Every even number greater than 454 is the sum of seven cubes

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Abstract. It is conjectured that every integer $N > 454$ is the sum of seven nonnegative cubes. We prove the conjecture when $N \equiv 2 \bmod 4$. This result, together with a recent proof for $4|N$, shows that the conjecture is true for all even N.

1 Introduction

Linnik [1943] showed that every sufficiently large natural number N is the sum of at most seven positive cubes. It has long been known that the set of $N > 0$ without such a representation contains

$$\{15,\ 22,\ 23,\ 50,\ 114,\ 167,\ 175,\ 186,\ 212,\ 231,\ 238,\ 239,\ 303,\ 364,\ 420,\ 428,\ 454\}, \tag{1}$$

and conjectured that (1) is the full set of exceptions. See the first section of [Ramaré 2007] for this history of this part of Waring's problem. The introduction of [Boklan-Elkies 2008] gave a briefer account; an even more abbreviated summary follows.

The first effective upper bound on the largest exception was exp(exp(13.94)), obtained in [McCurley 1984]; the upper bound now stands at exp(524) by the analytic sieve argument of [Ramaré 2007]. While any effective upper bound reduces the problem to a finite computation, exp(524) is still much too large to reach by such a computation in practice. But it is still useful to have a good lower bound on any exceptional N outside the known set (1), because known techniques for constructing seven-cube representations typically require N to be somewhat large. We shall use the bound $N \geq 2.5 \cdot 10^{26}$, proved in [Bertault-Ramaré-Zimmermann 1999]. The largest such bound reported is $\exp(78.7) > 1.5 \cdot 10^{34}$ [Deshouillers et al. 2000, pages 433–434], which is still very far below $\exp(524) > 3 \cdot 10^{224}$.

A different tack is to prove the conjecture under some congruence condition on N, in the hope that eventually every N might be covered by one such result. We know of two such theorems. The first [Bertault-Ramaré-Zimmermann 1999] shows that N is the sum of seven cubes if $N \equiv 0$ or $\pm 1 \bmod 9$ and N is an invertible cubic residue mod 37, and also that 37 could be replaced by some larger primes congruent to $1 \bmod 3$. The second [Boklan-Elkies 2008] proves that N is the sum of seven cubes if $4|N$ as long as N is outside the set (1) of known exceptions.

Here we prove this result for $2 \| N$, which together with [Boklan-Elkies 2008] establishes it whenever $2|N$:

Theorem. *If N is an even positive integer not in*

$$\{22,\ 50,\ 114,\ 186,\ 212,\ 238,\ 364,\ 420,\ 428,\ 454\} \tag{2}$$

then N is the sum of seven nonnegative cubes.

The proof adapts a technique used in earlier work on Waring's problem for cubes, dating back to the initial paper [Wieferich 1909] on sums of nine cubes, and including [Bertault-Ramaré-Zimmermann 1999, Boklan-Elkies 2008]. We need only consider $N \equiv 2 \bmod 4$ since the case $4|N$ was proved in [Boklan-Elkies 2008]. We use two new ingredients:

• We take coefficients $(a_1, a_2, a_3) = (1, 2, 5)$ in $Q = \sum_{i=1}^{3} a_i X_i^2$. As with other quadratic forms used in such constructions, this Q is diagonal and unique in its genus, but one doesn't expect a factor of 5 in $a_1 a_2 a_3$ when there is no condition on N mod 5. (Factors of 3 occur in [Boklan-Elkies 2008], but that is not surprising because of the special behavior of cubes in the p-adic integers $\mathbf{Z}_p$ for $p = 3$: they are all congruent to 0 or ± 1 mod 9.)

• Due to the structure of the set of positive integers not represented by Q, we must restrict the auxiliary parameter p to a residue class modulo 300. This modulus is beyond the range of the tables of [Ramaré-Rumely 1996]. Extending these tables to primes in congruence classes mod 300 would require a large computation with Dirichlet L-functions. Instead we replace the prime p by a product P of distinct primes each congruent to 5 mod 6. This retains the key property that every residue class has a cube root, while giving enough flexibility to reduce the use of [Ramaré-Rumely 1996] to primes in two arithmetic progressions mod 12. This refinement also lets us dispense with the factor β of [Boklan-Elkies 2008], since it can be included among the prime factors of P. It also streamlines or completes several other seven-cube constructions of this kind in [Boklan-Elkies 2010], where we prove that N is the sum of seven cubes if $N \equiv 0 \bmod 9$, $\pm 1 \bmod 18$, or $\pm 2 \bmod 9$. Note that the first two of these, together with the results of [Boklan-Elkies 2008] and the present paper, properly contain the theorem of [Bertault-Ramaré-Zimmermann 1999] by removing the additional hypothesis modulo 37 or a larger prime.

It is noted in [Boklan-Elkies 2008] that the construction actually produces a representation $N = \sum_{i=0}^{6} x_i^3$ with each x_i positive, not just nonnegative, once N is large enough, with $N = 2408$ probably being the last exception. The same is true here; indeed one can easily adjust the proof to produce a representation with $\max_i x_i / \min_i x_i$ uniformly bounded: it is enough to replace the bounds 1618 and 1786 on N/P^3 by $1618 + \delta$ and $1786 - \delta$, and and to change x_0 to $x_0' = x_0 + 6P$ if $x_0 < \delta^2 P$, for sufficiently small $\delta > 0$. The same can be done for the results in [Bertault-Ramaré-Zimmermann 1999] and [Boklan-Elkies 2008].

The rest of this paper is organized as follows. In the next section give the formulas (3,4,5) that represent N as a sum of seven cubes given suitable P, Q.

The following section derives conditions on P that guarantee that the criteria on Q can be satisfied (the new analysis here is in Lemma 1). Finally we prove (Lemma 2) that such P can be found if $N > 10^{20}$; this together with the bound $2.5 \cdot 10^{26}$ of [Bertault-Ramaré-Zimmermann 1999] completes the proof of the theorem.

2 From N to Q

Given $N \equiv 2 \bmod 4$ with N large enough, namely $N > 10^{20}$, we shall construct a representation of N as the sum of seven cubes. We may assume $5^3 \nmid N$, because we may write $N = 5^{3e} N_0$ with $5^3 \nmid N_0$, and use a seven-cube decomposition $N_0 = \sum_{i=0}^{6} x_i^3$ to write N as the sum of the cubes of $5^e x_i$. If N_0 is in the exceptional set (1), but $e > 0$ (so $N \neq N_0$), we use a decomposition of $5^3 N_0$ to the same effect; indeed for each of these 17 values of N_0 a direct computation shows that five cubes suffice, as do seven positive cubes (this is used to represent N as a sum of seven positive cubes, and is contained in the computation reported in [Boklan-Elkies 2008] that suggests all $N > 2408$ have such a representation).

We start from the usual six-cube identity (see e.g. [Boklan-Elkies 2008, Lemma 1]), taking $(c_1, c_2, c_3) = (4, 5, 8)$ to find

$$(4P + X_1)^3 + (4P - X_1)^3 + (5P + X_2)^3 + (5P - X_2)^3 + (8P + X_3)^3 + (8P - X_3)^3$$
$$= 1402P^3 + 6PQ_1, \qquad (3)$$

where $Q_1 := 4X_1^2 + 5X_2^2 + 8X_3^2$. We deduce that if P is a positive integer, and X_1, X_2, X_3 are integers such that $|X_1| < 4P$, $|X_2| < 5P$, and $|X_3| < 8P$, then for any positive integer x_0 we have a representation of

$$N = x_0^3 + 1402P^3 + 6PQ_1 \qquad (4)$$

as a sum of seven positive cubes.

In our setting, $N \equiv 2 \bmod 4$, so we must take x_0 even. We shall require that P be odd. Then $1402P^3 \equiv 2 \bmod 4$, so $6PQ_1$ is a multiple of 4, whence Q_1 is even. Therefore $2|X_2$, from which $4|Q_1$. We thus have $Q_1 = 4Q$ where Q is the diagonal quadratic form

$$Q := X_1^2 + 2X_3^2 + 5(X_2/2)^2. \qquad (5)$$

Dickson proved that this Q represents all nonnegative integers except those of the form $5^{2k}(25n \pm 10)$, see [Dickson 1927, p.69, Theorem IX].[1] We shall show that for $N > 10^{20}$, we can choose P such that there is an x_0 that makes Q a positive integer, not congruent to 0 or $\pm 10 \bmod 25$, with $Q < 16P^2$. Then $4Q < 4^3 P^2$, so for each $i = 1, 2, 3$ we have

$$4X_i^2 \leq c_i X_i^2 \leq Q_1 = 4Q < 4^3 P^2, \qquad (6)$$

so $|X_i| < 4P \leq c_i P$, and the X_i satisfy the inequalities that make each term positive in the seven-cube representation of N obtained from formulas (3,4).

[1] The theorem statement follows the proof on pages 67–69; the notation G for this form is on page 63.

3 The conditions on P

Taking $Q_1 = 4Q$ in (4) and solving for Q yields

$$Q = \frac{N - x_0^3 - 1402P^3}{24P}. \tag{7}$$

We noted already that x_0 must be even and P odd. We now see that P must also be chosen so that

$$P \equiv \frac{N}{2} \bmod 4 \tag{8}$$

to make the numerator of (7) divisible by 8. To make Q integral, it remains to choose x_0 so that $3P | N - x_0^3 - 1402P^3$. To that end, we require that P be a product of distinct primes each congruent to $5 \bmod 6$. Then every residue mod $6P$ has a cube root. We choose for x_0 the smallest positive solution of $x_0^3 \equiv N - 1402P^3 \bmod 6P$. Then x_0 is an even number in $(0, 6P]$, and Q is an integer in

$$\left(\frac{N - (1402 + 6^3)P^3}{24P}, \frac{N - 1402P^3}{24P} \right]. \tag{9}$$

Therefore $Q > 0$ provided $N/P^3 > 1402 + 6^3 = 1618$, and $Q < 16P^2$ provided $N/P^3 < 1402 + 24 \cdot 16 = 1786$. Thus we seek P in the interval $(AN^{1/3}, BN^{1/3})$ where $A = 1786^{-1/3}$ and $B = 1618^{-1/3}$, with $B/A = (893/809)^{1/3} > 1.033$. (These estimates correspond to Proposition 1 in [Boklan-Elkies 2008].)

To the condition (8) on $P \bmod 4$, we next add a condition mod 25 to assure that Q is congruent to neither 0 nor $\pm 10 \bmod 25$, from which it will follow that Q is represented by the diagonal quadratic form with coefficients $1, 2, 5$.

Lemma 1. *For any $N \not\equiv 0 \bmod 5^3$ there exist at least two nonzero choices of $b \in \mathbf{Z}/25\mathbf{Z}$ such that if $P \equiv b \bmod 25$ then (7) yields a value of Q not congruent to 0 or $\pm 10 \bmod 25$.*

(Note that since b is not the zero residue the congruence $P \equiv b \bmod 25$ cannot force P to have a repeated prime factor.)

Proof: If $5|N$ we shall choose either $b = \pm 5$ or $b = \pm 10$. Since $5|P$, also $5|x_0$, and then the numerator of (7) is

$$N - x_0^3 - 1402P^3 \equiv N \bmod 5^3.$$

In particular, if $25 \nmid N$ then either $b = \pm 5$ or $b = \pm 10$ works, because the numerator is not a multiple of 25, so its quotient by $24P$ is not a multiple of 5, and thus lies outside the forbidden congruence classes mod 25. If $25|N$ then we choose $b = \pm 5$ if $N \equiv 25$ or $100 \bmod 5^3$, and $b = \pm 10$ if $N \equiv 50$ or $75 \bmod 5^3$ (recall that $5^3 \nmid N$). Then $Q \equiv N/(24P) \equiv \pm 5 \bmod 25$, so again Q is outside the exceptional set for $X_1^2 + 2X_2^2 + 5X_3^2$.

Finally, if $5 \nmid N$, we first choose $b \bmod 5$ so that $1402b^3 \equiv N \bmod 5$, then choose $b \bmod 25$ so that

$$\frac{N - 1402b^3}{24b} \equiv \pm 5 \bmod 25. \tag{10}$$

The equation for b mod 5 has a solution because every integer is a cube mod 5; since N is not a multiple of 5, neither is b. The choice of b mod 5 guarantees that $5|Q$ if and only if $5|x_0$, in which case $Q \equiv (N - 1402b^3)/(24b) \bmod 5^3$. Thus (10) means that if $5|Q$ then $Q \equiv \pm 5 \bmod 25$. We claim that each sign arises for a unique choice of b mod 25, which we call $b_\pm$. We tabulate b_+ and b_- for each of the 20 possible residues of N mod 25:

N mod 25	1	2	3	4	6	7	8	9	11	12	13	14	16	17	18	19	21	22	23	24
b_+	2	21	9	18	22	1	14	13	17	6	19	8	12	11	24	3	7	16	4	23
b_-	7	6	24	13	2	11	4	8	22	16	9	3	17	21	14	23	12	1	19	18

Table 1: b_+ and b_- for each N mod 25

This completes the proof of Lemma 1. $\square$

Remark: The existence and uniqueness of $b_\pm$ in each case can be understood as follows. Fix an arbitrary b_0 such that $N - 1402b_0^3 \equiv 0 \bmod 5$, and let $Q_0 = (N - 1402b_0^3)/(24b_0)$. Then $b = b_0 + 5\beta$ yields $Q \equiv Q_0 + 5Q_0'\beta \bmod 25$ where

$$Q_0' = \frac{-N - 2 \cdot 1402b_0^3}{24b_0^2}$$

is the 5-adic derivative of Q at b_0 (so the linear approximation $Q_0 + 5Q_0'\beta$ is within $O(5^2)$ of the correct value). But a function of the form $(A - Bb^3)/b$ cannot vanish at some b together with its derivative $(-A - 2B^3)/b^2$ modulo 5 (or indeed mod p for any prime $p \neq 3$). Since $Q_0 \equiv 0 \bmod 5$, then, $Q_0' \not\equiv 0 \bmod 5$, so $Q_0 + 5Q_0'\beta \bmod 25$ runs over all lifts of Q_0 to $\mathbf{Z}/25\mathbf{Z}$ as β varies mod 5.

4 The existence of P

It remains to prove:

Lemma 2. *Suppose* $N \equiv 2 \bmod 4$ *with* $125 \nmid N$, *and* $N > 10^{20}$. *Choose* b *according to Lemma 1. Then there exists* $P \in (AN^{1/3}, BN^{1/3})$ *that is a product of distinct primes congruent to* $5 \bmod 6$ *with* $P \equiv b \bmod 25$ *and* $P \equiv (N/2) \bmod 4$.

Proof: Suppose first that $N > 10^{26}$. We then choose P as follows. Let $\mathcal{P}$ be the set of squarefree integers each of whose prime factors is congruent to 5 mod 6. Choose a finite subset $\mathcal{P}_0 \subset \mathcal{P}$ such that:

- All elements of $\mathcal{P}_0$ are congruent mod 4, say to r_0;
- $\mathcal{P}_0$ contains a representative of each nonzero class mod 25;
- $\max(\mathcal{P}_0)/\min(\mathcal{P}_0) = 1 + \epsilon_0$ with ϵ_0 small;
- $\max(\mathcal{P}_0)$ is as small as possible given ϵ_0.

We shall take $P = P_0 p$ with $P_0 \in \mathcal{P}_0$ and p a prime greater than $\max(\mathcal{P}_0)$ that is congruent to 5 mod 6 and to $r_0(N/2)$ mod 4. We choose p first, and then select

P_0 so that $P \equiv b \bmod 25$ with b depending on $N \bmod 25$ according to Lemma 1. Then P is a product of distinct primes each congruent to $5 \bmod 6$, and is in a suitable class mod 4 and mod 25 to guarantee the success of our construction as long as $1618 < N/P^3 < 1786$. This last condition, in turn, will be satisfied provided that

$$p \in (p_{\min}, p_{\max}) := \left(\frac{(N/1786)^{1/3}}{\mathcal{P}_{\min}}, \frac{(N/1618)^{1/3}}{\mathcal{P}_{\max}} \right), \tag{11}$$

an interval whose endpoints' ratio is

$$\frac{p_{\max}}{p_{\min}} = \frac{(1786/1618)^{1/3}}{\mathcal{P}_{\max}/\mathcal{P}_{\min}} > \frac{1.033}{1+\epsilon_0}. \tag{12}$$

A quick computer search finds the choice

$$\mathcal{P}_0 = \mathcal{P} \cap (1 + 4\mathbf{Z}) \cap [26141, 26669], \tag{13}$$

with $\epsilon_0 = 528/26141 < 0.0202$ and $1.033/(1+\epsilon_0) > 1.0125$. We tabulate the 38 elements $P_0 \in \mathcal{P}_0$, sorted by the remainder $\overline{P}_0$ of $P_0 \bmod 25$, together with the prime factorization of those $P_0 \in \mathcal{P}_0$ that are not prime:

$\overline{P}_0$	P_0	$\overline{P}_0$	P_0
1	$26401 = 17 \cdot 1553$; 26501	13	26513
2	26177; $26477 = 11 \cdot 29 \cdot 83$	14	26189; $26389 = 11 \cdot 2399$; 26489
3	26153; $26653 = 11 \cdot 2423$	15	$26365 = 5 \cdot 5273$; $26665 = 5 \cdot 5333$
4	$26329 = 113 \cdot 233$	16	26141 [min]
5	$26305 = 5 \cdot 5261$	17	26417
6	$26281 = 41 \cdot 641$	18	26393
7	26357	19	26669 [max]
8	26633	20	$26345 = 5 \cdot 11 \cdot 479$; $26545 = 5 \cdot 5309$
9	26309; $26609 = 11 \cdot 41 \cdot 59$	21	26321; $26521 = 11 \cdot 2411$
10	$26185 = 5 \cdot 5237$; $26485 = 5 \cdot 5297$	22	26297; 26597
11	26261; $26461 = 47 \cdot 563$; 26561	23	$26473 = 23 \cdot 1151$; 26573
12	26237	24	26249

Table 2: $P_0 \in [26141, 26669]$ for each nonzero $\overline{P}_0 \bmod 25$

Suppose first that $p_{\min} > 10^{10}$. We then take $k = 12$ in [Ramaré-Rumely 1996, Theorem 1], finding that for each $l \in (\mathbf{Z}/12\mathbf{Z})^*$ there exists a prime $p \equiv l \bmod 12$ such that $1 < p/p_{\min} < (1+\epsilon_{12})/(1-\epsilon_{12})$, with $\epsilon_{12} < 0.003$ according to [Ramaré-Rumely 1996, Table 1, p. 419]. This proves Lemma 2 for all

$$N > 1786(10^{10}\mathcal{P}_{\max})^3 = 3.38767\ldots \cdot 10^{46}. \tag{14}$$

If $\mathcal{P}_{\max} < p_{\min} < 10^{10}$, we apply the algorithm used in the Sublemma of [Boklan-Elkies 2008, Lemma 2] for primes congruent to $l \bmod 12$ for $l = 5$ and $l = 11$. We quickly find that for any $p_{\min} \in (26669, 10^{11})$ the interval $(p_{\min}, 1.006 p_{\min})$ contains at least one prime $p \equiv 5 \bmod 12$ and at least one prime $p \equiv 11 \bmod 12$. (Indeed in these two arithmetic progressions mod 12 the largest ratio between consecutive primes past $\mathcal{P}_{\max}$ is $35381/35201 < 1.00512$ for $l = 5$ and $45491/45263 < 1.00504$ for $l = 11$.) This extends the range of our proof from (14) down to

$$N > 1786\mathcal{P}_{\max}^6 = 6.42572\ldots \cdot 10^{29}. \tag{15}$$

[Note that this is already sufficient to prove our theorem, using the bound 10^{34} of [Ramaré 2007, Deshouillers et al. 2000].]

Finally, to bring the bound on N from (15) down to the claimed 10^{20}, we apply the same algorithm directly to P in each of the 48 odd residue classes mod 100 that is not a multiple of 25, searching not for primes congruent to 5 mod 6 but for elements of $\mathcal{P}$. We soon find that for any $P_{\min} \in ((10^{20}/1786)^{1/3}, \mathcal{P}_{\max})$ the interval $(P_{\min}, (1+\epsilon_0)P_{\min})$ contains at least one $P \in \mathcal{P}$ in each of those 48 classes. This completes the proof of Lemma 2 and of our theorem. ∎

Remarks: Indeed in each of those 48 classes the ratio between two consecutive $P \in \mathcal{P}$ never gets as large as 1.015. We could have dispensed with the step from (14) to (15) by extending the last computation past $\mathcal{P}_{\max}$ to 10^{10}; the required calculation would be much larger (more residue classes, and prime factorization rather than just primality testing), but still feasible. On the other hand we could not use the same argument to reduce the bound on N to the 10^{18} used in [Boklan-Elkies 2008], because the ratio 1.0389+ between the consecutive elements $92437 = 23 \cdot 4019$ and $96037 = 137 \cdot 701$ of $\mathcal{P} \cap (100\mathbf{Z} + 37)$ is too large. We could probably use the choice between b_+ and b_- (see Table 1) to extend the range of our construction down to 10^{18}, but not much lower.

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[2] The list was obtained by Jones in his doctoral thesis [Jones 1928], which though unpublished can be found on Jagy's quadratic-forms webpages, see the Bibliography. This list was first published in [Jones-Pall 1939], together with proofs but without the lists of excluded progressions. That information appears in [Dickson 1939, §58, Table 5, pages 112–113]; see also page 111 for the abbreviations A, B, ..., N for the sets appearing in this table. Jagy also reproduces this table online, see http://zakuski.math.utsa.edu/~kap/Forms/Dickson_Diagonal_1939.pdf .

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