DECOMPOSITION RESULTS FOR GRAM MATRIX DETERMINANTS

TEODOR BANICA AND STEPHEN CURRAN

ABSTRACT. We study the Gram matrix determinants for the groups S_n, O_n, B_n, H_n, for their free versions $S_n^+, O_n^+, B_n^+, H_n^+$, and for the half-liberated versions O_n^*, H_n^*. We first collect all the known computations of such determinants, along with complete and simplified proofs, and with generalizations where needed. We conjecture that all these determinants decompose as $D = \prod_\pi \varphi(\pi)$, with product over all associated partitions.

INTRODUCTION

We discuss in this paper the computation of certain advanced representation theory invariants, for the main examples of "easy quantum groups". These are the groups S_n, O_n, B_n, H_n, their free versions $S_n^+, O_n^+, B_n^+, H_n^+$, and the half-liberated versions O_n^*, H_n^*. Here S_n, O_n are the permutation and orthogonal groups, B_n is the bistochastic group consisting of orthogonal matrices whose rows and columns sum to 1, and $H_n = \mathbb{Z}_2 \wr S_n$ is the hyperoctahedral group. For a global introduction to these groups and quantum groups, we refer to our previous papers [4], [5], [6].

According to a paper of Weingarten [35], further processed and generalized by Collins [12], then Collins and Śniady [14], a number of advanced representation theory invariants of the quantum group are encoded in a certain associated matrix G_{kn}, called Gram matrix. For instance the inverse of the Gram matrix G_{kn} is the Weingarten matrix W_{kn}, whose knowledge allows the full computation of the Haar functional. See [35], [12], [14].

Among these invariants, the central object is the Gram matrix determinant, $\det(G_{kn})$. For instance the roots of $\det(G_{kn})$ are the poles of the Weingarten function W_{kn}, and the knowledge of these numbers clarifies the invertibility assumptions in [4], [5], [6].

The quantity $\det(G_{kn})$ appears in fact naturally in relation with many other questions, and its exact computation a well-known problem. A first purpose of the present work is to collect all the available formulae from the literature, and to write them down by using our unified "easy quantum group" formalism, along with complete, simplified proofs.

The basic example of such a formula is that for S_n, H_n, H_n^*. Here the Gram matrix is upper triangular, up to a simple determinant-preserving operation. The determinant,

2000 *Mathematics Subject Classification.* 46L54 (15A52, 33C80).

Key words and phrases. Orthogonal polynomial, Gram determinant.

computed by Jackson [23] and Lindström [29], is as follows:

$$\det(G_{kn}) = \prod_{\pi\in\mathcal{P}(k)} \frac{n!}{(n-|\pi|)!}$$

Here $\mathcal{P}(k)$ is the set of partitions associated to the quantum group, namely all partitions for S_n, all partitions with even blocks for H_n, and all partitions with blocks having the same number of odd and even legs for H_n^*, and $|.|$ is the number of blocks.

In the general case the situation is much more complicated. However, one may still wonder for a general decomposition result, of the following type:

$$\det(G_{kn}) = \prod_{\pi\in\mathcal{P}(k)} \varphi(\pi)$$

This question is of course quite vague, depending on how explicit we would like our functions φ to be. For instance a natural requirement would be that in the case of a liberation $G_n \to G_n^+$, the functions φ are related by an induction/restriction procedure.

This kind of specialized question appears to be quite difficult. In this paper we will obtain some preliminary decomposition results of the above type, as follows:

(1) For O_n, B_n, O_n^* a formula comes from the work of Collins-Matsumoto [13] and Zinn-Justin [38]. The natural decomposition here is over Young diagrams, and in principle one can pass to partitions by applying a certain surjective map.
(2) For $O_n^+, B_n^+, S_n^+, H_n^+$ we use the work of Di Francesco, Golinelli and Guitter [17], [18], [19], [20], Tutte [33] and Dahab [16]. We will obtain some evidence towards the existence of contributions $\varphi(\pi)$, of "trigonometric" nature.

We will make as well a number of speculations in relation with quantum group/planar algebra methods, and with spectral measure/orthogonal polynomial interpretations.

As a conclusion, there is still a lot of work to be done, mostly towards the conceptual understanding, at the level of Gram determinants, of the operation $G_n \to G_n^+$.

The paper is organized as follows: 1-2 are preliminary sections, in 3-4 we discuss the classical and half-liberated cases, and in 5-6 we discuss the free case. The final sections, 7-9, contain a number of speculations on the formulae, and a few concluding remarks.

Acknowledgements. This work was started at the Bedlewo 2009 workshop "Noncommutative harmonic analysis", and we are highly grateful to M. Bożejko for the invitation, and for several stimulating discussions. S.C. would like to thank the Paul Sabatier University and the Cergy-Pontoise University, where another part of this work was done. The work of T.B. was supported by the ANR grants "Galoisint" and "Granma", and the work of S.C. was partially supported by an NSF Postdoctoral Fellowship.

1. Easy quantum groups

Let $\mathcal{P}_s$ be the category of all partitions. That is, $\mathcal{P}_s(k,l)$ is the set of partitions between an upper row of k points and a lower row of l points, and the categorical operations are the horizontal and vertical concatenation, and the upside-down turning.

A category of partitions $\mathcal{P} \subset \mathcal{P}_s$ is by definition a collection of sets $\mathcal{P}(k,l) \subset \mathcal{P}_s(k,l)$, which is stable under the categorical operations. We have the following examples.

Proposition 1.1. *The following are categories of partitions:*

(1) $\mathcal{P}_o/\mathcal{P}_o^+$*: all pairings/all noncrossing pairings.*
(2) $\mathcal{P}_o^*$*: pairings with each string having an odd leg and an even leg.*
(3) $\mathcal{P}_b/\mathcal{P}_b^+$*: singletons plus pairings/noncrossing pairings.*
(4) $\mathcal{P}_s/\mathcal{P}_s^+$*: all partitions/all noncrossing partitions.*
(5) $\mathcal{P}_h/\mathcal{P}_h^+$*: partitions/noncrossing partitions with blocks of even size.*
(6) $\mathcal{P}_h^*$*: partitions with blocks having the same number of odd and even legs.*

Proof. This is clear from definitions. Note that $\mathcal{P}_g^\times$ corresponds via Tannakian duality [36], [37] to the easy quantum group $G^\times = (G_n^\times)$, with the notations in [4], [7]. □

We use the notation $\mathcal{P}(k) = \mathcal{P}(0,k)$. We denote by $\vee$ and $\wedge$ the set-theoretic sup and inf of partitions, always taken with respect to $\mathcal{P}_s$, and by $|.|$ the number of blocks.

Definition 1.2. *Associated to any category of partitions $\mathcal{P}$ and to any numbers $k, n \geq 0$ are the following matrices, with entries indexed by $\pi, \sigma \in \mathcal{P}(k)$:*

(1) *Gram matrix:* $G_{kn}(\pi,\sigma) = n^{|\pi\vee\sigma|}$.
(2) *Weingarten matrix:* $W_{kn} = G_{kn}^{-1}$.

In order for G_{kn} to be invertible, n must be big enough, and $n \geq k$ is known to be sufficient. The precise bounds depend on the category of partitions, and can be deduced from the various explicit formulae of $\det(G_{kn})$, to be given later on in this paper.

The interest in the above matrices comes from the fact that in the case $\mathcal{P} = \mathcal{P}_g^\times$, they describe the integration over the corresponding easy quantum group $G_n^\times$.

Theorem 1.3. *We have the Weingarten formula*

$$\int_{G_n^\times} u_{i_1j_1} \ldots u_{i_kj_k}\, du = \sum_{\pi,\sigma\in\mathcal{P}_g^\times(k)} \delta_\pi(i)\delta_\sigma(j) W_{kn}(\pi,\sigma)$$

where the δ symbols are 0 or 1, depending on whether the indices fit or not.

Proof. This follows by using a classical argument from [35], [14]. See [4], [7]. □

The exact computation of the Weingarten matrix is a quite subtle problem. A precise result is available only in the finite group case, where the formula is given in terms of the Möbius function μ on $\mathcal{P}$ as follows.

Proposition 1.4. *For S_n, H_n the Weingarten function is given by*

$$W_{kn}(\pi,\sigma)=\sum_{\tau\leq\pi\wedge\sigma}\mu(\tau,\pi)\mu(\tau,\sigma)\frac{(n-|\tau|)!}{n!}$$

and satisfies $W_{kn}(\pi,\sigma)=n^{-|\pi\wedge\sigma|}(\mu(\pi\wedge\sigma,\pi)\mu(\pi\wedge\sigma,\sigma)+O(n^{-1}))$.

Proof. The first assertion follows from the Weingarten formula: in that formula the integrals on the left are known, and this allows the computation of the right term, via the Möbius inversion formula. The second assertion follows from the first one. □

In the general case we have the following result, which is useful for applications.

Proposition 1.5. *For $\pi\leq\sigma$ we have the estimate*

$$W_{kn}(\pi,\sigma)=n^{-|\pi|}(\mu(\pi,\sigma)+O(n^{-1}))$$

and for π,σ arbitrary we have $W_{kn}(\pi,\sigma)=O(n^{|\pi\vee\sigma|-|\pi|-|\sigma|})$.

Proof. Once again this follows by using a classical argument, see [4]. □

2. Gram determinants

In this paper, we will be mainly interested in the computation of $\det(G_{kn})$. Let us being with some simple observations, coming from definitions.

Proposition 2.1. *Let $D_k(n)=\det(G_{kn})$, viewed as element of $\mathbb{Z}[n]$.*

(1) *D_k is monic, of degree $s_k=\sum_{\pi\in\mathcal{P}(k)}|\pi|$.*

(2) *We have $n^{b_k}|D_k$, where $b_k=\#\mathcal{P}(k)$.*

Proof. (1) This follows from $|\pi\vee\sigma|\leq|\pi|$, with equality if and only if $\sigma\leq\pi$. Indeed, from the inequality we get $\deg(D_k)\leq s_k$. Now the coefficient of n^{s_k} is the signed number of permutations $f:\mathcal{P}(k)\to\mathcal{P}(k)$ satisfying $f(\pi)\leq\pi$ for any π, and since there is only one such permutation, namely the identity, we obtain that this coefficient is 1.

(2) This is clear from the definition of D_k, and from $|\pi\vee\sigma|\geq 1$. □

The above result raises the question of computing the numbers $b_k=\#\mathcal{P}(k)$ and $s_k=\sum_{\pi\in\mathcal{P}(k)}|\pi|$. It is convenient here to introduce as well the related numbers $m_k=s_k/b_k$ and $a_k=2s_k-kb_k=(2m_k-k)b_k$, which will appear several times in what follows.

Proposition 2.2. *The numbers b_k,s_k,m_k,a_k are as follows:*

(1) *O_n,O_n^*,O_n^+: $b_{2l}=(2l)!!,l!$, $\frac{1}{l+1}\binom{2l}{l}$, $s_{2l}=lb_{2l}$, $m_{2l}=l$, $a_{2l}=0$.*

(2) *S_n: $b_k=Bell$, $s_k=b_{k+1}-b_k$, $m_k=\frac{b_{k+1}}{b_k}-1$, $a_k=2b_{k+1}-(k+2)b_k$.*

(3) *S_n^+: $b_k=\frac{1}{k+1}\binom{2k}{k}$, $s_k=\frac{1}{2}\binom{2k}{k}$, $m_k=\frac{k+1}{2}$, $a_k=b_k$.*

(4) *H_n^+: $b_{2l}=\frac{1}{2l+1}\binom{3l}{l}$, $s_{2l}=\binom{3l-1}{l-1}$, $m_{2l}=\frac{2l+1}{3}$, $a_{2l}=-2\binom{3l-1}{l-2}$.*

Proof. All these results are well-known. □

For the remaining quantum groups, namely B_n, B_n^+, H_n, H_n^*, the numbers b_k, s_k, m_k, a_k are given by quite complicated formulae. The best approach to their computation is via the trace of the Gram matrix, and its analytic interpretations.

So, let us first reformulate Proposition 2.1, in the following way.

Proposition 2.3. *With $D_k(n) = \det(G_{kn})$ and $T_k(t) = Tr(G_{kt})$, we have:*

(1) *$D_k(n) = n^{s_k}(1 + O(n^{-1}))$ as $n \to \infty$, where $s_k = T_k'(1)$.*
(2) *$D_k(n) = O(n^{b_k})$ as $n \to 0$, where $b_k = T_k(1)$.*

Proof. This is indeed just a reformulation of Proposition 2.1, using a variable t around 1. Note that in (2) we regard the variable n as a formal parameter, going to 0. □

The trace can be understood in terms of the associated Stirling numbers.

Proposition 2.4. *We have the formula*

$$T_k(t) = \sum_{r=1}^{k} S_{kr} t^r$$

where $S_{kr} = \#\{\pi \in \mathcal{P}(k) : |\pi| = r\}$ are the Stirling numbers.

Proof. This is clear from definitions. □

Another interpretation of the trace, analytic this time, is as follows.

Proposition 2.5. *For any $t \in (0, 1]$ we have the formula*

$$T_k(t) = \lim_{n\to\infty} \int_{G_n^\times} \chi_t^k$$

where $\chi_t = \sum_{i=1}^{[tn]} u_{ii}$ are the truncated characters of the quantum group.

Proof. As explained in [7], [4], this follows from the Weingarten formula. □

In general, the Stirling numbers S_{kr} and the trace $T_k(t)$ are given by quite complicated formulae, unless we are in the situation of one of the quantum groups in Proposition 2.2. Here these invariants are well-known in the O, S cases, and for H^+ we have:

$$T_{2l}(t) = \sum_{r=1}^{l} \frac{1}{r} \binom{l-1}{r-1} \binom{2l}{r-1} t^r$$

See [1]. In general now, the conceptual result concerns the asymptotic measures of truncated characters, i.e. the probability measures μ_t satisfying $T_k(t) = \int x^k d\mu_t(x)$.

Theorem 2.6. *The asymptotic measures of truncated characters are as follows:*

(1) *S_n/S_n^+: Poisson/free Poisson.*
(2) *O_n/O_n^+: Gaussian/semicircular.*
(3) *H_n/H_n^+: Bessel/free Bessel.*

(4) B_n/B_n^+: *shifted Gaussian/shifted semicircular.*
(5) O_n^*/H_n^*: *symmetrized Rayleigh/squeezed ∞-Bessel.*

Proof. The one-parameter measures in the statement are best found via a direct computation, by using classical and free cumulants. See [7], [4], [5]. □

3. The basic formula

We discuss now the explicit computation of the Gram determinants. The basic formula here, coming from the work of Jackson [23] and Lindström [29], is as follows.

Theorem 3.1. *For S_n, H_n, H_n^* we have*

$$\det(G_{kn}) = \prod_{\pi\in\mathcal{P}(k)} \frac{n!}{(n-|\pi|)!}$$

where $|.|$ is the number of blocks.

Proof. We use the fact that the partitions have the property of forming semilattices under $\vee$. The proof uses the upper triangularization procedure in [29] together with the explicit knowledge of the Möbius function on $\mathcal{P}(k)$ as in [23]. Consider the following matrix, obtained by making determinant-preserving operations:

$$G'_{kn}(\pi,\sigma) = \sum_{\pi\leq\tau} \mu(\pi,\tau) n^{|\tau\vee\sigma|}$$

It follows from the Möbius inversion formula that we have:

$$G'_{kn}(\pi,\sigma) = \begin{cases} n(n-1)\ldots(n-|\sigma|+1) & \text{if } \pi\leq\sigma \\ 0 & \text{if not} \end{cases}$$

Thus the matrix is upper triangular, and by computing the product on the diagonal we obtain the formula in the statement. □

A first remarkable feature of the above result is that the determinant for S_n, H_n, H_n^* can be computed from the trace: indeed, the Gram trace gives the Stirling numbers, which in turn give the Gram determinant. However, the connecting formula is quite complicated, so let us just record here an improvement of the first estimate in Proposition 2.3.

Proposition 3.2. *With $D_k(n) = \det(G_{kn})$ and $T_k(t) = Tr(G_{kt})$ we have*

$$D_k(n) = n^{s_k}\left(1 - \frac{z_k}{2}\,n^{-1} + O(n^{-2})\right)$$

where $s_k = T_k'(1)$ and $z_k = T_k''(1)$.

Proof. In terms of Stirling numbers, the formula in Theorem 3.1 reads:

$$D_k(n) = \prod_{r=1}^{k}\left(\frac{n!}{(n-r)!}\right)^{S_{kr}}$$

We use now the following basic estimate:

$$\frac{n!}{(n-r)!} = n^r\prod_{s=1}^{r-1}\left(1-\frac{s}{n}\right) = n^r\left(1-\frac{r(r-1)}{2}\,n^{-1}+O(n^{-2})\right)$$

Together with $T_k(t) = \sum_{r=1}^{k} S_{kr}t^r$, this gives the result. □

The above discussion raises the general question on whether the Gram determinant can be computed or not from the Gram trace, or from the measures in Theorem 2.6.

Since the connecting formula for S_n, H_n, H_n^* is already quite complicated, let us formulate for the moment a more modest conjecture, as follows.

Conjecture 3.3. *For any easy quantum group we have a formula of type*

$$\det(G_{kn}) = \prod_{\pi\in\mathcal{P}(k)}\varphi(\pi)$$

with the "contributions" being given by an explicit function $\varphi : \mathcal{P}(k) \to \mathbb{Q}(n)$.

This statement is of course quite vague, depending of the meaning of the above word "explicit". As already mentioned, one would expect φ to come from the Gram trace, or from the Stirling numbers, or, even better, from the measures in Theorem 2.6. Such a decomposition could potentially clarify the behavior of the Gram determinants under the "liberation" procedure $G \to G^+$.

This kind of general question appears to be quite difficult. In what follows we will obtain some evidence towards such general decomposition results.

4. The orthogonal case

We discuss now the cases O, B, O^*. Here the combinatorics is that of the Young diagrams. We denote by $|.|$ the number of boxes, and we use quantity f^λ, which gives the number of standard Young tableaux of shape λ.

Theorem 4.1. *For O_n we have*

$$\det(G_{kn}) = \prod_{|\lambda|=k/2} f_n(\lambda)^{f^{2\lambda}}$$

where $f_n(\lambda) = \prod_{(i,j)\in\lambda}(n+2j-i-1)$.

Proof. This follows from the results of Collins and Matsumoto [13] and Zinn-Justin [38]. Indeed, it is known from [38] that the Gram matrix is diagonalizable, as follows:

$$G_{kn} = \sum_{|\lambda|=k/2} f_n(\lambda) P_{2\lambda}$$

Here $1 = \Sigma P_{2\lambda}$ is the standard partition of unity associated to the Young diagrams having $k/2$ boxes, and the coefficients $f_n(\lambda)$ are those in the statement. Now since we have $Tr(P_{2\lambda}) = f^{2\lambda}$, this gives the result. □

Theorem 4.2. *For B_n we have*

$$\det(G_{kn}) = n^{a_k} \prod_{|\lambda|\leq k/2} f_n(\lambda)^{\binom{k}{2|\lambda|} f^{2\lambda}}$$

where $a_k = \sum_{\pi\in\mathcal{P}(k)}(2|\pi| - k)$, and $f_n(\lambda) = \prod_{(i,j)\in\lambda}(n + 2j - i - 2)$.

Proof. We recall from [7] that we have an isomorphism $B_n \simeq O_{n-1}$, given by $u = v + 1$, where u, v are the fundamental representations of B_n, O_{n-1}. We get:

$$Fix(u^{\otimes k}) = Fix\left((v+1)^{\otimes k}\right) = Fix\left(\sum_{r=0}^{k} \binom{k}{r} v^{\otimes r}\right)$$

Now if we denote by $\det', f'$ the objects in Theorem 4.1, we obtain:

$$\det(G_{kn}) = n^{a_k} \prod_{r=1}^{k} \det{}'(G_{r,n-1})^{\binom{k}{r}} = n^{a_k} \prod_{r=1}^{k} \left(\prod_{|\lambda|=r/2} f'_{n-1}(\lambda)^{f^{2\lambda}} \right)^{\binom{k}{r}}$$

This gives the formula in the statement. □

Theorem 4.3. *For O_n^* we have*

$$\det(G_{kn}) = \prod_{|\lambda|=k/2} f_n(\lambda)^{f^{\lambda^2}}$$

where $f_n(\lambda) = \prod_{(i,j)\in\lambda}(n + j - i)$.

Proof. We use the isomorphism of projective versions $PO_n^* = PU_n$, established in [8]. This isomorphism shows that the Gram matrices for O_n^* are the same as those for U_n. But for U_n it is known from [38] that the Gram matrix is diagonalizable, as follows:

$$G_{kn} = \sum_{|\lambda|=k/2} f_n(\lambda) P_\lambda$$

Here $1 = \Sigma P_\lambda$ is the standard partition of unity associated to the Young diagrams having $k/2$ boxes, and the coefficients $f_n(\lambda)$ are those in the statement. Now since we have $Tr(P_\lambda) = f^{\lambda^2}$, this gives the result. □

Observe that the above results provide a kind of answer to Conjecture 3.3, but with the Young diagrams contributing to the determinant, instead of the partitions. The remaining problems are to find the relevant surjective map from diagrams to partitions, and to see if the above formulae further simplify by using this surjective map.

5. Meander determinants

In this section we discuss the computation of the Gram matrix determinant, in the free cases $O_n^+, B_n^+, S_n^+, H_n^+$. Let P_r be the Chebycheff polynomials, given by $P_0 = 1, P_1 = n$ and $P_{r+1} = nP_r - P_{r-1}$. Consider also the following numbers, depending on $k, r \in \mathbb{Z}$:

$$f_{kr} = \binom{2k}{k-r} - \binom{2k}{k-r-1}$$

We set $f_{kr} = 0$ for $k \notin \mathbb{Z}$. The following key result was proved in [19].

Theorem 5.1. *For O_n^+ we have*

$$\det(G_{kn}) = \prod_{r=1}^{[k/2]} P_r(n)^{d_{k/2,r}}$$

where $d_{kr} = f_{kr} - f_{k+1,r}$.

Proof. As already mentioned, the result is from [19]. We present below a short proof. The result holds when k is odd, all the exponents being 0, so we assume that k is even.

Step 1. We use a general formula of type $G_{kn}(\pi, \sigma) =< f_\pi, f_\sigma >$.

Let Γ be a locally finite bipartite graph, with distinguished vertex 0 and adjacency matrix A, and let μ be an eigenvector of A, with eigenvalue n. Let L_k be the set of length k loops $l = l_1 \ldots l_k$ based at 0, and $H_k = span(L_k)$. For $\pi \in \mathcal{P}_{o^+}(k)$ define $f_\pi \in H_k$ by:

$$f_\pi = \sum_{l \in L_k} \left(\prod_{i \sim_\pi j} \delta(l_i, l_j^o) \gamma(l_i) \right) l$$

Here $e \to e^o$ is the edge reversing, and the "spin factor" is $\gamma = \sqrt{\mu(t)/\mu(s)}$, where s, t are the source and target of the edges. The point is that we have $G_{kn}(\pi, \sigma) =< f_\pi, f_\sigma >$. We refer to [30], [24], [21] for full details regarding this formula.

Step 2. With a suitable choice of (Γ, μ), we obtain a fomula of type $G_{kn} = T_{kn} T_{kn}^t$.

Indeed, let us choose $\Gamma = \mathbb{N}$ to be the Cayley graph of O_n^+, and the eigenvector entries $\mu(r)$ to be the Chebycheff polynomials $P_r(n)$, i.e. the orthogonal polynomials for O_n^+.

In this case, we have a bijection $\mathcal{P}_{o^+}(k) \to L_k$, constructed as follows. For $\pi \in \mathcal{P}_{o^+}(k)$ and $0 \leq i \leq k$ we define $h_\pi(i)$ to be the number of $1 \leq j \leq i$ which are joined by π to a number strictly larger than i. We then define a loop $l(\pi) = l(\pi)_1 \ldots l(\pi)_k$, where $l(\pi)_i$ is the edge from $h_\pi(i-1)$ to $h_\pi(i)$. Consider now the following matrix:

$$T_{kn}(\pi, \sigma) = \prod_{i \sim_\pi j} \delta(l(\sigma)_i, l(\sigma)_j^o) \gamma(l(\sigma)_i)$$

We have $f_\pi = \sum_\sigma T_{kn}(\pi,\sigma)\cdot l(\sigma)$, so we obtain as desired $G_{kn} = T_{kn}T_{kn}^t$.

Step 3. We show that, with suitable conventions, T_{kn} is lower triangular.

Indeed, consider the partial order on $\mathcal{P}_{o^+}(k)$ given by $\pi \leq \sigma$ if $h_\pi(i) \leq h_\sigma(i)$ for $i = 1,\ldots,k$. Our claim is that $\sigma \not\leq \pi$ implies $T_{kn}(\pi,\sigma) = 0$.

Indeed, suppose that $\sigma \not\leq \pi$, and let j be the least number with $h_\sigma(j) > h_\pi(j)$. Note that we must have $h_\sigma(j-1) = h_\pi(j-1)$ and $h_\sigma(j) = h_\pi(j) + 2$. It follows that we have $i \sim_\pi j$ for some $i < j$. From the definitions of T_{kn} and $l(\sigma)$, if $T_{kn}(\pi,\sigma) \neq 0$ then we must have $h_\sigma(i-1) = h_\sigma(j) = h_\pi(j) + 2$. But we also have $h_\pi(i-1) = h_\pi(j)$, so that $h_\sigma(i-1) = h_\pi(i-1) + 2$, which contradicts the minimality of j.

Step 4. End of the proof, by computing the determinant of T_{kn}.

Since T_{kn} is lower triangular we have:

$$\det(T_{kn}) = \prod_\pi T_{kn}(\pi,\pi) = \prod_\pi \prod_{i\sim_\pi j} \sqrt{\frac{P_{h_{\pi(i)}}}{P_{h_{\pi(i)-1}}}} = \prod_{r=1}^{k/2} P_r^{e_{kr}/2}$$

Here the exponents appearing on the right are by definition as follows:

$$e_{kr} = \sum_\pi \sum_{i\sim_\pi j} \delta_{h_\pi(i),r} - \delta_{h_\pi(i),r+1}$$

Our claim now, which finishes the proof, is that for $1 \leq r \leq k/2$ we have:

$$\sum_\pi \sum_{i\sim_\pi j} \delta_{h_\pi(i)r} = f_{k/2,r}$$

Indeed, note that the left term counts the number of times that the edge $(r, r+1)$ appears in all loops in L_k. Define a shift operator S on the edges of Γ by $S(s,t) = (s+1,t+1)$. Given a loop $l = l_1\ldots l_k$ and $1 \leq s \leq k$ with $l_s = (r, r+1)$, define a path $S^r(l_s)\ldots S^r(l_k)l_{s-1}^o\ldots l_1^o$. Observe that this is a path in Γ from $2r$ to 0 whose first edge is $(2r, 2r+1)$ and first reaches $r-1$ after $k-s+1$ steps.

Conversely, given a path $f_1\ldots f_k$ in Γ from $2r$ to 0 whose first edge is $(2r, 2r+1)$ and first reaches $r-1$ after s steps, define a loop $f_k^o\ldots f_s^o S^{-r}(f_1)\ldots S^{-r}(f_{s-1})$. Observe that this is a loop in Γ based at 0 whose $k-s+1$ edge is $(r, r+1)$.

These two operations are inverse to each other, so we have established a bijection between k-loops in Γ based at 0 whose s-th edge is $(r, r+1)$ and k-paths in Γ from $2r$ to 0 whose first edge is $(2r, 2r+1)$ and which first reach $r-1$ after $k-s+1$ steps.

It follows that the left hand side is equal to the number of paths in $\Gamma = \mathbb{N}$ from $2r$ to 0 whose first edge is $(2r, 2r+1)$. By the usual reflection trick, this is the difference of binomials defining $f_{k/2,r}$, and we are done. □

We use the notation $a_k = \sum_{\pi\in\mathcal{P}(k)}(2|\pi| - k)$, which already appeared in section 2.

Theorem 5.2. *For B_n^+ we have:*

$$\det(G_{kn}) = n^{a_k}\prod_{r=1}^{[k/2]} P_r(n-1)^{\sum_{l=1}^{[k/2]}\binom{k}{2l}d_{lr}}$$

Proof. We have $B_n^+ \simeq O_{n-1}^+$, see e.g. [32], so we can use the same method as in the proof of Theorem 4.2. By using prime exponents for the various O_n^+-related objects, we get:

$$\det(G_{kn}) = n^{a_k}\prod_{l=1}^{[k/2]}\det{}'(G_{2l,n-1})^{\binom{k}{2l}} = n^{a_k}\prod_{l=1}^{[k/2]}\left(\prod_{r=1}^{l}P_r(n-1)^{d_{lr}}\right)^{\binom{k}{2l}}$$

Together with Theorem 5.1, this gives the formula in the statement. □

Theorem 5.3. *For S_n^+ we have:*

$$\det(G_{kn}) = (\sqrt{n})^{a_k}\prod_{r=1}^{k} P_r(\sqrt{n})^{d_{kr}}$$

Proof. Let $\pi \to \widetilde{\pi}$ be the "cabling" operation, obtained by collapsing neighbors. According to the results of Kodiyalam-Sunder [27] and Chen-Przytycki [11], we have:

$$|\pi \vee \sigma| = k/2 + 2|\widetilde{\pi} \vee \widetilde{\sigma}| - |\widetilde{\pi}| - |\widetilde{\sigma}|$$

In terms of Gram matrices we get $G_{kn} = D_{kn}G'_{2k,\sqrt{n}}D_{kn}$, where $D_{kn} = diag(n^{|\widetilde{\pi}|/2-k/4})$, and where G' is the Gram matrix for O_n^+, so the result follows from Theorem 5.1. □

Theorem 5.4. *For H_n^+ we have the formula*

$$\det(G_{kn}) = (\sqrt{n})^{a_k}\prod_{r=1}^{[k/2]} P_r(\sqrt{n})^{2d'_{k/2,r}}$$

with $d'_{sr} = f'_{sr} - f'_{s,r+1}$, where $f'_{sr} = \binom{3s}{s-r} - \binom{3s}{s-r-1}$ for $s \in \mathbb{Z}$, $f'_{sr} = 0$ for $s \notin \mathbb{Z}$.

Proof. According to [2], the diagrams for H_n^+ are the "cablings" of the Fuss-Catalan diagrams [10], so we can use the same method as in the previous proof. So, by using the above formula from [27], [11], we have $G_{kn} = D_{kn}G'_{2k,\sqrt{n}}D_{kn}$, where $D_{kn} = diag(n^{|\widetilde{\pi}|/2-k/4})$, and where G' is the Gram determinant for the Fuss-Catalan algebra. But this latter determinant was computed by Di Francesco in [18], and this gives the result. □

6. Algebraic manipulations

In this section we perform some algebraic manipulations on the formulae found in the previous sections. Consider the quantity $a_k = \sum_{\pi\in\mathcal{P}(k)}(2|\pi|-k)$, which already appeared, several times. Then n^{a_k} is a true "contribution", in the sense of Conjecture 3.3.

We will prove here that a n^{a_k} factor appears naturally, in all the 10 formulae of Gram determinants. This can be regarded as a piece of evidence towards Conjecture 3.3.

In the classical and half-liberated cases there is no need for supplementary work in order to isolate this n^{a_k} factor, and the unified result is as follows.

Theorem 6.1. *In the classical and half-liberated cases, we have*

$$\begin{aligned}
S_n, H_n, H_n^*: \quad \det(G_{kn}) &= n^{a_k}\prod_{\pi\in\mathcal{P}(k)}\frac{n^{k-2|\pi|}n!}{(n-|\pi|)!}\\
O_n: \quad \det(G_{kn}) &= n^{a_k}\prod_{|\lambda|=k/2} f_n(\lambda)^{f^{2\lambda}}\\
B_n: \quad \det(G_{kn}) &= n^{a_k}\prod_{|\lambda|\leq k/2} f_n'(\lambda)^{\binom{k}{2|\lambda|}f^{2\lambda}}\\
O_n^*: \quad \det(G_{kn}) &= n^{a_k}\prod_{|\lambda|=k/2} f_n''(\lambda)^{f^{\lambda^2}}
\end{aligned}$$

where $f_n^\circ(\lambda)=\prod_{(i,j)\in\lambda}(n+j-i+\varphi^\circ)$, *with* $\varphi=j-1,\varphi'=j-2,\varphi''=0$.

Proof. This is a reformulation of the results in section 4, by using $a_k=0$ for O_n,O_n^*. □

In order to process the formulae in section 5, we need the following technical result.

Proposition 6.2. *The Chebycheff polynomials* P_r *have the following properties:*

(1) $P_r(n-1)=Q_r(n)$, *with* $Q_0=1,Q_1=n-1$ *and* $Q_{r+1}=(n-1)Q_r-Q_{r-1}$.
(2) $P_{2l}(n)=R_{2l}(n^2)$, *with* $R_0=1,R_2=n-1$ *and* $R_{2l+2}=(n-2)R_{2l}-R_{2l-2}$.
(3) $P_{2l+1}(n)=nR_{2l+1}(n^2)$, *with* $R_1=1,R_3=n-2$ *and* $R_{2l+3}=(n-2)R_{2l+1}-R_{2l-1}$.
(4) $P_{2l}(n)=n^{-l}S_{2l}(n^2)^{1/2}$, *with* $S_0=1,S_2=n(n-1)^2$ *and so on.*
(5) $P_{2l+1}(n)=n^{-l}S_{2l+1}(n^2)^{1/2}$, *with* $S_1=n,S_3=n^2(n-2)^2$ *and so on.*

Proof. This is routine. As pointed out in section 7 below, Q_r are the orthogonal polynomials for B_n^+, and R_{2l} are the orthogonal polynomials for S_n^+. As for the polynomials S_r, these are some technical objects, introduced in relation with the H_n^+ computation. □

Theorem 6.3. *In the free cases, we have*

$$O_n^+: \quad \det(G_{kn}) = n^{a_k}\prod_{r=1}^{[k/2]} P_r(n)^{d^1_{k/2,r}}$$

$$B_n^+: \quad \det(G_{kn}) = n^{a_k}\prod_{r=1}^{[k/2]} Q_r(n)^{\sum_{l=1}^{[k/2]}\binom{k}{2l}d^1_{lr}}$$

$$S_n^+: \quad \det(G_{kn}) = n^{a_k}\prod_{r=1}^{k} R_r(n)^{d^1_{kr}}$$

$$H_n^+: \quad \det(G_{kn}) = n^{a_k}\prod_{r=1}^{[k/2]} S_r(n)^{d^2_{k/2,r}}$$

where $d^i_{kr} = f^i_{kr} - f^i_{k,r+1}$, *with* $f^i_{kr} = \binom{(i+1)k}{k-r} - \binom{(i+1)k}{k-r-1}$ *for* $k \in \mathbb{Z}$, *and* $f^i_{kr} = 0$ *for* $k \notin \mathbb{Z}$.

Proof. The O_n^+ formula is the one in Theorem 5.1, with a $n^{a_k} = 1$ factor inserted.

The B_n^+ formula is the one in Theorem 5.2, with $P_r(n-1)$ replaced by $Q_r(n)$.

For the S_n^+ formula, we use Theorem 5.3. By replacing the Chebycheff polynomials P_{2l}, P_{2l+1} by the polynomials R_{2l}, R_{2l+1} from Proposition 6.2, we get:

$$\det(G_{kn}) = (\sqrt{n})^{a_k}\prod_{r=1}^{k} P_r(\sqrt{n})^{d^1_{kr}} = (\sqrt{n})^{a_k}\sqrt{n}^{\sum_{l=1}^{[(k+1)/2]} d^1_{k,2l-1}}\prod_{r=1}^{k} R_r(n)^{d_{kr}}$$

Now recall from Proposition 2.2 that $a_k = \frac{1}{k+1}\binom{2k}{k}$. On the other hand a direct computation gives $\sum_{l=1}^{[(k+1)/2]} d^1_{k,2l-1} = \frac{1}{k+1}\binom{2k}{k}$, so we get the formula in the statement.

For the H_n^+ formula we use a similar method. With $k = 2l$, Theorem 5.4 gives:

$$\det(G_{2l,n}) = (\sqrt{n})^{a_{2l}}\prod_{r=1}^{l} P_r(\sqrt{n})^{2d^2_{lr}} = (\sqrt{n})^{a_{2l}}(\sqrt{n})^{-2\sum_{s=2}^{l}[s/2]d^2_{ls}}\prod_{r=1}^{l} S_r(n)^{d^2_{lr}}$$

Now recall from Proposition 2.2 that $a_{2l} = -2\binom{3l-1}{l-2}$. On the other hand a direct computation gives $\sum_{s=2}^{l}[s/2]d^2_{ls} = \binom{3l-1}{l-2}$, so we get the formula in the statement. □

As a conclusion, the formulae in Theorem 6.1 and Theorem 6.3 are an intermediate step towards a general decomposition result of type $\det(G_{kn}) = \prod_{\pi\in\mathcal{P}(k)}\varphi(\pi)$. We will come back to the question of finding such a general decomposition result in section 8 below.

7. Orthogonal polynomials

We present here a speculation in the free case, in relation with orthogonal polynomials. As we will see, this speculation works for S_n^+, O_n^+, B_n^+, but doesn't work for H_n^+.

Definition 7.1. *The orthogonal polynomials for a real probability measure μ are the polynomials $Q_0, Q_1, Q_2, \ldots$ satisfying the following conditions:*

(1) $Q_k(n) = n^k + a_1 n^{k-1} + \ldots + a_{k-1} n + a_k$, *with* $a_i \in \mathbb{R}$.
(2) *For any* $k \neq l$ *we have* $\int Q_k(n) Q_l(n)\, d\mu(n) = 0$.

The orthogonal polynomials can be constructed by using a recursive formula, of type $Q_{k+1} = (n - \alpha_k)Q_k - \beta_k Q_{k-1}$. Here the parameters $\alpha_k, \beta_k \in \mathbb{R}$ are uniquely determined by the linear equations coming from the fact that Q_{k+1} must be orthogonal to n^{k-1}, n^k.

More precisely, by solving these two equations we obtain the following formulae, where the integral sign denotes the integration with respect to μ:

$$\alpha_k = \frac{\int n^{k+1} Q_k}{\int n^k Q_k} - \frac{\int n^k Q_{k-1}}{\int n^{k-1} Q_{k-1}}, \quad \beta_k = \frac{\int n^k Q_k}{\int n^{k-1} Q_{k-1}}$$

The numbers α_k, β_k are called Jacobi parameters of the sequence $\{Q_k\}$. Since $Q_0 = 1$, in order to describe $\{Q_k\}$ we just need to specify Q_1, and the Jacobi parameters.

The orthogonal polynomials for an easy quantum group are by definition those for the asymptotic measure of the main character, given in Theorem 2.6.

Proposition 7.2. *The basic orthogonal polynomials are as follows:*

(1) O_n: *here* $Q_1 = n$ *and* $Q_{k+1} = nQ_k - kQ_{k-1}$.
(2) B_n: *here* $Q_1 = n - 1$ *and* $Q_{k+1} = (n-1)P_k - kQ_{k-1}$.
(3) O_n^*: *here* $Q_1 = n$ *and* $Q_{k+1} = nQ_k - [(k+1)/2]Q_{k-1}$.
(4) S_n: *here* $Q_1 = n - 1$ *and* $Q_{k+1} = (n-k-1)Q_k - kQ_{k-1}$.
(5) O_n^+: *here* $Q_1 = n$ *and* $Q_{k+1} = nQ_k - Q_{k-1}$.
(6) B_n^+: *here* $Q_1 = n - 1$ *and* $Q_{k+1} = (n-1)Q_k - Q_{k-1}$.
(7) S_n^+: *here* $Q_1 = n - 1$ *and* $Q_{k+1} = (n-2)Q_k - Q_{k-1}$.

Proof. This result is well-known, and easy to deduce from definitions. Note that all the polynomials in the above statement are versions of the polynomials appearing in (1,3,4,5), which are respectively the Hermite, Charlier and Chebycheff polynomials. □

Let us go back now to the considerations in section 6. The polynomials R_{2l} appearing in Proposition 6.2 are the orthogonal polynomials for S_n^+, and it is natural to call $\{R_n | n \in \mathbb{N}\}$ the family of "extended orthogonal polynomials" for S_n^+.

Theorem 7.3. *In the O_n^+, B_n^+, S_n^+ cases we have a formula of type*

$$\det(G_{kn}) = n^{a_k} \prod_{r=1}^{k} Q_r(n)^{d_{kr}}$$

with $d_{kr} \in \mathbb{N}$, where $Q_r(n)$ are the corresponding extended orthogonal polynomials.

Proof. This follows from Theorem 6.3 and Proposition 7.2. □

Regarding now H_n^+, the combinatorics here is that of the Fuss-Catalan algebra [10], see [1], [2]. Since μ is symmetric, the orthogonal polynomials are given by $Q_1 = n$ and $Q_{k+1} = nQ_k - \beta_k Q_{k-1}$, where $\beta_k = \gamma_k/\gamma_{k-1}$, with $\gamma_k = \int n^k P_k$. The data is as follows:

	1	2	3	4	5	6	7	8
c_k	1	3	12	55	273	1428	7752	43263
γ_k	1	2	3	11/2	26/3	170/11	17.19/13	19.23/10
β_k	1	2	3/2	11/6	52/33	15.17/11.13	11.19/130	13.23/170

This suggests the following general formula:

$$\beta_k = \begin{cases} \dfrac{3(3k-1)(3k+2)}{4(2k-1)(2k+1)} & (k \text{ even}) \\ \\ \dfrac{3(3k-2)(3k+1)}{4(2k-1)(2k+1)} & (k \text{ odd}) \end{cases}$$

The problem can be probably investigated by using techniques from [22], [28], [31]. Our main problem is of course: what is the analogue of Theorem 7.3 for H_n^+?

Let us also mention that the computation of the orthogonal polynomials for H_n, H_n^* looks like a quite difficult problem. Probably the good framework here is that of the quantum groups $H_n^{(s)}$ from [4], because at $s = 2, \infty$ we have H_n, H_n^*.

We have as well the following question: is there a quantum group/planar algebra proof of Theorem 7.3, in the cases B_n^+, S_n^+? For O_n^+ this was done in Theorem 5.1.

8. More manipulations

We have seen in the previous section that the quantum group H_n^+ is somehow of a more complicated nature than the other quantum groups under consideration.

In this section we restrict attention to O_n^+, B_n^+, S_n^+, and we further rearrange the formulae in Theorem 6.3. The idea comes from the formula of O_n^+. Indeed, the numbers f_{kr} for O_n^+ count the $\mathcal{P}_{o^+}$ diagrams with $2r$ upper points and $2k$ lower points, with the property that each upper point is paired with a lower point. This kind of diagrams, called "epi" in the paper of Jones, Shlyakhtenko and Walker [26], have the following generalization.

Definition 8.1. *Let $\mathcal{P}$ be a category of partitions, and let $0 \leq r \leq k$.*

1. *We let $\mathcal{P}^r(k)$ be the set of partitions $\sigma \in \mathcal{P}(r, k)$, with $0 \leq r \leq k$, such that each upper point is connected to lower points only, and to at least one of them.*
2. *The elements of $\mathcal{P}^r(k)$ are called "epi". We let $\mathcal{P}^+(k) = \cup_{r=0}^k \mathcal{P}^r(k)$. For an epi $\sigma \in \mathcal{P}^r(k)$, we denote by $r(\sigma) = r$ the number of its upper legs.*

With these notations, we can now state and prove our main result. This is a global formula for the Gram determinants associated to the quantum groups O_n^+, B_n^+, S_n^+.

Theorem 8.2. *For O_n^+, B_n^+, S_n^+ we have the formula*

$$\det(G_{kn}) = n^{a_k} \prod_{\sigma\in\mathcal{P}^+(k)} \frac{F_{r(\sigma)}}{F_{r(\sigma)-1}}$$

where $F_r = P_{r/2}, Q_{r/2}, R_r$ are the corresponding extended orthogonal polynomials.

Proof. Observe first that the F_r quantities in the statement make indeed sense. This is because the epi for O_n^+, B_n^+ must have an even number of upper legs.

(1) For O_n^+ we have $f_{sr}^1 = \#\mathcal{P}^{2r}(2s)$, so the formula in Theorem 6.3 becomes:

$$\det(G_{kn}) = n^{a_k} \prod_{r=0}^{[k/2]} P_r(n)^{f_{k/2,r}-f_{k/2,r+1}} = n^{a_k} \prod_{r=0}^{[k/2]} P_r(n)^{\#\mathcal{P}^{2r}(k)-\#\mathcal{P}^{2r+2}(k)}$$

Now since we have $\mathcal{P}^+(k) = \cup_{r=0}^{[k/2]}\mathcal{P}^{2r}(k)$, we obtain the formula in the statement:

$$\det(G_{kn}) = n^{a_k} \prod_{r=0}^{[k/2]} \left(\prod_{\sigma\in\mathcal{P}^{2r}(k)} \frac{Q_r(n)}{P_{r-1}(n)} \right) = n^{a_k} \prod_{\sigma\in\mathcal{P}^+(k)} \frac{P_{r(\sigma)/2}(n)}{P_{r(\sigma)/2-1}(n)}$$

(2) For B_n^+ the epi have, according to our definitions, singletons only in the lower row. Thus these epi can be counted as function of those for O_n^+, and we get:

$$\det(G_{kn}) = n^{a_k} \prod_{r=0}^{[k/2]} Q_r(n)^{\sum_{l=1}^{[k/2]}\binom{k}{2l}d_{lr}^1} = n^{a_k} \prod_{r=0}^{[k/2]} Q_r(n)^{\#\mathcal{P}^{2r}(k)-\#\mathcal{P}^{2r+2}(k)}$$

A similar manipulation as in (1) gives now the formula in the statement.

(3) For S_n^+ the epi are in standard bijection (via fatenning/collapsing of neighbors) with the epi for O_n^+. Thus the formula in Theorem 6.3 becomes:

$$\det(G_{kn}) = n^{a_k} \prod_{r=0}^{k} R_r(n)^{d_{kr}^1} = n^{a_k} \prod_{r=0}^{k} R_r(n)^{\#\mathcal{P}^{r}(k)-\#\mathcal{P}^{2r+1}(k)}$$

Once again, a similar manipulation as in (1) gives the formula in the statement. □

Observe that the quantum group H_n^+ cannot be included into the above general theorem, and this for 2 reasons: first, because the orthogonal polynomial interpretation of the polynomials appearing in Theorem 6.3. fails, cf. the previous section, and second, because the epi interpretation of the exponents appearing in Theorem 6.3 seems to fail as well.

9. Concluding remarks

We have seen in this paper that the Gram matrix determinants have a natural interpretation in the easy quantum group framework, developed in [7], [4], [5], [6]. The known computations, that we partly extended, simplified, or rearranged in this paper, provide a complete set of formulae for the main examples of easy quantum groups.

Our conjecture is that these Gram determinants should have general decompositions of type $\det(G_{kn}) = \prod_{\pi\in\mathcal{P}(k)} \varphi(\pi)$. More precisely, the situation here is as follows:

(1) For S_n, H_n, H_n^* the conjecture holds, with $\varphi(\pi) = n!/(n-|\pi|)!$.
(2) For O_n, B_n, O_n^* we have a decomposition result, but over Young diagrams.
(3) For O_n^+, B_n^+, S_n^+ we have a decomposition result, but over the associated epi.

The remaining problem is to find the correct surjective maps for (2,3), i.e. the correct surjections from diagrams/epi to partitions. Of course, this question is not very clearly formulated. The main problem is probably to understand the behavior of the Gram matrix determinants in relation with the liberation operation $G_n \to G_n^+$. Indeed, we expect in this situation the contributions φ to be related by a kind of induction/restriction procedure.

In addition to the concrete computations performed in this paper, let us mention that there are as well some quite heavy, abstract methods, that we haven't really tried yet. First, the inclusion $G_n \subset G_n^+$ gives rise to a planar algebra module in the sense of Jones [25], and our above "liberation conjecture" can be understood as saying that the Gram matrix combinatorics behaves well with respect to this planar module structure. And second, modulo the orthogonal polynomial issues discussed in the previous section, some useful tools should come from the analytic theory of the Bercovici-Pata bijection [9].

References

[1] T. Banica, S.T. Belinschi, M. Capitaine and B. Collins, Free Bessel laws, *Canad. J. Math.*, to appear.
[2] T. Banica, J. Bichon and B. Collins, The hyperoctahedral quantum group, *J. Ramanujan Math. Soc.* **22** (2007), 345–384.
[3] T. Banica and B. Collins, Integration over compact quantum groups, *Publ. Res. Inst. Math. Sci.* **43** (2007), 277–302.
[4] T. Banica, S. Curran and R. Speicher, Classification results for easy quantum groups, *Pacific. J. Math.* **247** (2010), 1–26.
[5] T. Banica, S. Curran and R. Speicher, Stochastic aspects of easy quantum groups, *Probab. Theory Related Fields*, to appear.
[6] T. Banica, S. Curran and R. Speicher, De Finetti theorems for easy quantum groups, `arxiv: 0907.3314`.
[7] T. Banica and R. Speicher, Liberation of orthogonal Lie groups, *Adv. Math.* **222** (2009), 1461–1501.
[8] T. Banica and R. Vergnioux, Invariants of the half-liberated orthogonal group, *Ann. Inst. Fourier*, to appear.
[9] H. Bercovici and V. Pata, Stable laws and domains of attraction in free probability theory, *Ann. of Math.* **149** (1999), 1023–1060.
[10] D. Bisch and V.F.R. Jones, Algebras associated to intermediate subfactors, *Invent. Math.* **128** (1997), 89–157.
[11] Q. Chen and J. Przytycki, The Gram matrix of a Temperley-Lieb algebra is similar to the matrix of chromatic joins, *Commun. Contemp. Math.* **10** (2008), 849–855.
[12] B. Collins, Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability, *Int. Math. Res. Not.* **17** (2003), 953–982.
[13] B. Collins and S. Matsumoto, On some properties of orthogonal Weingarten functions, *J. Math. Phys.* **50** (2009), 1–18.

[14] B. Collins and P. Śniady, Integration with respect to the Haar measure on unitary, orthogonal and symplectic groups, *Comm. Math. Phys.* **264** (2006), 773–795.
[15] A. Copeland, F. Schmidt and R. Simion, Note on two determinants with interesting factorizations, *Discrete Math.* **256** (2002), 449–458.
[16] R. Dahab, The Birkhoff-Lewis equation, PhD dissertation, University of Waterloo, 1993.
[17] P. Di Francesco, Meander determinants, *Comm. Math. Phys.* **191** (1998), 543–583.
[18] P. Di Francesco, Folding and coloring problems in mathematics and physics, *Bull. Amer. Math. Soc.* **37** (2000), 251–307.
[19] P. Di Francesco, O. Golinelli and E. Guitter, Meanders and the Temperley-Lieb algebra, *Comm. Math. Phys.* **186** (1997), 1–59.
[20] P. Di Francesco, O. Golinelli and E. Guitter, Meanders: exact asymptotics, *Nucl. Phys. B* **570** (2000) 699–712.
[21] A. Guionnet, V.F.R. Jones and D. Shlyakhtenko, Random matrices, free probability, planar algebras and subfactors, *Proceedings NCG*, to appear.
[22] M. Hinz and W. Młotkowski, Multiplicative free square of the free Poisson measure and examples of free symmetrization, *Colloq. Math.* **119** (2010), 127–136.
[23] D. Jackson, The lattice of noncrossing partitions and the Birkhoff-Lewis equations, *European J. Combin.* **15** (1994), 245–250.
[24] V.F.R. Jones, The planar algebra of a bipartite graph, in "Knots in Hellas '98" (2000), 94–117.
[25] V.F.R. Jones, The annular structure of subfactors, *Monogr. Enseign. Math.* **38** (2001), 401–463.
[26] V.F.R. Jones, D. Shlyakhtenko and K. Walker, An orthogonal approach to the subfactor of a planar algebra, *Pacific J. Math.* **246** (2010), 187–197.
[27] V. Kodiyalam and V.S. Sunder, Temperley-Lieb and non-crossing partition planar algebras, *Contemp. Math.* **456** (2008), 61–72.
[28] F. Lehner, Cumulants, lattice paths, and orthogonal polynomials, *Discrete Math.* **270** (2003), 177–191.
[29] B. Lindstöm, Determinants on semilattices, *Proc. Amer. Math. Soc.* **20** (1969), 207–208.
[30] P. Martin, Potts models and related problems in statistical mechanics, World Scientific Publ. (1991).
[31] W. Młotkowski, Combinatorial relation between free cumulants and Jacobi parameters, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **12** (2009), 291–306.
[32] S. Raum, Isomorphisms and fusion rules of orthogonal free quantum groups and their complexifications, `arxiv:1006.2979`.
[33] W.T. Tutte, The matrix of chromatic joins, *J. Combin. Theory Ser. B* **57** (1993), 269–288.
[34] D.V. Voiculescu, K.J. Dykema and A. Nica, Free random variables, AMS (1992).
[35] D. Weingarten, Asymptotic behavior of group integrals in the limit of infinite rank, *J. Math. Phys.* **19** (1978), 999–1001.
[36] S.L. Woronowicz, Compact matrix pseudogroups, *Comm. Math. Phys.* **111** (1987), 613–665.
[37] S.L. Woronowicz, Tannaka-Krein duality for compact matrix pseudogroups. Twisted SU(N) groups, *Invent. Math.* **93** (1988), 35–76.
[38] P. Zinn-Justin, Jucys-Murphy elements and Weingarten matrices, *Lett. Math. Phys.* **91** (2010), 119–127.

T.B.: Department of Mathematics, Cergy-Pontoise University, 95000 Cergy-Pontoise, France. `teodor.banica@u-cergy.fr`

S.C.: Department of Mathematics, University of California, Los Angeles, CA 90095, USA. `curransr@math.ucla.edu`