MAXIMAL 0-1-FILLINGS OF MOON POLYOMINOES WITH RESTRICTED CHAIN LENGTHS AND RC-GRAPHS

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Abstract. We show that maximal 0-1-fillings of moon polynomials, with restricted chain lengths, can be identified with certain rc-graphs, also known as pipe dreams. In particular, this exhibits a connection between maximal 0-1-fillings of Ferrers shapes and Schubert polynomials. Moreover, it entails a bijective proof showing that the number of maximal fillings of a stack polyomino S with no north-east chains longer than k depends only on k and the multiset of column heights of S.

1. Introduction

1.1. Triangulations, multitriangulations and 0-1-fillings. The systematic study of 0-1-fillings of polyominoes with restricted chain lengths likely originates in an article by Jakob Jonsson [5]. At first, he was interested in a generalisation of triangulations, where the objects under consideration are maximal sets of diagonals of the n-gon, such that at most k diagonals are allowed to cross mutually. Thus, in the case $k = 1$ one recovers ordinary triangulations. He realised these objects as fillings of the staircase shaped polyomino with row-lengths $n-1, n-2, \dots, 1$ with zeros and ones. The condition that at most k diagonals cross mutually then translates into the condition that the longest north-east chain in the filling has length k, see Definition 2.3. Instead of studying fillings of the staircase shape only, he went on to consider more general shapes which he called *stack* and *moon polyominoes*, see Definition 2.2 and Figure 1.

For stack polyominoes he was able to prove that the number of maximal fillings depends only on k and the multiset of heights of the columns, not on the particular shape of the polyomino. He conjectured that this statement holds more generally for moon polyominoes, which was eventually proved by the author [13] using a technique introduced

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by Christian Krattenthaler [8] based on Sergey Fomin's growth diagrams for the Robinson-Schensted-Knuth correspondence. However, the proof given there is not fully bijective: what one would hope for is a correspondence between fillings of any two moon polyominoes that differ only by a permutation of the columns. This article is a step towards this goal.

1.2. **RC-graphs and the subword complex.** RC-graphs (for 'reduced word compatible sequence graphs', see [1], also known as 'pipe dreams' see [7]) were introduced by Sergey Fomin and Anatol Kirillov [3] to prove various properties of Schubert polynomials. Namely, for a given permutation w, the Schubert polynomial $\mathfrak{S}_w$ can be regarded as the generating function of rc-graphs, see Remark 4.

A different point of view is to consider them as facets of a certain simplicial complex. Let w_0 be the long permutation $n \cdots 21$, and consider its reduced factorisation

$$Q = s_{n-1} \cdots s_2 s_1 \; s_{n-1} \cdots s_3 s_2 \; \cdots\cdots \; s_{n-1} s_{n-2} \; s_{n-1}.$$

Then the subword complex associated to Q and w introduced by Allen Knutson and Ezra Miller [7, 6] has as facets those subwords of Q that are reduced factorisations of w. Subword complexes enjoy beautiful topological properties, which are transferred by the main theorem of this article to the simplicial complex of 0-1-fillings, as observed by Christian Stump [16], see also the article by Luis Serrano and Christian Stump [14].

The intimate connection between maximal fillings and rc-graphs demonstrated by the main theorem of this article, Theorem 3.2, *should* not have come as a surprise. Indeed, Sergey Fomin and Anatol Kirillov [4] established a connection between reduced words and reverse plane partitions already thirteen years ago, which is not much less than the case of Ferrers shapes in Theorem 4.3. They even pointed towards the possibility of a bijective proof using the Edelman-Greene correspondence.

More recently, the connection between Schubert polynomials and triangulations was noticed by Alexander Woo [17]. Vincent Pilaud and Michel Pocchiola [11] discovered rc-graphs (under the name 'beam arrangements') more generally for multitriangulations, however, they were unaware of the theory of Schubert polynomials. In particular, Theorem 3.18 of Vincent Pilaud's thesis [10] (see also Theorem 21 of [11]) is a variant of our Theorem 3.2 for multitriangulations.

Finally, Christian Stump and the author of the present article became aware of an article by Vincent Pilaud and Francisco Santos [12] that describes the structure of multitriangulations in terms of so-called k-stars (introduced by Harold Coxeter). We then decided to translate this concept to the language of fillings, and discovered pipe dreams yet again.

2. Definitions

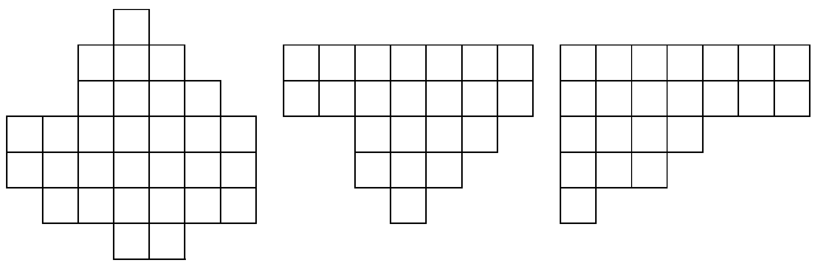


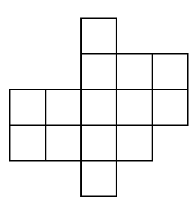
Figure 1. a moon-polyomino, a stack-polyomino and a Ferrers diagram

2.1. **Polyominoes.**

Definition 2.1. A *polyomino* is a finite subset of the quarter plane $\mathbb{N}^2$, where we regard an element of the subset as a cell. A *column* of a polyomino is the set of cells along a vertical line, a *row* is the set of cells along a horizontal line. We are using 'English' (or matrix) conventions for the indexing of the rows and columns of polyominoes: the top row and the left-most column have index 1.

The polyomino is *convex*, if for any two cells in a column (rsp. row), the elements of $\mathbb{N}^2$ in between are also cells of the polyomino. It is *intersection-free*, if any two columns are *comparable*, *i.e.*, the set of row coordinates of cells in one column is contained in the set of row coordinates of cells in the other. Equivalently, it is intersection-free, if any two rows are comparable.

For example, the polyomino



is convex, but not intersection-free, since the first and the last columns are incomparable.

Definition 2.2. A *moon polyomino* (or L-convex polyomino) is a convex, intersection-free polyomino. Equivalently we can require that any two cells of the polyomino can be connected by a path consisting of neighbouring cells in the polyomino, that changes direction at most once. A *stack polyomino* is a moon-polyomino where all columns start at the same level. A *Ferrers diagram* is a stack-polyomino with weakly decreasing row widths $\lambda_1, \lambda_2, \dots, \lambda_n$, reading rows from top to bottom.

Because a moon-polyomino is intersection free, the set of rows of maximal length in a moon polyomino must be consecutive. We call the set of rows including these and the rows above the *top half* of the polyomino. Similarly, the set of columns of maximal length, and all columns to the right of these, is the *right half* of the polyomino. The intersection of the top and the right half is the *top right quarter* of M.

2.2. **Fillings and Chains.**

Definition 2.3. A 0-1-*filling* of a polyomino is an assignment of the numbers 0 and 1 to the cells of the polyomino. Cells containing 0 are also called *empty*.

A *north-east chain* is a sequence of non-zero entries in a filling such that the smallest rectangle containing all its elements is completely contained in the moon polyomino, and such that for any two of its elements, one is strictly to the right and strictly above the other.

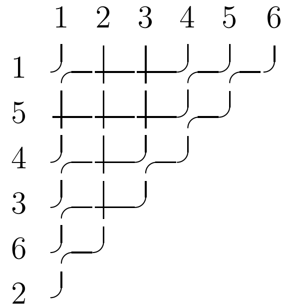
As it turns out, it is more convenient to draw dots instead of ones and leave cells filled with zeros empty.

Definition 2.4. $\mathcal{F}^{ne}_{01}(M,k)$ is the set of maximal 0-1-fillings of the moon polyomino M having length of the longest north-east chain equal to k. $\mathcal{F}^{ne}_{01}(M,k,\mathbf{r})$ is the subset of $\mathcal{F}^{ne}_{01}(M,k)$ consisting of those fillings that have exactly $\mathbf{r}_i$ zero entries in row i.

For any filling in $\mathcal{F}^{ne}_{01}(M,k)$, and an empty cell ϵ, there must be a chain C such that replacing the 0 with 1 in ϵ, and adding ϵ to C, would make C into a $(k+1)$-chain. In this situation, we say that C is a *maximal chain for* ϵ.

Remark 1. Note that extending the first k rows and columns of a Ferrers diagram does not affect the set $\mathcal{F}^{ne}_{01}$, which is why we choose to fix the number of zero entries instead of entries equal to one, which might seem more natural at first glance.

Remark 2. For the staircase shape $\lambda_0 = (n-1, \dots, 1)$, the set $\mathcal{F}^{ne}_{01}(\lambda_0, k)$ has a particularly beautiful interpretation, namely as the set of k-triangulations of the n-gon. More precisely, label the vertices of the n-gon clockwise from 1 to n, and identify a cell of the shape in row i



	1	2	3	4	5	6
1	•	+	+	•	•	•
5	+	+	+	•	•	
4	•	+	•	•		
3	•	+	•			
6	•	•				
2	•					

	1	2	3	4	5	6
1	•	2	3	•	•	•
5	2	3	4	•	•	
4	•	4	•	•		
3	•	5	•			
6	•	•				
2	•					

FIGURE 2. a reduced pipe dream associated to the reduced factorisation $s_3s_2s_4s_3s_2s_4s_5$ of $1, 5, 4, 3, 6, 2$.

and column j with the pair $(n-i+1, j)$ of vertices. Thus, the entries in the filling equal to one define a set of diagonals of the n-gon. It is not hard to check that a north-east chain of length k in the filling corresponds to a set of k mutually crossing diagonals in the n-gon.

2.3. **Pipe dreams.**

Definition 2.5. A *pipe dream* for a permutation w is a filling of a the quarter plane $\mathbb{N}^2$, regarding each element of $\mathbb{N}^2$ as a cell, *elbow joints* ╯╭ and a finite number of *crosses* ┼, such that a pipe entering from above in column i exits to the left from row $w(i)$. A pipe dream is *reduced* if each pair of pipes crosses at most once, it is then also called *rc-graph*. $\mathcal{RC}(w)$ is the set of reduced pipe dreams for w, and $\mathcal{RC}(w, \mathbf{r})$ is the subset of $\mathcal{RC}(w)$ having precisely $\mathbf{r}_i$ crosses in row i.

Remark 3. Every pipe dream in $\mathcal{RC}(w)$ is associated to a reduced factorisation of w as follows: replace each cross appearing in row i and column j of the pipe dream with the elementary transposition $(i+j-1, i+j)$. Then the reduced factorisation of w is given by the sequence of transpositions obtained by reading each row of the pipe dream from right to left, and the rows from top to bottom. An example can be found in Figure 2.

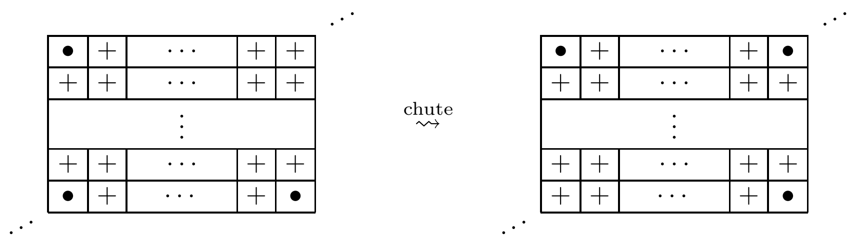
Usually it will be more convenient to draw dots instead of elbow joints and sometimes to omit crosses. We will do so without further notice.

Remark 4. Using reduced pipe dreams, it is possible to define the Schubert polynomial $\mathfrak{S}_w$ for the permutation w in a very concrete way. For a reduced pipe dream $D \in \mathcal{RC}(w)$, define $x^D = \prod_{(i,j)\in D} x_i$, where the product runs over all crosses in the pipe dream. Then the Schubert polynomial is just the generating function for pipe dreams:

$$\mathfrak{S}_w = \sum_{R \in \mathcal{RC}(w)} x^D$$

The following operation on pipe dreams, in a slightly less general form, was introduced by Nantel Bergeron and Sara Billey [1]. It will be the main tool in the proof of Theorem 3.2.

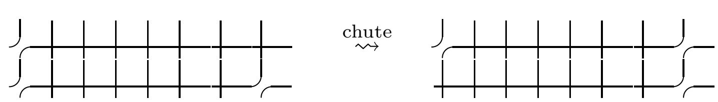
Definition 2.6. Let $D \in \mathcal{RC}(w)$ be a pipe dream. Then a *chute move* is a modification of D of the following form:



More formally, a *chutable rectangle* is a connected rectangle r inside a pipe dream D, with at least two columns and two rows, such that all but the following three locations of r are crosses: the north-west, south-west, and south-east corners. Applying a *chute move* to D is accomplished by placing a '+' in the south-west corner of a chutable rectangle r and removing the '+' from the north-east corner of r. We call the inverse operation *inverse chute move*.

Lemma 2.7 ([1], Lemma 3.5)**.** *The set $\mathcal{RC}(w)$ of reduced pipe dreams for w is closed under chute moves.*

Proof. The pictorial description of chute moves immediately implies that the permutation associated to the pipe dream remains unchanged. For example, here is the picture associated with a two rowed chute move:



□

Remark 5. It follows that chute moves define a partial order on $\mathcal{RC}(w)$, where D is covered by E if there is a chute move transforming E into D. Nantel Bergeron and Sara Billey restricted their attention to two rowed chute moves. For this case, their main theorem states that the poset defined by chute moves has a unique maximal element, namely

$$D_{top}(w) = \left\{(c, j) : c \leq \#\{i : i < w_j^{-1}, w_i > j\}\right\}.$$

It is easy to see that considering general chute moves, the poset has also a unique minimal element, namely

$$D_{bot}(w) = \{(i,c) : c \leq \#\{j : j > i, w_j < w_i\}\}\,.$$

In the next section, we will show a statement similar in spirit to the main theorem of Nantel Bergeron and Sara Billey for the more general chute moves defined above.

3. Maximal Fillings of Moon Polyominoes and Pipe Dreams

Consider a filling in $\mathcal{F}^{ne}_{01}(M,k)$. Replacing zeros with crosses, and all cells containing ones as well as all cells not in M with elbow joints, we clearly obtain a pipe dream. We will see in this section that it is in fact reduced.

Even without that knowledge, we can speak of chute moves applied to fillings in $\mathcal{F}^{ne}_{01}(M,k)$. However, a priori it is not clear under which condititions the result of such a move is again a filling in $\mathcal{F}^{ne}_{01}(M,k)$. In particular, we have to deal with the fact that under this identification all cells outside M are also filled with *elbow joints*, corresponding to *ones*. Of course, to determine the set of north-east chains, we have to consider the original filling and the boundary of M, and disregard elbow joints outside.

Similar to the article of Nantel Bergeron and Sara Billey, we will also consider two special fillings $D_{bot}(M,k)$ and $D_{top}(M,k)$. These will turn out to be the minimal and the maximal element in the poset having elements $\mathcal{F}^{ne}_{01}(M,k)$, where one filling is smaller than another if it can be obtained by applying chute moves to the latter.

Definition 3.1. Let M be a moon polyomino and $k \geq 0$. Then $D_{top}(M,k) \in \mathcal{F}^{ne}_{01}(M,k)$ is obtained by putting ones into all cells that can be covered by any rectangle of size at most $k \times k$, which is completely contained in the moon polyomino, and that touches the boundary of M with its lower-left corner.

Similarly, $D_{bot}(M,k) \in \mathcal{F}^{ne}_{01}(M,k)$ is obtained by putting ones into all cells that can be covered by any rectangle of size at most $k\times k$, which is completely contained in the moon polyomino, and that touches the boundary of M with its upper-right corner.

We can now state the main theorem of this article:

Theorem 3.2. *Let M be a moon polyomino and $k \geq 0$. The set $\mathcal{F}^{ne}_{01}(M,k,\mathbf{r})$ can be identified with the set of reduced pipe dreams $\mathcal{RC}\big(w(M,k),\mathbf{r}\big)$ having all crosses inside of M, for some permutation depending only on*

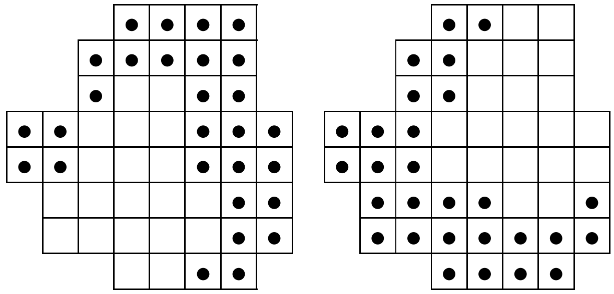


FIGURE 3. The special fillings $D_{bot}(M,k)$ and $D_{top}(M,k)$ for $k=2$ of a moon polyomino.

M and k, as follows: replace zeros with crosses, and all cells containing ones as well as all cells not in M with elbow joints.

More precisely, the set $\mathcal{F}^{ne}_{01}(M,k)$ is an interval in the poset of reduced pipe dreams $\mathcal{RC}\big(w(M,k)\big)$, with minimal element $D_{bot}(M,k)$ and maximal element $D_{top}(M,k)$.

As already remarked in the introduction, various versions of this theorem were independently proved by various authors, by various methods. Certainly the most general of these is Theorem 2.6 in the article by Luis Serrano and Christian Stump [14]. However, we are not aware of another proof for the property that $\mathcal{F}^{ne}_{01}(M,k)$ is in fact an interval in the bigger set of reduced pipe dreams.

Let us first state a very basic property of chute moves as applied to fillings:

Lemma 3.3. *Let M be a moon polyomino. Chute moves and their inverses applied to a filling in $\mathcal{F}^{ne}_{01}(M,k)$ produce another filling in $\mathcal{F}^{ne}_{01}(M,k)$, whenever all zero entries remain in M.*

Proof. We only have to check that chain lengths are preserved, which is not hard. □

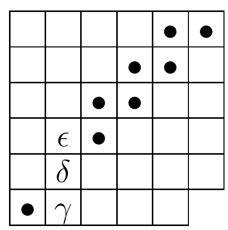
Most of what remains of this section is devoted to prove that there is precisely one filling in $\mathcal{F}^{ne}_{01}(M,k)$, that does not admit a chute move such that the result is again in $\mathcal{F}^{ne}_{01}(M,k)$, namely $D_{bot}(M,k)$, and precisely one filling that does not admit an inverse chute move with the same property, namely $D_{top}(M,k)$.

Although the strategy itself is actually very simple, this appears to be relatively delicate, so we split the proof into a few auxiliary lemmas. Let us fix k, a moon polyomino M, and a maximal filling $D\in\mathcal{F}^{ne}_{01}(M,k)$ different from $D_{bot}(M,k)$. We will then explicitely locate a chutable rectangle. Note that maximality of the filling will play a crucial role throughout. The first lemma is used to show that certain cells of the polyomino must be empty, because otherwise the filling would contain a chain of length $k+1$:

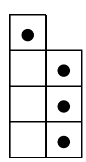
Lemma 3.4 (Chain induction). *Consider a maximal filling of a moon polyomino. Let ϵ be an empty cell such that all cells below ϵ in the same column are empty too, except possibly those that are below the lowest cell of the column left of ϵ. Assume that for* each *cell δ of these, there is a maximal chain for δ strictly north-east of δ. Then there is a maximal chain for ϵ strictly north-east of ϵ.*

Similarly, let ϵ be an empty cell such that all cells left of ϵ in the same row are empty too, except possibly those that are left of the left-most cell of the row below ϵ. Assume that for each *cell δ of these, there is a maximal chain for δ strictly north-east of δ. Then there is a maximal chain for ϵ strictly north-east of ϵ.*

Remark 6. Note that for the conclusion of Lemma 3.4 to hold, it is important that there is a maximal chain north-east of *all cells* below ϵ:

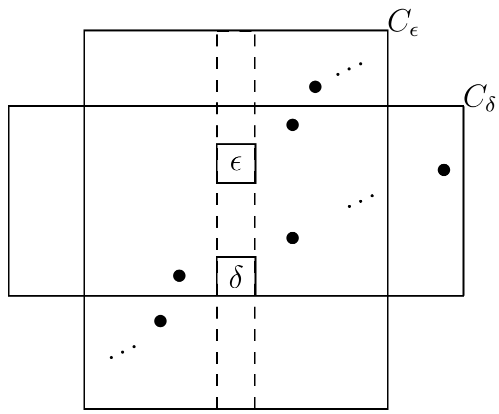


Although in the (for clarity non-maximal) filling above there is a 4-chain north-east of δ, there is no such chain north-east of ϵ - and indeed no such chain north-east of γ. In particular, we really have to assume that *all* cells below ϵ are empty. As the following example for $k=1$ demonstrates, it is equally necessary, that the filling is maximal:



Proof. Assume on the contrary that there is no maximal chain for ϵ north-east of ϵ. Consider a maximal chain C_ϵ for ϵ that has as many elements north-east of ϵ as possible. Let δ be the cell in the same column as ϵ, below ϵ, in the same row as the top entry of C_ϵ which is south-east of ϵ. By assumption, there is a maximal chain C_δ for δ north-east of δ. We have to consider two cases:

If the widest rectangle containing C_ϵ is not as wide as the smallest rectangle containing C_δ, then the entry of C_ϵ to the left of δ would extend C_δ to a $(k+1)$-chain, which is not allowed:



If the smallest rectangle containing C_ϵ is at least as wide as the widest rectangle containing C_δ, then we obtain a maximal chain for ϵ north-east of ϵ by induction. Let $c_\epsilon^1, c_\epsilon^2, \dots$ be the sequence of elements of C_ϵ north-east of ϵ, and $c_\delta^1, c_\delta^2, \dots$ the sequence of elements of C_δ north-east of δ. We will show that c_ϵ^i must be strictly north and weakly west of c_δ^i, for all i. Thus, the elements $c_\epsilon^1, c_\epsilon^2, \dots$ together with the elements of C_δ outside the smallest rectangle containing C_ϵ form a maximal chain for ϵ north-east of ϵ.

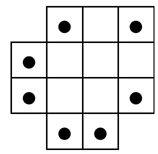
c_ϵ^1 is strictly north of c_δ^1, since otherwise C_δ would be a maximal chain for ϵ. c_ϵ^1 cannot be strictly east of c_δ^1, since in this case c_δ^1 together with C_ϵ would be a $(k+1)$-chain.

Suppose now that c_ϵ^{i-1} is strictly north and weakly west of c_δ^{i-1}. c_δ^i cannot be strictly north-east of c_ϵ^{i-1}, since this would yield a k-chain north-east of ϵ. c_δ^i must be strictly east of c_ϵ^{i-1}, since c_δ^i is strictly east of c_δ^{i-1}, which in turn is weakly east of c_ϵ^{i-1} by the induction hypothesis. Thus, c_ϵ^{i-1} is weakly north and strictly west of c_δ^i.

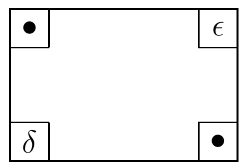
c_ϵ^i cannot be strictly north-east of c_δ^i, since then the elements of C_ϵ south-west of ϵ together with the elements $c_\delta^1, \dots, c_\delta^i$ and $c_\epsilon^i, c_\epsilon^{i+1}, \dots$ would form a $(k+1)$-chain. Finally, c_ϵ^i must be strictly north of c_δ^i, since c_ϵ^i is strictly north of c_ϵ^{i-1}, which in turn is weakly north of c_δ^i. □

Lemma 3.5. *Consider a maximal filling of a moon polyomino. Suppose that there is a rectangle with at least two columns and two rows completely contained in the polyomino, with all cells empty except the north-west, south-east and possibly the south-west corners. Then the south-west corner is indeed non-empty, i.e., the rectangle is chutable.*

Remark 7. Note that we must insist that the south-west corner of the rectangle is part of the polyomino. Here is a maximal filling with $k = 1$, where the three cells in the south-west do not form a chutable rectangle, since the south-west corner is missing:



Proof. Suppose on the contrary that the cell in the south-west corner is empty, too. Then, the situation is as in the following picture:



Since the filling is maximal, but the cells δ and ϵ are empty, there must be maximal chains for these cells. The corresponding rectangles must not cover any of the two cells containing ones, since that would imply the existence of a $(k+1)$-chain. Thus, any maximal chain for δ must be strictly north-east of δ, and any maximal chain for ϵ must be strictly south-west of ϵ. Since the polyomino is intersection free, the top row of the rectangle containing the maximal chain for ϵ is either contained in the bottom row of the rectangle containing the maximal chain for δ, or vice versa. In both cases, we have a contradiction. □

The next lemma parallels the main Lemma 3.6 in the article by Nantel Bergeron and Sara Billey [1]:

Lemma 3.6. *Consider a maximal filling of a moon polyomino. Suppose that there is a cell γ containing a 1 with an empty cell ϵ in the neighbouring cell to its right, such that there are at least as many cells above γ as above ϵ. Then the filling contains a chutable rectangle.*

Similarly, suppose that there is a cell γ containing a 1 with an empty cell ϵ in the neighbouring cell below it, such that there are at least as many cells right of γ as right of ϵ. Then the filling contains a chutable rectangle.

Proof. Suppose that all of the cells in the column containing ϵ are empty, which are below ϵ and weakly above the bottom cell of the column containing γ. There must then be a maximal chain for the lowest cell in this region, that is north-east of it. By Lemma 3.4, we conclude that there is also a maximal chain for ϵ north-east of ϵ. However, then the 1 in the cell left of ϵ together with this chain yields a $(k+1)$-chain, since the rectangle containing the maximal chain for ϵ extends by hypothesis to the column above the cell left of ϵ.

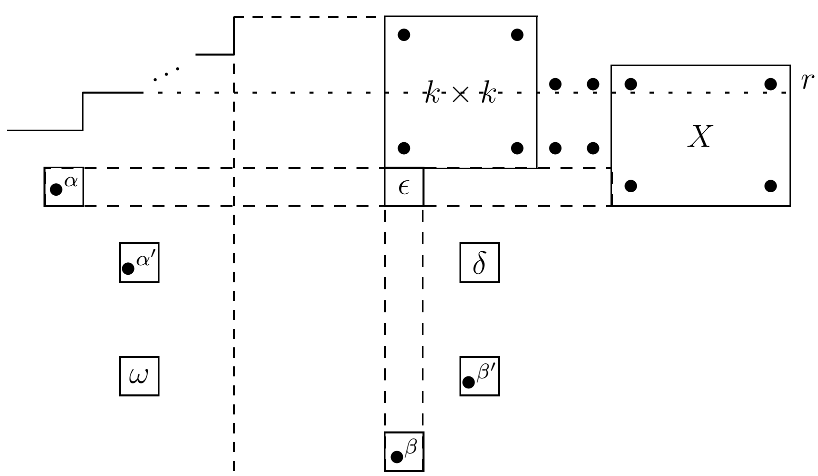
We can thus apply Lemma 3.5 to the following rectangle: the south-east corner being the top non-empty cell below ϵ, and the north-west corner being the lowest cell containing a 1 in the column of γ, strictly above the chosen south-east corner. □

Finally, the main statement follows from a careful analysis of fillings different from $D_{bot}(M,k)$, repeatedly applying the previous lemmas to exclude obstructions to the existence of a chutable rectangle:

Theorem 3.7. *Any maximal filling other than $D_{bot}(M,k)$ admits a chute move, and any maximal filling other than $D_{top}(M,k)$ admits an inverse chute move.*

Proof. Suppose that all cells in the top-right quarter of M that contain a 1 in $D_{bot}(M,k)$ also contain a 1 in the filling F at hand. It follows, that all cells that are empty in $D_{bot}(M,k)$ are empty in F, too, because there is a maximal chain for each of them. Thus, in this case $F = D_{bot}(M,k)$.

Otherwise, consider the set of left-most cells in the top-right quarter, that contain a 1 in $D_{bot}(M,k)$ but are empty in F, and among those the top cell, ϵ. If its left or lower neighbour contains a 1, we can apply Lemma 3.6 and are done. Otherwise, we have to find a rectangle as in the hypothesis of Lemma 3.5. The difficulty in this undertaking is to prove that the lower left corner is indeed part of the polyomino. To ease the understanding of the argument, we will frequently refer to the following sketch:



By construction, there is a $k \times k$ square filled with ones just above ϵ, and there cannot be any k-chain north-east of ϵ. This implies in particular that the top cell in the left-most column of the polyomino must be lower than the top row of the $k \times k$ square, because otherwise there could not be any maximal chain for ϵ.

By Lemma 3.4, there must therefore be a non-empty cell left of ϵ, which we label α, and a non-empty cell below ϵ, which we label β. Note that there may be entries to the right of ϵ, in the same column, which are non-empty. However, we can assume that to the right of the first such entry all other cells in this row are non-empty, too, because otherwise we could apply Lemma 3.6.

We can now construct a chutable rectangle: let β' be the top cell containing a 1 below an empty cell weakly to the right of ϵ, and if there are several, the left-most. Also, let α' be the lowest cell among the right-most containing a 1, which are weakly below α, but strictly above β'. Let δ be the cell in the same row as α' and the same column as β'. Let ω be the cell in the same column as α' and the same row as β' – we have to show that ω is in fact part of the polyomino. We can then apply Lemma 3.5 to the rectangle defined by δ' and the first non-empty cell to the right of ω, in the same row.

To achieve our goal, we show that there cannot be a maximal chain for δ north-east of δ. Suppose on the contrary that there is such a chain. At least its top-right element must be in a row (denoted r in the sketch) above the top cell of the column containing α': otherwise, α' together with this chain would form a $(k+1)$-chain. In the sketch, the non-empty cells that are implied are indicated by the rectangle denoted X.

Consider the bottom-left element of a maximal chain for ϵ. It cannot be strictly north of ω: in this case, it would also have to be strictly west of ω, since there are by construction no non-empty cells to the east. Therefore, the chain for ϵ has to be below the top row of the column containing ω and α', which in turn implies that it can be extended to a $(k+1)$-chain using the cells in X.

Suppose therefore, that the bottom-left element of a maximal chain for ϵ is weakly below ω. Since our sole goal is to show that ω is inside the polyomino, we only have to consider the case that the element is strictly right of ω. However, then we know that there is a maximal chain for the right neighbour of α which is not completely north-east of that cell. Since the rectangle enclosing this chain certainly starts left of the row r, and the polyomino is intersection free, it extends to the right border of the rectangle denoted X, and thus contains a $(k+1)$-chain.

We have shown that a maximal chain for δ must have some elements south-east of δ. It is now easy to see similarly as in the proof of

Lemma 3.5, that this implies that ω is indeed part of the polyomino, and in fact contains a 1. □

Proof of Theorem 3.2. All pipe dreams in $\mathcal{RC}(w)$ contained in M are maximal 0-1 fillings of M, since they can be generated by applying sequences of chute moves to $D_{top}(M, k)$.

Since we can apply chute moves to any maximal 0-1-filling of M except $D_{bot}(M, k)$, all such fillings arise in this fashion. (We have to remark here, that in case the pipe dream associated to some filling would not be reduced, applying chute moves eventually exhibits that the filling was not maximal.) Together with Lemma 3.3, this implies that all fillings $F_{01}^{ne}(M, k)$ have the same associated permutation.

We remark that, additionally, this procedure implies that all maximal 0-1-fillings of M have the same number of entries equal to zero, *i.e.*, the simplicial complex of fillings is pure. □

4. Applying the Edelman-Greene correspondence

Using the identification described in the previous section, we can apply a correspondence due to Paul Edelman and Curtis Greene [2], that associates pairs of tableaux to reduced factorisations of permutations. This in turn will yield the desired bijective proof of Jakob Jonsson's result at least for stack polyominoes.

We remark that the main result of this section was obtained for Ferrers shapes earlier by Luis Serrano and Christian Stump [14], using a relatively similar proof strategy. However, since the details are different, we believe it is useful to repeat it here.

Theorem 4.1 (Paul Edelman and Curtis Greene [2], Richard Stanley [15], Alain Lascoux and Marcel-Paul Schützenberger [9])**.** *There is a bijection between pairs of words reduced factorisations of a permutation w and pairs (P, Q) of Young tableaux of the same shape, such that P is column strict with reading word reduced equivalent to w, and Q is standard. Moreover, if w is vexillary, i.e.,* 2143*-avoiding, the tableau P is the same for all reduced factorisations of w.*

It turns out that the permutations associated to moon polyominoes are indeed vexillary:

Proposition 4.2. *For any moon-polyomino M and any k, the permutation $w(M, k)$ is vexillary.*

Remark 8. There are vexillary permutations which do not correspond to moon polyominoes. For example, the only two reduced pipe dreams for the permutation $4, 2, 5, 1, 3$ are as follows:

	1	2	3	4	5
4	+	+	+	•	•
2	+	•	+	•	
5	+	•	•		
1	•	•			
3	•				

and

	1	2	3	4	5
4	+	+	+	•	•
2	+	•	•	•	
5	+	+	•		
1	•	•			
3	•				

Proof. It is sufficient to prove the claim for $k = 0$, since the empty cells in the filling $D_{top}(M, k)$ for any k again form a moon polyomino. Thus, suppose that the permutation associated to M is not vexillary. Then we have indices $i < j < k < \ell$ such that $w(j) < w(i) < w(\ell) < w(k)$. It follows that the pipes entering in columns i and j from above cross, and so do the two pipes entering in columns k and ℓ, and thus correspond to cells of the moon polyomino. Since any two cells in the moon polyomino can be connected by a path of neighbouring cells changing direction at most once, there is a third cell where either the pipes entering from i and ℓ or from j and k cross, which is impossible. □

Theorem 4.3 (for Ferrers shapes, Luis Serrano and Christian Stump [14])**.** *Consider the set $\mathcal{F}^{ne}_{01}(S, k, \mathbf{r})$, where S is a stack polyomino. Let μ_i be the number of cells the i^{th} row of S is indented to the right, and suppose that $\mu_1 = \cdots = \mu_k = \mu_{k+1} = 0$.*

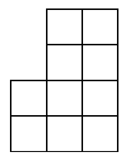
Let u be the word $1^{\mathbf{r}_1}, 2^{\mathbf{r}_2}, \dots$ and let v be the reduced factorisation of w associated to a given pipe dream. Then the Edelman-Greene correspondence applied to the pair of words (u, v) induces a bijection between $F^{ne}_{01}(S, k, \mathbf{r})$ and the set of pairs (P, Q) of Young tableaux satisfying the following conditions:

- *the common shape of P and Q is the multiset of column heights of the empty cells in $D_{top}(S, k)$,*
- *the first row of P equals $(k+1, k+2+\mu_{k+2}, k+3+\mu_{k+3}, \dots)$, and the entries in columns are consecutive,*
- *Q has type $\{1^{\mathbf{r}_1}, 2^{\mathbf{r}_2}, \dots)$, and entries in column i are at most $i + k$.*

Thus, the common shape of P and Q encodes the row lengths of S, the entries of the first column of P encode the left border of S, and the entries of Q encode the filling.

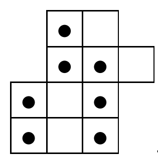
Remark 9. In particular, this theorem implies an explicit bijection between the sets $\mathcal{F}^{ne}_{01}(S_1, k, \mathbf{r})$ and $\mathcal{F}^{ne}_{01}(S_2, k, \mathbf{r})$, given that the multisets of column heights of S_1 and S_2 coincide.

Curiously, the most natural generalisation of the above theorem to moon polyominoes is not true. Namely, one may be tempted to replace the condition on Q by requiring that the entries of Q are between Q_{top} and Q_{bot} component-wise. However, this fails already for $k = 1$ and the shape



,

with $P = \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 5 \\ \cline{1-1}\end{array}$, $Q_{top} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$ and $Q_{bot} = \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 4 \\ \cline{1-1}\end{array}$. In this case, the tableau $Q = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$ has preimage



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Proof. In view of Proposition 4.2, to obtain the tableau P it is enough to insert the reduced word given by the filling $D_{top}(S, k)$ using the Edelman-Greene correspondence, which is not hard for stack polyominoes.

It remains to prove that the entries in column i of Q are at most $i+k$ precisely if (u, v) comes from a filling in $\mathcal{F}^{ne}_{01}(S, k)$. To this end, observe that the shape of the first i columns of P equals the shape of the tableau obtained after inserting the pair of words $((u_1, u_2, \dots, u_\ell), (v_1, v_2, \dots, v_\ell))$, where ℓ is such that $u_\ell \leq k + i$ and $u_{\ell+1} > k + i$.

Namely, this is the case if and only if the first $i + k + \mu_{i+k+1}$ positions of the permutation corresponding to $(v_1, v_2, \dots, v_\ell)$ coincide with those of the permutation w corresponding to v itself, as can be seen by considering $D_{top}(w)$, whose empty cells form again a stack polyomino.

This in turn is equivalent to all letters v_m being at least $k + i + 1 + \mu_{k+i+1}$ for $m > \ell$, *i.e.*, whenever the corresponding empty cell of the filling occurs in a row below the $(i + k)^{\text{th}}$ of S, and thus, when it is inside S. □

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References

[1] Nantel Bergeron and Sara Billey, *RC-graphs and Schubert polynomials*, Experiment. Math. **2** (1993), no. 4, 257–269.

[2] Paul Edelman and Curtis Greene, *Balanced tableaux*, Adv. in Math. **63** (1987), no. 1, 42–99, 10.1016/0001-8708(87)90063-6.

[3] Sergey Fomin and Anatol N. Kirillov, *The Yang-Baxter equation, symmetric functions, and Schubert polynomials*, Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993), vol. 153, 1996, 10.1016/0012-365X(95)00132-G, pp. 123–143.

[4] ———, *Reduced words and plane partitions*, J. Algebraic Combin. **6** (1997), no. 4, 311–319, 10.1023/A:1008694825493.

[5] Jakob Jonsson, *Generalized triangulations and diagonal-free subsets of stack polyominoes*, Journal of Combinatorial Theory, Series A **112** (2005), no. 1, 117–142.

[6] Allen Knutson and Ezra Miller, *Subword complexes in Coxeter groups*, Adv. Math. **184** (2004), no. 1, 161–176, 10.1016/S0001-8708(03)00142-7.

[7] ———, *Gröbner geometry of Schubert polynomials*, Ann. of Math. (2) **161** (2005), no. 3, 1245–1318, 10.4007/annals.2005.161.1245.

[8] Christian Krattenthaler, *Growth diagrams, and increasing and decreasing chains in fillings of Ferrers shapes*, Advances in Applied Mathematics **37** (2006), no. 3, 404–431, math.CO/0510676.

[9] Alain Lascoux and Marcel-Paul Schützenberger, *Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux*, Comptes Rendus des Séances de l'Académie des Sciences. Série I. Mathématique **295** (1982), no. 11, 629–633.

[10] Vincent Pilaud, *Multitriangulations, pseudotriangulations et quelques problèmes de réalisation de polytopes*, Ph.D. thesis, Université Paris Diderot Paris 7, 2010, math.CO/1009.1605.

[11] Vincent Pilaud and Michel Pocchiola, *Multi-pseudotriangulations*, 25th European Workshop on Computational Geometry, EuroCG'09, 2009, http://2009.eurocg.org/abstracts.pdf, pp. 227–230.

[12] Vincent Pilaud and Francisco Santos, *Multitriangulations as complexes of star polygons*, Discrete Comput. Geom. **41** (2009), no. 2, 284–317, 10.1007/s00454-008-9078-6.

[13] Martin Rubey, *Increasing and Decreasing Sequences in Fillings of Moon Polyominoes*, *To appear in* Advances in Applied Mathematics, math.CO/0604140.

[14] Luis Serrano and Christian Stump, *Maximal fillings of moon polyominoes, simplicial complexes, and Schubert polynomials*, Preprint (2010), math.CO/1009.4690.

[15] Richard P. Stanley, *On the number of reduced decompositions of elements of Coxeter groups*, European Journal of Combinatorics **5** (1984), no. 4, 359–372.
[16] Christian Stump, *A new perspective on k-triangulations*, Preprint (2010), math.CO/1009.4101.
[17] Alexander Woo, *Catalan numbers and schubert polynomials for $w = 1(n+1)...2$*, Preprint (2004), math.CO/0407160.

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