On the flux problem in the theory of steady Navier–Stokes equations with nonhomogeneous boundary conditions[*]

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Abstract

We study the nonhomogeneous boundary value problem for Navier–Stokes equations of steady motion of a viscous incompressible fluid in a two–dimensional bounded multiply connected domain $\Omega = \Omega_1 \setminus \overline{\Omega}_2$, $\overline{\Omega}_2 \subset \Omega_1$. We prove that this problem has a solution if the flux $\mathcal{F}$ of the boundary datum through $\partial\Omega_2$ is nonnegative (outflow condition).

1 Introduction

Let Ω be a bounded multiply connected domain in $\mathbb{R}^n$, $n = 2, 3$, with Lipschitz boundary $\partial\Omega$ consisting of N disjoint components Γ_j, i.e. $\partial\Omega = \Gamma_1 \cup \ldots \cup \Gamma_N$ and $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$. In Ω consider the stationary Navier–Stokes system with nonhomogeneous boundary conditions

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{a} & \text{on } \partial\Omega. \end{cases} \tag{1}$$

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Starting from the famous J. Leray's paper [22] published in 1933, problem (1) was a subject of investigation in many papers [1], [3]–[11], [14]–[19], [23], [31], [33], [34]. The continuity equation (1_2) implies the necessary compatibility condition for the solvability of problem (1):

$$\int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,dS = \sum_{j=1}^{N} \int\limits_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,dS = 0, \tag{2}$$

where $\mathbf{n}$ is a unit vector of the outward (with respect to Ω) normal to $\partial\Omega$. However, for a long time the existence of a weak solution $\mathbf{u}\in W^{1,2}(\Omega)$ to problem (1) was proved only under the condition

$$\mathcal{F}_j = \int\limits_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,dS = 0, \qquad j=1,2,\ldots,N, \tag{3}$$

(see [22], [18], [7], [34] [19], etc.). Condition (3) requires the fluxes $\mathcal{F}_j$ of the boundary datum $\mathbf{a}$ to be zero separately on all components Γ_j of the boundary $\partial\Omega$, while the compatibility condition (2) means only that the total flux is zero. Thus, (3) is stronger than (2) and (3) does not allow the presence of sinks and sources.

Problem (1), (3) was first studied by J. Leray [22] who initiated two different approaches to prove its solvability. In both approaches the problem is reduced to an operator equation with a compact operator and the existence of a fixed–point is obtained by using the Leray–Schauder theorem. The main difference in these approaches is in getting an a priori estimate of the solution. The first method uses the extension of boundary data $\mathbf{a}$ into Ω as $\mathbf{A}(\varepsilon,x) = curl\,\big(\zeta(\varepsilon,x)\mathbf{b}(x)\big)$, where $\zeta(\varepsilon,x)$ is Hopf's cut–off function [14]. For such extension there holds an estimate (see, e.g., [19])

$$\Big|\int\limits_{\Omega} (\mathbf{v}\cdot\nabla)\mathbf{A}\cdot\mathbf{v}\,dx\Big| \le \varepsilon c \int\limits_{\Omega} |\nabla\mathbf{v}|^2\,dx \quad \forall\ \mathbf{v}\in \mathring{W}^{1,2}(\Omega), \tag{4}$$

with c being independent of ε and $\varepsilon > 0$ taken sufficiently small (so that $\varepsilon c < \nu$). Obviously, the extension of the boundary data in the form of *curl* is possible only if condition (3) is satisfied. A. Takashita [31] has constructed a counterexample showing that estimate (4) is false whatever the choice of the extension $\mathbf{A}$ can be, if the condition (3) is not valid. Thus, the first approach may be applied only when (3) is valid.

The second approach is to prove an a priory estimate by contradiction. Such arguments also could be found in the book of O.A. Ladyzhenskaya [19].

Later, a slight modification of this argument was proposed independently by L.V. Kapitanskii and K. Pileckas [15], and by Ch.J. Amick [1]. This modification has the advantage that it allows to take any solenoidal extension of the boundary data and requires (unlike Hopf's construction) only the Lipschitz regularity of the boundary $\partial\Omega$. We should mention that the method used in [1], [15] was already contained in the basic paper of J.Leray [22]. In [15] the solvability of problem (1) was proved by this method only under "stronger" condition (3), while in [1] was constructed a class of plane domains with special symmetry on Ω and on $\mathbf{a} = \big(a_1, a_2\big)$, where problem (1) is solvable for arbitrary fluxes $\mathcal{F}_j$, assuming only condition (2). More precisely, it is proved in [1] that problem (1) has at least one solution for all values of $\mathcal{F}_j$, if $\Omega \subset \mathbb{R}^2$ is symmetric with respect to the x_1–axis and all components Γ_j intersect the line $\{x : \ x_2 = 0\}$, a_1 is an even function, while a_2 is an odd function with respect to x_2. Note that Amick's result was proved by contradiction and does not contain an effective a priori estimate for the Dirichlet integral of the solution. An effective estimate for the solution of the Navier–Stokes problem with the above symmetry conditions was first obtained by H. Fujita [8] (see also [26]). Recently V.V. Pukhnachev has established an analogous estimate for the solution to problem (1) in the case of three–dimensional stationary fluid motion with two mutually perpendicular planes of symmetry (private communication).

The assumption on $\mathcal{F}_j$ to be zero (see (3)) was relaxed in [10] where it is shown that problem (1) still admits a solution provided that $|\mathcal{F}_j|$ are sufficiently small[1]. In [3] estimates for $|\mathcal{F}_j|$ are expressed in terms of simple geometric characteristics of Ω which can be easily verified for arbitrary domains. These results have been extended to solutions corresponding to boundary data in Lebesgue's spaces in [27]. As far as exterior domains are concerned, the hypothesis of zero flux at the boundary has been replaced by the assumptions of small flux in [28].

An interesting contribution to the Navier–Stokes problem is due to H.Fujita and H. Morimoto [9] (see also [29]). They studied problem (1) in a domain Ω with two components of the boundary Γ_1 and Γ_2. Assuming that $\mathbf{a} = \mathcal{F}\nabla u_0 + \boldsymbol{\alpha}$, where $\mathcal{F} \in \mathbb{R}$, u_0 is a harmonic function, and $\boldsymbol{\alpha}$ satisfies condition (3), they proved that there is a countable subset $\mathcal{N} \subset \mathbb{R}$ such that if $\mathcal{F} \notin \mathcal{N}$ and $\boldsymbol{\alpha}$ is small (in a suitable norm), then system (1) has a weak solution. Moreover, if $\Omega \subset \mathbb{R}^2$ is an annulus and $u_0 = \log|x|$, then $\mathcal{N} = \emptyset$.

To the best of our knowledge this is the state of art of the Navier–Stokes

[1] As far as we are aware, the idea of requiring smallness of $|\mathcal{F}_j|$ instead of its vanishing appears for the first time in [6] (see also [7]).

problem with nonhomogeneous boundary conditions in bounded multiply connected domains. As a consequence, the fundamental question whether problem (1) is solvable for all values of $\mathcal{F}_j$ (Leray's problem) is still open despite of efforts of many mathematicians.

In this paper we study problem (1) in a plane domain

$$\Omega = \Omega_1 \setminus \overline{\Omega}_2, \quad \overline{\Omega}_2 \subset \Omega_1, \tag{5}$$

where Ω_1 and Ω_2 are bounded simply connected domains of $\mathbb{R}^2$ with Lipschitz boundaries $\partial\Omega_1 = \Gamma_1$, $\partial\Omega_2 = \Gamma_2$. Without loss of generality we may assume that $\Omega_2 \supset \{x \in \mathbb{R}^2 : |x| < 1\}$. Since Ω has only two components of the boundary, condition (2) may be rewritten in the form

$$\mathcal{F} = \int\limits_{\Gamma_2} \mathbf{a} \cdot \mathbf{n}\, dS = -\int\limits_{\Gamma_1} \mathbf{a} \cdot \mathbf{n}\, dS \tag{6}$$

($\mathbf{n}$ is an outward normal with respect to the domain Ω). Using some suggestions from [1], we prove that problem (1) is solvable without any restriction on the value of $|\mathcal{F}|$ provided $\mathcal{F} \geq 0$ (outflow condition). Note that this is the first result on Leray's problem which does not require smallness or symmetry conditions of the data.

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2 Notation and preliminary results

Everywhere in the paper $\Omega = \Omega_1 \setminus \overline{\Omega}_2 \subset \mathbb{R}^2$ is a bounded domain defined above by (5). We assume that the boundary $\partial\Omega$ is Lipschitz [2]. We use standard notations for function spaces: $C(\overline{\Omega})$, $C(\partial\Omega)$, $W^{k,q}(\Omega)$, $\mathring{W}^{k,q}(\Omega)$, $W^{\alpha,q}(\partial\Omega)$, where $\alpha \in (0,1), k \in \mathbb{N}_0, q \in [1,+\infty]$. $\mathcal{H}^1(\mathbb{R}^2)$ denotes the Hardy space on $\mathbb{R}^2$. In our notation we do not distinguish function spaces for scalar and vector valued functions; it is clear from the context whether we use scalar or vector (or tensor) valued function spaces. $H(\Omega)$ is subspace of all divergence free vector fields from $\mathring{W}^{1,2}(\Omega)$ with the norm

$$\|\mathbf{u}\|_{H(\Omega)} = \|\nabla \mathbf{u}\|_{L^2(\Omega)}.$$

[2] $\partial\Omega$ is Lipschitz, if for every $\xi \in \partial\Omega$, there is a neighborhood of ξ in which $\partial\Omega$ is the graph of a Lipschitz continuous function (defined on an open interval).

Note that for function $\mathbf{u} \in H(\Omega)$ the norm $\| \cdot \|_{H(\Omega)}$ is equivalent to $\| \cdot \|_{W^{1,2}(\Omega)}$.

Let us collect auxiliary results that we shall use below to prove the solvability of problem (1).

Lemma 1. *Let Ω be a bounded domain with Lipschitz boundary. If $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ and*

$$\int\limits_{\partial\Omega} \mathbf{a} \cdot \mathbf{n} \, dS = 0,$$

then there exists a divergence free extension $\mathbf{A} \in W^{1,2}(\Omega)$ of $\mathbf{a}$ such that

$$\|\mathbf{A}\|_{W^{1,2}(\Omega)} \leq c\|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}. \tag{7}$$

Lemma 1 is well known (see [20]).

Lemma 2. (see [30]). *Let Ω be a bounded domain with Lipschitz boundary and let $R(\boldsymbol{\eta})$ be a continuous linear functional defined on $\mathring{W}^{1,2}(\Omega)$. If*

$$R(\boldsymbol{\eta}) = 0 \qquad \forall \ \boldsymbol{\eta} \in H(\Omega),$$

then there exists a function $p \in L^2(\Omega)$ with $\int\limits_{\Omega} p(x)\, dx = 0$ such that

$$R(\boldsymbol{\eta}) = \int\limits_{\Omega} p \operatorname{div} \boldsymbol{\eta} \, dx \qquad \forall \ \boldsymbol{\eta} \in \mathring{W}^{1,2}(\Omega).$$

Moreover, $\|p\|_{L^2(\Omega)}$ is equivalent to $\|R\|_{(\mathring{W}^{1,2}(\Omega))^*}$.

Lemma 3. *Let $f \in \mathcal{H}^1(\mathbb{R}^2)$ and let*

$$J(x) = \int\limits_{\mathbb{R}^2} \log|x-y| \, f(y) \, dy. \tag{8}$$

Then

(i) $J \in C(\mathbb{R}^2)$;

(ii) $\nabla J \in L^2(\mathbb{R}^2),\ D^\alpha J \in L^1(\mathbb{R}^2), |\alpha| = 2.$

Lemma 3 is well known; a proof of the property (i) could be found in [32] (see Theorem 5.12 and Corollary 12.12 at p. 82–83), and the property (ii) is proved, for example, in [2] (see Theorem 5.13, p. 208).

Lemma 4. *Let* $\mathbf{w} \in W^{1,2}(\mathbb{R}^2)$ *and* $\operatorname{div} \mathbf{w} = 0$. *Then*

$$\operatorname{div}\big[(\mathbf{w}\cdot\nabla)\mathbf{w}\big] = \sum_{i,j=1}^{2} \frac{\partial w_i}{\partial x_j}\frac{\partial w_j}{\partial x_i} \in \mathcal{H}^1(\mathbb{R}^2).$$

Lemma 4 follows from div-curl lemma with two cancelations (see, e.g., Theorem II.1 in [5]).

Lemma 5. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with Lipschitz boundary and let* $h \in C(\partial\Omega)$. *If* h *could be extended into domain* Ω *as a function* $H \in W^{1,2}(\Omega)$, *then there exists a unique weak solution* $v \in W^{1,2}(\Omega)$ *of the problem*

$$\left\{\begin{array}{rcll} -\Delta v &=& 0 & \text{in } \Omega, \\ v &=& h & \text{on } \partial\Omega, \end{array}\right. \tag{9}$$

such that $v \in C(\overline{\Omega})$.

The proof of Lemma 5 could be found in [24] (see also Theorem 4.2 in [21]). Note that not every continuous on $\partial\Omega$ function h could be extended into Ω as a function H from $W^{1,2}(\Omega)$. If this is the case, then there exists a weak solution v of (9) satisfying only $v \in W^{1,2}_{loc}(\Omega) \cap C(\overline{\Omega})$ (see Chapter II in [21]).

3 Euler equation

In this section we collect some properties of a solution to the Euler system

$$\left\{\begin{array}{rcl} (\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla p &=& 0, \\ \operatorname{div} \mathbf{w} &=& 0, \end{array}\right. \tag{10}$$

that are used below to prove the main result of the paper.

Assume that $\mathbf{w} \in W^{1,2}(\Omega)$ and $p \in W^{1,2}(\Omega)$ satisfy the Euler equations (10) for almost all $x \in \Omega$ and let $\int\limits_{\Gamma_i} \mathbf{w} \cdot \mathbf{n} dS = 0,\ i = 1, 2$. Then there exists a continuous stream function $\psi \in W^{2,2}(\Omega)$ such that $\nabla\psi = (-w_2, w_1)$. Denote by $\Phi = p + \frac{|\mathbf{w}|^2}{2}$ the total head pressure corresponding to the solution $(\mathbf{w}, p)$. Obviously, $\Phi \in W^{1,s}(\Omega)$ for all $s \in [1, 2)$. By direct calculations one can easily get the identity

$$\nabla\Phi \equiv \Big(\frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2}\Big)\big(w_2, -w_1\big) = (\Delta\psi)\nabla\psi. \tag{11}$$

If all functions are smooth, from this identity the classical Bernoulli law follows immediately: *the total head pressure $\Phi(x)$ is constant along any streamline of the flow.*

In the general case the following assertion holds.

Lemma 6.[16]. *Let $\mathbf{w} \in W^{1,2}(\Omega)$ and $p \in W^{1,2}(\Omega)$ satisfy the Euler equations (10) for almost all $x \in \Omega$ and let $\int\limits_{\Gamma_i} \mathbf{w} \cdot \mathbf{n} dS = 0,\ i = 1, 2$. Then for any connected set $K \subset \overline{\Omega}$ such that*

$$\psi(x)\big|_K = const, \tag{12}$$

the identity

$$\Phi(x) = const \quad \mathfrak{H}^1 - almost \quad everywhere \quad on \quad K \tag{13}$$

holds. Here $\mathfrak{H}^1$ denotes one-dimensional Hausdorff measure[3]*.*

In particular, if $\mathbf{w} = 0$ on $\partial\Omega$ (in the sense of trace), then the pressure $p(x)$ is constant on $\partial\Omega$. Note that $p(x)$ could take different constant values $p_j = p(x)\big|_{\Gamma_j}, j = 1, 2$, on different components Γ_j of the boundary $\partial\Omega$.

Here and henceforth we understand connectedness in the sense of general topology. Note that the proof of the above lemma is based on classical results of [17] and on recent results obtained in [4]. The last statement of Lemma 6 was proved in [15] (see Lemma 4) and in [1] (see Theorem 2.2).

Lemma 7. *Let $(\mathbf{w}, p)$ satisfy the Euler equations (10) for almost all $x \in \Omega$, $\mathbf{w} \in W^{1,2}(\Omega)$ and $\mathbf{w}(x)\big|_{\partial\Omega} = 0$. Then*

$$p \in C(\overline{\Omega}) \cap W^{1,2}(\Omega). \tag{14}$$

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[3] $\mathfrak{H}^1(F) = \lim\limits_{t \to 0+} \mathfrak{H}^1_t(F)$, where $\mathfrak{H}^1_t(F) = \inf\{\sum\limits_{i=1}^{\infty} \mathrm{diam} F_i : \mathrm{diam} F_i \le t, F \subset \bigcup\limits_{i=1}^{\infty} F_i\}$.

Proof. From Euler equations (10) it follows that $p \in W^{1,s}(\Omega)$ for any $s \in [1,2)$ and

$$\|p\|_{W^{1,s}(\Omega)} \le c\|\mathbf{w}\|^2_{H(\Omega)}.$$

Multiply (10) by $\boldsymbol{\varphi} = \nabla\xi$, where $\xi \in C_0^\infty(\Omega)$:

$$\int\limits_\Omega \nabla p \cdot \nabla\xi \, dx = -\int\limits_\Omega \big(\mathbf{w}\cdot\nabla\big)\mathbf{w}\cdot\nabla\xi \, dx \quad \forall \xi \in C_0^\infty(\Omega).$$

Thus, $p \in W^{1,q}(\Omega)$ is the unique weak solution of the boundary value problem for the Poisson equations

$$\begin{cases} -\Delta p &= \operatorname{div}\big[\big(\mathbf{w}\cdot\nabla\big)\mathbf{w}\big] \quad & \text{in } \ \Omega, \\ p(x) &= p_1 & \text{on } \ \Gamma_1, \\ p(x) &= p_2 & \text{on } \ \Gamma_2. \end{cases} \tag{15}$$

According to Lemma 4, $\operatorname{div}\big[\big(\mathbf{w}\cdot\nabla\big)\mathbf{w}\big] \in \mathcal{H}^1(\mathbb{R}^2)$ (here we assume that $\mathbf{w} \in H(\Omega)$ is extended by zero to $\mathbb{R}^2$). Define the function $J_1(x)$ by the formula

$$J_1(x) = -\frac{1}{2\pi}\int\limits_{\mathbb{R}^2} \log|x-y| \operatorname{div}_y \big[\big(\mathbf{w}(y)\cdot\nabla_y\big)\mathbf{w}(y)\big] dy.$$

In virtue of Lemma 3, $J_1 \in C(\mathbb{R}^2)$, $\nabla J_1 \in L^2(\mathbb{R}^2)$, $D^\alpha J_1 \in L^1(\mathbb{R}^2), |\alpha| = 2$. Since $-\Delta J_1(x) = \operatorname{div}\big[\big(\mathbf{w}\cdot\nabla\big)\mathbf{w}\big]$ in $\mathbb{R}^2$, we get for $J_2(x) = p(x) - J_1(x)$ the following problem

$$\begin{cases} -\Delta J_2 &= 0 & \text{in } \ \Omega, \\ J_2\big|_{\partial\Omega} &= j_2 - j_1 & \text{on } \ \partial\Omega, \end{cases} \tag{16}$$

where $j_1(x) = J_1(x)\big|_{\partial\Omega}$,

$$j_2(x) = \begin{cases} p_1 & \text{on } \ \Gamma_1, \\ p_2 & \text{on } \ \Gamma_2. \end{cases}$$

The function j_1 is a trace on $\partial\Omega$ of $J_1 \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$, while $j_2 \in C(\partial\Omega)$ and j_2 obviously could be extended to Ω as a function from $W^{1,2}(\Omega)$. Thus, by Lemma 5 problem (16) has a unique weak solution $J_2 \in W^{1,2}(\Omega)$ such that $J_2 \in C(\overline{\Omega})$. By uniqueness $p(x) = J_1(x) + J_2(x)$. Hence, $p \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$.

We say that the function $f \in W^{1,s}(\Omega)$ satisfies a *weak one-side maximum principle locally* in Ω, if

$$\operatorname*{ess\,sup}_{x\in\Omega'} f(x) \le \operatorname*{ess\,sup}_{x\in\partial\Omega'} f(x) \tag{17}$$

holds for any strictly interior subdomain Ω' ($\overline{\Omega'} \subset \Omega$) with the boundary $\partial\Omega'$ that does not contain singleton connected components. (In (17) negligible sets are the sets of 2–dimensional Lebesgue measure zero in the left ess sup, and the sets of 1–dimensional Hausdorff measure zero in the right ess sup.) If (17) holds for any $\Omega' \subset \Omega$ with the boundary $\partial\Omega'$ not containing singleton connected components, then we say that $f \in W^{1,s}(\Omega)$ satisfies a *weak one-side maximum principle* in Ω (since the boundary $\partial\Omega$ is Lipschitz, we can take $\Omega' = \Omega$ in (17)).

Lemma 8. [16]. *Let $\mathbf{w} \in W^{1,2}(\Omega)$ and $p \in W^{1,2}(\Omega)$ satisfy the Euler equations (10) for almost all $x \in \Omega$ and let $\int\limits_{\Gamma_i} \mathbf{w}\cdot\mathbf{n}dS = 0,\ i = 1,2$. Assume that there exists a sequence of functions $\{\Phi_\mu\}$ such that $\Phi_\mu \in W^{1,s}_{loc}(\Omega)$ and $\Phi_\mu \rightharpoonup \Phi$ in the space $W^{1,s}_{loc}(\Omega)$ for all $s \in (1,2)$. If all Φ_μ satisfy the weak one-side maximum principle locally in Ω, then Φ satisfies the weak one-side maximum principle in Ω.*

In particular, if $\mathbf{w}\big|_{\partial\Omega} = 0$, then

$$\operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) \le \operatorname*{ess\,sup}_{x\in\partial\Omega} \Phi(x) = \max\{p_1, p_2\}. \tag{18}$$

The proof of the above lemma is based on Lemma 6, classical results of [17], and on recent results obtained in [4]. Note that the weaker version of Lemma 8 was proved by Ch. Amick [1] (see Theorem 3.2 and Remark thereafter).

4 Existence theorem

Let us consider Navier–Stokes problem (1) in the domain Ω defined by (5) and assume that $\partial\Omega$ is at least Lipschitz. If the boundary datum $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ and $\mathbf{a}$ satisfies the condition (6), i.e.,

$$\int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,dS = \int\limits_{\Gamma_1} \mathbf{a}\cdot\mathbf{n}\,dS + \int\limits_{\Gamma_2} \mathbf{a}\cdot\mathbf{n}\,dS = 0,$$

then by Lemma 1 there exists a divergence free extension $\mathbf{A} \in W^{1,2}(\Omega)$ of $\mathbf{a}$ and there holds estimate (7). Using this fact and standard results (see, e.g. [19]) we can find a weak solution $\mathbf{U} \in W^{1,2}(\Omega)$ of the Stokes problem such that $\mathbf{U} - \mathbf{A} \in H(\Omega)$ and

$$\nu \int\limits_{\Omega} \nabla \mathbf{U} \cdot \nabla \boldsymbol{\eta} \, dx = 0 \quad \forall \, \boldsymbol{\eta} \in H(\Omega). \tag{19}$$

Moreover,

$$\|\mathbf{U}\|_{W^{1,2}(\Omega)} \le c \|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}. \tag{20}$$

By a *weak solution* of problem (1) we understand a function $\mathbf{u}$ such that $\mathbf{w} = \mathbf{u} - \mathbf{A} \in H(\Omega)$ and satisfies the integral identity

$$\nu \int\limits_{\Omega} \nabla \mathbf{w} \cdot \nabla \boldsymbol{\eta} \, dx - \int\limits_{\Omega} \big((\mathbf{w} + \mathbf{U}) \cdot \nabla\big) \boldsymbol{\eta} \cdot \mathbf{w} \, dx - \int\limits_{\Omega} \big(\mathbf{w} \cdot \nabla\big) \boldsymbol{\eta} \cdot \mathbf{U} \, dx$$

$$= \int\limits_{\Omega} \big(\mathbf{U} \cdot \nabla\big) \boldsymbol{\eta} \cdot \mathbf{U} \, dx \qquad \forall \boldsymbol{\eta} \in H(\Omega). \tag{21}$$

We shall prove the following

Theorem 1. *Assume that* $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ *and let condition* (6) *be fulfilled. If* $\mathcal{F} = \int\limits_{\Gamma_2} \mathbf{a} \cdot \mathbf{n} \, dS \ge 0$, *then problem* (1) *admits at least one weak solution.*

Proof. 1. We follow a contradiction argument of J. Leray [22]. Although, this argument was used also in many other papers (e.g. [18], [19], [15], [1]), we reproduce here, for the reader convenience, some details of it. It is well known (e.g. [19]) that integral identity (21) is equivalent to an operator equation in the space $H(\Omega)$ with a compact operator, and, therefore, in virtue of the Leray–Schauder fixed–point theorem, to prove the existence of a weak solution to Navier–Stokes problem (1) it is sufficient to show that all possible solutions of the integral identity

$$\nu \int\limits_{\Omega} \nabla \mathbf{w} \cdot \nabla \boldsymbol{\eta} \, dx - \lambda \int\limits_{\Omega} \big((\mathbf{w} + \mathbf{U}) \cdot \nabla\big) \boldsymbol{\eta} \cdot \mathbf{w} \, dx - \lambda \int\limits_{\Omega} \big(\mathbf{w} \cdot \nabla\big) \boldsymbol{\eta} \cdot \mathbf{U} \, dx$$

$$= \lambda \int\limits_{\Omega} \big(\mathbf{U} \cdot \nabla\big) \boldsymbol{\eta} \cdot \mathbf{U} \, dx \qquad \forall \, \boldsymbol{\eta} \in H(\Omega) \tag{22}$$

are uniformly bounded (with respect to $\lambda \in [0, \nu^{-1}]$) in $H(\Omega)$. Assume this is false. Then there exist sequences $\{\lambda_k\}_{k\in\mathbb{N}} \subset [0, \nu^{-1}]$ and $\{\mathbf{w}_k\}_{k\in\mathbb{N}} \in H(\Omega)$ such that

$$\nu \int\limits_{\Omega} \nabla \mathbf{w}_k \cdot \nabla \boldsymbol{\eta}\, dx - \lambda_k \int\limits_{\Omega} \big((\mathbf{w}_k + \mathbf{U}) \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{w}_k\, dx - \lambda_k \int\limits_{\Omega} \big(\mathbf{w}_k \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{U}\, dx$$

$$= \lambda_k \int\limits_{\Omega} \big(\mathbf{U} \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{U}\, dx \qquad \forall\, \boldsymbol{\eta} \in H(\Omega), \tag{23}$$

and

$$\lim_{k\to\infty} \lambda_k = \lambda_0 \in [0, \nu^{-1}], \quad \lim_{k\to\infty} J_k = \lim_{k\to\infty} \|\mathbf{w}_k\|_{H(\Omega)} = \infty. \tag{24}$$

Let us take in (23) $\boldsymbol{\eta} = J_k^{-2}\mathbf{w}_k$ and denote $\widehat{\mathbf{w}}_k = J_k^{-1}\mathbf{w}_k$. Since

$$\int\limits_{\Omega} \big((\mathbf{w}_k + \mathbf{U}) \cdot \nabla\big)\mathbf{w}_k \cdot \mathbf{w}_k\, dx = 0,$$

we get

$$\nu \int\limits_{\Omega} |\nabla \widehat{\mathbf{w}}_k|^2\, dx = \lambda_k \int\limits_{\Omega} \big(\widehat{\mathbf{w}}_k \cdot \nabla\big)\widehat{\mathbf{w}}_k \cdot \mathbf{U}\, dx + J_k^{-1}\lambda_k \int\limits_{\Omega} \big(\mathbf{U} \cdot \nabla\big)\widehat{\mathbf{w}}_k \cdot \mathbf{U}\, dx. \tag{25}$$

Since $\|\widehat{\mathbf{w}}_k\|_{H(\Omega)} = 1$, there exists a subsequence $\{\widehat{\mathbf{w}}_{k_l}\}$ converging weakly in $H(\Omega)$ to a vector field $\widehat{\mathbf{w}} \in H(\Omega)$. Because of the compact imbedding

$$H(\Omega) \hookrightarrow L^r(\Omega) \quad \forall\, r \in (1, \infty),$$

the subsequence $\{\widehat{\mathbf{w}}_{k_l}\}$ converges strongly in $L^r(\Omega)$. Therefore, we can pass to a limit as $k_l \to \infty$ in equality (25). As a result we obtain

$$\nu = \lambda_0 \int\limits_{\Omega} \big(\widehat{\mathbf{w}} \cdot \nabla\big)\widehat{\mathbf{w}} \cdot \mathbf{U}\, dx. \tag{26}$$

2. Let us return to integral identity (23). Consider the functional

$$R_k(\boldsymbol{\eta}) = \int\limits_{\Omega} \Big(\nu \nabla \mathbf{w}_k \cdot \nabla \boldsymbol{\eta} - \lambda_k \big((\mathbf{w}_k + \mathbf{U}) \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{w}_k - \lambda_k \big(\mathbf{w}_k \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{U}\Big)\, dx$$

$$-\lambda_k \int\limits_{\Omega} \big(\mathbf{U} \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{U}\, dx \qquad \forall\, \boldsymbol{\eta} \in \mathring{W}^{1,2}(\Omega).$$

Obviously, $R_k(\boldsymbol{\eta})$ is a linear functional, and using (20) and the imbedding theorem, we obtain

$$\big|R_k(\boldsymbol{\eta})\big| \leq c\Big(\|\mathbf{w}_k\|_{H(\Omega)} + \|\mathbf{w}_k\|^2_{H(\Omega)} + \|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}\Big)\|\boldsymbol{\eta}\|_{H(\Omega)},$$

with constant c independent of k. It follows from (23) that

$$R_k(\boldsymbol{\eta}) = 0 \qquad \forall\, \boldsymbol{\eta} \in H(\Omega).$$

Therefore, by Lemma 2, there exist functions $p_k \in \widehat{L}^2(\Omega) = \{q \in L^2(\Omega) : \int\limits_\Omega q(x)\,dx = 0\}$ such that

$$R_k(\boldsymbol{\eta}) = \int\limits_\Omega p_k \operatorname{div} \boldsymbol{\eta}\,dx \qquad \forall\, \boldsymbol{\eta} \in \mathring{W}^{1,2}(\Omega),$$

and

$$\|p_k\|_{L^2(\Omega)} \leq c\Big(\|\mathbf{w}_k\|_{H(\Omega)} + \|\mathbf{w}_k\|^2_{H(\Omega)} + \|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}\Big). \tag{27}$$

The pair $\big(\mathbf{w}_k, p_k\big)$ satisfies the integral identity

$$\nu \int\limits_\Omega \nabla\mathbf{w}_k \cdot \nabla\boldsymbol{\eta}\,dx - \lambda_k \int\limits_\Omega \big((\mathbf{w}_k + \mathbf{U}) \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{w}_k\,dx - \lambda_k \int\limits_\Omega \big(\mathbf{w}_k \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{U}\,dx$$

$$-\lambda_k \int\limits_\Omega \big(\mathbf{U} \cdot \nabla\big)\boldsymbol{\eta} \cdot \mathbf{U}\,dx = \int\limits_\Omega p_k \operatorname{div} \boldsymbol{\eta}\,dx \qquad \forall\, \boldsymbol{\eta} \in \mathring{W}^{1,2}(\Omega). \tag{28}$$

Let $\mathbf{u}_k = \mathbf{w}_k + \mathbf{U}$. Then identity (28) takes the form (see (19))

$$\nu \int\limits_\Omega \nabla\mathbf{u}_k \cdot \nabla\boldsymbol{\eta}\,dx - \int\limits_\Omega p_k \operatorname{div} \boldsymbol{\eta}\,dx = -\lambda_k \int\limits_\Omega \big(\mathbf{u}_k \cdot \nabla\big)\mathbf{u}_k \cdot \boldsymbol{\eta}\,dx \ \ \forall\, \boldsymbol{\eta} \in \mathring{W}^{1,2}(\Omega).$$

Thus, $\big(\mathbf{u}_k, p_k\big)$ might be considered as a weak solution to the Stokes problem

$$\begin{cases} -\nu\Delta\mathbf{u}_k + \nabla p_k &= \mathbf{f}_k \qquad \text{in} \ \ \Omega, \\ \operatorname{div} \mathbf{u}_k &= 0 \qquad \text{in} \ \ \Omega, \\ \mathbf{u}_k &= \mathbf{a} \qquad \text{on} \ \ \partial\Omega, \end{cases}$$

with the right–hand side $\mathbf{f}_k = -\lambda_k\big(\mathbf{u}_k \cdot \nabla\big)\mathbf{u}_k$. Obviously, $\mathbf{f}_k \in L^s(\Omega)$ for $s \in (1, 2)$ and

$$\|\mathbf{f}_k\|_{L^s(\Omega)} \leq c\big\|\big(\mathbf{u}_k \cdot \nabla\big)\mathbf{u}_k\big\|_{L^s(\Omega)} \leq c\|\mathbf{u}_k\|_{L^{2s/(2-s)}(\Omega)}\|\nabla\mathbf{u}_k\|_{L^2(\Omega)}$$

$$\le c\Big(\big(\|\mathbf{w}_k\|_{H(\Omega)} + \|\mathbf{U}\|_{W^{1,2}(\Omega)}\big)^2\Big) \le c\Big(\|\mathbf{w}_k\|^2_{H(\Omega)} + \|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}\Big),$$

where c is independent of k. By well known local regularity results for the Stokes system (see [19], [11]) we have $\mathbf{w}_k \in W^{2,s}_{loc}(\Omega)$, $p_k \in W^{1,s}_{loc}(\Omega)$, and the estimate

$$\|\mathbf{w}_k\|_{W^{2,s}(\Omega')} + \|p_k\|_{W^{1,s}(\Omega')} \le c\Big(\|\mathbf{f}_k\|_{L^s(\Omega)} + \|\mathbf{u}_k\|_{W^{1,2}(\Omega)} + \|p_k\|_{L^2(\Omega)}\Big)$$

$$\le c\Big(\|\mathbf{w}_k\|^2_{H(\Omega)} + \|\mathbf{w}_k\|_{H(\Omega)} + \|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)} + \|\mathbf{a}\|^2_{W^{1/2,2}(\partial\Omega)}\Big), \qquad (29)$$

holds, where Ω' is arbitrary domain with $\overline{\Omega}' \subset \Omega$ and the constant c depends on dist$(\Omega', \partial\Omega)$ but not on k.

Denote $\widehat{p}_k = J_k^{-2} p_k$. It follows from (27) and (29) that

$$\|\widehat{p}_k\|_{L^2(\Omega)} \le const, \quad \|\widehat{p}_k\|_{W^{1,s}(\Omega')} \le const$$

for any $\overline{\Omega}' \subset \Omega$ and $s \in (1,2)$. Hence, from $\{\widehat{p}_{k_l}\}$ can be extracted a subsequence, still denoted by $\{\widehat{p}_{k_l}\}$, which converges weakly in $\widehat{L}^2(\Omega)$ and $W^{1,s}_{loc}(\Omega)$ to some function $\widehat{p} \in W^{1,s}_{loc}(\Omega) \cap \widehat{L}^2(\Omega)$. Let $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$. Taking in (28) $\boldsymbol{\eta} = J_{k_l}^{-2}\boldsymbol{\varphi}$ and letting $k_l \to \infty$ yields

$$-\lambda_0 \int\limits_\Omega \big(\widehat{\mathbf{w}} \cdot \nabla\big)\boldsymbol{\varphi} \cdot \widehat{\mathbf{w}}\, dx = \int\limits_\Omega \widehat{p}\, \mathrm{div}\varphi\, dx \quad \forall \boldsymbol{\varphi} \in C_0^\infty(\Omega).$$

Integrating by parts in the last equality, we derive

$$\lambda_0 \int\limits_\Omega \big(\widehat{\mathbf{w}} \cdot \nabla\big)\widehat{\mathbf{w}} \cdot \boldsymbol{\varphi}\, dx = -\int\limits_\Omega \nabla\widehat{p} \cdot \varphi\, dx \quad \forall \boldsymbol{\varphi} \in C_0^\infty(\Omega). \qquad (30)$$

Hence, the pair $\big(\widehat{\mathbf{w}}, \widehat{p}\big)$ satisfies for almost all $x \in \Omega$ the Euler equations

$$\left\{\begin{array}{rcl} \lambda_0\big(\widehat{\mathbf{w}} \cdot \nabla\big)\widehat{\mathbf{w}} + \nabla\widehat{p} & = & 0, \\ \mathrm{div}\,\widehat{\mathbf{w}} & = & 0, \end{array}\right. \qquad (31)$$

and $\widehat{\mathbf{w}}\big|_{\partial\Omega} = 0$. By Lemmas 6 and 7, $\widehat{p} \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$ and the pressure $\widehat{p}(x)$ is constant on Γ_1 and Γ_2.

Denote by $\widehat{p}_1$ and $\widehat{p}_2$ values of $\widehat{p}(x)$ on Γ_1 and Γ_2, respectively. Multiplying equations (31) by $\mathbf{U}$ and integrating by parts, we derive

$$\lambda_0 \int\limits_\Omega \big(\widehat{\mathbf{w}} \cdot \nabla\big)\widehat{\mathbf{w}} \cdot \mathbf{U}\, dx = -\int\limits_\Omega \nabla\widehat{p} \cdot \mathbf{U}\, dx = -\int\limits_{\partial\Omega} \widehat{p}\, \mathbf{a} \cdot \mathbf{n}\, dS$$

$$= -\widehat{p}_1 \int\limits_{\Gamma_1} \mathbf{a}\cdot\mathbf{n}\, dS - \widehat{p}_2 \int\limits_{\Gamma_2} \mathbf{a}\cdot\mathbf{n}\, dS = \mathcal{F}(\widehat{p}_1 - \widehat{p}_2) \tag{32}$$

(see formula (6)). If either $\mathcal{F} = 0$ or $\widehat{p}_1 = \widehat{p}_2$, it follows from (32) that

$$\lambda_0 \int\limits_{\Omega} \big(\widehat{\mathbf{w}}\cdot\nabla\big)\widehat{\mathbf{w}}\cdot\mathbf{U}\, dx = 0. \tag{33}$$

The last relation contradicts equality (26). Therefore, the norms $\|\mathbf{w}\|_{H(\Omega)}$ of all possible solutions to identity (22) are uniformly bounded with respect to $\lambda \in [0, \nu^{-1}]$ and by Leray–Schauder fixed–point theorem problem (1) admits at least one weak solution $\mathbf{u} \in W^{1,2}(\Omega)$.

3. Up to this point our arguments were standard and followed those of Leray [22] (see also [15] and [1]). However, by the our assumptions $\mathcal{F} > 0$ and, in general, $\widehat{p}_2 \neq \widehat{p}_1$ (see a counterexample in [1]). Thus, (33) may be false. In order to prove that $\widehat{p}_1$ and $\widehat{p}_2$ do coincide in the case $\mathcal{F} > 0$, we use the property of $\big(\widehat{\mathbf{w}}, \widehat{p}\big)$ to be a limit (in some sense) of solutions to the Navier–Stokes equations. Note that the possibility of using this fact was already pointed up by Amick [1].

Let $\Phi_{k_l} = p_{k_l} + \frac{\lambda_{k_l}}{2}|\mathbf{u}_{k_l}|^2$, where $\mathbf{u}_{k_l} = \mathbf{w}_{k_l} + \mathbf{U}$, be a total head pressures corresponding to the solutions $\big(\mathbf{w}_{k_l}, p_{k_l}\big)$ of identities (25). Then $\Phi_{k_l} \in W^{2,s}_{loc}(\Omega)$, $s \in (1,2)$, satisfy almost everywhere in Ω the equations

$$\nu\Delta\Phi_{k_l} - \lambda_{k_l}\big(\mathbf{u}_{k_l}\cdot\nabla\big)\Phi_{k_l} = \nu\Big(\frac{\partial u_{1k_l}}{\partial x_2} - \frac{\partial u_{2k_l}}{\partial x_1}\Big)^2.$$

It is well known [12], [13] (see also [25]) that for Φ_{k_l} one-side maximum principle holds locally (since the boundary is only Lipschitz, Φ_{k_l} do not have second derivatives up to the boundary). Set $\widehat{\Phi}_{k_l} = J^{-2}_{k_l}\Phi_{k_l}$. It follows from (27), (29) that the sequence $\widehat{\Phi}_{k_l}$ weakly converges to $\widehat{\Phi} = \widehat{p} + \frac{\lambda_0}{2}|\widehat{\mathbf{u}}|^2$ in $L^2(\Omega)\cap W^{1,s}_{loc}(\Omega)$, $s \in (1,2)$. Therefore, by Lemma 8, $\widehat{\Phi}$ satisfies the weak one-sided maximum principle and

$$\operatorname*{ess\,sup}_{x\in\Omega} \widehat{\Phi}(x) \le \operatorname*{ess\,sup}_{x\in\partial\Omega} \widehat{\Phi}(x) = \max\{\widehat{p}_1, \widehat{p}_2\} \tag{34}$$

(see (18)).

We conclude from equalities (26) and (32)

$$(\widehat{p}_1 - \widehat{p}_2)\mathcal{F} = \nu > 0.$$

So, if $\mathcal{F} > 0$, then

$$\widehat{p}_2 < \widehat{p}_1. \tag{35}$$

Now, it follows from (34), (35) that

$$\int\limits_\Omega \widehat{\Phi}(x)\,dx \le \operatorname*{ess\,sup}_{x\in\Omega} \widehat{\Phi}(x)|\Omega| \le \widehat{p}_1|\Omega|, \tag{36}$$

where $|\Omega|$ means the measure of Ω.

On the other hand, from equation (31_1) we obtain the identity

$$0 = x\cdot\nabla\widehat{p}(x) + \lambda_0 x\cdot\big(\widehat{\mathbf{w}}(x)\cdot\nabla\big)\widehat{\mathbf{w}}(x) = \operatorname{div}\big[x\,\widehat{p}(x) + \lambda_0\big(\widehat{\mathbf{w}}(x)\cdot x\big)\widehat{\mathbf{w}}(x)\big]$$

$$-\widehat{p}(x)\operatorname{div} x - \lambda_0|\widehat{\mathbf{w}}(x)|^2 = \operatorname{div}\big[x\,\widehat{p}(x) + \lambda_0\big(\widehat{\mathbf{w}}(x)\cdot x\big)\widehat{\mathbf{w}}(x)\big] - 2\widehat{\Phi}(x).$$

Integrating this identity we derive

$$2\int\limits_\Omega \widehat{\Phi}(x)\,dx = \int\limits_{\partial\Omega} \widehat{p}(x)\big(x\cdot\mathbf{n}\big)\,dS = \widehat{p}_1\int\limits_{\Gamma_1}\big(x\cdot\mathbf{n}\big)\,dS + \widehat{p}_2\int\limits_{\Gamma_2}\big(x\cdot\mathbf{n}\big)\,dS$$

$$= \widehat{p}_1\int\limits_{\Omega_1}\operatorname{div} x\,dx - \widehat{p}_2\int\limits_{\Omega_2}\operatorname{div} x\,dx = 2\big(\widehat{p}_1|\Omega_1| - \widehat{p}_2|\Omega_2|\big).$$

Hence,

$$\int\limits_\Omega \widehat{\Phi}(x)\,dx = \widehat{p}_1|\Omega_1| - \widehat{p}_2|\Omega_2| = \widehat{p}_1|\Omega| + \big(\widehat{p}_1 - \widehat{p}_2\big)|\Omega_2|. \tag{37}$$

Inequalities (36) and (37) yield

$$\widehat{p}_1 \le \widehat{p}_2.$$

This contradicts inequality (35). Thus, all solutions of integral identity (22) are uniformly bounded in $H(\Omega)$ and by the Leray–Schauder fixed–point theorem there exists at least one weak solution of problem (1). $\square$

Remark 2. Let $\Omega = \{x : 1 < |x| < 2\}$ be the annulus and let (r,θ) be polar coordinates in $\mathbb{R}^2$. If $f \in C_0^\infty(1,2)$, then the pair $\widehat{\mathbf{w}} = \big(\widehat{w}_r, \widehat{w}_\theta\big)$ and $\widehat{p}$ with

$$\widehat{w}_r(r,\theta) = 0, \quad \widehat{w}_\theta(r,\theta) = f(r), \quad \widehat{p}(r,\theta) = \lambda_0\int\limits_1^r \frac{f^2(t)}{t}\,dt \tag{38}$$

satisfy both equations (31) and the boundary condition $\widehat{\mathbf{w}}\big|_{\partial\Omega} = 0$ ($\widehat{w}_r$ and $\widehat{w}_\theta$ are components of the velocity field in polar coordinate system). On the other hand,

$$0 = \widehat{p}(x)\big|_{r=1} \neq \widehat{p}(x)\big|_{r=2} = \lambda_0 \int\limits_1^2 \frac{f^2(t)}{t}\,dt > 0.$$

This simple example, due to Ch.J. Amick [1] (see also [10], v. II, p. 59), shows that, in general, the pressure $\widehat{p}$ corresponding to the solution of Euler equations (31) could have not equal constant values on different components of the boundary.

It is interesting to observe that for the solution like (38) necessarily holds $\widehat{p}_1 > \widehat{p}_2$. Indeed, writing the Euler equations (31) in polar coordinates and integrating over Ω yields

$$\lambda_0 \int\limits_\Omega \frac{\widehat{w}_\theta^2(r)}{r}\,drd\theta = \lambda_0 \int\limits_\Omega \frac{f^2(r)}{r}\,drd\theta = \int\limits_\Omega \frac{\partial p(r)}{\partial r}\,drd\theta = \widehat{p}_1 - \widehat{p}_2 > 0. \quad (39)$$

The solution (38) cannot be a limit of solutions to Navier–Stokes problem (in the sense described in the proof of Theorem 1). If it is so, then we conclude from (26), (32) and (39) that $\mathcal{F} > 0$. But this, as it is proved in Theorem 1, leads to a contradiction.

We emphasize that in the case when $\mathcal{F} < 0$ (inflow condition) problem (1) remains unsolved. However, in this case we do not know any counterexample showing that for the solution of Euler equations (31) the inequality $\widehat{p}_2 > \widehat{p}_1$ holds.

It is well known (see [3], [10]) that independently of the sign of $\mathcal{F}$ problem (1) has a solution, if $|\mathcal{F}|$ is sufficiently small. Using this result Theorem 1 can be strengthened as follows

Theorem 2. *Assume that* $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ *and let condition* (6) *be fulfilled. Then there exists* $\mathcal{F}_0 > 0$ *such that for any* $\mathcal{F} \in (-\mathcal{F}_0, +\infty)$ *problem* (1) *admits at least one weak solution.*

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