathbb{R}} \rho^{\alpha+2}w^4\ dxd\tau + \varepsilon\int_0^t\int_{\mathbb{R}} \rho^{\alpha}u_x^2\ dxd\tau \\
&\le C(t) + \varepsilon\int_0^t\int_{\mathbb{R}} \rho^{\alpha+2}w^2\ dxd\tau \\
&\le \begin{cases} C(t) + \varepsilon\int_0^t\int_{\mathbb{R}} \rho^{\beta}w^2\ dxd\tau, \text{ if } \ 2\alpha+2 \le \beta \\ C(t) + \int_0^t \sup_{x\in supp\{w\}} \varepsilon\rho^{\beta}(x,\tau)\int_{\mathbb{R}} \rho^{2\alpha+2-\beta}w^2\ dxd\tau, \ \text{ if } \ 2\alpha+2 > \beta \end{cases} \qquad (2.29) \\
&\le C(t) + C(t)\int_0^t\int_{\mathbb{R}} \rho^{2\alpha+2-\beta}w^2\ dxd\tau \\
&\le C(t) + \delta\int_0^t\int_{\mathbb{R}} \rho^{\gamma+1}w^2\ dxd\tau.
\end{aligned}$$

Here we have used the fact $2\alpha + 2 - \beta < \gamma + 1$ for $\gamma > 1$.

By Lemma 2.1 and the Hölder inequality, we obtain

$$\begin{aligned}
|\int_{-\infty}^{x} \rho u w dy| \ &\le \ \int_{supp\{w\}} |\rho u| dy \\
&= \ \left(\int_{supp\{w\}} \rho dy\right)^{\frac{1}{2}} \left(\int_{supp\{w\}} \rho u^2 dy\right)^{\frac{1}{2}} \le C(t). \qquad (2.30)
\end{aligned}$$

Then it follows that

$$
\begin{aligned}
&|\int_{\mathbb{R}} \left(\rho w \int_{-\infty}^{x} \rho u w \; dy\right) dx| + |\int_{\mathbb{R}} \left(\rho_0 w \int_{-\infty}^{x} \rho_0 u_0 w \; dy\right) dx| && (2.31)\\
&\quad + |\int_0^t \int_{\mathbb{R}} \left(\rho u w_x \int_{-\infty}^{x} \rho u w \; dy\right) dx d\tau| \le C(t). && (2.32)
\end{aligned}
$$

Similarly, we have

$$
|\int_0^t \int_{\mathbb{R}} \left(\rho w \int_{-\infty}^{x} (\rho u^2 + p(\rho)) w_x \; dy\right) dx d\tau| \le C(t), \tag{2.33}
$$

and

$$
\begin{aligned}
&|\varepsilon \int_0^t \int_{\mathbb{R}} \left(\rho w \int_{-\infty}^{x} \rho^\alpha u_x w_x \; dy\right) dx d\tau| \\
&\le \varepsilon \int_0^t \int_{\mathbb{R}} \left(\rho w \int_{\mathbb{R}} \rho^\alpha |u_x||w_x| \; dy\right) dx d\tau \\
&\le \varepsilon \int_0^t \left(\int_{\mathbb{R}} \rho w \; dx\right)\left(\int_{\mathbb{R}} \rho^\alpha u_x^2 \; dy + \int_{\mathbb{R}} \rho^\alpha w_x^2 \; dy\right) d\tau \\
&\le C(t)
\end{aligned} \tag{2.34}
$$

Substituting (2.29), (2.31) $-$ (2.34) into (2.25) and noticing the smallness of δ, we proved Lemma2.3.

Lemma 2.4 *Suppost that* $\frac{2}{3} \le \alpha \le \gamma$, *and* $(\rho_0(x), u_0(x))$ *satisfy the conditions in the Lemmas 2.1, 2.2, 2.3. Furthermore, assume that for some* $M_0 > 0$ *independent of* ε, *the following*

$$
\int_{\mathbb{R}} \rho_0(x) \, |u_0(x) - \bar{u}(x)| \, dx \le M_0 < \infty, \tag{2.35}
$$

holds. Then for any compact set $K \subset \mathbb{R}$, *it holds that*

$$
\int_0^t \int_K \rho^{\gamma+\theta} + \rho |u|^3 \; dx d\tau \le C(t, K), \tag{2.36}
$$

where $C = C(E_0, E_1, M_0, t, K) > 0$ *is independent of* ε.

Proof. First, we introduce a useful result about the entropy pair, see [36] for details. Taking $\psi^\star(w) = \frac{1}{2} w|w|$, then there exists a positive constant $C > 0$, depending only on $\gamma > 1$, such that the corresponding entropy pair $(\eta^\star, q^\star) = (\eta^{\psi^\star}, q^{\psi^\star})$ satisfies

$$
\begin{cases}
|\eta^\star(\rho, u)| \le (\rho |u|^2 + \rho^\gamma), \\
q^\star(\rho, u) \ge C^{-1}\left(\rho |u|^3 + \rho^{\gamma+\theta}\right), \\
|\eta^\star_m(\rho, u)| \le C\left(|u| + \rho^\theta\right), \\
|\eta^\star_{mm}(\rho, u)| \le C \rho^{-1},
\end{cases}
\quad \text{for all } \rho \ge 0 \text{ and } u \in \mathbb{R}. \tag{2.37}
$$

If $\eta_m^\star$ is regarded as the function of (ρ, u), we have

$$\begin{cases} |\eta_{mu}^\star(\rho, u)| \leq C, \\ |\eta_{m\rho}^\star(\rho, u)| \leq C\rho^{\theta-1}, \end{cases} \quad \text{for all } \rho \geq 0 \text{ and } u \in \mathbb{R}. \tag{2.38}$$

For this weak entropy pair $(\eta^\star, q^\star)$, we note that

$$\eta^\star(\rho, 0) = \eta_\rho^\star(\rho, 0) = 0, \quad q^\star(\rho, 0) = \frac{\theta}{2}\rho^{3\theta+1} \int_{\mathbb{R}} |s|^3[1-s^2]_+^\lambda \, ds$$

and

$$\eta_m^\star(\rho, 0) = \beta\rho^\theta, \text{ with } \beta := \int_{\mathbb{R}} |s|[1-s^2]_+^\lambda \, ds.$$

Taylor expansion implies

$$\eta^\star(\rho, m) = \beta\rho^\theta m + r(\rho, m), \tag{2.39}$$

with

$$r(\rho, m) \leq C\rho u^2. \tag{2.40}$$

for some constant $C > 0$. Now we introduce a new entropy pair $(\hat{\eta}, \hat{q})$,

$$\hat{\eta}(\rho, m) = \eta^\star(\rho, m - \rho u^-), \quad \hat{q}(\rho, m) = q^\star(\rho, m - \rho u^-) + u^-\eta^\star(\rho, m - \rho u^-),$$

which satisfies

$$\begin{cases} \hat{\eta}(\rho, m) = \beta\rho^{\theta+1}(u - u^-) + r(\rho, \rho(u - u^-)), \\ r(\rho, \rho(u - u^-)) \leq C\rho(u - u^-)^2. \end{cases} \tag{2.41}$$

Integrating $(1.1)_1 \times \hat{\eta}_\rho + (1.1)_2 \times \hat{\eta}_m$ over $(0, t) \times (-\infty, x)$, we obtain

$$\begin{aligned} &\int_{-\infty}^x (\hat{\eta}(\rho, m) - \hat{\eta}(\rho_0, m_0)) \, dy + \int_0^t q^\star(\rho, \rho(u - \bar{u})) + u^-\eta^\star(\rho, \rho(u - \bar{u})) \, d\tau \\ &= tq^\star(\rho^-, 0) + \varepsilon \int_0^t \hat{\eta}_m \rho^\alpha u_x \, d\tau - \varepsilon \int_0^t \int_{-\infty}^x (\hat{\eta}_{mu}\rho^\alpha u_x^2 + \hat{\eta}_{m\rho}\rho^\alpha \rho_x u_x) \, dyd\tau. \end{aligned} \tag{2.42}$$

Utilizing (2.38), we have the following estimates

$$|\varepsilon \int_0^t \int_{-\infty}^x \hat{\eta}_{mu}\rho^\alpha u_x^2 dyd\tau| \leq C\varepsilon \int_0^t \int_{\mathbb{R}} \rho^\alpha u_x^2 dyd\tau \leq C(t), \tag{2.43}$$

$$\begin{aligned} &|\varepsilon \int_0^t \int_{-\infty}^x \hat{\eta}_{m\rho}\rho^\alpha \rho_x u_x dyd\tau| \leq C\varepsilon \int_0^t \int_{\mathbb{R}} \rho^{\theta-1}\rho^\alpha |\rho_x u_x| \, dyd\tau \\ &\leq C\varepsilon \int_0^t \int_{\mathbb{R}} \rho^\alpha u_x^2 dyd\tau + C\varepsilon \int_0^t \int_{\mathbb{R}} \rho^{\alpha+\gamma-3}\rho_x^2 \, dyd\tau \leq C(t). \end{aligned} \tag{2.44}$$

Substituting (2.43) and (2.44) into (2.42), then integrating the result with respect to x over K and using (2.37), we obtain

$$\begin{aligned}
&\int_0^t \int_K \rho^{\alpha+\gamma} + \rho|u-u^-|^3 \, dxd\tau \\
&\leq C(t) + C\int_0^t \int_K |\eta^\star(\rho, \rho(u-\bar{u}))| \, dxd\tau + C\varepsilon \int_0^t \int_K \rho^\alpha |u_x||u| \, dxd\tau \qquad (2.45)\\
&+ C\varepsilon \int_0^t \int_K \rho^{\alpha+\theta}|u_x| \, dxd\tau \; + 2\sup_{\tau\in[0,t]} \left| \int_K \left(\int_{-\infty}^x \hat{\eta}(\rho(y,\tau), (\rho u)(y,\tau)) \, dy \right) dx \right|.
\end{aligned}$$

Applying Lemma 2.1, it is easy to get

$$\int_0^t \int_K |\eta^\star(\rho, \rho(u-\bar{u}))| \, dxd\tau \leq C(t). \qquad (2.46)$$

Now Cauchy-Schwartz inequality and (2.28) lead to

$$\begin{aligned}
\varepsilon \int_0^t \int_K \rho^{\alpha+\theta}|u_x| \, dxd\tau &\leq C\varepsilon \int_0^t \int_K \rho^\alpha u_x^2 \, dxd\tau \; + C\varepsilon \int_0^t \int_K \rho^{\alpha+2\theta} \, dxd\tau \\
&\leq C(t) + C(t) \int_0^t \int_K \rho^{\frac{\gamma-1}{2}} \, dxd\tau \\
&\leq C(t). \qquad (2.47)
\end{aligned}$$

Noticing that $3\beta > 3\alpha - 2$ and (2.28), we have

$$\begin{aligned}
\varepsilon \int_0^t \int_K \rho^\alpha |u_x||u| \, dxd\tau &\leq \frac{1}{2}\varepsilon \int_0^t \int_K \rho^\alpha u_x^2 \, dxd\tau \; + \frac{1}{2}\varepsilon \int_0^t \int_K \rho^\alpha u^2 \, dxd\tau \\
&\leq C(t) + C(\delta)\varepsilon^3 \int_0^t \int_K \rho^{3\alpha-2} \, dxd\tau + \delta \int_0^t \int_K \rho|u|^3 \, dxd\tau \\
&\leq C(t) + \varepsilon^3 \int_0^t \int_K \rho^{3\beta} \, dxd\tau + \delta \int_0^t \int_K \rho|u|^3 \, dxd\tau \\
&\leq C(t) + \delta \int_0^t \int_K \rho|u|^3 \, dxd\tau \qquad (2.48)
\end{aligned}$$

where δ is small enough which will be determined later.

Now we estimate the last term on the right hand side of (2.45). (1.1) implies that

$$(\rho u - \rho u^-)_t + (\rho u^2 + p(\rho) - \rho u u^-)_x = \varepsilon(\rho^\alpha u_x)_x. \qquad (2.49)$$

Integrating (2.49) over $[0,t] \times (-\infty, x)$ for $x \in K$, we obtain

$$\begin{aligned}
\int_{-\infty}^x \rho(u-u^-) \, dy &= \int_{-\infty}^x \rho_0(u_0 - u^-) \, dy - \int_0^t (\rho u^2 + p(\rho) - \rho u u^- - p(\rho^-)) \, d\tau \\
&+ \varepsilon \int_0^t \rho^\alpha u_x \, d\tau. \qquad (2.50)
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\Big|\int_{-\infty}^{x}\hat{\eta}(\rho(y,\tau),(\rho u)(y,\tau))\,dy\Big|\\
&\le\Big|\int_{-\infty}^{x}(\hat{\eta}(\rho,\rho u)-\beta\rho^{\theta+1}(u-\bar{u}))\,dy\Big|+\Big|\int_{-\infty}^{x}\beta\rho^{\theta+1}(u-u^{-})\,dy\Big|\\
&\le\Big|\int_{-\infty}^{x}r(\rho,\rho(u-u^{-}))\,dy\Big|+\Big|\int_{-\infty}^{x}\beta(\rho^{\theta}-(\rho^{-})^{\theta})\rho(u-u^{-})\,dy\Big|\\
&\qquad+\beta(\rho^{-})^{\theta}\Big|\int_{-\infty}^{x}\rho(u-u^{-})\,dy\Big|\\
&\le C(t)+\beta(\rho^{-})^{\theta}\Big|\int_{-\infty}^{x}\rho(u-u^{-})\,dy\Big|,
\end{aligned}
\tag{2.51}
$$

which, together with (2.35), Lemma2.1-Lemma2.3 and (2.50), implies that

$$
\int_{K}\Big|\int_{-\infty}^{x}\hat{\eta}(\rho(y,\tau),m(y,\tau))dy\Big|\,dx\le C(t).\tag{2.52}
$$

Now, if one chooses δ small enough, then substitutes (2.52), (2.46), (2.48) and (2.47) into (2.45), the proof of Lemma 2.4 follows.

Remark 2.2 *In the uniform estimates above, we have required that $\frac{2}{3}\le\alpha\le\gamma$, and the initial functions $(\rho_0^{\varepsilon}(x),u_0^{\varepsilon}(x))$ satisfy*

(i) $\rho_0^{\varepsilon}(x)>0,\quad \int_{\mathbb{R}}\rho_0^{\varepsilon}(x)|u_0^{\varepsilon}(x)-\bar{u}(x)|dx\le M_0<\infty$;

(ii) The total mechanical energy with respect to $(\bar{\rho},\bar{u})$ is finite:

$$
\int_{\mathbb{R}}\frac{1}{2}\rho_0^{\varepsilon}(x)\left|u_0^{\varepsilon}(x)-\bar{u}(x)\right|^2+e^{*}\left(\rho_0^{\varepsilon}(x),\bar{\rho}(x)\right)dx=:E_0<\infty;
$$

(iii) $\varepsilon^2\int_{\mathbb{R}}\frac{|\rho_{0x}^{\varepsilon}(x)|^2}{\rho_0^{\varepsilon}(x)^{3-2\alpha}}dx\le E_1<\infty$.

(iv) M_0, E_0, E_1 are independent of ε. These conditions are essential parts of Condition 1 in section 1. We remark here that the limit of the functions satisfying the conditions i)-iv) is very general, including a wide class of L^{∞} functions with finite energy and may contain vacuum. It is obvious that the above limit can serve as the initial data of isentropic gas dynamics for the existence of finite energy solutions. We also note that the condition (iii) is slightly weaker than the corresponding one in [10] near vacuum. We refer to [10] for further details.

3 $H_{loc}^{-1}(\mathbb{R}_+^2)-$ Compactness

In this section we will use the uniform estimates obtained in the previous section to prove the following key Lemma, which states the $H_{loc}^{-1}(\mathbb{R}_+^2)-$compactness of the approximate solution sequence.

Lemma 3.1 *Let $\frac{2}{3} \leq \alpha \leq \gamma$, $\psi \in C_0^2(\mathbb{R})$, (η^ψ, q^ψ) be a weak entropy pair generated by ψ. Then for the solutions $(\rho^\varepsilon, u^\varepsilon)$ with $m^\varepsilon = \rho^\varepsilon u^\varepsilon$ of Navier-Stokes equations (1.1)–(1.2), the following sequence*

$$\eta^\psi(\rho^\varepsilon, m^\varepsilon)_t + q^\psi(\rho^\varepsilon, m^\varepsilon)_x \text{ are compact in } H^{-1}_{loc}(\mathbb{R}^2_+) \tag{3.1}$$

Proof. In order to prove this lemma, we first introduce the following results for the entropy pair (η^ψ, q^ψ) generated by $\psi \in C_0^2(\mathbb{R})$, and see [10] for details.

For a C^2 function $\psi : \mathbb{R} \to \mathbb{R}$, compactly supported on the interval $[a, b]$, we have

$$supp\{\eta^\psi\}, supp\{q^\psi\} \subset \left\{(\rho, m) = (\rho, \rho u) : u + \rho^\theta \geq a, \quad u - \rho^\theta \leq b\right\}. \tag{3.2}$$

Furthermore, there exists a constant $C_\psi > 0$ such that, for any $\rho \geq 0$ and $u \in \mathbb{R}$, we have

(i) For $\gamma \in (1, 3]$,

$$|\eta^\psi(\rho, m)| + |q^\psi(\rho, m)| \leq C_\psi \rho. \tag{3.3}$$

(ii) For $\gamma \in (3, +\infty)$,

$$|\eta^\psi(\rho, m)| \leq C_\psi \rho, \quad |q^\psi(\rho, m)| \leq C_\psi(\rho + \rho^{\theta+1}). \tag{3.4}$$

(iii) If η^ψ is considered as a function of (ρ, m), $m = \rho u$, then

$$|\eta^\psi_m(\rho, m)| + |\rho \eta^\psi_{mm}(\rho, m)| \leq C_\psi, \tag{3.5}$$

and, if η^ψ_m is considered as a function of (ρ, u), then

$$|\eta^\psi_{mu}(\rho, \rho u)| + |\rho^{1-\theta} \eta^\psi_{m\rho}(\rho, \rho u)| \leq C_\psi. \tag{3.6}$$

Now we are going to prove the lemma.

A direct computation on $(1.1)_1 \times \eta^\psi_\rho(\rho^\varepsilon, m^\varepsilon) + (1.1)_2 \times \eta^\psi_m(\rho^\varepsilon, m^\varepsilon)$ gives

$$\begin{aligned} \eta^\psi(\rho^\varepsilon, m^\varepsilon)_t + q^\psi(\rho^\varepsilon, m^\varepsilon)_x = \varepsilon(\eta^\psi_m(\rho^\varepsilon, m^\varepsilon)(\rho^\varepsilon)^\alpha u^\varepsilon_x)_x - \varepsilon \eta^\psi_{mu}(\rho^\varepsilon, m^\varepsilon)(\rho^\varepsilon)^\alpha (u^\varepsilon_x)^2 \\ -\varepsilon \eta^\psi_{m\rho}(\rho^\varepsilon, m^\varepsilon)(\rho^\varepsilon)^\alpha \rho^\varepsilon_x u^\varepsilon_x \end{aligned} \tag{3.7}$$

Let $K \subset \mathbb{R}$ be compact, using (3.6) and Cauchy-Schwartz inequality, we have

$$\begin{aligned} &\varepsilon \int_0^t \int_K |\eta^\psi_{mu}(\rho^\varepsilon, m^\varepsilon)(\rho^\varepsilon)^\alpha|(u^\varepsilon_x)^2 + |\eta^\psi_{m\rho}(\rho^\varepsilon, m^\varepsilon)(\rho^\varepsilon)^\alpha \rho^\varepsilon_x u^\varepsilon_x| \, dxdt \\ &\leq C\varepsilon \int_0^t \int_K (\rho^\varepsilon)^\alpha (u^\varepsilon_x)^2 \, dxd\tau + C\varepsilon \int_0^t \int_K (\rho^\varepsilon)^{\alpha+\gamma-3} (\rho^\varepsilon_x)^2 \, dxd\tau \\ &\leq C(t). \end{aligned} \tag{3.8}$$

This implies that

$$-\varepsilon \eta^\psi_{mu}(\rho^\varepsilon, m^\varepsilon)(\rho^\varepsilon)^\alpha (u^\varepsilon_x)^2 - \varepsilon \eta^\psi_{m\rho}(\rho^\varepsilon, m^\varepsilon)(\rho^\varepsilon)^\alpha \rho^\varepsilon_x u^\varepsilon_x \text{ are bounded in } L^1([0, T] \times K), \tag{3.9}$$

and thus it is compact in $W_{loc}^{-1,p_1}(\mathbb{R}_+^2)$, for $1 < p_1 < 2$.

Moreover, noticing $|\eta_m^\psi(\rho^\varepsilon, \rho^\varepsilon u^\varepsilon)| \le C_\psi$, we have

$$\begin{aligned}
&\int_0^t \int_K \left(\varepsilon \eta_m^\psi(\rho^\varepsilon, m^\varepsilon)(\rho^\varepsilon)^\alpha u_x^\varepsilon\right)^{\frac{4}{3}} dxdt \\
&\le \int_0^t \int_K \varepsilon^{\frac{4}{3}} (\rho^\varepsilon)^{\frac{4\alpha}{3}} |u_x^\varepsilon|^{\frac{4}{3}} dxdt \\
&\le C\varepsilon^{\frac{4}{3}} \int_0^t \int_K (\rho^\varepsilon)^\alpha |u_x^\varepsilon|^2 dxdt + C\varepsilon^{\frac{4}{3}} \int_0^t \int_K (\rho^\varepsilon)^{2\alpha} dxdt \\
&\le C(T,K)\varepsilon^{\frac{1}{3}} + C\varepsilon^{\frac{4}{3}} \int_0^t \int_K (\rho^\varepsilon)^{\gamma+1} dxdt \\
&\le C(T,K)\varepsilon^{\frac{1}{3}} \to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned} \tag{3.10}$$

Then(3.10) and (3.9) yield that

$$\eta^\psi(\rho^\varepsilon, m^\varepsilon)_t + q^\psi(\rho^\varepsilon, m^\varepsilon)_x \text{ are compact in } W_{loc}^{-1,p_2}(\mathbb{R}_+^2) \text{ for some } 1 < p_2 < 2. \tag{3.11}$$

On the other hand, using the estimates in (3.3)–(3.4) and Lemma 2.1–Lemma 2.4, we have

$$\eta^\psi(\rho^\varepsilon, m^\varepsilon), q^\psi(\rho^\varepsilon, m^\varepsilon) \text{ are uniformly bounded in } L_{loc}^{p_3}(\mathbb{R}_+^2) \text{ for } p_3 > 2, \tag{3.12}$$

where $p_3 = \gamma + 1 > 2$ when $\gamma \in (1, 3]$; and $p_3 = \frac{\gamma+\theta}{1+\theta} > 2$ when $\gamma > 3$. This yields that,

$$\eta^\psi(\rho^\varepsilon, m^\varepsilon)_t + q^\psi(\rho^\varepsilon, m^\varepsilon)_x \text{ are uniformly bounded in } W_{loc}^{-1,p_3}(\mathbb{R}_+^2). \tag{3.13}$$

Then (3.11) and (3.13) implies Lemma 3.1.

4 Proof of Theorem

Proof of Theorem 1.1. From Lemmas 2.1–2.4 and the compactness estimate Lemma 3.1, we have verified the conditions (i)-(iii) of Theorem 1.2 for the sequence of solutions $(\rho^\varepsilon, m^\varepsilon)$. Basing on Theorem 1.2, there is a subsequence $(\rho^\varepsilon, m^\varepsilon)$(still denoted as $(\rho^\varepsilon, m^\varepsilon)$) and a pair of measurable functions (ρ, m) such that

$$(\rho^\varepsilon, m^\varepsilon) \to (\rho, m), \quad a.e\ \varepsilon \to 0. \tag{4.1}$$

It is easy to check that (ρ, m) is a finite-energy entropy solution (ρ, m) to the Cauchy problem (1.3) with initial data $(\rho_0, \rho_0 u_0)$ for the isentropic Euler equations with $\gamma > 1$. Therefore, the proof of Theorem 1.1 is completed.

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