ISOMETRIC IMMERSIONS OF THE HYPERBOLIC PLANE INTO THE HYPERBOLIC SPACE

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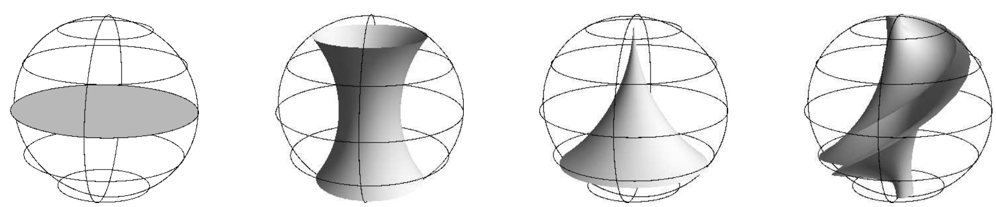
Abstract. In this paper, we parametrize the space of isometric immersions of the hyperbolic plane into the hyperbolic 3-space in terms of null-causal curves in the space of oriented geodesics. Moreover, we characterize "ideal cones" (i.e., cones whose vertices are on the ideal boundary) by behavior of their mean curvature.

Introduction

Consider isometric immersions of $\tilde{\Sigma}^n(c)$ into $\tilde{\Sigma}^{n+1}(c)$, where $\tilde{\Sigma}^m(c)$ denotes the simply connected m-dimensional space form of constant sectional curvature c. Such immersions are only cylinders [HN] in the Euclidean case ($c = 0$). In the spherical case ($c > 0$), such immersions are only totally geodesic embeddings [OS]. On the other hand, in the hyperbolic case ($c < 0$), it is well-known that there are nontrivial examples of such isometric immersions [N, F, AH] (see Figure 1 for the case of $n = 2$).



(A) totally geodesic (B) Example 3.7 (C) Example 3.8 (D) Example 3.9

Figure 1. Examples constructed by Nomizu [N] (see Section 3).

We denote by $\boldsymbol{H}^n = \tilde{\Sigma}^n(-1)$ the n-dimensional hyperbolic space, that is, the complete simply connected and connected Riemannian manifold of constant curvature -1. Nomizu [N] and Ferus [F] showed that, for a given C^∞ totally geodesic foliation of codimension 1 in $\boldsymbol{H}^n$, there is a family of isometric immersions of $\boldsymbol{H}^n$ into $\boldsymbol{H}^{n+1}$ without umbilic points such that, for each immersion, the foliation defined by its asymptotic distribution coincides with the given foliation. Furthermore, Abe, Mori and Takahashi [AMT] parametrized the space of isometric immersions of $\boldsymbol{H}^n$ into $\boldsymbol{H}^{n+1}$ by a family of properly chosen countably many $\boldsymbol{R}^n$-valued functions.

In this paper, we shall give another parametrization in the case of $n = 2$: we represent isometric immersions of $\boldsymbol{H}^2$ into $\boldsymbol{H}^3$ by curves in the space $L\boldsymbol{H}^3$ of oriented geodesics in $\boldsymbol{H}^3$. Moreover, we characterize certain asymptotic behavior of such immersions in terms of their mean curvature.

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More precisely, an isometric immersion of $\boldsymbol{H}^2$ into $\boldsymbol{H}^3$ is a complete *extrinsically flat* surface in $\boldsymbol{H}^3$, that is, a complete surface whose extrinsic curvature vanishes. It is known that a complete extrinsically flat surface is *ruled*, i.e., a locus of a 1-parameter family of geodesics in $\boldsymbol{H}^3$ [P] (see Proposition 3.2). Hence, we shall deal with extrinsically flat ruled surfaces: *developable* surfaces in $\boldsymbol{H}^3$. On the other hand, it is well-known that the space of oriented geodesics $L\boldsymbol{H}^3$ has two significant geometric structures: the natural complex structure J [Hi, GG] and the para-complex structure P [KK, Ka, Ki]. Recently, Salvai [S] determined the family of metrics $\{\mathcal{G}_\theta\}_{\theta\in S^1}$ each of which is invariant under the action of the identity component of the isometry group of $\boldsymbol{H}^3$. Each metric $\mathcal{G}_\theta$ is of neutral signature, Kähler with respect to J and para-Kähler with respect to P. In this paper, we especially focus on two neutral metrics $\mathcal{G}^{\mathrm{r}} = \mathcal{G}_0$ and $\mathcal{G}^{\mathrm{i}} = \mathcal{G}_{\pi/2}$ in $\{\mathcal{G}_\theta\}_{\theta\in S^1}$. In Section 2, we shall investigate the relationships among J, P, $\{\mathcal{G}_\theta\}_{\theta\in S^1}$ and the canonical symplectic form on $L\boldsymbol{H}^3$, and give a characterization of $\mathcal{G}^{\mathrm{i}}$ and $\mathcal{G}^{\mathrm{r}}$ (Proposition 2.1). In Section 3, we introduce a representation formula for developable surfaces in $\boldsymbol{H}^3$ in terms of *null-causal curves* (Proposition 3.6):

Theorem I. *A curve in $L\boldsymbol{H}^3$ which is null with respect to $\mathcal{G}^{\mathrm{i}}$ and causal with respect to $\mathcal{G}^{\mathrm{r}}$ generates a developable surface in $\boldsymbol{H}^3$. Conversely, any developable surface generated by complete geodesics in $\boldsymbol{H}^3$ is given in this manner.*

Here, a regular curve in a pseudo-Riemannian manifold is called *null* (resp. *causal*) if every tangent vector gives null (resp. timelike or null) direction. In Section 4, we shall investigate curves in $L\boldsymbol{H}^3$ which are null with respect to both $\mathcal{G}^{\mathrm{r}}$ and $\mathcal{G}^{\mathrm{i}}$. Such curves generate cones whose vertices are on the ideal boundary, which we call *ideal cones* (Proposition 4.2). On the other hand, on each asymptotic curve γ on a complete developable surface, the mean curvature is proportional to $e^{\pm t}$ or $1/\cosh t$, where t denotes the arc length parameter of γ (Lemma 3.3). Based on this fact, a complete developable surface is said to be *of exponential type*, if the mean curvature is proportional to $e^{\pm t}$ on each asymptotic curve in the non umbilic point set (see Definition 4.5). Then we have the following

Theorem II. *A real-analytic developable surface of exponential type is an ideal cone.*

The assumption of “real-analyticity” cannot be removed (see Example 4.8).

As mentioned before, complete flat surfaces in the Euclidean 3-space $\boldsymbol{R}^3$ are only cylinders. However, if we admit *singularities*, there are a lot of interesting examples. Murata and Umehara [MU] investigated the global geometric properties of a class of flat surfaces with singularities in $\boldsymbol{R}^3$, so-called *flat fronts*. On the other hand, there is another generalization of ruled (resp. developable) surfaces in $\boldsymbol{R}^3$: *horocyclic* (resp. *horospherical flat horocyclic*) surfaces in $\boldsymbol{H}^3$ (for more details, see [IST, TT]).

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1. Preliminaries

1.1. Hyperbolic 3-space.

We denote by $\boldsymbol{L}^4$ the Lorentz-Minkowski 4-space with the Lorentz metric

$$\left\langle {}^t(x_0, x_1, x_2, x_3), {}^t(y_0, y_1, y_2, y_3) \right\rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3,$$

where ^{t} denotes the transposition. Then the hyperbolic 3-space is given by

$$\boldsymbol{H}^3 = \left\{ \boldsymbol{x} = {}^t(x_0, x_1, x_2, x_3) \in \boldsymbol{L}^4 \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1,\ x_0 > 0 \right\} \tag{1.1}$$

with the induced metric from $\boldsymbol{L}^4$, which is a complete simply connected and connected Riemannian 3-manifold with constant sectional curvature -1. We identify $\boldsymbol{L}^4$ with the set of 2×2 Hermitian matrices $\mathrm{Herm}(2) = \{X^* = X\}$ ($X^* := {}^t\overline{X}$) by

$$\boldsymbol{L}^4 \ni {}^t(x_0, x_1, x_2, x_3) \longleftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + i x_2 \\ x_1 - i x_2 & x_0 - x_3 \end{pmatrix} \in \mathrm{Herm}(2)$$

with the metric

$$\langle X, Y \rangle = -\frac{1}{2} \operatorname{trace}(X\tilde{Y}), \qquad \langle X, X \rangle = -\det X,$$

where $\tilde{Y}$ is the cofactor matrix of Y. Under this identification, the hyperbolic 3-space $\boldsymbol{H}^3$ is represented as

$$\boldsymbol{H}^3 = \{ p \in \mathrm{Herm}(2) \,|\, \det p = 1,\ \operatorname{trace} p > 0 \}. \tag{1.2}$$

We call this realization of $\boldsymbol{H}^3$ the *Hermitian model*. We fix the basis $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ of $\mathrm{Herm}(2)$ as

$$\sigma_0 = \mathrm{id}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3}$$

In the Hermitian model, the cross product at $T_p\boldsymbol{H}^3$ is given by

$$X \times Y = \frac{i}{2}(Xp^{-1}Y - Yp^{-1}X), \tag{1.4}$$

for $X, Y \in T_p\boldsymbol{H}^3$ (cf. [KRSUY, (3 - 1)]). The special linear group $\mathrm{SL}(2, \boldsymbol{C})$ acts isometrically and transitively on $\boldsymbol{H}^3$ by

$$\boldsymbol{H}^3 \ni p \longmapsto apa^* \in \boldsymbol{H}^3, \tag{1.5}$$

where $a \in \mathrm{SL}(2, \boldsymbol{C})$. The isotropy subgroup of $\mathrm{SL}(2, \boldsymbol{C})$ at σ_0 is the special unitary group $\mathrm{SU}(2)$. Therefore we can identify

$$\boldsymbol{H}^3 = \mathrm{SL}(2, \boldsymbol{C}) / \mathrm{SU}(2) = \left\{ aa^* \,\middle|\, a \in \mathrm{SL}(2, \boldsymbol{C}) \right\}$$

in the usual way. Moreover, the identity component of the isometry group $\mathrm{Isom}_0(\boldsymbol{H}^3)$ is isomorphic to $\mathrm{PSL}(2, \boldsymbol{C}) := \mathrm{SL}(2, \boldsymbol{C})/\{\pm \mathrm{id}\}$.

1.2. **The unit tangent bundle.**

We denote by $U\boldsymbol{H}^3$ the unit tangent bundle of $\boldsymbol{H}^3$, which can be identified with

$$U\boldsymbol{H}^3 = \left\{ (p, v) \in \mathrm{Herm}(2) \times \mathrm{Herm}(2) \,\middle|\, \begin{array}{c} \det p = -\det v = 1, \\ \operatorname{trace} p > 0,\ \langle p, v \rangle = 0 \end{array} \right\}.$$

The projection

$$\pi : U\boldsymbol{H}^3 \ni (p, v) \longmapsto p \in \boldsymbol{H}^3 \tag{1.6}$$

gives a sphere bundle. The tangent space at $(p, v) \in U\boldsymbol{H}^3$ can be written by

$$T_{(p,v)}U\boldsymbol{H}^3 = \left\{ (X, V) \in \mathrm{Herm}(2) \times \mathrm{Herm}(2) \,\middle|\, \begin{array}{c} \langle p, X \rangle = \langle v, V \rangle = 0, \\ \langle p, V \rangle = -\langle X, v \rangle \end{array} \right\}. \tag{1.7}$$

The *canonical contact form* Θ on $U\boldsymbol{H}^3$ is given by

$$\Theta_{(p,v)}(X, V) = \langle X, v \rangle = -\langle p, V \rangle, \qquad (X, V) \in T_{(p,v)}U\boldsymbol{H}^3. \tag{1.8}$$

The isometric action of $\mathrm{SL}(2,\boldsymbol{C})$ on $\boldsymbol{H}^3$ as in (1.5) induces a transitive action on $U\boldsymbol{H}^3$ as

$$U\boldsymbol{H}^3 \ni (p,v) \longmapsto (apa^*, ava^*) \in U\boldsymbol{H}^3,$$

where $a \in \mathrm{SL}(2,\boldsymbol{C})$. The isotropy subgroup of $\mathrm{SL}(2,\boldsymbol{C})$ at $(\sigma_0,\sigma_3) \in U\boldsymbol{H}^3$ is

$$\left\{ \left(\begin{array}{cc} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{array} \right) \middle|\, \theta \in \boldsymbol{R}/2\pi\boldsymbol{Z} \right\}$$

which is isomorphic to the unitary group U(1), where σ_0 and σ_3 are as in (1.3). Hence we have

$$U\boldsymbol{H}^3 = \mathrm{SL}(2,\boldsymbol{C})/\,\mathrm{U}(1) = \left\{ (aa^*, a\sigma_3 a^*) \,\middle|\, a \in \mathrm{SL}(2,\boldsymbol{C}) \right\}. \tag{1.9}$$

1.3. **The space of oriented geodesics.**

The space $L\boldsymbol{H}^3$ of oriented geodesics in $\boldsymbol{H}^3$ is defined as the set of equivalence classes of unit speed geodesics in $\boldsymbol{H}^3$. Here, two unit speed geodesics $\gamma_1(t)$, $\gamma_2(t)$ in $\boldsymbol{H}^3$ are said to be *equivalent* if there exists $t_0 \in \boldsymbol{R}$ such that $\gamma_1(t+t_0) = \gamma_2(t)$. We denote by $[\gamma]$ the equivalence class represented by $\gamma(t)$. The set $L\boldsymbol{H}^3$ has a structure of a smooth 4-manifold. Moreover, if we denote by $\mathrm{SO}^+(1,1)$ the restricted Lorentz group, the projection

$$\hat{\pi} : U\boldsymbol{H}^3 \ni (p,v) \longmapsto [\gamma_{p,v}] \in L\boldsymbol{H}^3 \tag{1.10}$$

defines an $\mathrm{SO}^+(1,1)$-bundle, where $\gamma_{p,v}$ is the geodesic starting at $p \in \boldsymbol{H}^3$ with the initial velocity $v \in T_p\boldsymbol{H}^3$.

1.3.1. *The natural complex structure and a holomorphic coordinate system.*

Hitchin [Hi] constructed the natural complex structure J on $L\boldsymbol{H}^3$ (*minitwistor construction*). Here, we introduce a local holomorphic coordinate system (μ_1,μ_2) of the complex surface $(L\boldsymbol{H}^3, J)$ [GG]. We denote by $\partial\boldsymbol{H}^3$ the ideal boundary of $\boldsymbol{H}^3$, that is, the set of asymptotic classes of oriented geodesics. For a geodesic $\gamma = \gamma(t)$, set γ_+, $\gamma_- \in \partial\boldsymbol{H}^3$ as

$$\gamma_+ := \lim_{t\to\infty} \gamma(t), \qquad \gamma_- := \lim_{t\to-\infty} \gamma(t). \tag{1.11}$$

Evidently, γ_+ and γ_- are independent of choice of a representative of $[\gamma]$, and $(\gamma_+,\gamma_-) \in (\partial\boldsymbol{H}^3 \times \partial\boldsymbol{H}^3) \setminus \Delta$ holds, where Δ is the diagonal set of $\partial\boldsymbol{H}^3 \times \partial\boldsymbol{H}^3$. Conversely, for any distinct points a, $b \in \partial\boldsymbol{H}^3$, there exists a unique equivalence class $[\gamma] \in L\boldsymbol{H}^3$ such that $\gamma_+ = a$, $\gamma_- = b$. Thus, we can identify $L\boldsymbol{H}^3 = (\partial\boldsymbol{H}^3 \times \partial\boldsymbol{H}^3) \setminus \Delta$ as a set. Now we recall the *upper-half space model* of $\boldsymbol{H}^3$:

$$\boldsymbol{R}^3_+ = \left(\{(w,r) \in \boldsymbol{C}\times\boldsymbol{R} \mid r > 0\},\ \frac{dw d\bar{w} + dr^2}{r^2} \right). \tag{1.12}$$

A map

$$\Psi : \boldsymbol{H}^3 \ni \left(\begin{array}{cc} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{array} \right) \longmapsto \left(\frac{x_1 + ix_2}{x_0 - x_3}, \frac{1}{x_0 - x_3} \right) \in \boldsymbol{R}^3_+ \tag{1.13}$$

gives an isometry. The geodesics of $\boldsymbol{R}^3_+$ are divided into two types: straight lines parallel to the r-axis and semicircles perpendicular to the w-plane.

Identifying $\partial\boldsymbol{H}^3$ with the Riemann sphere $\hat{\boldsymbol{C}} := \boldsymbol{C} \cup \{\infty\}$, we may consider γ_+ and γ_- as points in $\hat{\boldsymbol{C}}$. Then we set an open subset $\mathcal{U}$ of $L\boldsymbol{H}^3$ as

$$\mathcal{U} := \left\{ [\gamma] \in L\boldsymbol{H}^3 \,\middle|\, \gamma_+ \neq 0,\ \gamma_- \neq \infty \right\}, \tag{1.14}$$

and complex numbers μ_1, μ_2 as

$$\mu_1 := -\gamma_-, \qquad \mu_2 := \frac{1}{\overline{\gamma_+}} \tag{1.15}$$

for $[\gamma] \in \mathcal{U}$ (see Figure 2). Georgiou and Guilfoyle [GG] proved that $(\mathcal{U}; (\mu_1, \mu_2))$ defines a local holomorphic coordinate system of $L\boldsymbol{H}^3$ compatible to the complex structure J, and the map $[\gamma] \longmapsto (\mu_1, \mu_2)$ extends to a biholomorphic map

$$(L\boldsymbol{H}^3, J) \xrightarrow{\sim} (\hat{\boldsymbol{C}} \times \hat{\boldsymbol{C}}) \setminus \hat{\Delta},$$

where $\hat{\Delta} = \{(\mu_1, \mu_2) \in \boldsymbol{C}^2 \mid 1 + \mu_1 \bar{\mu}_2 = 0\} \cup \{(0, \infty), (\infty, 0)\}$, so-called the reflected diagonal.

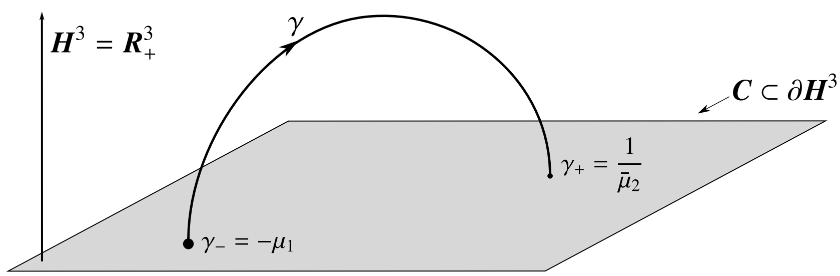

FIGURE 2. The holomorphic coordinate system (μ_1, μ_2).

Remark 1.1 (As a complex line bundle). Over the complex projective line $\boldsymbol{P}^1$, the map

$$\Pi : L\boldsymbol{H}^3 \ni [\gamma] \longmapsto \gamma_- \in \boldsymbol{P}^1$$

gives a complex line bundle. Each fiber of γ_- is $\boldsymbol{P}^1 \setminus \{\gamma_-\}$ which is identified with $\boldsymbol{C}$. It is easy to see that Π is a trivial bundle $\mathcal{O}_{\boldsymbol{P}^1}(0)$. On the other hand, the space $L\boldsymbol{R}^3$ of oriented geodesics in the Euclidean 3-space is biholomorphic to the holomorphic tangent bundle $T\boldsymbol{P}^1$ of $\boldsymbol{P}^1$ [GK]. That is $L\boldsymbol{R}^3 \cong \mathcal{O}_{\boldsymbol{P}^1}(2)$. This implies that $L\boldsymbol{H}^3$ is not isomorphic to $L\boldsymbol{R}^3$ as a line bundle over $\boldsymbol{P}^1$.

1.3.2. *The invariant metrics, Kähler and para-Kähler structures.*
The isometric action of $\mathrm{SL}(2, \boldsymbol{C})$ on $\boldsymbol{H}^3$ as in (1.5) induces an action on $\partial \boldsymbol{H}^3 = \hat{\boldsymbol{C}}$ as

$$\hat{\boldsymbol{C}} \ni z \longmapsto \frac{a_{11} z + a_{12}}{a_{21} z + a_{22}} \in \hat{\boldsymbol{C}},$$

where $a = (a_{ij}) \in \mathrm{SL}(2, \boldsymbol{C})$. This action induces a holomorphic and transitive action of $\mathrm{Isom}_0(\boldsymbol{H}^3) = \mathrm{PSL}(2, \boldsymbol{C})$ on $L\boldsymbol{H}^3 = (\hat{\boldsymbol{C}} \times \hat{\boldsymbol{C}}) \setminus \hat{\Delta}$ as

$$(\hat{\boldsymbol{C}} \times \hat{\boldsymbol{C}}) \setminus \hat{\Delta} \ni (\mu_1, \mu_2) \longmapsto \left(\frac{-a_{11}\mu_1 + a_{12}}{a_{21}\mu_1 - a_{22}}, \frac{\bar{a}_{22}\mu_2 + \bar{a}_{21}}{\bar{a}_{12}\mu_2 + \bar{a}_{11}} \right) \in (\hat{\boldsymbol{C}} \times \hat{\boldsymbol{C}}) \setminus \hat{\Delta}, \tag{1.16}$$

for $a = (a_{ij}) \in \mathrm{PSL}(2, \boldsymbol{C})$. If we set a $\boldsymbol{C}$-valued symmetric 2-tensor on $L\boldsymbol{H}^3$ as

$$\mathcal{G} := \frac{4\, d\mu_1 d\bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2}, \tag{1.17}$$

then it holds that

$$\mathcal{G}_\theta := \mathrm{Re}\left(e^{-i\theta} \mathcal{G}\right) = (\cos\theta)\, \mathcal{G}^{\mathrm{r}} + (\sin\theta)\, \mathcal{G}^{\mathrm{i}} \tag{1.18}$$

defines a pseudo-Riemannian metric on $L\boldsymbol{H}^3$ of neutral signature for each $\theta \in \boldsymbol{R}/2\pi\boldsymbol{Z}$, which is invariant under the action given in (1.16), where $\mathcal{G}^{\mathrm{r}}$ and $\mathcal{G}^{\mathrm{i}}$ are the neutral metrics given by the real and imaginary part of $\mathcal{G}$, respectively,

$$\mathcal{G}^{\mathrm{r}} := \frac{1}{2} \left\{ \frac{4\, d\mu_1 d\bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} + \frac{4\, d\mu_2 d\bar{\mu}_1}{(1 + \mu_2 \bar{\mu}_1)^2} \right\}, \quad \mathcal{G}^{\mathrm{i}} := \frac{1}{2i} \left\{ \frac{4\, d\mu_1 d\bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} - \frac{4\, d\mu_2 d\bar{\mu}_1}{(1 + \mu_2 \bar{\mu}_1)^2} \right\}. \tag{1.19}$$

Conversely, Salvai [S] proved that any pseudo-Riemannian metric on $L\boldsymbol{H}^3$ invariant under the action as in (1.16) is a constant multiple of $\mathcal{G}_\theta$ for some $\theta \in \boldsymbol{R}/2\pi\boldsymbol{Z}$. Thus we call $\mathcal{G}_\theta$ ($\theta \in \boldsymbol{R}/2\pi\boldsymbol{Z}$) *invariant metrics*. Any invariant metric $\mathcal{G}_\theta$ is Kähler with respect to the natural complex structure

$$J\left(\frac{\partial}{\partial\mu_1}\right) = i\frac{\partial}{\partial\mu_1}, \qquad J\left(\frac{\partial}{\partial\mu_2}\right) = i\frac{\partial}{\partial\mu_2}. \tag{1.20}$$

On the other hand, a involutive $(1,1)$-tensor P on $L\boldsymbol{H}^3$ given as

$$P\left(\frac{\partial}{\partial\mu_1}\right) = -\frac{\partial}{\partial\mu_1}, \qquad P\left(\frac{\partial}{\partial\mu_2}\right) = \frac{\partial}{\partial\mu_2} \tag{1.21}$$

is a *para-Kähler structure* on $L\boldsymbol{H}^3$ for any $\mathcal{G}_\theta$. That is, for $[\gamma]$ in $L\boldsymbol{H}^3$, we have

$$\dim_{\boldsymbol{R}}\{X \in T_{[\gamma]}L\boldsymbol{H}^3 \,|\, P(X) = \pm X\} = 2, \quad \mathcal{G}_\theta(P\cdot, P\cdot) = -\mathcal{G}_\theta(\cdot,\cdot), \quad \nabla^L P = 0,$$

where ∇^L is the common Levi-Civita connection of $(L\boldsymbol{H}^3, \mathcal{G}_\theta)$ for all θ.

2. The Invariant Metrics and the Canonical Symplectic Form

In this section, we shall characterize two neutral metrics $\mathcal{G}^{\mathrm{r}}$ and $\mathcal{G}^{\mathrm{i}}$ given in (1.19): both the para-Kähler form of $(L\boldsymbol{H}^3, \mathcal{G}^{\mathrm{r}}, P)$ and the Kähler form of $(L\boldsymbol{H}^3, \mathcal{G}^{\mathrm{i}}, J)$ coincide with the twice of the canonical symplectic form on $L\boldsymbol{H}^3$ up to sign (Proposition 2.1). Moreover, identifying $L\boldsymbol{H}^3 = \mathrm{SL}(2,\boldsymbol{C})/\mathrm{GL}(1,\boldsymbol{C})$, we prove that $\mathcal{G}$ in (1.17) coincides with the $\boldsymbol{C}$-valued symmetric 2-tensor induced from the Killing form of the Lie algebra $\mathfrak{sl}(2,\boldsymbol{C})$ of $\mathrm{SL}(2,\boldsymbol{C})$ up to real constant multiplication (Proposition 2.3).

The canonical symplectic form.

Let ω be the *canonical symplectic form* on $L\boldsymbol{H}^3$, that is, ω is the symplectic form on $L\boldsymbol{H}^3$ satisfying

$$\hat{\pi}^*\omega = d\Theta, \tag{2.1}$$

where Θ is the canonical contact form given in (1.8) on the unit tangent bundle $U\boldsymbol{H}^3$, and $\hat{\pi} : U\boldsymbol{H}^3 \to L\boldsymbol{H}^3$ is the projection as in (1.10).

We denote by ω_J the Kähler form of $(L\boldsymbol{H}^3, \mathcal{G}^{\mathrm{i}}, J)$, and by ω_P the para-Kähler form of $(L\boldsymbol{H}^3, \mathcal{G}^{\mathrm{r}}, P)$, that is,

$$\omega_J = \mathcal{G}^{\mathrm{i}}(\cdot, J\cdot), \qquad \omega_P = \mathcal{G}^{\mathrm{r}}(\cdot, P\cdot). \tag{2.2}$$

Then we have the following

Proposition 2.1.

$$\omega_J = -\omega_P = 2\omega.$$

To prove this, we introduce metrics on $U\boldsymbol{H}^3$ and $L\boldsymbol{H}^3$ induced from the Killing form of $\mathfrak{sl}(2,\boldsymbol{C})$ considering $U\boldsymbol{H}^3$ and $L\boldsymbol{H}^3$ as homogeneous spaces of $\mathrm{SL}(2,\boldsymbol{C})$.

The Killing form of $\mathfrak{sl}(2,\boldsymbol{C})$.

Let B be the half of the Killing form of the Lie algebra $\mathfrak{sl}(2,\boldsymbol{C})$ of $\mathrm{SL}(2,\boldsymbol{C})$, i.e.,

$$B(X,Y) = 2\,\mathrm{trace}(XY), \qquad X, Y \in \mathfrak{sl}(2,\boldsymbol{C}). \tag{2.3}$$

Then we set B^{r} and B^{i} to be the real and imaginary part of B, respectively:

$$B^{\mathrm{r}} := \mathrm{Re}\,B, \qquad B^{\mathrm{i}} := \mathrm{Im}\,B. \tag{2.4}$$

Remark 2.2. The special linear group $\mathrm{SL}(2,\boldsymbol{C})$ is the double cover of the restricted Lorentz group $\mathrm{SO}^+(1,3)$. The Killing form of the real Lie algebra of $\mathfrak{so}(1,3)$ of $\mathrm{SO}^+(1,3)$ coincides with a constant multiple of B^{r}.

The unit tangent bundle.

The tangent space of the unit tangent bundle $U\boldsymbol{H}^3 = \mathrm{SL}(2,\boldsymbol{C})/\,\mathrm{U}(1)$ as in (1.9) at $(\sigma_0,\sigma_3) \in U\boldsymbol{H}^3$ is identified with the orthogonal complement of the Lie algebra $\mathfrak{u}(1)$ of $\mathrm{U}(1)$ with respect to B^{r}, that is,

$$T_{(\sigma_0,\sigma_3)}U\boldsymbol{H}^3 = \mathfrak{u}(1)^\perp = \left\{ i\varepsilon\sigma_3 + h_\xi + v_\eta \;\middle|\; \varepsilon \in \boldsymbol{R},\ \xi,\eta \in \boldsymbol{C} \right\},$$

where σ_0, σ_3 are as in (1.3), and h_ξ, v_η are defined by

$$h_\xi = \begin{pmatrix} 0 & \xi \\ \overline{\xi} & 0 \end{pmatrix}, \qquad v_\eta = \begin{pmatrix} 0 & -\eta \\ \overline{\eta} & 0 \end{pmatrix}. \tag{2.5}$$

These notations are used since h_ξ, v_η are horizontal and vertical tangent vectors of the sphere bundle $\pi : U\boldsymbol{H}^3 \to \boldsymbol{H}^3$ given in (1.6), respectively. The restriction of B^{r} in (2.4) to $T_{(\sigma_0,\sigma_3)}U\boldsymbol{H}^3$ can be written by

$$B^{\mathrm{r}}(X,X) = 4(\varepsilon^2 + |\xi|^2 - |\eta|^2), \tag{2.6}$$

for $X = i\varepsilon\sigma_3 + h_\xi + v_\eta \in T_{(\sigma_0,\sigma_3)}U\boldsymbol{H}^3$. Thus B^{r} defines a pseudo-Riemannian metric B_U on $U\boldsymbol{H}^3$ of signature $(+,+,+,-,-)$. Moreover, the projection

$$\pi : (U\boldsymbol{H}^3, B_U) \longrightarrow (\boldsymbol{H}^3, \langle\ ,\ \rangle) \tag{2.7}$$

defined as in (1.6) is a pseudo-Riemannian submersion.

The space of oriented geodesics.

Consider the smooth and transitive action of $\mathrm{SL}(2,\boldsymbol{C})$ given as

$$L\boldsymbol{H}^3 \ni [\gamma] \longmapsto [a\gamma a^*] \in L\boldsymbol{H}^3,$$

for $a \in \mathrm{SL}(2,\boldsymbol{C})$, where $[a\gamma a^*]$ is the equivalence class of the geodesic $a\gamma(t)a^*$ for some representative γ of $[\gamma]$. Note that this action coincides with the action given in (1.16). If we denote by $\gamma_{\sigma_0,\sigma_3}$ the geodesic in $\boldsymbol{H}^3$ starting at σ_0 with initial velocity σ_3, then the isotropy subgroup of $\mathrm{SL}(2,\boldsymbol{C})$ at $[\gamma_0] := [\gamma_{\sigma_0,\sigma_3}] \in L\boldsymbol{H}^3$ is given by

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \;\middle|\; \lambda \in \boldsymbol{C}\setminus\{0\} \right\},$$

which is identified with the general linear group $\mathrm{GL}(1,\boldsymbol{C})$. Hence we have

$$L\boldsymbol{H}^3 = \mathrm{SL}(2,\boldsymbol{C})/\,\mathrm{GL}(1,\boldsymbol{C}) = \left\{ [a\gamma_0 a^*] \;\middle|\; a \in \mathrm{SL}(2,\boldsymbol{C}) \right\}. \tag{2.8}$$

Then the tangent space of $L\boldsymbol{H}^3$ at $[\gamma_0]$ is identified with the orthogonal complement of the Lie algebra $\mathfrak{gl}(1,\boldsymbol{C})$ of $\mathrm{GL}(1,\boldsymbol{C})$ with respect to B^{r}, that is,

$$T_{[\gamma_0]}L\boldsymbol{H}^3 = \mathfrak{gl}(1,\boldsymbol{C})^\perp = \left\{ h_\xi + v_\eta \;\middle|\; \xi,\eta \in \boldsymbol{C} \right\},$$

where h_ξ and v_η are horizontal and vertical vectors of $T_{(\sigma_0,\sigma_3)}U\boldsymbol{H}^3$ defined in (2.5). The restrictions to $T_{[\gamma_0]}L\boldsymbol{H}^3$ of B^{r} and B^{i} defined in (2.4) can be written by

$$B^{\mathrm{r}}(X,X) = 4(|\xi|^2 - |\eta|^2), \qquad B^{\mathrm{i}}(X,X) = 8\,\mathrm{Im}(\xi\overline{\eta}),$$

for $X = h_\xi + v_\eta \in T_{[\gamma_0]}L\boldsymbol{H}^3$, respectively. Thus B^{r} and B^{i} define pseudo-Riemannian metrics B^{r}_L and B^{i}_L on $L\boldsymbol{H}^3$ of neutral signature, respectively. Of course, the projection

$$\hat{\pi} : (U\boldsymbol{H}^3, B_U) \longrightarrow (L\boldsymbol{H}^3, B^{\mathrm{r}}_L) \tag{2.9}$$

defined in (1.10) is a pseudo-Riemannian submersion.

Let $B_L := B^{\mathrm{r}}_L + iB^{\mathrm{i}}_L$ be the $\boldsymbol{C}$-valued 2-tensor on $L\boldsymbol{H}^3 = \mathrm{SL}(2,\boldsymbol{C})/\,\mathrm{GL}(1,\boldsymbol{C})$ induced from B in (2.3). Then we have the following

Proposition 2.3. *For the the $\boldsymbol{C}$-valued symmetric 2-tensor $\mathcal{G}$ on $L\boldsymbol{H}^3$ defined in* (1.17)*, it follows that*

$$\mathcal{G} = -B_L.$$

Proof. It is enough to check the equality at $[\gamma_0] = [\gamma_{\sigma_0,\sigma_3}] \in L\boldsymbol{H}^3$ only. For a sufficiently small neighborhood $\mathcal{R}$ of the origin $o \in \boldsymbol{R}^4$, consider a map $\psi : \mathcal{R} \to \mathrm{SL}(2,\boldsymbol{C})$ given by

$$\psi(u_1,u_2,v_1,v_2) = \begin{pmatrix} 1 & u_1 - iv_2 + iu_2 - v_1 \\ u_1 - iv_2 - iu_2 + v_1 & 1 + (u_1 - iv_2)^2 + (u_2 + iv_1)^2 \end{pmatrix}. \tag{2.10}$$

This map ψ may be considered as a parametrization of $L\boldsymbol{H}^3 = \mathrm{SL}(2,\boldsymbol{C})/\,\mathrm{GL}(1,\boldsymbol{C})$ around $\psi(o) = [\gamma_0]$. For ξ, $\eta \in \boldsymbol{C}$, set

$$\vec{\boldsymbol{x}}_{\xi,\eta} := (\mathrm{Re}\,\xi)\left.\frac{\partial}{\partial u_1}\right|_o + (\mathrm{Im}\,\xi)\left.\frac{\partial}{\partial u_2}\right|_o + (\mathrm{Re}\,\eta)\left.\frac{\partial}{\partial v_1}\right|_o + (\mathrm{Im}\,\eta)\left.\frac{\partial}{\partial v_2}\right|_o \in T_o\mathcal{R}, \tag{2.11}$$

and $X := \psi_*(\vec{\boldsymbol{x}}_{\xi,\eta}) \in T_{[\gamma_0]}L\boldsymbol{H}^3$. Then we have $X = h_\xi + v_\eta$, and

$$B^{\mathrm{r}}_L(X,X) = B^{\mathrm{r}}(X,X) = 4(|\xi|^2 - |\eta|^2), \qquad B^{\mathrm{i}}_L(X,X) = B^{\mathrm{i}}(X,X) = 8\,\mathrm{Im}(\xi\bar{\eta}) \tag{2.12}$$

at $[\gamma_0] \in L\boldsymbol{H}^3$, where h_ξ, v_η are given in (2.5).

On the other hand, set $\hat{\psi} := \pi_1 \circ \psi : \mathcal{R} \to L\boldsymbol{H}^3$, where $\pi_1 : \mathrm{SL}(2,\boldsymbol{C}) \ni a \mapsto [a\gamma_0 a^*] \in L\boldsymbol{H}^3$. The coordinates (μ_1,μ_2) (see (1.15)) of $\hat{\psi}(u_1,u_2,v_1,v_2)$ can be calculated as

$$\mu_1(u_1,u_2,v_1,v_2) = -\frac{(u_1+iu_2)-(v_1+iv_2)}{1+(u_1-iv_2)^2+(u_2+iv_1)^2}, \quad \mu_2(u_1,u_2,v_1,v_2) = (u_1+iu_2)+(v_1+iv_2).$$

Then $\hat{X} := \hat{\psi}_*(\vec{\boldsymbol{x}}_{\xi,\eta}) \in T_{[\gamma_0]}L\boldsymbol{H}^3$ is given by

$$\hat{X} = (-\xi+\eta)\frac{\partial}{\partial\mu_1} + (\xi+\eta)\frac{\partial}{\partial\mu_2} + (-\bar{\xi}+\bar{\eta})\frac{\partial}{\partial\bar{\mu}_1} + (\bar{\xi}+\bar{\eta})\frac{\partial}{\partial\bar{\mu}_2}.$$

By (2.12), we have

$$\mathcal{G}^{\mathrm{r}}(\hat{X},\hat{X}) = -4(|\xi|^2 - |\eta|^2) = -B^{\mathrm{r}}_L(X,X), \qquad \mathcal{G}^{\mathrm{i}}(\hat{X},\hat{X}) = -8\,\mathrm{Im}(\xi\bar{\eta}) = -B^{\mathrm{i}}_L(X,X)$$

at $[\gamma_0] \in L\boldsymbol{H}^3$, where $\mathcal{G}^{\mathrm{r}}$ and $\mathcal{G}^{\mathrm{i}}$ are as in (1.19). □

Proof of Proposition 2.1.

By a similar calculation as in the proof of Proposition 2.3, the complex structure J in (1.20) and the para-complex structure P in (1.21) satisfy

$$J(h_\xi + v_\eta) = h_{i\xi} + v_{i\eta}, \qquad P(h_\xi + v_\eta) = h_\eta + v_\xi,$$

for a tangent vector $h_\xi + v_\eta \in T_{[\gamma_0]}L\boldsymbol{H}^3$. Thus by Proposition 2.3, the Kähler form ω_J and the para-Kähler form ω_P defined in (2.2) can be calculated as

$$\omega_P(X,Y) = -\omega_J(X,Y) = -2\,\mathrm{Re}(\xi\bar{\delta} - \eta\bar{\beta}), \tag{2.13}$$

where $X = h_\xi + v_\eta$, $Y = h_\beta + v_\delta \in T_{[\gamma_0]}L\boldsymbol{H}^3$.

To calculate the canonical symplectic form ω in (2.1), set $\tilde{\psi} := \pi_2 \circ \psi : \mathcal{R} \to U\boldsymbol{H}^3$, where ψ is the map in (2.10) and $\pi_2 : \mathrm{SL}(2,\boldsymbol{C}) \ni a \mapsto (aa^*, a\sigma_3 a^*) \in U\boldsymbol{H}^3$. Then the horizontal lifts of $X = h_\xi + v_\eta$, $Y = h_\beta + v_\delta \in T_{[\gamma_0]}L\boldsymbol{H}^3$ are given by $\tilde{X} := \tilde{\psi}_*(\vec{\boldsymbol{x}}_{\xi,\eta}) = (h_\xi, h_\eta)$,

$\tilde{Y} := \tilde{\psi}_*(\overrightarrow{\boldsymbol{x}}_{\beta,\delta}) = (h_\beta, h_\delta) \in T_{(\sigma_0,\sigma_3)}U\boldsymbol{H}^3$, where h_ξ, h_β, $\cdots$ are as in (1.7) and $\overrightarrow{\boldsymbol{x}}_{\xi,\eta}$, $\overrightarrow{\boldsymbol{x}}_{\beta,\delta}$ are given in (2.11). By (2.13), we have

$$\begin{aligned} 2\omega_{[\gamma_0]}(\tilde{X}, \tilde{Y}) &= 2d\Theta_{(\sigma_0,\sigma_3)}(\tilde{X}, \tilde{Y}) = \left\langle h_\xi, h_\delta \right\rangle - \left\langle h_\beta, h_\eta \right\rangle \\ &= 2\,\mathrm{Re}(\xi\bar{\delta} - \eta\bar{\beta}) = -\omega_P(X, Y) = \omega_J(X, Y) \end{aligned}$$

at $[\gamma_0] \in L\boldsymbol{H}^3$, where Θ denotes the canonical contact form in (1.8). □

Remark 2.4. The metric $\mathcal{G}^{\mathrm{i}} = \mathrm{Im}\,\mathcal{G}$ in (1.19) is the twice of the Kähler metric defined in [GG, Definition 12]. In fact, we defined $\mathcal{G}$ as in (1.17) so that the double fibration

$$\begin{array}{ccc} & (U\boldsymbol{H}^3 = \mathrm{SL}(2,\boldsymbol{C})/\,\mathrm{U}(1),\ B_U) & \\ \swarrow^{\pi} & & \searrow^{\hat{\pi}} \\ (\boldsymbol{H}^3 = \mathrm{SL}(2,\boldsymbol{C})/\,\mathrm{SU}(2),\ \langle\ ,\ \rangle) & & (L\boldsymbol{H}^3 = \mathrm{SL}(2,\boldsymbol{C})/\,\mathrm{GL}(1,\boldsymbol{C}),\ B_L^{\mathrm{r}} = -\mathcal{G}^{\mathrm{r}}) \end{array}$$

is compatible, that is, both π in (2.7) and $\hat{\pi}$ in (2.9) are pseudo-Riemannian submersions.

Remark 2.5 (A relationship to the Fubini-Study metric). Consider a holomorphic curve $F : \boldsymbol{P}^1 = \hat{\boldsymbol{C}} \to L\boldsymbol{H}^3$ given by $F|_{\boldsymbol{C}} : \boldsymbol{C} \ni \mu \longmapsto (\mu, \mu) \in L\boldsymbol{H}^3$. The image of F in $L\boldsymbol{H}^3$ can be considered as

$$L_o\boldsymbol{H}^3 = \left\{[\gamma] \in L\boldsymbol{H}^3 \,\middle|\, \gamma \text{ through the origin } o = (0,0,0) \in \boldsymbol{B}^3\right\},$$

where $\boldsymbol{B}^3$ denotes the *Poincaré ball model* of $\boldsymbol{H}^3$:

$$\boldsymbol{B}^3 = \left(\left\{(x,y,z) \in \boldsymbol{R}^3 \,\middle|\, x^2 + y^2 + z^2 < 1\right\},\ 4\frac{dx^2 + dy^2 + dz^2}{(1 - x^2 - y^2 - z^2)^2}\right).$$

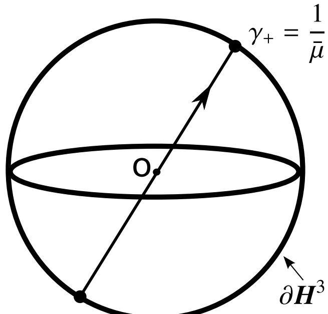

FIGURE 3. An oriented geodesic through the origin.

We call F or $L_o\boldsymbol{H}^3$ the *standard embedding* of $\boldsymbol{P}^1$. Moreover, if we equip on $\boldsymbol{P}^1$ the Fubini-Study metric g_{FS} of constant curvature 1, then the standard embedding

$$F : (\boldsymbol{P}^1, g_{FS}) \longrightarrow (L\boldsymbol{H}^3, \mathcal{G}^{\mathrm{r}})$$

is an isometric embedding. In fact, we defined $\mathcal{G}$ as the opposite sign of B_L (Proposition 2.3) because of this fact.

3. A Representation Formula for Developable Surfaces

In this section, we shall prove Theorem I in the introduction. First, we review fundamental facts on isometric immersions of $\boldsymbol{H}^2$ into $\boldsymbol{H}^3$ as surfaces in $\boldsymbol{H}^3$, and prove that isometric immersions of $\boldsymbol{H}^2$ into $\boldsymbol{H}^3$ are developable (Proposition 3.2). Then we shall prove Theorem I (Proposition 3.6).

3.1. **Isometric immersions and developable surfaces.**

In this paper, a *surface* in $\boldsymbol{H}^3$ is considered as an immersion f of a differentiable 2-manifold Σ into $\boldsymbol{H}^3$ (cf. (1.2)):

$$f : \Sigma \longrightarrow \boldsymbol{H}^3 \subset \boldsymbol{L}^4 = \mathrm{Herm}(2).$$

We denote by $g = f^* \langle\ ,\ \rangle$ *the first fundamental form* of f. For the unit normal vector field $\boldsymbol{\nu}$ of f, we denote by A and $I\!I$ the *shape operator* and the *second fundamental form* of f, respectively, that is, $A = -(f_*)^{-1} \circ \boldsymbol{\nu}_*$, $I\!I(V, W) = -\langle \boldsymbol{\nu}_*(V), f_*(W)\rangle$, where V and W are vector fields on Σ. Let k_1, k_2 be the *principal curvatures* of f, then the *extrinsic curvature* K_{ext} and the *mean curvature* H can be written as

$$K_{\mathrm{ext}} = k_1 k_2, \qquad H = \frac{k_1 + k_2}{2},$$

respectively. If we denote by K and ∇ the Gaussian curvature and the Levi-Civita connection of the Riemannian 2-manifold (Σ, g), respectively, then we have

$$K = -1 + K_{\mathrm{ext}}, \tag{3.1}$$

$$\nabla_V A(W) = \nabla_W A(V), \tag{3.2}$$

for vector fields V, W on Σ. We call (3.1) the *Gauss equation*, and (3.2) the *Codazzi equation*. A surface in $\boldsymbol{H}^3$ is said to be *extrinsically flat* if its extrinsic curvature is identically zero. By the Gauss equation, we have that an isometric immersion of $\boldsymbol{H}^2$ into $\boldsymbol{H}^3$ is a complete extrinsically flat surface.

On the other hand, any unit speed geodesic in $\boldsymbol{H}^3$ can be expressed as

$$\gamma_{p,v}(t) = p \cosh t + v \sinh t, \qquad (p, v) \in U\boldsymbol{H}^3.$$

Definition 3.1 (Ruled surfaces and developable surfaces). A *ruled surface* in $\boldsymbol{H}^3$ is a locus of 1-parameter family of geodesics in $\boldsymbol{H}^3$. For a ruled surface $f : \Sigma \to \boldsymbol{H}^3$, there exists a local coordinate system $\varphi = (s, t)$ of Σ such that

$$(f \circ \varphi^{-1})(s, t) = c(s) \cosh t + v(s) \sinh t,$$

where c is a curve in $\boldsymbol{H}^3$ and v is a unit normal vector field along c. A ruled surface is said to be *developable* if it is extrinsically flat.

Then we have the following

Proposition 3.2 ([P, Theorem 4])**.** *A complete extrinsically flat surface in $\boldsymbol{H}^3$ is developable.*

To show this, we first prove an analogue of *Massey's lemma* [Mas, Lemma 2] (cf. Remark 3.4). For a surface $f : \Sigma \to \boldsymbol{H}^3$, a curve in Σ is said to be *asymptotic* if each tangent space of the curve gives the kernel of the second fundamental form of f.

Lemma 3.3 (Hyperbolic Massey's lemma)**.** *For an extrinsically flat surface $f : \Sigma \to \boldsymbol{H}^3$, let $\mathcal{W}$ be the set of umbilic points of f and γ an asymptotic curve in the non umbilic point set $\mathcal{W}^c = \Sigma \setminus \mathcal{W}$. Then the mean curvature H of f satisfies*

$$\frac{\partial^2}{\partial t^2}\left(\frac{1}{H}\right) = \frac{1}{H},$$

on γ, where t denotes the arc length parameter of γ.

Proof. Take a non umbilic point $p \in \mathcal{W}^c$, and curvature line coordinate system (s, v) around p with v-curves asymptotic. Then the first and second fundamental forms g and $I\!I$ are expressed as $g = g_{11}ds^2 + g_{22}dv^2$, $I\!I = h_{11}ds^2$ $(h_{11} \neq 0)$, and hence the Codazzi equation (3.2) is equivalent to

$$\frac{\partial h_{11}}{\partial v} = \frac{h_{11}}{2g_{11}} \frac{\partial g_{11}}{\partial v}, \tag{3.3}$$

$$0 = \frac{h_{11}}{2g_{11}} \frac{\partial g_{22}}{\partial s}. \tag{3.4}$$

By (3.4), g_{22} depends only on v. Reparametrizing with $dt = \sqrt{g_{22}(v)}\,dv$, we obtain $g = g_{11}ds^2 + dt^2$, $I\!I = h_{11}ds^2$ $(h_{11} \neq 0)$. In this coordinate system, each t-curve is an asymptotic curve parametrized by arc length and the Gaussian curvature K of f is written as

$$K = -\frac{1}{\sqrt{g_{11}}} \frac{\partial^2 \sqrt{g_{11}}}{\partial t^2}.$$

Since f is extrinsically flat, the Gauss equation (3.1) yields

$$\frac{\partial^2 \sqrt{g_{11}}}{\partial t^2} = \sqrt{g_{11}}. \tag{3.5}$$

On the other hand, by (3.3), we have

$$\frac{\partial}{\partial t} \log \frac{h_{11}}{\sqrt{g_{11}}} = \frac{1}{h_{11}} \frac{\partial h_{11}}{\partial t} - \frac{1}{2g_{11}} \frac{\partial g_{11}}{\partial t} = 0,$$

and hence there exists a function $a = a(s)$ such that

$$h_{11}(s, t) = a(s)\sqrt{g_{11}(s, t)} \qquad (a(s) \neq 0).$$

Then the mean curvature H of f can be written as $H = a(s)/(2\sqrt{g_{11}})$. Besides (3.5), we have

$$\frac{\partial^2}{\partial t^2}\left(\frac{1}{H}\right) = \frac{\partial^2}{\partial t^2} \frac{2\sqrt{g_{11}}}{a(s)} = \frac{2}{a(s)} \frac{\partial^2}{\partial t^2} \sqrt{g_{11}} = \frac{2}{a(s)} \sqrt{g_{11}} = \frac{1}{H}.$$

□

Remark 3.4. Although original Massey's lemma [Mas, Lemma 2] is for flat surfaces in $\boldsymbol{R}^3$, we can generalize it for extrinsically flat surfaces in $\boldsymbol{S}^3$ in the same way. On the other hand, Murata and Umehara generalized Massey's lemma for a class of flat surfaces with singlarities (*flat fronts*) in $\boldsymbol{R}^3$ [MU, Lemma 1.15].

Proof of Proposition 3.2

Most part of this proof is a modification of the proof of Hartman-Nirenberg theorem given by Massey [Mas]. However, some part of the original Massey's proof is not valid for hyperbolic case, thus the final part of this proof is written carefully (see Claim below).

Let $f : \Sigma \to \boldsymbol{H}^3$ be a complete extrinsically flat surface and $\mathcal{W}$ the set of umbilic points of f. Since the restriction of f to $\mathcal{W}$ is a totally geodesic embedding, $f|_{\mathcal{W}}$ is ruled. By the proof of Lemma 3.3, for any non umbilic point in $\mathcal{W}^c = \Sigma \setminus \mathcal{W}$, there exists a local coordinate neighborhood $(U; (s, t))$ around the point such that

$$g = g_{11}ds^2 + dt^2, \qquad I\!I = h_{11}ds^2 \quad (h_{11} \neq 0).$$

Then it can be shown that the geodesic curvature of each t-curve vanishes anywhere. This means that any asymptotic curve in $\mathcal{W}^c$ is a part of geodesic in $\boldsymbol{H}^3$. For a fixed point

$q \in \mathcal{W}^c$, let $G(q)$ be the unique asymptotic curve in $\mathcal{W}^c$ passing through q. By Lemma 3.3, it follows that the mean curvature H is given by

$$H = \frac{1}{a\cosh t + b\sinh t} \tag{3.6}$$

on $G(q)$, where a, b are constants and t denotes the distance induced from the first fundamental form of f measured from q. If $G(q)$ intersects with the boundary $\partial\mathcal{W}$, the mean curvature H vanishes at $Q \in \partial\mathcal{W} \cap G(q)$, a contradiction. Thus any asymptotic curve in $\mathcal{W}^c$ does not intersect with the boundary of $\mathcal{W}^c$, and hence we have $f|_{\mathcal{W}^c}$ is ruled. It is sufficient to show the following

Claim . *$\partial\mathcal{W}$ is a disjoint union of geodesics in $\boldsymbol{H}^3$.*

Proof. For a point $p \in \partial\mathcal{W}$, there exists a sequence $\{p_n\}_{n\in\boldsymbol{N}}$ in $\mathcal{W}^c$ such that $\lim_{n\to\infty} p_n = p$. Let $G(p_n)$ be the unique asymptotic curve through $p_n \in \mathcal{W}^c$. Since $G(p_n)$ is a geodesic in H^3, we can express as $G(p_n)(t) = p_n \cosh t + v_n \sinh t$, with a unit tangent vector $v_n \in T_{p_n}\boldsymbol{H}^3$. We shall prove that there exists v of the limit of $\{v_n\}_{n\in\boldsymbol{N}}$, taking a subsequence, if necessary. Set $p_n = (p_{0_n}, \boldsymbol{p}_n)$, $v_n = (v_{0_n}, \boldsymbol{v}_n) \in \boldsymbol{L}^4 = \boldsymbol{R} \times \boldsymbol{R}^3$. Then we have

$$-p_{0_n}^2 + |\boldsymbol{p}_n|_E^2 = -1, \qquad -v_{0_n}^2 + |\boldsymbol{v}_n|_E^2 = 1, \qquad -p_{0_n}v_{0_n} + \langle \boldsymbol{p}_n, \boldsymbol{v}_n\rangle_E = 0,$$

for all $n \in \boldsymbol{N}$, where $\langle\cdot,\cdot\rangle_E$ is the Euclidean inner product of $\boldsymbol{R}^3$ and $|\cdot|_E$ is the associated Euclidean norm. By the Cauchy-Schwartz inequality,

$$|v_{0_n}| = \frac{1}{p_{0_n}}|\langle \boldsymbol{p}_n, \boldsymbol{v}_n\rangle_E| \leq \frac{1}{p_{0_n}}|\boldsymbol{p}_n|_E|\boldsymbol{v}_n|_E = \sqrt{\frac{p_{0_n}^2 - 1}{p_{0_n}^2}}\sqrt{v_{0_n}^2 + 1},$$

and we have

$$\frac{|v_{0_n}|}{\sqrt{v_{0_n}^2 + 1}} \leq \sqrt{1 - \frac{1}{p_{0_n}^2}} \leq 1, \tag{3.7}$$

for $n \in \boldsymbol{N}$. If $|v_{0_n}| \to \infty$,

$$\frac{|v_{0_n}|}{\sqrt{v_{0_n}^2 + 1}} \longrightarrow 1$$

holds and we have $p_{0_n} \to \infty$ by (3.7). But it contradicts with $\lim_{n\to\infty} p_n = p$. Thus there exists $R > 0$ such that $\{v_n\}_{n\in\boldsymbol{N}} \subset B(R)$, where $B(R) = \{{}^t(x_0, x_1, x_2, x_3) \in \boldsymbol{L}^4 \mid x_0^2 + x_1^2 + x_2^2 + x_3^2 \leq R\}$. If we set $\boldsymbol{S}_1^3 := \{\boldsymbol{x} \in \boldsymbol{L}^4 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 1\}$, we also have $\{v_n\}_{n\in\boldsymbol{N}} \subset \boldsymbol{S}_1^3 \cap B(R)$. Since $\boldsymbol{S}_1^3 \cap B(R)$ is compact, there exists a subsequence $\{v_{n_k}\} \subset \{v_n\}$ such that $\lim_{k\to\infty} v_{n_k} = v$ exists. Therefore we can define $G(p) = \lim_{n\to\infty} G(p_n) \subset \mathcal{W}^c \cup \partial\mathcal{W}$ as $\gamma_{p,v}$. If $G(p) \cap \mathcal{W}^c$ is non empty, take $q \in G(p) \cap \mathcal{W}^c$. Then $G(q) = G(p)$ and hence $G(q)$ through $p \in \partial\mathcal{W}$, a contradiction. Thus $G(p) \subset \partial\mathcal{W}$. □

As a corollary, we have the following

Corollary 3.5. *An isometric immersion of $\boldsymbol{H}^2$ into $\boldsymbol{H}^3$ is a complete developable surface in $\boldsymbol{H}^3$.*

3.2. **Proof of Theorem I.**

Since a ruled surface in $\boldsymbol{H}^3$ is a locus of 1-parameter family of geodesics, it gives a curve in the space of oriented geodesics $L\boldsymbol{H}^3$. Conversely, a curve in $L\boldsymbol{H}^3$ generates a ruled surface (it may have singularities) in $\boldsymbol{H}^3$. Here, we shall investigate the curves given by developable surfaces in $\boldsymbol{H}^3$. Let (μ_1, μ_2) be a point in $L\boldsymbol{H}^3$ as in (1.15). Then it corresponds to a equivalence class $[\gamma]$, where $\gamma(t)$ is expressed as

$$\gamma(t) = \frac{1}{|1+\mu_1\bar{\mu}_2|}\begin{pmatrix} e^t + e^{-t}|\mu_1|^2 & e^t\mu_2 - e^{-t}\mu_1 \\ e^t\bar{\mu}_2 - e^{-t}\bar{\mu}_1 & e^t|\mu_2|^2 + e^{-t}\end{pmatrix} \in \mathrm{Herm}(2). \tag{3.8}$$

A regular curve in a pseudo-Riemannian manifold is called *null* (resp. *causal*) if every tangent vector gives null (resp. timelike or null) direction. Recall that the neutral metrics $\mathcal{G}^{\mathrm{r}}$ and $\mathcal{G}^{\mathrm{i}}$ are defined in (1.19). Theorem I is a direct conclusion of the following

Proposition 3.6. *For a regular curve $\alpha(s) = (\mu_1(s), \mu_2(s)) : \boldsymbol{R} \supset I \to \mathcal{U} \subset L\boldsymbol{H}^3$ which is null with respect to $\mathcal{G}^{\mathrm{i}}$ and causal with respect to $\mathcal{G}^{\mathrm{r}}$, a map $f : I \times \boldsymbol{R} \to \boldsymbol{H}^3$ defined by*

$$f(s,t) = \frac{1}{|1+\mu_1(s)\bar{\mu}_2(s)|}\begin{pmatrix} e^t + e^{-t}|\mu_1(s)|^2 & e^t\mu_2(s) - e^{-t}\mu_1(s) \\ e^t\bar{\mu}_2(s) - e^{-t}\bar{\mu}_1(s) & e^t|\mu_2(s)|^2 + e^{-t}\end{pmatrix} \tag{3.9}$$

is a developable surface. Conversely, any developable surface generated by complete geodesics in $\boldsymbol{H}^3$ can be written locally in this manner.

Proof. By (3.8), a parametrization of the locus of α can be written by f as in (3.9). First we shall prove that if α is null with respect to $\mathcal{G}^{\mathrm{i}}$ and causal with respect to $\mathcal{G}^{\mathrm{r}}$, then f is an immersion. Set

$$\Lambda(s,t) := |f_s \times f_t|^2 = \frac{e^{2t}|\mu_2'|^2 + e^{-2t}|\mu_1'|^2}{|1+\mu_1\bar{\mu}_2|^2} - \frac{1}{2}\mathcal{G}^{\mathrm{r}}(\alpha', \alpha'), \tag{3.10}$$

where $' = d/ds$, $f_s = \partial f/\partial s$, $f_t = \partial f/\partial t$ and $\times$ denotes the cross product of $\boldsymbol{H}^3$ as in (1.4). Thus we have $\Lambda(s,t)$ is positive if $\mathcal{G}^{\mathrm{r}}(\alpha', \alpha')$ is negative. Consider the case $\mathcal{G}^{\mathrm{r}}(\alpha', \alpha') = 0$ at $s \in I$. Since α is null with respect to $\mathcal{G}^{\mathrm{i}}$, we have $|\mu_1'||\mu_2'| = 0$. The regularity of α shows that either $\mu_1' = 0$ or $\mu_2' = 0$ occurs. Without loss of generality, we may assume $\mu_1' = 0$. Then the regularity of α means $\mu_2' \neq 0$, and then $\Lambda(s,t) = e^{2t}|\mu_2'|^2/|1+\mu_1\bar{\mu}_2|^2$ is positive. Thus f is an immersion.

Next we shall show that f is extrinsically flat. The unit normal vector field $\boldsymbol{\nu}$ of f is given by

$$\boldsymbol{\nu}(s,t) = \frac{f_s \times f_t}{|f_s \times f_t|} = \frac{i}{|1+\mu_1\bar{\mu}_2|^3\sqrt{\Lambda(s,t)}}\begin{pmatrix} a(s,t) & z(s,t) \\ -\bar{z}(s,t) & b(s,t)\end{pmatrix}, \tag{3.11}$$

where

$$\begin{aligned} a(s,t) &= 2i\,\mathrm{Im}\{e^t(1+\mu_1\bar{\mu}_2)\bar{\mu}_1\mu_2' - e^{-t}(1+\mu_2\bar{\mu}_1)\bar{\mu}_1\mu_1'\}, \\ b(s,t) &= -2i\,\mathrm{Im}\{e^t(1+\mu_1\bar{\mu}_2)\bar{\mu}_2\mu_2' - e^{-t}(1+\mu_2\bar{\mu}_1)\bar{\mu}_2\mu_1'\}, \\ z(s,t) &= -e^t\{(1+\mu_1\bar{\mu}_2)\mu_2' + (1+\mu_2\bar{\mu}_1)\mu_1\mu_2\bar{\mu}_2'\} + e^{-t}\{(1+\mu_2\bar{\mu}_1)\mu_1' + (1+\mu_1\bar{\mu}_2)\mu_1\mu_2\bar{\mu}_1'\}. \end{aligned}$$

Since

$$K_{\mathrm{ext}} = \frac{\langle f_s, \boldsymbol{\nu}_s\rangle\langle f_t, \boldsymbol{\nu}_t\rangle - \langle f_s, \boldsymbol{\nu}_t\rangle\langle f_t, \boldsymbol{\nu}_s\rangle}{\langle f_s, f_s\rangle\langle f_t, f_t\rangle - \langle f_s, f_t\rangle^2} \qquad \text{and} \qquad \mathcal{G}^{\mathrm{i}}(\alpha', \alpha') = \mathrm{Im}\,\frac{4\mu_1'\bar{\mu}_2'}{(1+\mu_1\bar{\mu}_2)^2},$$

we have

$$K_{\mathrm{ext}} = \frac{i}{\sqrt{\Lambda(s,t)}^3}\left\{\frac{\mu_1'\bar{\mu}_2'}{(1+\mu_1\bar{\mu}_2)^2} - \frac{\mu_2'\bar{\mu}_1'}{(1+\mu_2\bar{\mu}_1)^2}\right\} = \frac{-1}{2\sqrt{\Lambda(s,t)}^3}\mathcal{G}^{\mathrm{i}}(\alpha', \alpha'). \tag{3.12}$$

Therefore $\mathcal{G}^{\mathrm{i}}(\alpha',\alpha')=0$ if and only if $K_{\mathrm{ext}}=0$.

Conversely, for a ruled surface $\hat{f}:\Sigma\to\boldsymbol{H}^3$, there exists a 1-parameter family $\alpha=\alpha(s)$ of geodesics such that its locus coincides with the given surface $\hat{f}$. Using a suitable isometry, we may assume that the image of α is included in $\mathcal{U}$ in (1.14), that is,

$$\alpha:\boldsymbol{R}\supset I\ni s\longmapsto(\mu_1(s),\mu_2(s))\in\mathcal{U}\subset L\boldsymbol{H}^3.$$

Thus $\hat{f}$ is given by f as in (3.9) locally. We shall prove that, if the ruled surface $\hat{f}$ is developable, α is a regular curve which is null with respect to $\mathcal{G}^{\mathrm{i}}$ and causal with respect to $\mathcal{G}^{\mathrm{r}}$. If there exists a point such that $\alpha'=0$, $\hat{f}$ is not an immersion because of (3.10). Thus α is a regular curve. Moreover α is a null with respect to $\mathcal{G}^{\mathrm{i}}$ by (3.12). Then we shall prove α is causal with respect to $\mathcal{G}^{\mathrm{r}}$. If $\mathcal{G}^{\mathrm{r}}(\alpha',\alpha')>0$,

$$\mathcal{G}^{\mathrm{r}}(\alpha',\alpha')=\operatorname{Re}\frac{4\mu_1'\bar{\mu}_2'}{(1+\mu_1\bar{\mu}_2)^2}=\frac{4|\mu_1'||\mu_2'|}{|1+\mu_1\bar{\mu}_2|^2},$$

holds since $\mathcal{G}^{\mathrm{i}}(\alpha',\alpha')=0$. Then we have

$$\Lambda(s,t)=\frac{4|\mu_1'||\mu_2'|}{|1+\mu_1\bar{\mu}_2|^2}\sinh^2\left(t+\frac{1}{2}\log\frac{|\mu_2'|}{|\mu_1'|}\right),$$

and hence $\hat{f}$ has a singular point at $t=(\log|\mu_1'|-\log|\mu_2'|)/2$, a contradiction. □

3.3. **Examples.**

Nomizu [N] constructed fundamental examples of complete developable surfaces in $\boldsymbol{H}^3$ (cf. Figure 1 in the introduction).

Example 3.7 (Hyperbolic 2-cylinders, [N, Example 1]). Let $\boldsymbol{D}$ be the unit disc in $\boldsymbol{C}$. For a regular curve $\zeta(s):\boldsymbol{R}\to\boldsymbol{D}$, set

$$\alpha_1(s)=(-\zeta(s),\zeta(s)).$$

Then α_1 determines a regular curve in $L\boldsymbol{H}^3=(\hat{\boldsymbol{C}}\times\hat{\boldsymbol{C}})\setminus\hat{\Delta}$, which is null with respect to $\mathcal{G}^{\mathrm{i}}$ and causal with respect to $\mathcal{G}^{\mathrm{r}}$. Thus by Theorem I, the locus of α_1 is a developable surface, called *hyperbolic* 2-*cylinder*. Figure 1 (B) shows an example of $\zeta(s)=e^{is}/3$.

Example 3.8 (Ideal cones, [N, Example 2]). For a regular curve $\mu(s):\boldsymbol{R}\to\boldsymbol{C}$, set

$$\alpha_2(s)=(\mu(s),0).$$

Then α_2 determines a regular curve in $L\boldsymbol{H}^3=(\hat{\boldsymbol{C}}\times\hat{\boldsymbol{C}})\setminus\hat{\Delta}$, which is null with respect to both $\mathcal{G}^{\mathrm{i}}$ and $\mathcal{G}^{\mathrm{r}}$. Thus by Theorem I, the locus of α_2 is a developable surface. Figure 1 (C) shows an example of $\mu(s)=e^{is}/2$. We will see this example more precisely in Section 4.

Example 3.9 (Rectifying developables of helices, [N, Example 3]). For constants κ, $\tau\in\boldsymbol{R}\setminus\{0\}$, set $a_\pm:=\sqrt{(\kappa\pm1)^2+\tau^2}$, $A_\pm:=\sqrt{\pm(1-\kappa^2-\tau^2)+a_+a_-}$ and $\alpha_3:\boldsymbol{R}\to\boldsymbol{C}^2$ as

$$\alpha_3(s)=\Bigg(\kappa\frac{4\sqrt{2}\sqrt{\kappa^2+\tau^2}i+4\tau A_-}{(\sqrt{2}\sqrt{\kappa^2+\tau^2}i+4\tau A_+)(a_++a_-)^2+4\kappa A_-}\exp\left(\frac{A_++iA_-}{\sqrt{2}}s\right),$$
$$\frac{1}{\kappa}\frac{(\sqrt{2}\sqrt{\kappa^2+\tau^2}-\tau A_+)(a_++a_-)^2-4\kappa A_-}{4\sqrt{2}\sqrt{\kappa^2+\tau^2}i+4\tau A_--(a_++a_-)^2A_+}\exp\left(\frac{-A_++iA_-}{\sqrt{2}}s\right)\Bigg).$$

Then α_3 determines a regular curve in $L\boldsymbol{H}^3=(\hat{\boldsymbol{C}}\times\hat{\boldsymbol{C}})\setminus\hat{\Delta}$, which is null with respect to $\mathcal{G}^{\mathrm{i}}$ and causal with respect to $\mathcal{G}^{\mathrm{r}}$. Thus by Theorem I, the locus of α_3 is a developable surface. In fact, this is a rectifying developable [N] of the helix of constant curvature κ and torsion τ in $\boldsymbol{H}^3$. Figure 1 (D) shows an example of $\kappa=\tau=1$.

4. Ideal Cones and Behavior of the Mean Curvature

In this section, we shall prove Theorem II in the introduction. First, we define "ideal cones", determine the corresponding curves in $L\boldsymbol{H}^3$ and investigate behavior of their mean curvature. Next, we introduce the notion of developable surfaces *of exponential type* in $\boldsymbol{H}^3$. Finally, we prove Theorem II.

4.1. **Null curves and ideal cones.**

Definition 4.1 (Ideal cones). We call a complete developable surface in $\boldsymbol{H}^3$ an *ideal cone*, if it is a locus of 1-parameter family of geodesics sharing one side end as a same point in the ideal boundary. The shared point is called *vertex*.

Proposition 4.2. *An ideal cone gives a curve in $L\boldsymbol{H}^3$ which is null with respect to both $\mathcal{G}^{\mathrm{i}}$ and $\mathcal{G}^{\mathrm{r}}$. Conversely, if the locus of a curve in $L\boldsymbol{H}^3$ which is null with respect to both $\mathcal{G}^{\mathrm{i}}$ and $\mathcal{G}^{\mathrm{r}}$ is complete, then the locus is an ideal cone.*

Proof. Without loss of generality, we may assume the vertex of the ideal cone is $\infty \in \partial\boldsymbol{H}^3$. Then the curve $\alpha(s) = (\mu_1(s), \mu_2(s)) \in (\hat{\boldsymbol{C}} \times \hat{\boldsymbol{C}}) \setminus \hat{\Delta} = L\boldsymbol{H}^3$ given by the ideal cone satisfies $\mu_2(s) = 0$. Hence $\mathcal{G}^{\mathrm{r}}(\alpha', \alpha') = \mathcal{G}^{\mathrm{i}}(\alpha', \alpha') = 0$ holds. Conversely, a curve $\alpha(s) = (\mu_1(s), \mu_2(s))$ in $L\boldsymbol{H}^3$ is null with respect to $\mathcal{G}^{\mathrm{i}}$ if and only if $\mathcal{G}(\alpha', \alpha')$ is always real. Moreover if α is null with respect to $\mathcal{G}^{\mathrm{r}}$, we have

$$\mathcal{G}(\alpha', \alpha') = \frac{\mu_1'(s)\bar{\mu}_2'(s)}{(1 + \mu_1(s)\bar{\mu}_2(s))^2} = 0, \tag{4.1}$$

for all s. By the regularity of α, (4.1) holds if and only if either $\mu_1'(s)$ vanishes identically or so does $\mu_2'(s)$. This means the locus of α is a ruled surface which is asymptotic to a point in the ideal boundary. □

Remark 4.3. By Proposition 4.2, it follows that a complete *ruled* surface which is a locus of 1-parameter family of geodesics sharing one side end as a same point in the ideal boundary is necessarily developable, that is, an ideal cone. If the vertex is $\infty \in \partial\boldsymbol{H}^3$, the shape of ideal cone is a cylinder over a plane curve in the upper half space $\boldsymbol{R}^3_+$ (cf. Figure 4).

(a) in the Poincaré ball model (b) in the upper half space model

Figure 4. An ideal cone whose vertex at ∞.

Now we shall investigate behavior of the mean curvature of ideal cones.

Proposition 4.4. *For an ideal cone f, let γ be an asymptotic curve of the non umbilic point set of f such that γ_+ is the vertex of f, and let t be the arc length parameter of γ. Then the mean curvature H of f is proportional to e^t on γ.*

Proof. Without loss of generality, we may assume the vertex of f is $\infty \in \partial \boldsymbol{H}^3$. Then the curve α in $L\boldsymbol{H}^3$ corresponding to f is given by $\alpha(s) = (\mu(s), 0)$ on $\mathcal{U} \subset L\boldsymbol{H}^3$. By the representation formula (3.9), f can be written as

$$f(s,t) = \begin{pmatrix} e^t + e^{-t}|\mu(s)|^2 & -e^{-t}\mu(s) \\ -e^{-t}\overline{\mu}(s) & e^{-t} \end{pmatrix}. \tag{4.2}$$

Then the induced metric $g = f^* \langle \ , \ \rangle$ is

$$g = e^{-2t}|\mu'|^2 ds^2 + dt^2. \tag{4.3}$$

Now we shall see that $\mu(s)$ can be considered as an Euclidean plane curve as follows. By the isometry $\Psi : \boldsymbol{H}^3 \to \boldsymbol{R}^3_+$ as in (1.13), f is transferred to $(\Psi \circ f)(s,t) = (\mu(s), e^t) \in \boldsymbol{R}^3_+$, that is, the cylinder over the plane curve $\mu(s) \in \boldsymbol{C}$. Set $\Omega := \{(w, 1) \,|\, w \in \boldsymbol{C}\} \subset \boldsymbol{R}^3_+$, a complete flat surface in $\boldsymbol{R}^3_+$ so-called the *horosphere* through $(0, 1)$ and ∞. Thus Ω can be considered as the Euclidean plane. Then the intersection of f and Ω is parametrized by $(\Psi \circ f)(s, 0) = (\mu(s), 1)$. Thus we can consider μ as a curve in the Euclidean plane Ω.

If we take the arc length parameter s of the curve μ in Ω, the induced metric g in (4.3) is written as $g = e^{-2t}ds^2 + dt^2$. Since the unit normal vector field $\boldsymbol{\nu}$ of f can be expressed by

$$\boldsymbol{\nu}(s,t) = \begin{pmatrix} 2\,\mathrm{Im}(\overline{\mu}\mu') & i\mu' \\ -i\overline{\mu}' & 0 \end{pmatrix},$$

the second fundamental form $I\!I$ of f is written as $I\!I = e^{-t}\,\mathrm{Im}(\mu'\overline{\mu}'')ds^2 = -e^{-t}\kappa_E(s)ds^2$, where κ_E is the curvature of μ in the Euclidean plane Ω. Therefore the mean curvature H of f is given by $H(s,t) = -e^t\kappa_E(s)/2$. □

4.2. **Developable surfaces of exponential type.**

Here we shall investigate behavior of the mean curvature of *complete* developable surfaces. For a complete developable surface $f : \Sigma \to \boldsymbol{H}^3$, let $p \in \Sigma$ be a non umbilic point. Then there exists a unique asymptotic curve γ through p which is a geodesic in $\boldsymbol{H}^3$. By hyperbolic Massey's lemma (Lemma 3.3), it holds that

$$\frac{1}{H} = P\cosh t + Q \sinh t$$

on γ (see (3.6)), where P and Q are constants and t is the arc length parameter of γ. Without loss of generality, we may assume P is positive. Then

$$\frac{1}{H} = \begin{cases} \sqrt{P^2 - Q^2}\cosh\left(t + \dfrac{1}{2}\log\dfrac{P+Q}{P-Q}\right) & (\text{if } P > |Q|), \\ Pe^{\pm t} & (\text{if } P = |Q|), \\ \sqrt{Q^2 - P^2}\sinh\left(t + \dfrac{1}{2}\log\dfrac{Q+P}{Q-P}\right) & (\text{if } P < |Q|). \end{cases}$$

Completeness of f implies that t varies from $-\infty$ to ∞. But in the third case, the mean curvature diverges at some $t \in \boldsymbol{R}$, a contradiction. Hence only the first and the second cases can happen, that is, the mean curvature H of a complete developable surface is proportional to exponential function or hyperbolic secant function on each asymptotic curves with respect to the arc length parameter.

Definition 4.5 (Developable surfaces of exponential type). A complete developable surface is said to be *of exponential type* if it is not totally umbilic and the mean curvature is proportional to $e^{\pm t}$ on each asymptotic curves in the set of non umbilic points, where t is the arc length parameter of the asymptotic curve.

Proposition 4.4 says that non totally umbilic ideal cones are developable surfaces of exponential type.

4.3. **Proof of Theorem II.**

Definition 4.6 (Asymptotics of geodesics). Two unit speed geodesics γ_1, γ_2 in $\boldsymbol{H}^3$ are said to be *asymptotic* if $\{d(\gamma_1(t),\gamma_2(t)) \mid t>0\}$ is bounded from above, where d denotes the hyperbolic distance.

For (p,v), $(q,w) \in U\boldsymbol{H}^3$, it is known that the geodesics

$$\gamma_{p,v}(t) = p\cosh t + v\sinh t, \qquad \gamma_{q,w}(t) = q\cosh t + w\sinh t$$

are asymptotic if and only if $\langle p+v, q+w\rangle = 0$ holds.

Theorem II in the introduction is proved directly by the following

Proposition 4.7. *A developable surface of exponential type whose umbilic point set has no interior is an ideal cone. That is, asymptotic curves of such a surface are asymptotic to each other.*

Let $f:\Sigma\to\boldsymbol{H}^3$ be a developable surface of exponential type whose umbilic point set has no interior. We may assume Σ is simply connected, taking the universal cover $\boldsymbol{H}^2$, if necessary. Here, we consider $\boldsymbol{H}^2$ as the hyperboloid in the Lorentz-Minkowski 3-space $\boldsymbol{L}^3$. The proof is divided into three steps (Claims 1–3).

Claim 1. *There exists a global coordinate system $\varphi=(s,t):\Sigma=\boldsymbol{H}^2\to\boldsymbol{R}^2$ such that*

$$(f\circ\varphi^{-1})(s,t) = c(s)\cosh t + v(s)\sinh t \tag{4.4}$$

holds, the induced metric g and the second fundamental form $I\!I$ of f are given by

$$g = g_{11}(s,t)ds^2 + dt^2, \qquad I\!I = e^t\delta(s)g_{11}(s,t)ds^2,$$

respectively, where δ is a smooth function of s.

Proof. Since the umbilic point set of f has no interior, the proof of Proposition 3.2 implies that each connected component of umbilic point set is a geodesic in $\boldsymbol{H}^3$. Thus by the proof of Lemma 3.3, we can find a coordinate neighborhood $(U;(s,t))\subset\boldsymbol{H}^2$ such that U is open dense in $\boldsymbol{H}^2$ and $g=g_{11}(s,t)ds^2+dt^2$ hold on U. By taking $t\mapsto t+\text{constant}$, if necessary, each coordinate system (s,t) can be joined smoothly over the umbilic point set. □

Claim 2. *The vector field $v(s)$ in* (4.4) *is expressed as*

$$v(s) = \frac{\boldsymbol{n}(s)+\delta(s)\boldsymbol{b}(s)}{\sqrt{1+\{\delta(s)\}^2}}, \tag{4.5}$$

where $\boldsymbol{n}$ and $\boldsymbol{b}$ denotes the principal and binormal normal vector field of the curve c in $\boldsymbol{H}^3$, respectively. Furthermore, the curvature κ and the torsion τ of c satisfy

$$\kappa(s) = \sqrt{1+\{\delta(s)\}^2}, \qquad \tau(s) = \frac{\delta'(s)}{1+\{\delta(s)\}^2}. \tag{4.6}$$

Proof. We may assume the curve c in $\boldsymbol{H}^3$ is parametrized by the arc length s. Let β be the curve in $\boldsymbol{H}^2$ which is the inverse image of the curve c by f. By changing the orientation of β, if necessary, we may assume the unit normal vector N of β in $\boldsymbol{H}^2$ satisfies

$$f_*(N) = v. \tag{4.7}$$

Then the map $Y : \boldsymbol{R}^2 \to \boldsymbol{H}^2 \subset \boldsymbol{L}^3$ defined by

$$Y(s,t) = \beta(s)\cosh t + N(s)\sinh t$$

gives a parametrization of $\boldsymbol{H}^2$. Let $\boldsymbol{\nu}$ be the unit normal vector field of f. Then the shape operator A of f satisfies $A(Y_s) = \delta(s)e^t Y_s$, $A(Y_t) = \boldsymbol{0}$. Let κ_β be the geodesic curvature of β and ∇ the Levi-Civita connection of $\boldsymbol{H}^2$. By the Frenet formula for the curve β in $\boldsymbol{H}^2$,

$$\nabla_s N = N'(s) = -\kappa_\beta(s)\beta'(s) \tag{4.8}$$

holds, where we consider N is the $\boldsymbol{L}^3$-valued function and $N' = dN/ds$, etc. Thus we have $Y_s := \partial Y/\partial s = (\cosh t - \kappa_\beta(s)\sinh t)\beta'(s)$, and hence

$$\nabla_t Y_s = \frac{\sinh t - \kappa_\beta(s)\cosh t}{\cosh t - \kappa_\beta(s)\sinh t} Y_s$$

holds. Since the shape operator A of f satisfies the Codazzi equation (3.2), it follows that

$$\boldsymbol{0} = (\nabla_t A)(Y_s) - (\nabla_s A)(Y_t) = \nabla_t(\delta(s)e^t Y_s) = \left(1 + \frac{\sinh t - \kappa_\beta(s)\cosh t}{\cosh t - \kappa_\beta(s)\sinh t}\right)\delta(s)e^t Y_s,$$

where $Y_t = \partial Y/\partial t$. Substituting $t = 0$ into this, we have that

$$\kappa_\beta(s) = 1 \tag{4.9}$$

for s in $\boldsymbol{R}$, that is, β is congruent to the horocycle.

Next, we shall calculate the principal normal vector field $\boldsymbol{n}$, the binormal vector field $\boldsymbol{b}$, curvature κ and torsion τ of the curve c in $\boldsymbol{H}^3$. Let D be the Levi-Civita connection of $\boldsymbol{H}^3$. By (4.8) and (4.9), $\nabla_s\beta'(s) = N(s)$ holds. Moreover, by (4.7), it holds that

$$\begin{aligned} D_s c'(s) &= f_*(\nabla_s\beta'(s)) + I\!I(\beta'(s), \beta'(s))\boldsymbol{\nu}(s,0) \\ &= f_*(N(s)) + \delta(s)\boldsymbol{\nu}(s,0) = \upsilon(s) + \delta(s)\boldsymbol{\nu}(s,0), \end{aligned}$$

and hence we have

$$\kappa(s) = \left|D_s c'(s)\right| = \sqrt{1 + \{\delta(s)\}^2}, \qquad \boldsymbol{n}(s) = \frac{D_s c'(s)}{\kappa(s)} = \frac{\upsilon(s) + \delta(s)\boldsymbol{\nu}(s,0)}{\sqrt{1 + \{\delta(s)\}^2}}.$$

If we denote by $\boldsymbol{e}(s) = c'(s)$ the unit tangent vector field of c, $\boldsymbol{b}(s)$ is obtained as

$$\boldsymbol{b}(s) = \boldsymbol{e}(s) \times \boldsymbol{n}(s) = \frac{\boldsymbol{\nu}(s,0) - \delta(s)\upsilon(s)}{\sqrt{1 + \{\delta(s)\}^2}},$$

where $\times$ is the cross product in $\boldsymbol{H}^3$ (cf. (1.4)). Since

$$\begin{cases} D_s\boldsymbol{\nu}(s,0) = -f_*(A(Y_s)(s,0)) = -f_*(\delta(s)Y_s(s,0)) = -\delta(s)\boldsymbol{e}(s) \\ D_s\upsilon(s) = -f_*(\nabla_s N) - \langle A(N), \beta'\rangle\,\boldsymbol{\nu}(s,0) = f_*(-\beta'(s)) = -\boldsymbol{e}(s), \end{cases}$$

we have

$$D_s\boldsymbol{b}(s) = \boldsymbol{b}'(s) = -\frac{\delta'(s)}{1 + \{\delta(s)\}^2}\frac{\upsilon(s) + \delta(s)\boldsymbol{\nu}(s,0)}{\sqrt{1 + \{\delta(s)\}^2}} = -\frac{\delta'(s)}{1 + \{\delta(s)\}^2}\boldsymbol{n}(s).$$

Thus the torsion τ of c is given as in (4.6). Since the unit vector field $\upsilon(s)$ is included in the normal plane of c and satisfies

$$\langle\upsilon(s), \boldsymbol{n}(s)\rangle = \frac{1}{\sqrt{1 + \{\delta(s)\}^2}}, \qquad \langle\upsilon(s), \boldsymbol{b}(s)\rangle = -\frac{\delta(s)}{\sqrt{1 + \{\delta(s)\}^2}},$$

we have that $\upsilon(s)$ is the form given in (4.5). □

Claim 3. *Any two asymptotic curves are asymptotic to each other in the sense of Definition 4.6.*

Proof. Under the notations in Claim 1 and 2, we have

$$(f\circ\varphi^{-1})(s,t) = c(s)\cosh t + \frac{\boldsymbol{n}(s)+\delta(s)\boldsymbol{b}(s)}{\kappa(s)}\sinh t.$$

For $s\in\boldsymbol{R}$, set $\gamma_s(t) := (f\circ X)(s,t)$. It is sufficient to prove that, for fixed $s_0\in\boldsymbol{R}$, the function

$$\rho : \boldsymbol{R}\ni s\longmapsto \left\langle c(s)+\frac{\boldsymbol{n}(s)+\delta(s)\boldsymbol{b}(s)}{\kappa(s)}, c(s_0)+\frac{\boldsymbol{n}(s_0)+\delta(s_0)\boldsymbol{b}(s_0)}{\kappa(s_0)}\right\rangle\in\boldsymbol{R},$$

is equivalently zero. Using the Frenet-Serret formula

$$\boldsymbol{e}'(s) = c(s)+\kappa(s)\boldsymbol{n}(s),\qquad \boldsymbol{n}'(s) = -\kappa(s)\boldsymbol{e}(s)+\tau(s)\boldsymbol{b}(s),\qquad \boldsymbol{b}'(s) = -\tau(s)\boldsymbol{n}(s)$$

for the curve c in $\boldsymbol{H}^3$, we have

$$\frac{d}{ds}\left(c(s)+\frac{\boldsymbol{n}(s)+\delta(s)\boldsymbol{b}(s)}{\kappa(s)}\right) = \frac{\kappa(s)\tau(s)\delta(s)-\kappa'(s)}{\kappa^2(s)}\boldsymbol{n}(s) + \frac{\kappa(s)\tau(s)-\kappa(s)\delta'(s)+\kappa'(s)\delta(s)}{\kappa^2(s)}\boldsymbol{b}(s). \tag{4.10}$$

On the other hand, we have

$$\kappa(s)\tau(s)\delta(s)-\kappa'(s) = \kappa(s)\tau(s)-\kappa(s)\delta'(s)+\kappa'(s)\delta(s) = 0,$$

by (4.6) in Claim 2. Substituting this into (4.10), we have $\rho'(s) = 0$ for all s. Besides $\rho(s_0) = 0$, we obtain $\rho(s) = 0$ for all s. □

4.4. **A non-real-analytic example.**

Example 4.8. The assumption of analyticity in Theorem II cannot be removed since non-real-analytic developable surfaces of exponential type might have more than one asymptotic points. Figure 5 shows an example asymptotic to distinct two points in the ideal boundary.

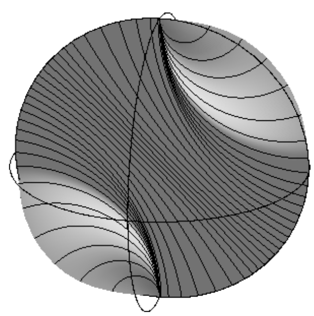

FIGURE 5. A non-real-analytic developable surface of exponential type asymptotic to 0 and ∞.

The corresponding curve $\alpha(s)$ in $L\boldsymbol{H}^3$ is given by $\alpha(s) = (x_1(s)+iy_1(s), x_2(s)+iy_2(s))$, where

$$x_1(s) = \begin{cases} 0 & (s\le -1)\\ (\sqrt{2}-1)(s+1)/(1+e^{\frac{1}{s}+\frac{1}{s+1}}) & (-1<s<0)\\ (\sqrt{2}-1)(s+1) & (0\le s),\end{cases}\qquad y_1(s) = \begin{cases} 0 & (s\le\sqrt{2})\\ 2e^{\frac{\sqrt{2}+1}{\sqrt{2}-s}} & (\sqrt{2}<s),\end{cases}$$

$$x_2(s) = \begin{cases} (\sqrt{2}-1)(1-s) & (s \le 0) \\ (\sqrt{2}-1)(1-s)/(1+e^{\frac{1}{1-s}-\frac{1}{s}}) & (0 < s < 1) \\ 0 & (1 \le s), \end{cases} \qquad y_2(s) = \begin{cases} 2e^{\frac{\sqrt{2}+1}{\sqrt{2}-s}} & (s \le -\sqrt{2}) \\ 0 & (-\sqrt{2} < s). \end{cases}$$

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